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MORE MORPHISMS BETWEEN BUNDLE GERBES

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ABSTRACT. Usually bundle gerbes are considered as objects of a 2-groupoid, whose 1-morphisms, called stable isomorphisms, are all invertible. I introduce new 1-morphisms which include stable isomorphisms, trivializations and bundle gerbe modules. They fit into the structure of a 2-category of bundle gerbes, and lead to natural definitions of surface holonomy for closed surfaces, surfaces with boundary, and unoriented closed surfaces.

Introduction

From several perspectives it becomes clear that bundle gerbes are objects in a 2-category: from the bird's-eye view of algebraic geometry, where gerbes appear as some kind of stack, or in topology, where they appear as one possible categorification of a line bundle, but also from a worm's-eye view on the definitions of bundle gerbes and their morphisms, which show that there have to be morphisms between the morphisms.

In [Ste00] a 2-groupoid is defined, whose objects are bundle gerbes, and whose 1-morphisms are stable isomorphisms. To explain a few details, recall that bundle gerbes are defined using surjective submersions $\pi : Y \to M$, and that a stable isomorphism $\mathcal{A} : \mathcal{G}_1 \to \mathcal{G}_2$ between two bundle gerbes $\mathcal{G}_1$ and $\mathcal{G}_2$ with surjective submersions $\pi_1 : Y_1 \to M$ and $\pi_2 : Y \to M$ consists of a certain line bundle A over the fibre product $Y_1 \times_M Y_2$. 2-morphisms between stable isomorphisms are morphisms $\beta : A \to A'$ of those line bundles, obeying a compatibility constraint. Many examples of surjective submersions arise from open covers $\{U_\alpha\}_{\alpha\in A}$ of M by taking Y to be the disjoint union of the open sets U_α and π to be the projection $(x, \alpha) \mapsto x$. From this point of view, fibre products $Y_1 \times_M Y_2$ correspond the common refinement of two open covers. So, the line bundle A of a stable isomorphism lives over the common refinement of the open covers of the two bundle gerbes.

Difficulties with this definition of stable isomorphisms arise when two stable isomorphisms $\mathcal{A} : \mathcal{G}_1 \to \mathcal{G}_2$ and $\mathcal{A}' : \mathcal{G}_2 \to \mathcal{G}_3$ are going to be composed: one has to define a line bundle $\tilde{A}$ over $Y_1 \times_M Y_3$ using the line bundles A over $Y_1 \times_M Y_2$ and A' over $Y_2 \times_M Y_3$. In [Ste00] this problem is solved using descent theory for line bundles.

In this note, I present another definition of 1-morphisms between bundle gerbes (Definition 1.4). Compared to stable isomorphisms, their definition is relaxed in two aspects:

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1) the line bundle is replaced by a certain vector bundle of rank possibly higher than one.

2) this vector bundle is defined over a smooth manifold Z with surjective submersion $\zeta : Z \to Y_1 \times_M Y_2$. In terms of open covers, the vector bundle lives over a refinement of the common refinement of the open covers of the two bundle gerbes.

Stable isomorphisms appear as a particular case of this relaxed definition. I also give a generalized definition of 2-morphisms between such 1-morphisms (Definition 1.6). Two goals are achieved by this new type of morphisms between bundle gerbes. First, relaxation 1) produces many 1-morphisms which are not invertible, in contrast to the stable isomorphisms in [Ste00]. To be more precise, a 1-morphism is invertible if and only if its vector bundle has rank 1 (Proposition 1.13). The non-invertible 1-morphisms provide a new formulation of left and right bundle gerbe modules (Definition 3.6). Second, relaxation 1) erases the difficulties with the composition of 1-morphisms: the vector bundle $\tilde{A}$ of the composition $\mathcal{A}' \circ \mathcal{A}$ is just defined to be the tensor product of the vector bundles A and A' of the two 1-morphisms pulled back to $Z \times_{Y_2} Z'$, where A over Z is the vector bundle of $\mathcal{A}$ and A' over Z' is the vector bundle of $\mathcal{A}'$. The composition defined like that is strictly associative (Proposition 1.8). This way we end up with a strictly associative 2-category $\mathfrak{BGrb}(M)$ of bundle gerbes over M. The aim of this note is to show that a good understanding of this 2-category can be useful.

This note is organized as follows. Section 1 contains the definitions and properties of the 2-category $\mathfrak{BGrb}(M)$ of bundle gerbes over M. We also equip this 2-category with a monoidal structure, pullbacks and a duality. Section 2 relates our new definition of 1-morphisms between bundle gerbes to the one of a stable isomorphism: two bundle gerbes are isomorphic objects in $\mathfrak{BGrb}(M)$ if and only if they are stably isomorphic (Corollary 2.2). In section 3 we present a unified view on important structure related to bundle gerbes in terms of the new morphisms of the 2-category $\mathfrak{BGrb}(M)$:

a) a *trivialization* of a bundle gerbe $\mathcal{G}$ is a 1-isomorphism $\mathcal{A} : \mathcal{G} \to \mathcal{I}_\rho$ from $\mathcal{G}$ to a trivial bundle gerbe $\mathcal{I}_\rho$ given by a 2-form ρ on M.

b) a *bundle gerbe module* of a bundle gerbe $\mathcal{G}$ is a (not necessarily invertible) 1-morphism $\mathcal{E} : \mathcal{G} \to \mathcal{I}_\omega$ from $\mathcal{G}$ to a trivial bundle gerbe $\mathcal{I}_\omega$.

c) a *Jandl structure* on a bundle gerbe $\mathcal{G}$ over M is a triple $(k, \mathcal{A}, \varphi)$ of an involution k of M, a 1-isomorphism $\mathcal{A} : k^*\mathcal{G} \to \mathcal{G}^*$ and a certain 2-morphism $\varphi : k^*\mathcal{A} \Rightarrow \mathcal{A}^*$.

Then we demonstrate how this understanding in combination with the properties of the 2-category $\mathfrak{BGrb}(M)$ can be employed to give convenient definitions of surface holonomy. For this purpose we classify the morphisms between trivial bundle gerbes: there is an equivalence of categories

$$\mathfrak{Hom}(\mathcal{I}_{\rho_1}, \mathcal{I}_{\rho_2}) \cong \mathfrak{Bun}_{\rho_2 - \rho_1}(M)$$

between the morphism category between the trivial bundle gerbes $\mathcal{I}_{\rho_1}$ and $\mathcal{I}_{\rho_2}$ and the category of vector bundles over M for which the trace of the curvature gives the 2-form $\rho_2 - \rho_1$ times its rank.

The interpretation of bundle gerbe modules and Jandl structures in terms of morphisms between bundle gerbes is one step to understand the relation between two approaches to two-dimensional conformal field theories: on the one hand the Lagrangian approach with a metric and a bundle gerbe $\mathcal{G}$ being the relevant structure [GR02] and on the other hand the algebraic approach in which a special symmetric Frobenius algebra object A in a modular tensor category $\mathcal{C}$ plays this role [FRS02]. Similarly as bundle gerbes, special symmetric Frobenius algebra objects in $\mathcal{C}$ form a 2-category, called $\mathcal{F}rob_{\mathcal{C}}$. In both approaches it is well-known how boundary conditions have to be imposed. In the Lagrangian approach one chooses a D-brane: a submanifold Q of the target space together with a bundle gerbe module for the bundle gerbe $\mathcal{G}$ restricted to Q [Gaw05]. In the algebraic approach one chooses a 1-morphism from A to the tensor unit I of $\mathcal{C}$ (which is trivially a special symmetric Frobenius algebra object) in the 2-category $\mathcal{F}rob_{\mathcal{C}}$ [SFR06]. Now that we understand a gerbe module as a 1-morphism from $\mathcal{G}$ to $\mathcal{I}_\omega$ we have found a common principle in both approaches. A similar success is obtained for unoriented conformal field theories. In the Lagrangian approach, the bundle gerbe $\mathcal{G}$ has to be endowed with a Jandl structure [SSW05], which is in particular a 1-isomorphism $k^*\mathcal{G} \to \mathcal{G}^*$ to the dual bundle gerbe $\mathcal{G}^*$. In the algebraic approach one has to choose a certain algebra isomorphism $A \to A^{\mathrm{op}}$ from A to the opposed algebra [FRS04].

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Conventions. Let us fix the following conventions for the whole article: by *vector bundle* I refer to a complex vector bundle of finite rank, equipped with a hermitian structure and with a connection respecting this hermitian structure. Accordingly, a *morphism of vector bundles* is supposed to respect both the hermitian structures and the connections. In particular, a *line bundle* is a vector bundle in the above sense of rank one. The symmetric monoidal category $\mathfrak{Bun}(M)$, which is formed by all vector bundles over a smooth manifold M and their morphisms in the above sense, is for simplicity tacitly replaced by an equivalent strict tensor category.

1. The 2-Category of Bundle Gerbes

Summarizing, the 2-category $\mathfrak{BGrb}(M)$ of bundle gerbes over a smooth manifold M consists of the following structure:

1. A class of objects – bundle gerbes over M.

2. A morphism category $\mathfrak{Hom}(\mathcal{G}, \mathcal{H})$ for each pair $\mathcal{G}, \mathcal{H}$ of bundle gerbes, whose objects are called 1-morphisms and are denoted by $\mathcal{A} : \mathcal{G} \to \mathcal{H}$, and whose morphisms are called 2-morphisms and are denoted $\beta : \mathcal{A}_1 \Rightarrow \mathcal{A}_2$.

3. A composition functor

$$\circ \;:\; \mathfrak{Hom}(\mathcal{H},\mathcal{K}) \times \mathfrak{Hom}(\mathcal{G},\mathcal{H}) \longrightarrow \mathfrak{Hom}(\mathcal{G},\mathcal{K})$$

   for each triple $\mathcal{G},\mathcal{H},\mathcal{K}$ of bundle gerbes.

4. An identity 1-morphism $\mathrm{id}_{\mathcal{G}} : \mathcal{G} \to \mathcal{G}$ for each bundle gerbe $\mathcal{G}$ together with natural 2-isomorphisms

$$\rho_{\mathcal{A}} \;:\; \mathrm{id}_{\mathcal{H}} \circ \mathcal{A} \Longrightarrow \mathcal{A} \quad \text{and} \quad \lambda_{\mathcal{A}} \;:\; \mathcal{A} \circ \mathrm{id}_{\mathcal{G}} \Longrightarrow \mathcal{A}$$

   associated to every 1-morphism $\mathcal{A} : \mathcal{G} \to \mathcal{H}$.

This structure satisfies the axioms of a strictly associative 2-category:

(2C1) For three 1-morphisms $\mathcal{A} : \mathcal{G}_1 \to \mathcal{G}_2$, $\mathcal{A}' : \mathcal{G}_2 \to \mathcal{G}_3$ and $\mathcal{A}'' : \mathcal{G}_3 \to \mathcal{G}_4$, the composition functor satisfies

$$\mathcal{A}'' \circ (\mathcal{A}' \circ \mathcal{A}) \;=\; (\mathcal{A}'' \circ \mathcal{A}') \circ \mathcal{A}.$$

(2C2) For 1-morphisms $\mathcal{A} : \mathcal{G}_1 \to \mathcal{G}_2$ and $\mathcal{A}' : \mathcal{G}_2 \to \mathcal{G}_3$, the 2-isomorphisms $\lambda_{\mathcal{A}}$ and $\rho_{\mathcal{A}}$ satisfy the equality

$$\mathrm{id}_{\mathcal{A}'} \circ \rho_{\mathcal{A}} \;=\; \lambda_{\mathcal{A}'} \circ \mathrm{id}_{\mathcal{A}}$$

as 2-morphisms from $\mathcal{A}' \circ \mathrm{id}_{\mathcal{G}_2} \circ \mathcal{A}$ to $\mathcal{A}' \circ \mathcal{A}$.

The following two subsections contain the definitions of the structure of the 2-category $\mathfrak{BGrb}(M)$. The two axioms are proved in Propositions 1.8 and 1.11. The reader who is not interested in these details may directly continue with section 3.

1.1. Objects and Morphisms. The definition of the objects of the 2-category $\mathfrak{BGrb}(M)$ – the bundle gerbes over M – is the usual one, just like, for instance, in [Mur96, Ste00, GR02]. Given a surjective submersion $\pi : Y \to M$ we use the notation $Y^{[k]} := Y \times_M \ldots \times_M Y$ for the k-fold fibre product, which is again a smooth manifold. Here we consider fibre products to be strictly associative for simplicity. For the canonical projections between fibre products we use the notation $\pi_{i_1 \ldots i_k} : Y^{[n]} \to Y^{[k]}$.

1.2. Definition. *A bundle gerbe $\mathcal{G}$ over a smooth manifold M consists of the following structure:*

1. *a surjective submersion $\pi : Y \to M$,*

2. *a line bundle L over $Y^{[2]}$,*

3. *a 2-form $C \in \Omega^2(Y)$, and*

4. an isomorphism

$$\mu \;:\; \pi_{12}^*L \otimes \pi_{23}^*L \longrightarrow \pi_{13}^*L$$

of line bundles over $Y^{[3]}$.

This structure has to satisfy two axioms:

(G1) The curvature of L is fixed by

$$\mathrm{curv}(L) = \pi_2^*C - \pi_1^*C.$$

(G2) μ is associative in the sense that the diagram

$$\begin{tikzcd}
\pi_{12}^*L \otimes \pi_{23}^*L \otimes \pi_{34}^*L \arrow[r, "\pi_{123}^*\mu\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\pi_{234}^*\mu"'] & \pi_{13}^*L \otimes \pi_{34}^*L \arrow[d, "\pi_{134}^*\mu"] \\
\pi_{12}^*L \otimes \pi_{24}^*L \arrow[r, "\pi_{124}^*\mu"'] & \pi_{14}^*L
\end{tikzcd}$$

of isomorphisms of line bundles over $Y^{[4]}$ is commutative.

To give an example of a bundle gerbe, we introduce trivial bundle gerbes. Just as for every 1-form $A \in \Omega^1(M)$ there is the (topologically) trivial line bundle over M having this 1-form as its connection 1-form, we find a trivial bundle gerbe for every 2-form $\rho \in \Omega^2(M)$. Its surjective submersion is the identity $\mathrm{id} : M \to M$, and its 2-form is ρ. Its line bundle over $M \times_M M \cong M$ is the trivial line bundle with the trivial connection, and its isomorphism is the identity isomorphism between trivial line bundles. Now, axiom (G1) is satisfied since $\mathrm{curv}(L) = 0$ and $\pi_1 = \pi_2 = \mathrm{id}_M$. The associativity axiom (G2) is surely satisfied by the identity isomorphism. Thus we have defined a bundle gerbe, which we denote by $\mathcal{I}_\rho$.

It should not be unmentioned that the geometric nature of bundle gerbes allows explicit constructions of all (bi-invariant) bundle gerbes over all compact, connected and simple Lie groups [GR02, Mei02, GR03]. It becomes in particular essential that a surjective submersion $\pi : Y \to M$ is more general than an open cover of M.

An important consequence of the existence of the isomorphism μ in the structure of a bundle gerbe $\mathcal{G}$ is that the line bundle L restricted to the image of the diagonal embedding $\Delta : Y \to Y^{[2]}$ is canonically trivializable (as a line bundle with connection):

1.3. Lemma. *There is a canonical isomorphism $t_\mu : \Delta^*L \to 1$ of line bundles over Y, which satisfies*

$$\pi_1^*t_\mu \otimes \mathrm{id} = \Delta_{112}^*\mu \quad \textit{and} \quad \mathrm{id} \otimes \pi_2^*t_\mu = \Delta_{122}^*\mu$$

as isomorphisms of line bundles over $Y^{[2]}$, where $\Delta_{112} : Y^{[2]} \to Y^{[3]}$ duplicates the first and $\Delta_{122} : Y^{[2]} \to Y^{[3]}$ duplicates the second factor.

Proof. The isomorphism t_μ is defined using the canonical pairing with the dual line bundle L^* (which is strict by convention) and the isomorphism μ:

$$\Delta^*L = \Delta^*L \otimes \Delta^*L \otimes \Delta^*L^* \xrightarrow{\Delta^*\mu\otimes\mathrm{id}} \Delta^*L \otimes \Delta^*L^* = 1 \tag{1}$$

The two claimed equations follow from the associativity axiom (G2) by pullback of the diagram along Δ_{1222} and Δ_{1112} respectively. $\square$

Now we define the category $\mathfrak{Hom}(\mathcal{G}_1, \mathcal{G}_2)$ for two bundle gerbes $\mathcal{G}_1$ and $\mathcal{G}_2$, to whose structure we refer by the same letters as in Definition 1.2 but with indices 1 or 2 respectively.

1.4. Definition. *A 1-morphism $\mathcal{A} : \mathcal{G}_1 \to \mathcal{G}_2$ consists of the following structure:*

1. *a surjective submersion $\zeta : Z \to Y_1 \times_M Y_2$,*
2. *a vector bundle A over Z, and*
3. *an isomorphism*
$$\alpha \ : \ L_1 \otimes \zeta_2^*A \longrightarrow \zeta_1^*A \otimes L_2 \tag{2}$$
of vector bundles over $Z \times_M Z$.

This structure has to satisfy two axioms:

(1M1) The curvature of A obeys
$$\frac{1}{n}\mathrm{tr}(\mathrm{curv}(A)) = C_2 - C_1,$$
where n is the rank of the vector bundle A.

(1M2) The isomorphism α is compatible with the isomorphisms μ_1 and μ_2 of the gerbes $\mathcal{G}_1$ and $\mathcal{G}_2$ in the sense that the diagram

$$\begin{tikzcd}
\zeta_{12}^*L_1 \otimes \zeta_{23}^*L_1 \otimes \zeta_3^*A \arrow[r, "\mu_1\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\zeta_{23}^*\alpha"'] & \zeta_{13}^*L_1 \otimes \zeta_3^*A \arrow[dd, "\zeta_{13}^*\alpha"] \\
\zeta_{12}^*L_1 \otimes \zeta_2^*A \otimes \zeta_{23}^*L_2 \arrow[d, "\zeta_{12}^*\alpha\otimes\mathrm{id}"'] & \\
\zeta_1^*A \otimes \zeta_{12}^*L_2 \otimes \zeta_{23}^*L_2 \arrow[r, "\mathrm{id}\otimes\mu_2"'] & \zeta_1^*A \otimes \zeta_{13}^*L_2
\end{tikzcd}$$

of isomorphisms of vector bundles over $Z \times_M Z \times_M Z$ is commutative.

Here we work with the following simplifying notation: we have not introduced notation for the canonical projections $Z \to Y_1$ and $Z \to Y_2$, accordingly we don't write pullbacks with these maps. So in (2), where the line bundles L_i are pulled back along the induced map $Z^{[2]} \to Y_i^{[2]}$ for $i = 1, 2$ and also in axiom (1M1) which is an equation of 2-forms on Z.

As an example of a 1-morphism, we define the identity 1-morphism

$$\mathrm{id}_{\mathcal{G}} \ : \ \mathcal{G} \longrightarrow \mathcal{G} \tag{3}$$

of a bundle gerbe $\mathcal{G}$ over M. It is defined by $Z := Y^{[2]}$, the identity $\zeta := \mathrm{id}_Z$, the line bundle L of $\mathcal{G}$ over Z and the isomorphism λ defined by

$$\pi_{13}^*L \otimes \pi_{34}^*L \xrightarrow{\pi_{134}^*\mu} \pi_{14}^*L \xrightarrow{\pi_{124}^*\mu^{-1}} \pi_{12}^*L \otimes \pi_{24}^*L, \tag{4}$$

where we identified $Z^{[2]} = Y^{[4]}$, $\zeta_2 = \pi_{34}$ and $\zeta_1 = \pi_{12}$. Axiom (1M1) is the same as axiom (G1) for the bundle gerbe $\mathcal{G}$ and axiom (1M2) follows from axiom (G2).

The following lemma introduces an important isomorphism of vector bundles associated to every 1-morphism, which will be used frequently in the definition of the structure of $\mathfrak{BGrb}(M)$ and also in section 2.

1.5. LEMMA. *For any 1-morphism $\mathcal{A} : \mathcal{G}_1 \to \mathcal{G}_2$ there is a canonical isomorphism*

$$\mathrm{d}_{\mathcal{A}} \ : \ \zeta_1^*A \longrightarrow \zeta_2^*A$$

of vector bundles over $Z^{[2]} = Z \times_P Z$, where $P := Y_1 \times_M Y_2$, with the following properties:

a) It satisfies the cocycle condition

$$\zeta_{13}^*\mathrm{d}_{\mathcal{A}} = \zeta_{23}^*\mathrm{d}_{\mathcal{A}} \circ \zeta_{12}^*\mathrm{d}_{\mathcal{A}}$$

as an equation of isomorphisms of vector bundles over $Z^{[3]}$.

b) The diagram

$$\begin{tikzcd} L_1 \otimes \zeta_3^*A \arrow[r, "\zeta_{13}^*\alpha"] \arrow[d, "\mathrm{id}\otimes\zeta_{34}^*\mathrm{d}_{\mathcal{A}}"'] & \zeta_1^*A \otimes L_2 \arrow[d, "\zeta_{12}^*\mathrm{d}_{\mathcal{A}}\otimes\mathrm{id}"] \\ L_1 \otimes \zeta_4^*A \arrow[r, "\zeta_{24}^*\alpha"'] & \zeta_2^*A \otimes L_2 \end{tikzcd}$$

of isomorphisms of vector bundles over $Z^{[2]} \times_M Z^{[2]}$ is commutative.

Proof. Notice that the isomorphism α of $\mathcal{A}$ restricted from $Z \times_M Z$ to $Z \times_P Z$ gives an isomorphism

$$\alpha|_{Z\times_P Z} \ : \ \Delta^*L_1 \otimes \zeta_2^*A \longrightarrow \zeta_1^*A \otimes \Delta^*L_2. \tag{5}$$

By composition with the isomorphisms t_{μ_1} and t_{μ_2} from Lemma 1.3 we obtain the isomorphism $\mathrm{d}_{\mathcal{A}}$:

$$\zeta_1^*A \xrightarrow{\mathrm{id}\otimes t_{\mu_2}^{-1}} \zeta_1^*A \otimes \Delta^*L_2 \xrightarrow{\alpha|_{Z\times_P Z}^{-1}} \Delta^*L_1 \otimes \zeta_2^*A \xrightarrow{t_{\mu_1}\otimes\mathrm{id}} \zeta_2^*A. \tag{6}$$

The cocycle condition a) and the commutative diagram b) follow both from axiom (1M2) for $\mathcal{A}$ and the properties of the isomorphisms t_{μ_1} and t_{μ_2} from Lemma 1.3. □

Now that we have defined the objects of $\mathfrak{Hom}(\mathcal{G}_1, \mathcal{G}_2)$, we come to its morphisms. For two 1-morphisms $\mathcal{A}_1 : \mathcal{G}_1 \to \mathcal{G}_2$ and $\mathcal{A}_2 : \mathcal{G}_1 \to \mathcal{G}_2$, consider triples

$$(W, \omega, \beta_W) \tag{7}$$

consisting of a smooth manifold W, a surjective submersion $\omega : W \to Z_1 \times_P Z_2$, where again $P := Y_1 \times_M Y_2$, and a morphism $\beta_W : A_1 \to A_2$ of vector bundles over W. Here we work again with the convention that we don't write pullbacks along the unlabelled canonical projections $W \to Z_1$ and $W \to Z_2$. The triples (7) have to satisfy one axiom (2M): the morphism β_W has to be compatible with isomorphism α_1 and α_2 of the 1-morphisms $\mathcal{A}_1$ and $\mathcal{A}_2$ in the sense that the diagram

$$\begin{tikzcd} L_1 \otimes \omega_2^*A_1 \arrow[r, "\alpha_1"] \arrow[d, "1\otimes\omega_2^*\beta_W"'] & \omega_1^*A_1 \otimes L_2 \arrow[d, "\omega_1^*\beta_W\otimes 1"] \\ L_1 \otimes \omega_2^*A_2 \arrow[r, "\alpha_2"'] & \omega_1^*A_2 \otimes L_2 \end{tikzcd} \tag{8}$$

of morphisms of vector bundles over $W \times_M W$ is commutative. On the set of all triples (7) satisfying this axiom we define an equivalence relation according to that two triples (W, ω, β_W) and $(W', \omega', \beta_{W'})$ are equivalent, if there exists a smooth manifold X with surjective submersions to W and W' for which the diagram

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ W \arrow[dr, "\omega"'] & & W' \arrow[dl, "\omega'"] \\ & Z_1 \times_P Z_2 & \end{tikzcd} \tag{9}$$

of surjective submersions is commutative, and the morphisms β_W and $\beta_{W'}$ coincide when pulled back to X.

1.6. Definition. *A 2-morphism $\beta : \mathcal{A}_1 \Rightarrow \mathcal{A}_2$ is an equivalence class of triples* (W, ω, β_W) *satisfying axiom (2M).*

As an example of a 2-morphism we define the identity 2-morphism $\mathrm{id}_{\mathcal{A}} : \mathcal{A} \Rightarrow \mathcal{A}$ associated to every 1-morphism $\mathcal{A} : \mathcal{G}_1 \to \mathcal{G}_2$. It is defined as the equivalence class of the triple $(Z^{[2]}, \mathrm{id}_{Z^{[2]}}, \mathrm{d}_{\mathcal{A}})$ consisting of the fibre product $Z^{[2]} = Z \times_P Z$, the identity $\mathrm{id}_{Z[2]}$ and the isomorphism $\mathrm{d}_{\mathcal{A}} : \zeta_1^* A \to \zeta_2^* A$ of vector bundles over $Z^{[2]}$ from Lemma 1.5. Axiom (2M) for this triple is proven with Lemma 1.5 b).

Now we have defined objects and morphisms of the morphism category $\mathfrak{Hom}(\mathcal{G}_1, \mathcal{G}_2)$, and we continue with the definition the composition $\beta' \bullet \beta$ of two 2-morphisms $\beta : \mathcal{A}_1 \Rightarrow \mathcal{A}_2$ and $\beta' : \mathcal{A}_2 \Rightarrow \mathcal{A}_3$. It is called vertical composition in agreement with the diagrammatical notation

$$\begin{tikzcd} \mathcal{G}_1 \arrow[r, bend left=60, "\mathcal{A}_1"{name=U}] \arrow[r, "\mathcal{A}_2"{name=M}] \arrow[r, bend right=60, "\mathcal{A}_3"'{name=D}] & \mathcal{G}_2 \arrow[from=U, to=M, Rightarrow, "\beta"] \arrow[from=M, to=D, Rightarrow, "\beta'"] \end{tikzcd} . \tag{10}$$

We choose representatives (W, ω, β_W) and $(W', \omega', \beta_{W'})$ and consider the fibre product $\tilde{W} := W \times_{Z_2} W'$ with its canonical surjective submersion $\tilde{\omega} : \tilde{W} \to Z_1 \times_P Z_3$, where again $P := Y_1 \times_M Y_2$. By construction we can compose the pullbacks of the morphisms β_W and $\beta_{W'}$ to $\tilde{W}$ and obtain a morphism

$$\beta_{W'} \circ \beta_W \ : \ A_1 \longrightarrow A_3 \tag{11}$$

of vector bundles over $\tilde{W}$. From axiom (2M) for β and β' the one for the triple $(\tilde{W}, \tilde{\omega}, \beta_{W'} \circ \beta_W)$ follows. Furthermore, the equivalence class of this triple is independent of the choices of the representatives of β and $\beta'$and thus defines the 2-morphism $\beta' \bullet \beta$. The composition $\bullet$ of the category $\mathfrak{Hom}(\mathcal{G}_1, \mathcal{G}_2)$ defined like this is associative.

It remains to check that the 2-isomorphism $\mathrm{id}_{\mathcal{A}} : \mathcal{A} \Rightarrow \mathcal{A}$ defined above is the identity under the composition $\bullet$. Let $\beta : \mathcal{A} \Rightarrow \mathcal{A}'$ be a 2-morphism and (W, ω, β_W) a representative. The composite $\beta \bullet \mathrm{id}_{\mathcal{A}}$ can be represented by the triple $(W', \omega', \beta \circ \mathrm{d}_{\mathcal{A}})$ with $W' = Z \times_P W$, where $\omega' : W' \to Z \times_P Z'$ is the identity on the first factor and the projection $W \to Z'$ on the second one. We have to show, that this triple is equivalent to the original representative (W, ω, β_W) of β. Consider the fibre product

$$X := W \times_{(Z \times_P Z')} W' \cong W \times_{Z'} W, \tag{12}$$

so that condition (9) is satisfied. The restriction of the commutative diagram (8) of morphisms of vector bundles over $W \times_M W$ from axiom (2M) for β to X gives rise to the commutative diagram

$$\begin{tikzcd} \zeta_2^* A \arrow[r, "\mathrm{d}_{\mathcal{A}}^{-1}"] \arrow[d, "\omega_2^* \beta_W"'] & \zeta_1^* A \arrow[d, "\omega_1^* \beta_W"] \\ A' \arrow[r, "\Delta^* \mathrm{d}_{\mathcal{A}'}^{-1}"'] & A' \end{tikzcd} \tag{13}$$

of morphisms of vector bundles over X, where $\mathrm{d}_{\mathcal{A}}$ and $\mathrm{d}_{\mathcal{A}'}$ are the isomorphisms determined by the 1-morphisms $\mathcal{A}$ and $\mathcal{A}'$ according to Lemma 1.5. Their cocycle condition from Lemma 1.5 a) implies $\Delta^*\mathrm{d}_{\mathcal{A}'} = \mathrm{id}$, so that diagram (13) is reduced to the equality $\omega_2^*\beta_W \circ \mathrm{d}_{\mathcal{A}} = \omega_1^*\beta_W$ of isomorphisms of vector bundles over X. This shows that the triples (W, ω, β_W) and $(W', \omega', \beta_W \circ \mathrm{d}_{\mathcal{A}})$ are equivalent and we have $\beta \bullet \mathrm{id}_{\mathcal{A}} = \beta$. The equality $\mathrm{id}_{\mathcal{A}'} \bullet \beta = \beta$ follows analogously.

Now the definition of the morphism category $\mathfrak{Hom}(\mathcal{G}_1, \mathcal{G}_2)$ is complete. A morphism in this category, i.e. a 2-morphism $\beta : \mathcal{A} \Rightarrow \mathcal{A}'$, is invertible if and only if the morphism $\beta_W : A \to A'$ of any representative (W, ω, β_W) of β is invertible. Since – following our convention – morphism of vector bundles respect the hermitian structures, β_W is invertible if and only if the ranks of the vector bundles of the 1-morphisms $\mathcal{A}$ and $\mathcal{A}'$ coincide. In the following, we call two 1-morphisms $\mathcal{A} : \mathcal{G}_1 \to \mathcal{G}_2$ and $\mathcal{A}' : \mathcal{G}_1 \to \mathcal{G}_2$ isomorphic, if there exists a 2-isomorphism $\beta : \mathcal{A} \Rightarrow \mathcal{A}'$ between them.

1.7. The Composition Functor. Let $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ be three bundles gerbes over M. We define the composition functor

$$\circ \; : \; \mathfrak{Hom}(\mathcal{G}_2, \mathcal{G}_3) \times \mathfrak{Hom}(\mathcal{G}_1, \mathcal{G}_2) \longrightarrow \mathfrak{Hom}(\mathcal{G}_1, \mathcal{G}_3) \tag{14}$$

on objects in the following way. Let $\mathcal{A} : \mathcal{G}_1 \to \mathcal{G}_2$ and $\mathcal{A}' : \mathcal{G}_2 \to \mathcal{G}_3$ be two 1-morphisms. The composed 1-morphism

$$\mathcal{A}' \circ \mathcal{A} \; : \; \mathcal{G}_1 \longrightarrow \mathcal{G}_3 \tag{15}$$

consists of the fibre product $\tilde{Z} := Z \times_{Y_2} Z'$ with its canonical surjective submersion $\tilde{\zeta} : \tilde{Z} \to Y_1 \times_M Y_3$, the vector bundle $\tilde{A} := A \otimes A'$ over $\tilde{Z}$, and the isomorphism

$$\tilde{\alpha} := (\mathrm{id}_{\zeta_1^*A} \otimes \alpha') \circ (\alpha \otimes \mathrm{id}_{\zeta_2'^*A'}) \tag{16}$$

of vector bundles over $\tilde{Z} \times_M \tilde{Z}$.

Indeed, this defines a 1-morphism from $\mathcal{G}_1$ to $\mathcal{G}_3$. Recall that if ∇_A and $\nabla_{A'}$ denote the connections on the vector bundles A and A', the tensor product connection ∇ on $A \otimes A'$ is defined by

$$\nabla(\sigma \otimes \sigma') = \nabla_A(\sigma) \otimes \sigma' + \sigma \otimes \nabla_{A'}(\sigma') \tag{17}$$

for sections $\sigma \in \Gamma(A)$ and $\sigma' \in \Gamma(A')$. If we take n to be the rank of A and n' the rank of A' the curvature of the tensor product vector bundle is

$$\mathrm{curv}(A \otimes A') = \mathrm{curv}(A) \otimes \mathrm{id}_{n'} + \mathrm{id}_n \otimes \mathrm{curv}(A'). \tag{18}$$

Hence its trace

$$\begin{aligned} \frac{1}{nn'}\mathrm{tr}(\mathrm{curv}(\tilde{A})) &= \frac{1}{n}\mathrm{tr}(\mathrm{curv}(A)) + \frac{1}{n'}\mathrm{tr}(\mathrm{curv}(A')) \\ &= C_2 - C_1 + C_3 - C_2 \\ &= C_3 - C_1 \end{aligned} \tag{19}$$

satisfies axiom (1M1). Notice that equation (19) involves unlabeled projections from $\tilde{Z}$ to Y_1, Y_2 and Y_3, where the one to Y_2 is unique because $\tilde{Z}$ is the fibre product over Y_2. Furthermore, $\tilde{\alpha}$ is an isomorphism

$$\begin{tikzcd} L_1 \otimes \tilde{\zeta}_2^*\tilde{A} \arrow[r, equal] & L_1 \otimes \zeta_2^*A \otimes \zeta_2'^*A' \arrow[d, "\alpha\otimes\mathrm{id}"] & \\ & \zeta_1^*A \otimes L_2 \otimes \zeta_2'^*A' \arrow[d, "\mathrm{id}\otimes\alpha'"] & \\ & \zeta_1^*A \otimes \zeta_1'^*A' \otimes L_3 \arrow[r, equal] & \tilde{\zeta}_1^*\tilde{A} \otimes L_3. \end{tikzcd} \tag{20}$$

Axiom (1M2) follows from axioms (1M2) for $\mathcal{A}$ and $\mathcal{A}'$.

1.8. Proposition. *The composition of 1-morphisms is strictly associative: for three 1-morphisms $\mathcal{A} : \mathcal{G}_1 \to \mathcal{G}_2$, $\mathcal{A}' : \mathcal{G}_2 \to \mathcal{G}_3$ and $\mathcal{A}'' : \mathcal{G}_3 \to \mathcal{G}_4$ we have*

$$(\mathcal{A}'' \circ \mathcal{A}') \circ \mathcal{A} = \mathcal{A}'' \circ (\mathcal{A}' \circ \mathcal{A}).$$

Proof. By definition, both 1-morphism $(\mathcal{A}'' \circ \mathcal{A}') \circ \mathcal{A}$ and $\mathcal{A}'' \circ (\mathcal{A}' \circ \mathcal{A})$ consist of the smooth manifold $X = Z \times_{Y_2} Z' \times_{Y_3} Z''$ with the same surjective submersion $X \to Y_1 \times_M Y_4$. On X, they have the same vector bundle $A \otimes A' \otimes A''$, and finally the same isomorphism

$$(\mathrm{id} \otimes \mathrm{id} \otimes \alpha'') \circ (\mathrm{id} \otimes \alpha' \otimes \mathrm{id}) \circ (\alpha \otimes \mathrm{id} \otimes \mathrm{id}) \tag{21}$$

of vector bundles over $X \times_M X$. $\square$

Now we have to define the functor $\circ$ on 2-morphisms. Let $\mathcal{A}_1, \mathcal{A}_1' : \mathcal{G}_1 \to \mathcal{G}_2$ and $\mathcal{A}_2, \mathcal{A}_2' : \mathcal{G}_2 \to \mathcal{G}_3$ be 1-morphisms between bundle gerbes. The functor $\circ$ on morphisms is called horizontal composition due to the diagrammatical notation

$$\begin{tikzcd} \mathcal{G}_1 \arrow[r, bend left, "\mathcal{A}_1"] \arrow[r, bend right, "\mathcal{A}_1'"'] \arrow[r, phantom, "\Downarrow\beta_1"] & \mathcal{G}_2 \arrow[r, bend left, "\mathcal{A}_2"] \arrow[r, bend right, "\mathcal{A}_2'"'] \arrow[r, phantom, "\Downarrow\beta_2"] & \mathcal{G}_3 \end{tikzcd} = \begin{tikzcd} \mathcal{G}_1 \arrow[r, bend left, "\mathcal{A}_2\circ\mathcal{A}_1"] \arrow[r, bend right, "\mathcal{A}_2'\circ\mathcal{A}_1'"'] \arrow[r, phantom, "\Downarrow\beta_2\circ\beta_1"] & \mathcal{G}_3 \end{tikzcd}. \tag{22}$$

Recall that the compositions $\mathcal{A}_2\circ\mathcal{A}_1$ and $\mathcal{A}_2'\circ\mathcal{A}_1'$ consist of smooth manifolds $\tilde{Z} = Z_1 \times_{Y_2} Z_2$ and $\tilde{Z}' = Z_1' \times_{Y_2} Z_2'$ with surjective submersions to $P := Y_1 \times_M Y_3$, of vector bundles $\tilde{A} := A_1 \otimes A_2$ over $\tilde{Z}$ and $\tilde{A}' := A_1' \otimes A_2'$ over $\tilde{Z}'$, and of isomorphisms $\tilde{\alpha}$ and $\tilde{\alpha}'$ over $\tilde{Z} \times_M \tilde{Z}$ and $\tilde{Z}' \times_M \tilde{Z}'$.

To define the composed 2-morphism $\beta_2 \circ \beta_1$, we first need a surjective submersion

$$\omega \; : \; W \longrightarrow \tilde{Z} \times_P \tilde{Z}'. \tag{23}$$

We choose representatives $(W_1, \omega_1, \beta_{W_1})$ and $(W_2, \omega_2, \beta_{W_2})$ of the 2-morphisms β_1 and β_2 and define

$$W := \tilde{Z} \times_P (W_1 \times_{Y_2} W_2) \times_P \tilde{Z}' \tag{24}$$

with the surjective submersion $\omega := \tilde{z} \times \tilde{z}'$ projecting on the first and the last factor. Then, we need a morphism $\beta_W : \tilde{z}^*\tilde{A} \to \tilde{z}'^*\tilde{A}'$ of vector bundles over W. Notice that we have maps

$$u \;:\; W_1 \times_{Y_2} W_2 \longrightarrow \tilde{Z} \quad \text{and} \quad u' \;:\; W_1 \times_{Y_2} W_2 \longrightarrow \tilde{Z}' \tag{25}$$

such that we obtain surjective submersions

$$\tilde{z} \times u \;:\; W \longrightarrow \tilde{Z}^{[2]} \quad \text{and} \quad u' \times \tilde{z}' \;:\; W \longrightarrow \tilde{Z}'^{[2]}. \tag{26}$$

Recall from Lemma 1.5 that the 1-morphisms $\mathcal{A}_2 \circ \mathcal{A}_1$ and $\mathcal{A}'_2 \circ \mathcal{A}'_1$ define isomorphisms $\mathrm{d}_{\mathcal{A}_2\circ\mathcal{A}_1}$ and $\mathrm{d}_{\mathcal{A}'_2\circ\mathcal{A}'_1}$ of vector bundles over $\tilde{Z}^{[2]}$ and $\tilde{Z}'^{[2]}$, whose pullbacks to W along the above maps are isomorphisms

$$\mathrm{d}_{\mathcal{A}_2\circ\mathcal{A}_1} \;:\; \tilde{z}^*\tilde{A} \longrightarrow u^*\tilde{A} \quad \text{and} \quad \mathrm{d}_{\mathcal{A}'_2\circ\mathcal{A}'_1} \;:\; u'^*\tilde{A}' \longrightarrow \tilde{z}'^*\tilde{A}' \tag{27}$$

of vector bundles over W. Finally, the morphisms β_{W_1} and β_{W_2} give a morphism

$$\tilde{\beta} := \beta_{W_1} \otimes \beta_{W_2} \;:\; u^*\tilde{A} \longrightarrow u'^*\tilde{A}' \tag{28}$$

of vector bundles over W so that the composition

$$\beta_W := \mathrm{d}_{\mathcal{A}'_2\circ\mathcal{A}'_1} \circ \tilde{\beta} \circ \mathrm{d}_{\mathcal{A}_2\circ\mathcal{A}_1} \tag{29}$$

is a well-defined morphism of vector bundles over W. Axiom (2M) for the triple (W, ω, β_W) follows from Lemma 1.5 b) for $\mathcal{A}_2 \circ \mathcal{A}_1$ and $\mathcal{A}'_2 \circ \mathcal{A}'_1$ and from the axioms (2M) for the representatives of β_1 and β_2. Furthermore, the equivalence class of (W, ω, β_W) is independent of the choices of the representatives of β_1 and β_2.

1.9. Lemma. *The assignment $\circ$, defined above on objects and morphisms, is a functor*

$$\circ \;:\; \mathfrak{Hom}(\mathcal{G}_2, \mathcal{G}_3) \times \mathfrak{Hom}(\mathcal{G}_1, \mathcal{G}_2) \longrightarrow \mathfrak{Hom}(\mathcal{G}_1, \mathcal{G}_3).$$

Proof. i) The assignment $\circ$ respects identities, i.e. for 1-morphisms $\mathcal{A}_1 : \mathcal{G}_1 \to \mathcal{G}_2$ and $\mathcal{A}_2 : \mathcal{G}_2 \to \mathcal{G}_3$,

$$\mathrm{id}_{\mathcal{A}_2} \circ \mathrm{id}_{\mathcal{A}_1} = \mathrm{id}_{\mathcal{A}_2\circ\mathcal{A}_1}. \tag{30}$$

To show this we choose the defining representatives $(W_1, \mathrm{id}, \mathrm{d}_{\alpha_1})$ of $\mathrm{id}_{\mathcal{A}_1}$ and $(W_2, \mathrm{id}, \mathrm{d}_{\alpha_2})$ of $\mathrm{id}_{\mathcal{A}_2}$, where $W_1 = Z_1 \times_{(Y_1\times_M Y_2)} Z_1$ and $W_2 = Z_2 \times_{(Y_2\times_M Y_3)} Z_2$. Consider the diffeomorphism

$$f : W_1 \times_{Y_2} W_2 \to \tilde{Z} \times_{Y_1\times_M Y_2\times_M Y_3} \tilde{Z} : (z_1, z'_1, z_2, z'_2) \mapsto (z_1, z_2, z'_1, z'_2), \tag{31}$$

where $\tilde{Z} = Z_1 \times_{Y_2} Z_2$. From the definitions of the isomorphisms $\mathrm{d}_{\mathcal{A}_1}$, $\mathrm{d}_{\mathcal{A}_2}$ and $\mathrm{d}_{\mathcal{A}_2\circ\mathcal{A}_2}$ we conclude the equation

$$\mathrm{d}_{\mathcal{A}_1} \otimes \mathrm{d}_{\mathcal{A}_2} = f^*\mathrm{d}_{\mathcal{A}_2\circ\mathcal{A}_1} \tag{32}$$

of isomorphisms of vector bundles over $W_1 \times_{Y_2} W_2$. The horizontal composition $\mathrm{id}_{\mathcal{A}_2} \circ \mathrm{id}_{\mathcal{A}_1}$ is canonically represented by the triple (W, ω, β_W) where W is defined in (24) and β_W is defined in (29). Now, the diffeomorphism f extends to an embedding $f : W \to \tilde{Z}^{[4]}$ into the four-fold fibre product of $\tilde{Z}$ over $P = Y_1 \times_M Y_3$, such that $\omega : W \to \tilde{Z}^{[2]}$ factorizes over f,

$$\omega = \tilde{\zeta}_{14} \circ f. \tag{33}$$

From (29) and (32) we obtain

$$\begin{aligned} \beta_W &= \mathrm{d}_{\mathcal{A}_2 \circ \mathcal{A}_1} \circ (\mathrm{d}_{\mathcal{A}_1} \otimes \mathrm{d}_{\mathcal{A}_2}) \circ \mathrm{d}_{\mathcal{A}_2 \circ \mathcal{A}_1} \\ &= f^*(\tilde{\zeta}_{34}^* \mathrm{d}_{\mathcal{A}_2 \circ \mathcal{A}_1} \circ \tilde{\zeta}_{23}^* \mathrm{d}_{\mathcal{A}_2 \circ \mathcal{A}_1} \circ \tilde{\zeta}_{12}^* \mathrm{d}_{\mathcal{A}_2 \circ \mathcal{A}_1}). \end{aligned} \tag{34}$$

The cocycle condition for $\mathrm{d}_{\mathcal{A}_2 \circ \mathcal{A}_1}$ from Lemma 1.5 a) and (33) give

$$\beta_W = f^* \tilde{\zeta}_{14}^* \mathrm{d}_{\mathcal{A}_2 \circ \mathcal{A}_1} = \omega^* \mathrm{d}_{\mathcal{A}_2 \circ \mathcal{A}_1}. \tag{35}$$

We had to show that the triple (W, ω, β_W) which represents $\mathrm{id}_{\mathcal{A}_2} \circ \mathrm{id}_{\mathcal{A}_2}$ is equivalent to the triple $(\tilde{Z}^{[2]}, \mathrm{id}, \mathrm{d}_{\mathcal{A}_2 \circ \mathcal{A}_1})$ which defines the identity 2-morphism $\mathrm{id}_{\mathcal{A}_2 \circ \mathcal{A}_1}$. For the choice $X := W$ with surjective submersions $\mathrm{id} : X \to W$ and $\omega : X \to \tilde{Z}^{[2]}$, equation (35) shows exactly this equivalence.

ii) The assignment $\circ$ respects the composition $\bullet$, i.e. for 2-morphisms $\beta_i : \mathcal{A}_i \Rightarrow \mathcal{A}_i'$ and $\beta_i' : \mathcal{A}_i' \Rightarrow \mathcal{A}_i''$ between 1-morphisms $\mathcal{A}_i$, $\mathcal{A}_i'$ and $\mathcal{A}_i''$ from $\mathcal{G}_i$ to $\mathcal{G}_{i+1}$, everything for $i = 1, 2$, we have an equality

$$(\beta_2' \bullet \beta_2) \circ (\beta_1' \bullet \beta_1) = (\beta_2' \circ \beta_1') \bullet (\beta_2 \circ \beta_1) \tag{36}$$

of 2-morphisms from $\mathcal{A}_2 \circ \mathcal{A}_1$ to $\mathcal{A}_2'' \circ \mathcal{A}_1''$. This equality is also known as the compatibility of vertical and horizontal compositions. To prove it, let us introduce the notation $\tilde{Z} := Z_1 \times_{Y_2} Z_2$, and analogously $\tilde{Z}'$ and $\tilde{Z}''$, furthermore we write $P := Y_1 \times_M Y_3$. Notice that the 2-morphism on the left hand side of (36) is represented by a triple (V, ν, β_V) with

$$V = \tilde{Z} \times_P (\tilde{W}_1 \times_{Y_2} \tilde{W}_2) \times_P \tilde{Z}'', \tag{37}$$

where the fibre products $\tilde{W}_i := W_i \times_{Z_i'} W_i'$ arise from the vertical compositions $\beta_i' \bullet \beta_i$. The surjective submersion $\nu : V \to \tilde{Z} \times_P \tilde{Z}''$ is the projection on the first and the last factor, and

$$\beta_V = \mathrm{d}_{\mathcal{A}_2'' \circ \mathcal{A}_1''} \circ ((\beta_1' \circ \beta_1) \otimes (\beta_2' \circ \beta_2)) \circ \mathrm{d}_{\mathcal{A}_2 \circ \mathcal{A}_1} \tag{38}$$

is a morphism of vector bundles over V. The 2-morphism on the right hand side of (36) is represented by the triple $(V', \nu', \beta_{V'})$ with

$$\begin{aligned} V' &= (\tilde{Z} \times_P (W_1 \times_{Y_2} W_2) \times_P \tilde{Z}') \times_{\tilde{Z}'} (\tilde{Z}' \times_P (W_1' \times_{Y_2} W_2') \times_P \tilde{Z}'') \\ &\cong \tilde{Z} \times_P (W_1 \times_{Y_2} W_2) \times_P \tilde{Z}' \times_P (W_1' \times_{Y_2} W_2') \times_P \tilde{Z}'', \end{aligned} \tag{39}$$

where ν' is again the projection on the outer factors, and

$$\beta_{V'} = \mathrm{d}_{\mathcal{A}_2'' \circ \mathcal{A}_1''} \circ (\beta_1' \otimes \beta_2') \circ \mathrm{d}_{\mathcal{A}_2' \circ \mathcal{A}_1'} \circ (\beta_1 \otimes \beta_2) \circ \mathrm{d}_{\mathcal{A}_2 \circ \mathcal{A}_1}, \tag{40}$$

where we have used the cocycle condition for $\mathrm{d}_{\mathcal{A}_2'\circ\mathcal{A}_1'}$ from Lemma 1.5 b).

We have to show that the triples (V,ν,β_V) and $(V',\nu',\beta_{V'})$ are equivalent. Consider the fibre product

$$X := V \times_{\tilde{Z}\times_P\tilde{Z}''} V' \tag{41}$$

with surjective submersions $v : X \to V$ and $v' : X \to V'$. To show the equivalence of the two triples, we have to prove the equality

$$v^*\beta_V = v'^*\beta_{V'}. \tag{42}$$

It is equivalent to the commutativity of the outer shape of the following diagram of isomorphisms of vector bundles over X:

$$\begin{tikzcd}
& A_1\otimes A_2 \arrow[dl, "\mathrm{d}_{\mathcal{A}_2\circ\mathcal{A}_1}"'] \arrow[dr, "\mathrm{d}_{\mathcal{A}_2\circ\mathcal{A}_1}"] & \\
v^*(A_1\otimes A_2) \arrow[rr, "\mathrm{d}_{\mathcal{A}_2\circ\mathcal{A}_1}"] \arrow[dd, "\beta_1\otimes\beta_2"'] & & v'^*(A_1\otimes A_2) \arrow[d, "\beta_1\otimes\beta_2"] \\
& & v'^*(A_1'\otimes A_2') \arrow[dd, "\mathrm{d}_{\mathcal{A}_2'\circ\mathcal{A}_1'}"] \\
v^*(A_1'\otimes A_2') \arrow[urr, "\mathrm{d}_{\mathcal{A}_2'\circ\mathcal{A}_1'}"] \arrow[drr, "\mathrm{d}_{\mathcal{A}_2'\circ\mathcal{A}_1'}"'] \arrow[dd, "\beta_1'\otimes\beta_2'"'] & & \\
& & v'^*(A_1'\otimes A_2') \arrow[d, "\beta_1'\otimes\beta_2'"] \\
v^*(A_1''\otimes A_2'') \arrow[rr, "\mathrm{d}_{\mathcal{A}_2''\circ\mathcal{A}_1''}"] \arrow[dr, "\mathrm{d}_{\mathcal{A}_2''\circ\mathcal{A}_1''}"'] & & v'^*(A_1''\otimes A_2'') \arrow[dl, "\mathrm{d}_{\mathcal{A}_2''\circ\mathcal{A}_1''}"] \\
& A_1''\otimes A_2'' &
\end{tikzcd} \tag{43}$$

The commutativity of the outer shape of this diagram follows from the commutativity of its five subdiagrams: the triangular ones are commutative due to the cocycle condition from Lemma 1.5 a), and the commutativity of the foursquare ones follows from axiom (2M) of the 2-morphisms. $\square$

To finish the definition of the 2-category $\mathfrak{BGrb}(M)$ we have to define the natural 2-isomorphisms $\lambda_{\mathcal{A}} : \mathcal{A}\circ\mathrm{id}_{\mathcal{G}} \Rightarrow \mathcal{A}$ and $\rho_{\mathcal{A}} : \mathrm{id}_{\mathcal{G}'}\circ\mathcal{A} \Rightarrow \mathcal{A}$ for a given 1-morphism $\mathcal{A} : \mathcal{G} \to \mathcal{G}'$, and we have to show that they satisfy axiom (2C2). We define the 1-morphism $\mathcal{A}\circ\mathrm{id}_{\mathcal{G}}$ as follows: it has the canonical surjective submersion from $\tilde{Z} = Y^{[2]}\times_Y Z \cong Y\times_M Z$ to $P := Y\times_M Y'$ and the vector bundle $L\otimes A$ over $\tilde{Z}$. Consider

$$W := \tilde{Z}\times_P Z \cong Z\times_{Y'} Z \tag{44}$$

and the identity $\omega := \mathrm{id}_W$. Under this identification, let us consider the restriction of the isomorphism α of the 1-morphism $\mathcal{A}$ from $Z\times_M Z$ to $W = Z\times_{Y'} Z$. If $s : W \to W$ denotes the exchange of the two factors, we obtain an isomorphism

$$s^*\alpha|_W \;:\; L\otimes\zeta_1^*A \longrightarrow \zeta_2^*A\otimes\Delta^*L' \tag{45}$$

of vector bundles over W. By composition with the canonical trivialization of the line bundle Δ^*L' from Lemma 1.3 it gives an isomorphism

$$\lambda_W := (\mathrm{id} \otimes t_{\mu'}) \circ s^*\alpha|_W \;:\; L \otimes \zeta_1^*A \longrightarrow \zeta_2^*A \tag{46}$$

of vector bundles over W. The axiom (2M) for the triple (W, ω, λ_W) follows from axiom (1M2) for the 1-morphism $\mathcal{A}$ and from the properties of $t_{\mu'}$ from Lemma 1.3. So, $\lambda_{\mathcal{A}}$ is defined to be the equivalence class of this triple. The definition of $\rho_{\mathcal{A}}$ goes analogously: we take $W = Z \times_Y Z$ and obtain by restriction the isomorphism

$$\alpha|_W \;:\; \Delta^*L \otimes \zeta_2^*A \longrightarrow \zeta_1^*A \otimes L'. \tag{47}$$

Then, the 2-isomorphism $\rho_{\mathcal{A}}$ is defined by the triple (W, ω, ρ_W) with the isomorphism

$$\rho_W := (t_\mu \otimes \mathrm{id}) \circ \alpha|_W^{-1} \;:\; \zeta_1^*A \otimes L' \longrightarrow \zeta_2^*A \tag{48}$$

of vector bundles over W.

1.10. Lemma. *The 2 -isomorphisms $\lambda_{\mathcal{A}}$ and $\rho_{\mathcal{A}}$ are natural in $\mathcal{A}$, i.e. for any 2-morphism $\beta : \mathcal{A} \Rightarrow \mathcal{A}'$ the naturality squares*

$$\begin{tikzcd} \mathrm{id}_{\mathcal{G}'} \circ \mathcal{A} \arrow[r, Rightarrow, "\rho_{\mathcal{A}}"] \arrow[d, Rightarrow, "\mathrm{id}_{\mathrm{id}_{\mathcal{G}'}} \circ \beta"'] & \mathcal{A} \arrow[d, Rightarrow, "\beta"] \\ \mathrm{id}_{\mathcal{G}'} \circ \mathcal{A}' \arrow[r, Rightarrow, "\rho_{\mathcal{A}'}"'] & \mathcal{A}' \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} \mathcal{A} \circ \mathrm{id}_{\mathcal{G}} \arrow[r, Rightarrow, "\lambda_{\mathcal{A}}"] \arrow[d, Rightarrow, "\beta \circ \mathrm{id}_{\mathrm{id}_{\mathcal{G}}}"'] & \mathcal{A} \arrow[d, Rightarrow, "\beta"] \\ \mathcal{A}' \circ \mathrm{id}_{\mathcal{G}} \arrow[r, Rightarrow, "\lambda_{\mathcal{A}'}"'] & \mathcal{A}' \end{tikzcd}$$

are commutative.

Proof. To calculate for instance the horizontal composition $\mathrm{id}_{\mathrm{id}_{\mathcal{G}'}} \circ \beta$ in the diagram on the left hand side first note that $\mathrm{id}_{\mathrm{id}_{\mathcal{G}'}}$ is canonically represented by the triple $(Y'^{[2]}, \mathrm{id}, \mathrm{id}_L)$. The isomorphism

$$\mathrm{d}_{\mathrm{id}_{\mathcal{G}'} \circ \mathcal{A}} : \tilde{\zeta}_1^*(A \otimes L') \to \tilde{\zeta}_2^*(A \otimes L'), \tag{49}$$

which appears in the definition of the horizontal composition, is an isomorphism of vector bundles over $\tilde{Z} \times_{Y \times_M Y'} \tilde{Z}$, where $\tilde{\zeta} : \tilde{Z} := Z \times_M Y' \to Y \times_M Y'$ is the surjective submersion of the composite $\mathrm{id}_{\mathcal{G}'} \circ \mathcal{A}$. Here it simplifies to

$$\mathrm{d}_{\mathrm{id}_{\mathcal{G}'} \circ \mathcal{A}} = (t_\mu \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\alpha^{-1} \otimes \mathrm{id}) \circ (1 \otimes \tilde{\zeta}_1^* \mu'^{-1}). \tag{50}$$

With these simplifications and with axiom (1M2) for $\mathcal{A}$ and $\mathcal{A}'$, the naturality squares reduce to the compatibility axiom (2M) of β with the isomorphisms α and α' of $\mathcal{A}$ and $\mathcal{A}'$ respectively. $\square$

It remains to show that the isomorphisms $\lambda_{\mathcal{A}}$ and $\rho_{\mathcal{A}}$ satisfy axiom (2C2) of a 2-category.

1.11. Proposition. *For 1-morphisms $\mathcal{A} : \mathcal{G}_1 \to \mathcal{G}_2$ and $\mathcal{A}' : \mathcal{G}_2 \to \mathcal{G}_3$, the 2-isomorphisms $\lambda_{\mathcal{A}}$ and $\rho_{\mathcal{A}}$ satisfy*

$$\mathrm{id}_{\mathcal{A}'} \circ \rho_{\mathcal{A}} = \lambda_{\mathcal{A}'} \circ \mathrm{id}_{\mathcal{A}}.$$

Proof. The equation to prove is an equation of 2-morphisms from $\mathcal{A}' \circ \mathrm{id}_{\mathcal{G}_2} \circ \mathcal{A}$ to $\mathcal{A}' \circ \mathcal{A}$. The first 1-morphism consists of the surjective submersion $\tilde{Z} := Z \times_M Z' \to P_{13}$, where we define $P_{ij} := Y_i \times_M Y_j$, further of the vector bundle $A \otimes L_2 \otimes A'$ over $\tilde{Z}$. The second 1-morphism $\mathcal{A}' \circ \mathcal{A}$ consists of the surjective submersion $\tilde{Z}' := Z \times_{Y_2} Z' \to P_{13}$ and the vector bundle $A \otimes A'$ over $\tilde{Z}'$. Let us choose the defining representatives for the involved 2-morphisms: we choose $(Z'^{[2]}, \mathrm{id}, \mathrm{d}_{\mathcal{A}'})$ for $\mathrm{id}_{\mathcal{A}'}$, with $W := Z \times_{Y_1} Z$ we choose (W, id, ρ_W) for $\rho_{\mathcal{A}}$, with $W' := Z' \times_{Y_3} Z'$ we choose $(W', \mathrm{id}, \lambda_{W'})$ for $\lambda_{\mathcal{A}'}$, and we choose $(Z^{[2]}, \mathrm{id}, \mathrm{d}_{\mathcal{A}'})$ for $\mathrm{id}_{\mathcal{A}}$.

Now, the horizontal composition $\mathrm{id}_{\mathcal{A}'} \circ \rho_{\mathcal{A}}$ is defined by the triple (V, ν, β_V) with

$$V = \tilde{Z} \times_{P_{13}} (W \times_{Y_2} Z'^{[2]}) \times_{P_{13}} \tilde{Z}', \tag{51}$$

the projection $\nu : V \to \tilde{Z} \times_{P_{13}} \tilde{Z}'$ on the first and the last factor, and the isomorphism

$$\beta_V = \mathrm{d}_{\mathcal{A}'\circ\mathcal{A}} \circ (\rho_W \otimes \mathrm{d}_{\mathcal{A}'}) \circ \mathrm{d}_{\mathcal{A}'\circ\mathrm{id}\circ\mathcal{A}} \tag{52}$$

of vector bundles over V. The horizontal composition $\lambda_{\mathcal{A}'} \circ \mathrm{id}_{\mathcal{A}}$ is defined by the triple $(V', \nu', \beta_{V'})$ with

$$V' = \tilde{Z} \times_{P_{13}} (Z^{[2]} \times_{Y_2} W') \times_{P_{13}} \tilde{Z}', \tag{53}$$

again the projection ν' on the first and the last factor, and the isomorphism

$$\beta_{V'} = \mathrm{d}_{\mathcal{A}'\circ\mathcal{A}} \circ (\mathrm{d}_{\mathcal{A}} \otimes \lambda_{W'}) \circ \mathrm{d}_{\mathcal{A}'\circ\mathrm{id}\circ\mathcal{A}} \tag{54}$$

of vector bundles over V.

To prove the proposition, we show that the triples (V, ν, β_V) and $(V', \nu', \beta_{V'})$ are equivalent. Consider the fibre product

$$X := V \times_{(\tilde{Z} \times_{P_{13}} \tilde{Z}')} V' \tag{55}$$

with surjective submersions $v : X \to V$ and $v' : X \to V'$. The equivalence of the two triples follows from the equation

$$v^*\beta_V = v'^*\beta' \tag{56}$$

of isomorphisms of vector bundles over X. It is equivalent to the commutativity of the outer shape of the following diagram of isomorphisms of vector bundles over X:

$$\begin{tikzcd}
& A \otimes L_2 \otimes A' \arrow[dl, "\mathrm{d}_{\mathcal{A}'\circ\mathrm{id}\circ\mathcal{A}}"'] \arrow[dr, "\mathrm{d}_{\mathcal{A}'\circ\mathrm{id}\circ\mathcal{A}}"] & \\
v^*(A \otimes L_2 \otimes A') \arrow[rr, "\mathrm{d}_{\mathcal{A}'\circ\mathrm{id}\circ\mathcal{A}}"] \arrow[d, "\rho_W \otimes \mathrm{d}_{\mathcal{A}'}"'] & & v'^*(A \otimes L_2 \otimes A') \arrow[d, "\mathrm{d}_{\mathcal{A}} \otimes \lambda_{W'}"] \\
v^*(A \otimes A') \arrow[rr, "\mathrm{d}_{\mathcal{A}'\circ\mathcal{A}}"] \arrow[dr, "\mathrm{d}_{\mathcal{A}'\circ\mathcal{A}}"'] & & v'^*(A \otimes A') \arrow[dl, "\mathrm{d}_{\mathcal{A}'\circ\mathcal{A}}"] \\
& A \otimes A' &
\end{tikzcd} \tag{57}$$

The diagram is patched together from three subdiagrams, and the commutativity of the outer shape follows because the three subdiagrams are commutative: the triangle diagrams are commutative due to the cocycle condition from Lemma 1.5 b) for the 1-morphisms $\mathcal{A}' \circ \mathrm{id}_{\mathcal{G}_2} \circ \mathcal{A}$ and $\mathcal{A}' \circ \mathcal{A}$ respectively. The commutativity of the rectangular diagram in the middle follows from Lemma 1.3 and from axioms (1M2) for $\mathcal{A}$ and $\mathcal{A}'$. $\square$

1.12. Invertible 1-Morphisms. In this subsection we address the question, which of the 1-morphisms of the 2-category $\mathfrak{BGrb}(M)$ are invertible. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two bundle gerbes over M. In a (strictly associative) 2-category, a 1-morphism $\mathcal{A} : \mathcal{G}_1 \to \mathcal{G}_2$ is called invertible or 1-isomorphism, if there is a 1-morphism $\mathcal{A}^{-1} : \mathcal{G}_2 \to \mathcal{G}_1$ in the opposite direction, together with 2-isomorphisms $i_l : \mathcal{A}^{-1} \circ \mathcal{A} \Rightarrow \mathrm{id}_{\mathcal{G}_1}$ and $i_r : \mathrm{id}_{\mathcal{G}_2} \Rightarrow \mathcal{A} \circ \mathcal{A}^{-1}$ such that the diagram

$$\begin{tikzcd}
\mathcal{A} \circ \mathcal{A}^{-1} \circ \mathcal{A} \arrow[r, Rightarrow, "\mathrm{id}_{\mathcal{A}} \circ i_l"] & \mathcal{A} \circ \mathrm{id}_{\mathcal{G}_1} \arrow[d, Rightarrow, "\lambda_{\mathcal{A}}"] \\
\mathrm{id}_{\mathcal{G}_2} \circ \mathcal{A} \arrow[u, Rightarrow, "i_r \circ \mathrm{id}_{\mathcal{A}}"] \arrow[r, Rightarrow, "\rho_{\mathcal{A}}"'] & \mathcal{A}
\end{tikzcd} \tag{58}$$

of 2-isomorphisms is commutative. The inverse 1-isomorphism $\mathcal{A}^{-1}$ is unique up to isomorphism.

Notice that if $\beta : \mathcal{A} \Rightarrow \mathcal{A}'$ is a 2-morphism between invertible 1-morphisms we can form a 2-morphism $\beta^{\#} : \mathcal{A}'^{-1} \Rightarrow \mathcal{A}^{-1}$ using the 2-isomorphisms i_r for $\mathcal{A}^{-1}$ and i_l for $\mathcal{A}'^{-1}$. Then, diagram (58) induces the equation $\mathrm{id}_{\mathcal{A}}^{\#} = \mathrm{id}_{\mathcal{A}^{-1}}$.

1.13. Proposition. *A 1-morphism $\mathcal{A} : \mathcal{G}_1 \to \mathcal{G}_2$ in $\mathfrak{BGrb}(M)$ is invertible if and only if the vector bundle of $\mathcal{A}$ is of rank* 1.

Proof. Suppose that $\mathcal{A}$ is invertible, and let n be the rank of its vector bundle. Let $\mathcal{A}^{-1}$ be an inverse 1-morphism with a vector bundle of rank m. By definition, the composed 1-morphisms $\mathcal{A} \circ \mathcal{A}^{-1}$ and $\mathcal{A}^{-1} \circ \mathcal{A}$ have vector bundles of rank nm, which has – to admit the existence of the 2-isomorphisms i_l and i_r – to coincide with the rank of the vector bundle of the identity 1-morphisms $\mathrm{id}_{\mathcal{G}_1}$ and $\mathrm{id}_{\mathcal{G}_2}$ respectively, which is 1. So $n = m = 1$. The other inclusion is shown below by an explicit construction of an inverse 1-morphism $\mathcal{A}^{-1}$ to a 1-morphism $\mathcal{A}$ with vector bundle of rank 1. □

Let a 1-morphism $\mathcal{A}$ consist of a surjective submersion $\zeta : Z \to Y_1 \times_M Y_2$, of a line bundle A over Z and of an isomorphism α of line bundles over $Z \times_M Z$. We explicitly construct an inverse 1-morphism $\mathcal{A}^{-1}$: it has the surjective submersion $Z \to Y_1 \times_M Y_2 \to Y_2 \times_M Y_1$, where the first map is ζ and the second one exchanges the factors, the dual line bundle A^* over Z and the isomorphism

$$\begin{tikzcd} L_2 \otimes \zeta_2^* A^* \arrow[r, equal] & \zeta_1^* A^* \otimes \zeta_1^* A \otimes L_2 \otimes \zeta_2^* A^* \arrow[d, "\mathrm{id} \otimes \alpha^{-1} \otimes \mathrm{id}"] & \\ & \zeta_1^* A^* \otimes L_1 \otimes \zeta_2^* A \otimes \zeta_2^* A^* \arrow[r, equal] & \zeta_1^* A^* \otimes L_1. \end{tikzcd} \tag{59}$$

Axiom (1M1) for the 1-morphism $\mathcal{A}^{-1}$ is satisfied because A^* has the negative curvature, and axiom (1M2) follows from the one for $\mathcal{A}$.

To construct the 2-isomorphism $i_l : \mathcal{A}^{-1} \circ \mathcal{A} \Rightarrow \mathrm{id}_{\mathcal{G}_1}$ notice that the 1-morphism $\mathcal{A}^{-1} \circ \mathcal{A}$ consists of the line bundle $\zeta_1^* A \otimes \zeta_2^* A^*$ over $\tilde{Z} = Z \times_{Y_2} Z$. We identify $\tilde{Z} \cong \tilde{Z} \times_P Y_1^{[2]}$, where $P = Y_1^{[2]}$, which allows us to choose a triple $(\tilde{Z}, \mathrm{id}_{\tilde{Z}}, \beta_{\tilde{Z}})$ defining i_l. In this triple, the isomorphism $\beta_{\tilde{Z}}$ is defined to be the composition

$$\zeta_1^* A \otimes \zeta_2^* A^* \xrightarrow{\mathrm{id} \otimes t_{\mu_2}^{-1} \otimes \mathrm{id}} \zeta_1^* A \otimes \Delta^* L_2 \otimes \zeta_2^* A^* \xrightarrow{\alpha^{-1} \otimes \mathrm{id}} L_1 \otimes \zeta_2^* A \otimes \zeta_2^* A^* = L_1. \tag{60}$$

Axiom (2M) for the isomorphism $\beta_{\tilde{Z}}$ follows from axiom (1M2) of $\mathcal{A}$, so that the triple $(\tilde{Z}, \mathrm{id}_{\tilde{Z}}, \beta_{\tilde{Z}})$ defines a 2-isomorphism $i_l : \mathcal{A}^{-1} \circ \mathcal{A} \Rightarrow \mathrm{id}_{\mathcal{G}_1}$. The 2-isomorphism $i_r : \mathrm{id}_{\mathcal{G}_2} \Rightarrow \mathcal{A} \circ \mathcal{A}^{-1}$ is constructed analogously: here we take the isomorphism

$$L_2 = \zeta_1^* A^* \otimes \zeta_1^* A \otimes L_2 \xrightarrow{\mathrm{id} \otimes \alpha^{-1}} \zeta_1^* A^* \otimes \Delta^* L_1 \otimes \zeta_2^* A \xrightarrow{\mathrm{id} \otimes t_{\mu_1} \otimes \mathrm{id}} \zeta_1^* A^* \otimes \zeta_2^* A. \tag{61}$$

of line bundles over W. Notice that by using the pairing $A^* \otimes A = 1$ we have used that A is a line bundle as assumed. Finally, the commutativity of diagram (58) follows from axiom (1M2) of $\mathcal{A}$.

Proposition 1.13 shows that we have many 1-morphisms in $\mathfrak{BGrb}(M)$ which are not invertible, in contrast to the 2-groupoid of bundle gerbes defined in [Ste00]. Notice that we have already benefited from the simple definition of the composition $\mathcal{A}^{-1} \circ \mathcal{A}$, which makes it also easy to see that it is compatible with the construction of inverse 1-morphisms $\mathcal{A}^{-1}$:

$$(\mathcal{A}_2 \circ \mathcal{A}_1)^{-1} = \mathcal{A}_1^{-1} \circ \mathcal{A}_2^{-1}. \tag{62}$$

1.14. Additional Structures. The 2-category of bundle gerbes has natural definitions of pullbacks, tensor products and dualities; all of them have been introduced for objects in [Mur96, MS00].

Pullbacks and tensor products of 1-morphisms and 2-morphisms can also be defined in a natural way, and we do not carry out the details here. Summarizing, the monoidal structure on $\mathfrak{BGrb}(M)$ is a strict 2-functor

$$\otimes \;:\; \mathfrak{BGrb}(M) \times \mathfrak{BGrb}(M) \longrightarrow \mathfrak{BGrb}(M), \tag{63}$$

for which the trivial bundle gerbe $\mathcal{I}_0$ is a strict tensor unit, i.e.

$$\mathcal{I}_0 \otimes \mathcal{G} = \mathcal{G} = \mathcal{G} \otimes \mathcal{I}_0. \tag{64}$$

The idea of the definition of $\otimes$ is to take fibre products of the involved surjective submersions, to pull back all the structure to this fibre product and then to use the monoidal structure of the category of vector bundles over that space. This was assumed to be strict, and so is $\otimes$. Pullbacks for the 2-category $\mathfrak{BGrb}(M)$ are implemented by strict monoidal 2-functors

$$f^* \;:\; \mathfrak{BGrb}(M) \longrightarrow \mathfrak{BGrb}(X) \tag{65}$$

associated to every smooth map $f : X \to M$ in the way that $g^* \circ f^* = (f \circ g)^*$ for a second smooth map $g : Y \to X$. The idea of its definition is, to pull back surjective submersions, for instance

$$\begin{tikzcd} f^{-1}Y \arrow[r, "\tilde{f}"] \arrow[d] & Y \arrow[d, "\pi"] \\ X \arrow[r, "f"'] & M \end{tikzcd} \tag{66}$$

and then pull back the structure over Y along the covering map $\tilde{f}$. The 2-functors $\otimes$ and f^* are all compatible with the assignment of inverses $\mathcal{A}^{-1}$ to 1-morphisms $\mathcal{A}$ from subsection 1.12:

$$f^*(\mathcal{A}^{-1}) = (f^*\mathcal{A})^{-1} \quad \text{and} \quad (\mathcal{A}_1 \otimes \mathcal{A}_2)^{-1} = \mathcal{A}_1^{-1} \otimes \mathcal{A}_2^{-1}. \tag{67}$$

Also the trivial bundle gerbes $\mathcal{I}_\rho$ behave naturally under pullbacks and tensor products:

$$f^*\mathcal{I}_\rho = \mathcal{I}_{f^*\rho} \quad \text{and} \quad \mathcal{I}_{\rho_1} \otimes \mathcal{I}_{\rho_2} = \mathcal{I}_{\rho_1+\rho_2}. \tag{68}$$

To define a duality we are a bit more precise, because this has yet not been done systematically in the literature. Even though we will strictly concentrate on what we need in section 3.8. For those purposes, it is enough to understand the duality as a strict 2-functor

$$()^* : \mathfrak{BGrb}(M)^{\mathrm{op}} \to \mathfrak{BGrb}(M) \tag{69}$$

where the opposed 2-category $\mathfrak{BGrb}(M)^{\mathrm{op}}$ has all 1-morphisms reversed, while the 2-morphisms are as before. This 2-functor will satisfy the identity

$$()^{**} = \mathrm{id}_{\mathfrak{BGrb}(M)}. \tag{70}$$

We now give the complete definition of the functor $()^*$ on objects, 1-morphisms and 2-morphisms. For a given bundle gerbe $\mathcal{G}$, the dual bundle gerbe $\mathcal{G}^*$ consists of the same surjective submersion $\pi : Y \to M$, the 2-form $-C \in \Omega^2(Y)$, the line bundle L^* over $Y^{[2]}$ and the isomorphism

$$\mu^{*-1} : \pi_{12}^* L^* \otimes \pi_{23}^* L^* \to \pi_{13}^* L^* \tag{71}$$

of line bundles over $Y^{[3]}$. This structure clearly satisfies the axioms of a bundle gerbe. We obtain immediately

$$\mathcal{G}^{**} = \mathcal{G} \quad \text{and} \quad (\mathcal{G} \otimes \mathcal{H})^* = \mathcal{H}^* \otimes \mathcal{G}^*, \tag{72}$$

and for the trivial bundle gerbe $\mathcal{I}_\rho$ the equation

$$\mathcal{I}_\rho^* = \mathcal{I}_{-\rho}. \tag{73}$$

For a 1-morphisms $\mathcal{A} : \mathcal{G}_1 \to \mathcal{G}_2$ consisting of a vector bundle A over Z with surjective submersion $\zeta : Z \to P$ with $P := Y_1 \times_M Y_2$ and of an isomorphism α of vector bundles over $Z \times_M Z$, we define the dual 1-morphism

$$\mathcal{A}^* \ : \ \mathcal{G}_2^* \longrightarrow \mathcal{G}_1^* \tag{74}$$

as follows: its surjective submersion is the pullback of ζ along the exchange map $s : P' \to P$, with $P' := Y_2 \times Y_1$; that is a surjective submersion $\zeta' : Z' \to P'$ and a covering map s_Z in the commutative diagram

$$\begin{tikzcd} Z' \arrow[r, "s_Z"] \arrow[d, "\zeta'"'] & Z \arrow[d, "\zeta"] \\ P' \arrow[r, "s"'] & P. \end{tikzcd} \tag{75}$$

The vector bundle of $\mathcal{A}^*$ is $A' := s_Z^* A$ over Z' and its isomorphism is

$$\begin{tikzcd} L_2^* \otimes \zeta_2'^* A' \arrow[r, equal] & L_2^* \otimes L_1 \otimes \zeta_2'^* s_Z^* A \otimes L_1^* \arrow[d, "\mathrm{id} \otimes \tilde{s}^* \alpha \otimes \mathrm{id}"] & \\ & L_2^* \otimes \zeta_1'^* s_Z^* A \otimes L_2 \otimes L_1^* \arrow[r, equal] & \zeta_1'^* A' \otimes L_1^*. \end{tikzcd} \tag{76}$$

Axiom (1M1) is satisfied since the dual bundle gerbes have 2-forms with opposite signs,

$$\mathrm{curv}(A') = s_Z^* \mathrm{curv}(A) = s_Z^*(C_2 - C_1) = C_2 - C_1 = (-C_1) - (-C_2). \tag{77}$$

Axiom (1M2) relates the isomorphism (76) to the isomorphisms μ_1^{*-1} and μ_2^{*-1} of the dual bundle gerbes. It can be deduced from axiom (1M2) of $\mathcal{A}$ using the following general fact, applied to μ_1^* and μ_2^*: the dual f^* of an isomorphism $f : L_1 \to L_2$ of line bundles coincides with the isomorphism

$$L_2^* = L_2^* \otimes L_1 \otimes L_1^* \xrightarrow{\mathrm{id}\otimes f\otimes\mathrm{id}} L_2^* \otimes L_2 \otimes L_1^* = L_1^*, \tag{78}$$

defined using the duality on line bundles.

Dual 1-morphisms defined like this have the properties

$$\mathcal{A}^{**} = \mathcal{A} \quad , \quad (\mathcal{A}' \circ \mathcal{A})^* = \mathcal{A}^* \circ \mathcal{A}'^* \quad \text{and} \quad (\mathcal{A}_1 \otimes \mathcal{A}_2)^* = \mathcal{A}_2^* \otimes \mathcal{A}_1^*. \tag{79}$$

Finally, for a 2-morphism $\beta : \mathcal{A}_1 \Rightarrow \mathcal{A}_2$ we define the dual 2-morphism

$$\beta^* \;:\; \mathcal{A}_1^* \Longrightarrow \mathcal{A}_2^* \tag{80}$$

in the following way. If β is represented by a triple (W, ω, β_W) with an isomorphism $\beta_W : A_1 \to A_2$ of vector bundles over W, we consider the pullback of $\omega : W \to Z_1 \times_P Z_2$ along $s_{Z_1} \times s_{Z_2} : Z_1' \times_{P'} Z_2' \to Z_1 \times_P Z_2$, where Z_1, Z_2' and P' are as in (75), and s_{Z_1} and s_{Z_2} are the respective covering maps. This gives a commutative diagram

$$\begin{tikzcd} W' \arrow[r, "s_W"] \arrow[d, "\omega'"'] & W \arrow[d, "\omega"] \\ Z_1' \times_{P'} Z_2' \arrow[r, "s_{Z_1} \times s_{Z_2}"'] & Z_1 \times_P Z_2. \end{tikzcd} \tag{81}$$

Now consider the triple $(W', \omega', s_W^*\beta_W)$ with the isomorphism

$$s_W^*\beta_W \;:\; s_{Z_1}^* A_1 \longrightarrow s_{Z_2}^* A_2 \tag{82}$$

of vector bundles over W'. It satisfies axiom (2M), and its equivalence class does not depend on the choice of the representative of β. So we define the dual 2-morphism β^* to be this class. Dual 2-morphisms are compatible with vertical and horizontal compositions

$$(\beta_2 \circ \beta_1)^* = \beta_1^* \circ \beta_2^* \quad \text{and} \quad (\beta \bullet \beta')^* = \beta^* \bullet \beta'^* \tag{83}$$

and satisfy furthermore

$$\beta^{**} = \beta \quad \text{and} \quad (\beta_1 \otimes \beta_2)^* = \beta_2^* \otimes \beta_1^*. \tag{84}$$

We can use adjoint 2-morphisms in the following situation: if $\mathcal{A} : \mathcal{G} \to \mathcal{H}$ is an invertible 1-morphism with inverse $\mathcal{A}^{-1}$ and associated 2-isomorphisms $i_l : \mathcal{A}^{-1} \circ \mathcal{A} \Rightarrow \mathrm{id}_{\mathcal{G}}$ and $i_r : \mathrm{id}_{\mathcal{H}} \Rightarrow \mathcal{A} \circ \mathcal{A}^{-1}$, their duals i_l^* and i_r^* show that $(\mathcal{A}^{-1})^*$ is an inverse to $\mathcal{A}^*$. Since inverses are unique up to isomorphism,

$$(\mathcal{A}^*)^{-1} \cong (\mathcal{A}^{-1})^*. \tag{85}$$

Summarizing, equations (72), (79), (83) and (84) show that $()^*$ is a monoidal strict 2-functor, which is strictly involutive. Let us finally mention that it is also compatible with pullbacks:

$$f^*(\mathcal{G}^*) = (f^*\mathcal{G})^* \quad , \quad f^*\mathcal{A}^* = (f^*\mathcal{A})^* \quad \text{and} \quad f^*\beta^* = (f^*\beta)^*. \tag{86}$$

2. Descent Theory for Morphisms

In this section we compare 1-morphisms between bundle gerbes in the sense of Definition 1.4 with 1-morphisms whose surjective submersion $\zeta : Z \to Y_1 \times_M Y_2$ is the identity, like in [Ste00]. For this purpose, we introduce the subcategory $\mathfrak{Hom}_{FP}(\mathcal{G}_1, \mathcal{G}_2)$ of the morphism category $\mathfrak{Hom}(\mathcal{G}_1, \mathcal{G}_2)$, where all smooth manifolds Z and W appearing in the definitions of 1- and 2-morphisms are equal to the fibre product $P := Y_1 \times_M Y_2$. Explicitly, an object in $\mathfrak{Hom}_{FP}(\mathcal{G}_1, \mathcal{G}_2)$ is a 1-morphism $\mathcal{A} : \mathcal{G}_1 \to \mathcal{G}_2$ whose surjective submersion is the identity id_P and a morphism in $\mathfrak{Hom}_{FP}(\mathcal{G}_1, \mathcal{G}_2)$ is a 2-morphism $\beta : \mathcal{A}_1 \Rightarrow \mathcal{A}_2$ which can be represented by the triple (P, ω, β) where $\omega : P \to P \times_P P \cong P$ is the identity.

2.1. Theorem. *The inclusion functor*

$$D \;:\; \mathfrak{Hom}_{FP}(\mathcal{G}_1, \mathcal{G}_2) \longrightarrow \mathfrak{Hom}(\mathcal{G}_1, \mathcal{G}_2)$$

is an equivalence of categories.

In the proof we will make use of the fact that vector bundles form a stack, i.e. fibred category satisfying a gluing condition. To make this gluing condition concrete, we define for a surjective submersion $\zeta : Z \to P$ a category $\mathfrak{Des}(\zeta)$ as follows. Its objects are pairs (A, d), where A is a vector bundle over Z and

$$d \;:\; \zeta_1^* A \longrightarrow \zeta_2^* A \tag{87}$$

is an isomorphism of vector bundles over $Z^{[2]}$ such that

$$\zeta_{13}^* d = \zeta_{23}^* d \circ \zeta_{12}^* d. \tag{88}$$

A morphism $\alpha : (A, d) \to (A', d')$ in $\mathfrak{Des}(\zeta)$ is an isomorphism $\alpha : A \to A'$ of vector bundles over Z such that the diagram

$$\begin{tikzcd} \zeta_1^* A \arrow[r, "\zeta_1^*\alpha"] \arrow[d, "d"'] & \zeta_1^* A' \arrow[d, "d'"] \\ \zeta_2^* A \arrow[r, "\zeta_2^*\alpha"'] & \zeta_2^* A' \end{tikzcd} \tag{89}$$

of isomorphisms of vector bundles over $Z^{[2]}$ is commutative. The composition of morphisms is just the composition of isomorphisms of vector bundles. Now, the gluing condition for the stack of vector bundles is that the pullback along ζ is an equivalence

$$\zeta^* \;:\; \mathfrak{Bun}(P) \longrightarrow \mathfrak{Des}(\zeta) \tag{90}$$

between the category $\mathfrak{Bun}(P)$ of vector bundles over P and the category $\mathfrak{Des}(\zeta)$.

Proof of Theorem 2.1. We show that the faithful functor D is an equivalence of categories by proving (a) that it is essentially surjective and (b) that the subcategory $\mathfrak{Hom}_{FP}(\mathcal{G}_1, \mathcal{G}_2)$ is full.

For (a) we have to show that for every 1-morphism $\mathcal{A} : \mathcal{G}_1 \to \mathcal{G}_2$ with arbitrary surjective submersion $\zeta : Z \to P$ there is an isomorphic 1-morphism $\mathcal{S}_\mathcal{A} : \mathcal{G}_1 \to \mathcal{G}_2$ with surjective submersion id_P. Notice that the isomorphism $\mathrm{d}_\mathcal{A} : \zeta_1^*A \to \zeta_2^*A$ of vector bundles over $Z^{[2]}$ from Lemma 1.5 satisfies the cocycle condition (88), so that $(A, \mathrm{d}_\mathcal{A})$ is an object in $\mathfrak{Des}(\zeta)$. Now consider the surjective submersion $\zeta^2 : Z \times_M Z \to P^{[2]}$. By Lemma 1.5 b) and under the identification of $Z^{[2]} \times_M Z^{[2]}$ with $(Z \times_M Z) \times_{P^{[2]}} (Z \times_M Z)$ the diagram

$$\begin{tikzcd} L_1 \otimes \zeta_2^*A \arrow[r, "\zeta_{12}^*\alpha"] \arrow[d, "1\otimes\zeta_{24}^*\mathrm{d}_\mathcal{A}"'] & \zeta_1^*A \otimes L_2 \arrow[d, "\zeta_{13}^*\mathrm{d}_\mathcal{A}\otimes 1"] \\ L_1 \otimes \zeta_4^*A \arrow[r, "\zeta_{34}^*\alpha"'] & \zeta_3^*A \otimes L_2 \end{tikzcd} \tag{91}$$

of isomorphisms of vector bundles over $(Z \times_M Z) \times_{P^{[2]}} (Z \times_M Z)$ is commutative, and shows that α is a morphism in $\mathfrak{Des}(\zeta^2)$. Now we use that ζ^* is an equivalence of categories: we choose a vector bundle S over P together with an isomorphism $\beta : \zeta^*S \to A$ of vector bundles over Z, and an isomorphism

$$\sigma \;:\; L_1 \otimes \zeta_2^*S \longrightarrow \zeta_1^*S \otimes L_2 \tag{92}$$

of vector bundles over $P \times_M P$ such that the diagram

$$\begin{tikzcd} L_1 \otimes \zeta_2^*\zeta^*S \arrow[r, "\zeta^*\sigma"] \arrow[d, "\mathrm{id}\otimes\zeta_2^*\beta"'] & \zeta_1^*\zeta^*S \otimes L_2 \arrow[d, "\zeta_1^*\beta\otimes\mathrm{id}"] \\ L_1 \otimes \zeta_2^*A \arrow[r, "\alpha"'] & \zeta_1^*A \otimes L_2 \end{tikzcd} \tag{93}$$

of isomorphisms of vector bundles over $Z\times_M Z$ is commutative. Since ζ is an equivalence of categories, the axioms of $\mathcal{A}$ imply the ones of the 1-morphism $\mathcal{S}_\mathcal{A}$ defined by the surjective submersion id_P, the vector bundle S over P and the isomorphism σ over $P^{[2]}$. Finally, the triple $(Z \times_P P, \mathrm{id}_Z, \beta)$ with $Z \cong Z \times_P P$ defines a 2-morphism $\mathcal{S}_\mathcal{A} \Rightarrow \mathcal{A}$, whose axiom (2M) is (93).

(b) We have to show that any morphism $\beta : \mathcal{A} \Rightarrow \mathcal{A}'$ in $\mathfrak{Hom}(\mathcal{G}_1, \mathcal{G}_2)$ between objects $\mathcal{A}$ and $\mathcal{A}'$ in $\mathfrak{Hom}_{FP}(\mathcal{G}_1, \mathcal{G}_2)$ is already a morphism in $\mathfrak{Hom}_{FP}(\mathcal{G}_1, \mathcal{G}_2)$. Let (W, ω, β_W) be any representative of β with a surjective submersion $\omega : W \to P$ and an isomorphism $\beta_W : \omega^*A \to \omega^*A'$ of vector bundles over W. The restriction of axiom (2M) for the triple (W, ω, β_W) from $W \times_M W$ to $W \times_P W$ shows $\omega_1^*\beta_W = \omega_2^*\beta_W$. This shows that β_W is a morphism in the descent category $\mathfrak{Des}(\omega)$. Let $\beta_P : A \to A'$ be an isomorphism of vector bundles over P such that

$$\omega^*\beta_P = \beta_W \tag{94}$$

Because ω is an equivalence of categories, the triple $(P, \mathrm{id}_P, \beta_P)$ defines a 2-morphism from $\mathcal{A}$ to $\mathcal{A}'$ being a morphism in $\mathfrak{Hom}_{FP}(\mathcal{G}_1, \mathcal{G}_2)$. Equation (94) shows that the triples $(P, \mathrm{id}_P, \beta_P)$ and (W, ω, β_W) are equivalent. $\square$

In the remainder of this section we present two corollaries of Theorem 2.1. First, and most importantly, we make contact to the notion of a stable isomorphism between bundle gerbes. By definition [MS00], a stable isomorphism is a 1-morphism, whose surjective submersion is the identity id_P on the fibre product of the surjective submersions of the two bundle gerbes, and whose vector bundle over P is a line bundle. From Proposition 1.13 and Theorem 2.1 we obtain

2.2. COROLLARY. *There exists a stable isomorphism $\mathcal{A} : \mathcal{G}_1 \to \mathcal{G}_2$ if and only if the bundle gerbes are isomorphic objects in $\mathfrak{BGrb}(M)$.*

It is shown in [MS00] that the set of stable isomorphism classes of bundle gerbes over M is a group (in virtue of the monoidal structure) which is isomorphic to the Deligne cohomology group $\mathrm{H}^2(M, \mathcal{D}(2))$. This is a very important fact which connects the theory of bundle gerbes to other theories of gerbes, for instance, to Dixmier-Douady sheaves of groupoids [Bry93]. Corollary 2.2 states that although our definition of morphisms differs from the one of [MS00], the bijection between isomorphism classes of bundle gerbes and the Deligne cohomology group persists.

Second, Theorem 2.1 admits to use existing classification results for 1-isomorphisms. Consider the full subgroupoid $\mathfrak{Aut}(\mathcal{G})$ of $\mathfrak{Hom}(\mathcal{G}, \mathcal{G})$ associated to a bundle gerbe $\mathcal{G}$, which consists of all 1-isomorphisms $\mathcal{A} : \mathcal{G} \to \mathcal{G}$, and all (necessarily invertible) 2-morphisms between those. From Theorem 2.1 and Lemma 2 of [SSW05] we obtain

2.3. COROLLARY. *The skeleton of the groupoid $\mathfrak{Aut}(\mathcal{G})$, i.e. the set of isomorphism classes of 1-isomorphisms $\mathcal{A} : \mathcal{G} \to \mathcal{G}$ is a torsor over the group $\mathrm{Pic}_0(M)$ of isomorphism classes of flat line bundles over M.*

In 2-dimensional conformal field theory, where a bundle gerbe $\mathcal{G}$ is considered to be a part of the background field, the groupoid $\mathfrak{Aut}(\mathcal{G})$ may be called the groupoid of gauge transformations of $\mathcal{G}$. The above corollary classifies such gauge transformation up to equivalence.

3. Some important Examples of Morphisms

To discuss holonomies of bundle gerbes, it is essential to establish an equivalence between the morphism categories between trivial bundle gerbes over M and vector bundles of certain curvature over M. Given two 2-forms ρ_1 and ρ_2 on M, consider the category $\mathfrak{Hom}_{FP}(\mathcal{I}_{\rho_1}, \mathcal{I}_{\rho_2})$. An object $\mathcal{A} : \mathcal{I}_{\rho_1} \to \mathcal{I}_{\rho_2}$ consists of the smooth manifold $Z = M$ with the surjective submersion $\zeta = \mathrm{id}_M$, a vector bundle A over M and an isomorphism $\alpha : A \to A$. Axiom (1M2) states

$$\frac{1}{n}\mathrm{tr}(\mathrm{curv}(A)) = \rho_2 - \rho_1 \tag{95}$$

with n the rank of A, and axiom (1M2) reduces to $\alpha^2 = \alpha$, which in turn means $\alpha = \mathrm{id}_A$. Together with Theorem 2.1, this defines a canonical equivalence of categories

$$\mathrm{Bun} \;:\; \mathfrak{Hom}(\mathcal{I}_{\rho_1}, \mathcal{I}_{\rho_2}) \longrightarrow \mathfrak{Bun}_{\rho_2 - \rho_1}(M), \tag{96}$$

where $\mathfrak{Bun}_\rho(M)$ is the category of vector bundles A over M whose curvature satisfies (95). Its following properties can directly be deduced from the definitions.

3.1. PROPOSITION. *The functor* Bun *respects the structure of the 2-category of bundle gerbes, namely:*

a) the composition of 1-morphisms,

$$\mathrm{Bun}(\mathcal{A}_2 \circ \mathcal{A}_1) = \mathrm{Bun}(\mathcal{A}_1) \otimes \mathrm{Bun}(\mathcal{A}_2) \quad \text{and} \quad \mathrm{Bun}(\mathrm{id}_{\mathcal{I}_\rho}) = 1.$$

b) the assignment of inverses to invertible 1-morphisms,

$$\mathrm{Bun}(\mathcal{A}^{-1}) = \mathrm{Bun}(\mathcal{A})^*.$$

c) the monoidal structure,

$$\mathrm{Bun}(\mathcal{A}_1 \otimes \mathcal{A}_2) = \mathrm{Bun}(\mathcal{A}_1) \otimes \mathrm{Bun}(\mathcal{A}_2).$$

d) pullbacks,

$$\mathrm{Bun}(f^*\mathcal{A}) = f^*\mathrm{Bun}(\mathcal{A}) \quad \text{and} \quad \mathrm{Bun}(f^*\beta) = f^*\mathrm{Bun}(\beta).$$

e) the duality

$$\mathrm{Bun}(\mathcal{A}^*) = \mathrm{Bun}(\mathcal{A}) \quad \text{and} \quad \mathrm{Bun}(\beta^*) = \mathrm{Bun}(\beta).$$

In the following subsections we see how the 2-categorial structure of bundle gerbes and the functor Bun can be used to give natural definitions of surface holonomy in several situations.

3.2. TRIVIALIZATIONS. We give the following natural definition of a trivialization.

3.3. DEFINITION. *A trivialization of a bundle gerbe $\mathcal{G}$ is a 1-isomorphism*

$$\mathcal{T} : \mathcal{G} \longrightarrow \mathcal{I}_\rho.$$

Let us write out the details of such a 1-isomorphism. By Theorem 2.1 we may assume that the surjective submersion of $\mathcal{T}$ is the identity id_P on $P := Y \times_M M \cong Y$ with projection π to M. Then, $\mathcal{T}$ consists further of a line bundle T over Y, and of an isomorphism $\tau : L \otimes \pi_2^*T \to \pi_1^*T$ of line bundles over $Y^{[2]}$. Axiom (1M2) gives $\pi_{13}^*\tau \circ (\mu \otimes \mathrm{id}) = \pi_{12}^*\tau \circ \pi_{23}^*\tau$ as an equation of isomorphisms of line bundles over $Y^{[3]}$. This is exactly the definition of a trivialization one finds in the literature [CJM02]. Additionally, axiom (1M2) gives $\mathrm{curv}(T) = \pi^*\rho - C$. If one does not specify ρ as a part of the definition of a trivialization, it is uniquely determined by this equation.

Trivializations are essential for the definition of holonomy around closed oriented surfaces.

3.4. Definition. *Let $\phi : \Sigma \to M$ be a smooth map from a closed oriented surface Σ to a smooth manifold M, and let $\mathcal{G}$ a bundle gerbe over M. Let*

$$\mathcal{T} \;:\; \phi^*\mathcal{G} \longrightarrow \mathcal{I}_\rho$$

be any trivialization. The holonomy of $\mathcal{G}$ around ϕ is defined as

$$\mathrm{hol}_{\mathcal{G}}(\phi) := \exp\left(\mathrm{i}\int_\Sigma \rho\right) \in U(1).$$

In this situation, the functor Bun is a powerful tool to prove that this definition does not depend on the choice of the trivialization: if $\mathcal{T}' : \phi^*\mathcal{G} \to \mathcal{I}_{\rho'}$ is another trivialization, the composition $\mathcal{T} \circ \mathcal{T}'^{-1} : \mathcal{I}_{\rho'} \to \mathcal{I}_\rho$ corresponds to a line bundle $\mathrm{Bun}(\mathcal{T} \circ \mathcal{T}'^{-1})$ over M with curvature $\rho - \rho'$. In particular, the difference between any two 2-forms ρ is a closed 2-form with integer periods and vanishes under the exponentiation in the definition of $\mathrm{hol}_{\mathcal{G}}(\phi)$.

3.5. Bundle Gerbe Modules. For oriented surfaces with boundary one has to choose additional structure on the boundary to obtain a well-defined holonomy. This additional structure is provided by a vector bundle twisted by the bundle gerbe $\mathcal{G}$ [Gaw05], also known as a $\mathcal{G}$-module. In our formulation, its definition takes the following form:

3.6. Definition. *Let $\mathcal{G}$ be a bundle gerbe over M. A left $\mathcal{G}$-module is a 1-morphism $\mathcal{E} : \mathcal{G} \to \mathcal{I}_\omega$, and a right $\mathcal{G}$-module is a 1-morphism $\mathcal{F} : \mathcal{I}_\omega \to \mathcal{G}$.*

Let us compare this definition with the original definition of (left) bundle gerbe modules in [BCM+02]. Assume – again by Theorem 2.1 – that a left $\mathcal{G}$-module $\mathcal{E} : \mathcal{G} \to \mathcal{I}_\omega$ has the surjective submersion id_P with $P \cong Y$. Then, it consists of a vector bundle E over Y and of an isomorphism $\epsilon : L \otimes \pi_2^*E \to \pi_1^*E$ of vector bundles over $Y^{[2]}$ which satisfies

$$\pi_{13}^*\epsilon \circ (\mu \otimes \mathrm{id}) = \pi_{23}^*\epsilon \circ \pi_{12}^*\epsilon \tag{97}$$

by axiom (1M2). The curvature of E is restricted by axiom (1M2) to

$$\frac{1}{n}\mathrm{tr}(\mathrm{curv}(E)) = \pi^*\omega - C \tag{98}$$

with n the rank of E.

The definition of bundle gerbe modules as 1-morphisms makes clear that left and right $\mathcal{G}$-modules form categories $\mathfrak{LMod}(\mathcal{G})$ and $\mathfrak{RMod}(\mathcal{G})$. This is useful for example to see that a 1-isomorphism $\mathcal{A} : \mathcal{G} \to \mathcal{G}'$ defines equivalences of categories

$$\mathfrak{LMod}(\mathcal{G}) \cong \mathfrak{LMod}(\mathcal{G}') \quad \text{and} \quad \mathfrak{RMod}(\mathcal{G}) \cong \mathfrak{RMod}(\mathcal{G}') \tag{99}$$

and that there are equivalences between left modules of $\mathcal{G}$ and right modules of $\mathcal{G}^*$ (and vice versa), by taking duals of the respective 1-morphisms. Moreover, for a trivial bundle gerbe $\mathcal{I}_\rho$ the categories $\mathfrak{LMod}(\mathcal{I}_\rho)$ and $\mathfrak{RMod}(\mathcal{I}_\rho)$ become canonically equivalent to the category $\mathfrak{Bun}(M)$ of vector bundles over M via the functor Bun. We can combine this result with the equivalences (99) applied to a trivialization $\mathcal{T} : \mathcal{G} \to \mathcal{I}_\rho$ of a bundle gerbe $\mathcal{G}$ over M. In detail, a left $\mathcal{G}$-module $\mathcal{E} : \mathcal{G} \to \mathcal{I}_\omega$ first becomes a left $\mathcal{I}_\rho$-module

$$\mathcal{E} \circ \mathcal{T}^{-1} \;:\; \mathcal{I}_\rho \longrightarrow \mathcal{I}_\omega \tag{100}$$

which in turn defines the vector bundle $E := \mathrm{Bun}(\mathcal{E} \circ \mathcal{T}^{-1})$ over M. The same applies to right $\mathcal{G}$-modules $\mathcal{F} : \mathcal{I}_\omega \to \mathcal{G}$ which defines a vector bundle $\bar{E} := \mathrm{Bun}(\mathcal{T} \circ \mathcal{F})$ over M.

A D-brane for the bundle gerbe $\mathcal{G}$ is a submanifold Q of M together with a left $\mathcal{G}|_Q$-module. Here $\mathcal{G}|_Q$ means the pullback of $\mathcal{G}$ along the inclusion $Q \hookrightarrow M$.

3.7. Definition. *Let $\mathcal{G}$ be a bundle gerbe over M with D-brane $(Q, \mathcal{E})$ and let $\phi : \Sigma \to M$ be a smooth map from a compact oriented surface Σ with boundary to M, such that $\phi(\partial\Sigma) \subset Q$. Let*

$$\mathcal{T} \;:\; \phi^*\mathcal{G} \longrightarrow \mathcal{I}_\rho$$

be any trivialization of the pullback bundle gerbe $\phi^*\mathcal{G}$ and let

$$E := \mathrm{Bun}(\phi^*\mathcal{E} \circ \mathcal{T}^{-1}) \tag{101}$$

be the associated vector bundle over $\partial\Sigma$. The holonomy of $\mathcal{G}$ around ϕ is defined as

$$\mathrm{hol}_{\mathcal{G},\mathcal{E}}(\phi) := \exp\left(\mathrm{i}\int_\Sigma \rho\right) \cdot \mathrm{tr}\left(\mathrm{hol}_E(\partial\Sigma)\right) \in \mathbb{C}.$$

The definition does not depend on the choice of the trivialization: for another trivialization $\mathcal{T}' : \phi^*\mathcal{G} \to \mathcal{I}_{\rho'}$ and the respective vector bundle $E' := \mathrm{Bun}(\mathcal{E} \circ \mathcal{T}'^{-1})$ we find by Proposition 3.1 a)

$$E' = \mathrm{Bun}(\mathcal{E} \circ \mathcal{T}'^{-1}) \cong \mathrm{Bun}(\mathcal{E} \circ \mathcal{T}^{-1} \circ \mathcal{T} \circ \mathcal{T}'^{-1}) = E \otimes \mathrm{Bun}(\mathcal{T} \circ \mathcal{T}'^{-1}). \tag{102}$$

Because isomorphic vector bundles have the same holonomies, and the line bundle $\mathrm{Bun}(\mathcal{T} \circ \mathcal{T}'^{-1})$ has curvature $\rho - \rho'$ we obtain

$$\mathrm{tr}\left(\mathrm{hol}_{E'}(\partial\Sigma)\right) = \mathrm{tr}\left(\mathrm{hol}_E(\partial\Sigma)\right) \cdot \exp\left(\mathrm{i}\int_\Sigma \rho - \rho'\right). \tag{103}$$

This shows the independence of the choice of the trivialization.

3.8. Jandl Structures. In this last section, we use the duality on the 2-category $\mathfrak{BGrb}(M)$ introduced in section 1.14 to define the holonomy of a bundle gerbe around unoriented, and even unorientable surfaces (without boundary). For this purpose, we explain the concept of a Jandl structure on a bundle gerbe $\mathcal{G}$, which has been introduced in [SSW05], in terms of 1- and 2-isomorphisms of the 2-category $\mathfrak{BGrb}(M)$.

3.9. Definition. *A Jandl structure $\mathcal{J}$ on a bundle gerbe $\mathcal{G}$ over M is a collection $(k, \mathcal{A}, \varphi)$ of an involution $k : M \to M$, i.e. a diffeomorphism with $k \circ k = \mathrm{id}_M$, a 1-isomorphism*

$$\mathcal{A} \,:\, k^*\mathcal{G} \longrightarrow \mathcal{G}^*$$

and a 2-isomorphism

$$\varphi \,:\, k^*\mathcal{A} \Longrightarrow \mathcal{A}^*$$

which satisfies the condition

$$k^*\varphi = \varphi^{*-1}.$$

Notice that the existence of the 2-isomorphism φ is only possible because $\mathcal{G}^{**} = \mathcal{G}$ from (72), and that the equation $k^*\varphi = \varphi^{*-1}$ only makes sense because $\mathcal{A}^{**} = \mathcal{A}$ from (79). Let us now discuss the relation between Definition 3.9 and the original definition of a Jandl structure from [SSW05]. For this purpose we elaborate the details. We denote the pullback of the surjective submersion $\pi : Y \to M$ along k by $\pi_k : Y_k \to M$; for simplicity we take $Y_k := Y$ and $\pi_k := k \circ \pi$. Now, we assume by Theorem 2.1 that the 1-isomorphism $\mathcal{A}$ consists of a line bundle A over $Y_k \times_M Y$. As smooth manifolds, we can identify $Y_k \times_M Y$ with $P := Y^{[2]}$; to have an identification as smooth manifolds with surjective submersions to M, we define the projection $p : P \to M$ by $p := \pi \circ \pi_2$. Under this identification, the exchange map $s : Y \times_M Y_k \to Y_k \times_M Y$ becomes an involution of P which lifts k,

$$\begin{tikzcd} P \arrow[r, "s"] \arrow[d, "p"'] & P \arrow[d, "p"] \\ M \arrow[r, "k"'] & M. \end{tikzcd} \tag{104}$$

The dual 1-isomorphism $\mathcal{A}^*$ has by definition the line bundle s^*A over P. Now, similarly as for the pullback of $\pi : Y \to M$ we denote the pullback of $p : P \to M$ by $p_k : P_k \to M$ and choose $P_k := P$ and $p_k := k \circ p$. This way, the pullback 1-isomorphism $k^*\mathcal{A}$ has the line bundle A over P. Again by Theorem 2.1, we assume that the 2-isomorphism φ can be represented by a triple $(P, \mathrm{id}_P, \varphi_P)$ with an isomorphism $\varphi_P : A \to s^*A$ of line bundles over P satisfying the compatibility axiom (2M) with the isomorphism α of $\mathcal{A}$:

$$\begin{tikzcd} L \otimes \zeta_2^*A \arrow[r, "\alpha"] \arrow[d, "\mathrm{id}\otimes\zeta_2^*\varphi_P"'] & \zeta_1^*A \otimes L \arrow[d, "\zeta_1^*\varphi_P\otimes\mathrm{id}"] \\ L \otimes \zeta_2^*s^*A \arrow[r, "s^*\alpha"'] & \zeta_1^*s^*A \otimes L \end{tikzcd} \tag{105}$$

The dual 2-isomorphism φ^* is given by $(P, \mathrm{id}_P, s^*\varphi_P)$, and the equation $\varphi = k^*\varphi^{*-1}$ becomes $\varphi_P = s^*\varphi_P^{-1}$. So, φ_P is an s-equivariant structure on A. This is exactly the original definition [SSW05]: a stable isomorphism $\mathcal{A} : k^*\mathcal{G} \to \mathcal{G}^*$, whose line bundle A is equipped with an s-equivariant structure which is compatible with the isomorphism α of $\mathcal{A}$ in the sense of the commutativity of diagram (105).

Defining a Jandl structure in terms of 1- and 2-morphisms has – just like for gerbe modules – several advantages. For example, it is easy to see that Jandl structures are compatible with pullbacks along equivariant maps, tensor products and duals of bundle gerbes. Furthermore, we have an obvious definition of morphisms between Jandl structures, which induces exactly the notion of equivalent Jandl structures we introduced in [SSW05].

3.10. DEFINITION. *A morphism $\beta : \mathcal{J} \to \mathcal{J}'$ between Jandl structures $\mathcal{J} = (k, \mathcal{A}, \varphi)$ and $\mathcal{J}' = (k, \mathcal{A}', \varphi')$ on the same bundle gerbe $\mathcal{G}$ over M with the same involution k is a 2-morphism*

$$\beta \;:\; \mathcal{A} \Longrightarrow \mathcal{A}'$$

which commutes with φ and φ' in the sense that the diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, Rightarrow, "\varphi"] \arrow[d, Rightarrow, "\beta"'] & k^*\mathcal{A}^* \arrow[d, Rightarrow, "k^*\beta^*"] \\ \mathcal{A}' \arrow[r, Rightarrow, "\varphi'"'] & k^*\mathcal{A}'^* \end{tikzcd}$$

of 2-morphisms is commutative.

Since $\mathcal{A}$ is invertible, every morphism of Jandl structures is invertible. We may thus speak of a groupoid $\mathfrak{Jdl}(\mathcal{G}, k)$ of Jandl structures on the bundle gerbe $\mathcal{G}$ with involution k. The skeleton of this groupoid has been classified [SSW05]: it forms a torsor over the group of flat k-equivariant line bundles over M. The following proposition relates these groupoids of Jandl structures on isomorphic bundle gerbes on the same space with the same involution. This relation is a new result, coming and benefiting very much from the 2-categorial structure of bundle gerbes we have developed.

3.11. PROPOSITION. *Any 1-isomorphism $\mathcal{B} : \mathcal{G} \to \mathcal{G}'$ induces an equivalence of groupoids*

$$J_{\mathcal{B}} \;:\; \mathfrak{Jdl}(\mathcal{G}', k) \longrightarrow \mathfrak{Jdl}(\mathcal{G}, k)$$

with the following properties:

a) any 2-morphism $\beta : \mathcal{B} \Rightarrow \mathcal{B}'$ induces a natural equivalence $J_{\mathcal{B}} \cong J_{\mathcal{B}'}$.

b) there is a natural equivalence $J_{\mathrm{id}_{\mathcal{G}}} \cong \mathrm{id}_{\mathfrak{Jdl}(\mathcal{G},k)}$.

c) it respects the composition of 1-morphisms in the sense that

$$J_{\mathcal{B}' \circ \mathcal{B}} = J_{\mathcal{B}} \circ J_{\mathcal{B}'}.$$

Proof. The functor $J_{\mathcal{B}}$ sends a Jandl structure $(k, \mathcal{A}, \varphi)$ on $\mathcal{G}'$ to the triple $(k, \mathcal{A}', \varphi')$ with the same involution k, the 1-isomorphism

$$\mathcal{A}' := \mathcal{B}^* \circ \mathcal{A} \circ k^*\mathcal{B} \;:\; k^*\mathcal{G} \longrightarrow \mathcal{G}^* \tag{106}$$

and the 2-isomorphism

$$\begin{tikzcd} k^*\mathcal{A}' \arrow[r, equal] & k^*\mathcal{B}^* \circ k^*\mathcal{A} \circ \mathcal{B} \arrow[d, Rightarrow, "\mathrm{id}_{k^*\mathcal{B}^*} \circ \varphi \circ \mathrm{id}_{\mathcal{B}}"] & \\ & k^*\mathcal{B}^* \circ \mathcal{A}^* \circ \mathcal{B} \arrow[r, equal] & k^*\mathcal{A}'^* \end{tikzcd} \tag{107}$$

where we use equation (79). The following calculation shows that $(k, \mathcal{A}', \varphi')$ is a Jandl structure:

$$\begin{aligned} k^*\varphi'^* &\stackrel{\text{def}}{=} k^*(\mathrm{id}_{k^*\mathcal{B}^*} \circ \varphi \circ \mathrm{id}_{\mathcal{B}})^* \\ &\stackrel{(79)}{=} \mathrm{id}_{k^*\mathcal{B}^*} \circ k^*\varphi^* \circ \mathrm{id}_{\mathcal{B}} \\ &= \mathrm{id}_{\mathcal{B}} \circ \varphi^{-1} \circ \mathrm{id}_{\mathcal{B}^*} \\ &\stackrel{\text{def}}{=} \varphi'^{-1}. \end{aligned} \tag{108}$$

A morphism β of Jandl structures on $\mathcal{G}'$ is sent to the morphism

$$J_{\mathcal{B}}(\beta) := \mathrm{id}_{\mathcal{B}^*} \circ \beta \circ \mathrm{id}_{k^*\mathcal{B}} \tag{109}$$

of the respective Jandl structures on $\mathcal{G}'$. The two axioms of the composition functor $\circ$ from Lemma 1.9 show that the composition of morphisms of Jandl structures is respected, so that $J_{\mathcal{B}}$ is a functor. It is an equivalence because $J_{\mathcal{B}^{-1}}$ is an inverse functor, where the natural equivalences $J_{\mathcal{B}^{-1}} \circ J_{\mathcal{B}} \cong \mathrm{id}$ and $J_{\mathcal{B}} \circ J_{\mathcal{B}^{-1}} \cong \mathrm{id}$ use the 2-isomorphisms i_r and i_l from section 1.12 associated to the inverse 1-morphism $\mathcal{B}^{-1}$.

To prove a), let $\beta : \mathcal{B} \Rightarrow \mathcal{B}'$ be a 2-morphism. We define the natural equivalence $J_{\mathcal{B}} \cong J_{\mathcal{B}'}$, which is a collection of morphisms $\beta_{\mathcal{J}} : J_{\mathcal{B}}(\mathcal{J}) \to J_{\mathcal{B}'}(\mathcal{J})$ of Jandl structures on $\mathcal{G}$ for any Jandl structure $\mathcal{J}$ on $\mathcal{G}'$ by

$$\beta_{\mathcal{J}} := \beta^* \circ \mathrm{id}_{\mathcal{A}} \circ k^*\beta. \tag{110}$$

This defines indeed a morphism of Jandl structures and makes the naturality square

$$\begin{tikzcd} J_{\mathcal{B}}(\mathcal{J}) \arrow[r, "\beta_{\mathcal{J}}"] \arrow[d, "J_{\mathcal{B}}(\beta)"'] & J_{\mathcal{B}'}(\mathcal{J}) \arrow[d, "J_{\mathcal{B}'}(\beta)"] \\ J_{\mathcal{B}}(\mathcal{J}') \arrow[r, "\beta_{\mathcal{J}'}"'] & J_{\mathcal{B}'}(\mathcal{J}') \end{tikzcd} \tag{111}$$

commutative. The natural equivalence for b) uses the 2-isomorphisms $\lambda_{\mathcal{A}}$ and $\rho_{\mathcal{A}}$ of the 2-category $\mathfrak{BGrb}(M)$ and the fact that $\mathrm{id}_{\mathcal{G}}^* = \mathrm{id}_{\mathcal{G}^*}$. Finally, c) follows from the definition of $J_{\mathcal{B}}$ and the fact that the duality functor $()^*$ respects the composition of 1-morphisms, see (79). $\square$

It is worthwhile to consider a Jandl structure $\mathcal{J} = (k, \mathcal{A}, \varphi)$ over a trivial bundle gerbe $\mathcal{I}_\rho$. By definition, this is a 1-isomorphism

$$\mathcal{A} \,:\, \mathcal{I}_{k^*\rho} \longrightarrow \mathcal{I}_{-\rho} \tag{112}$$

and a 2-isomorphism $\varphi : k^*\mathcal{A} \Rightarrow \mathcal{A}^*$ satisfying $\varphi = k^*\varphi^{*-1}$. Now we apply the functor Bun and obtain a line bundle $\hat{R} := \mathrm{Bun}(\mathcal{A})$ over M of curvature $-(\rho + k^*\rho)$ and an isomorphism $\hat{\varphi} := \mathrm{Bun}(\varphi) : k^*\hat{R} \to \hat{R}$ of line bundles over M which satisfies $\hat{\varphi} = k^*\hat{\varphi}^{-1}$, summarizing: a k-equivariant line bundle. So, the functor Bun induces an equivalence of groupoids

$$\mathrm{Bun}^k_\rho \,:\, \mathfrak{Jdl}(\mathcal{I}_\rho, k) \longrightarrow \mathfrak{LBun}^k_{-(\rho+k^*\rho)}(M) \tag{113}$$

between the groupoid of Jandl structures on $\mathcal{I}_\rho$ with involution k and the groupoid of k-equivariant line bundles over M with curvature $-(\rho + k^*\rho)$. In particular, if $\mathcal{G}$ is a bundle gerbe over M and $\mathcal{T} : \mathcal{G} \to \mathcal{I}_\rho$ a trivialization, we obtain a functor

$$\mathfrak{Jdl}(\mathcal{G}, k) \xrightarrow{\;J_{\mathcal{T}^{-1}}\;} \mathfrak{Jdl}(\mathcal{I}_\rho, k) \xrightarrow{\;\mathrm{Bun}^k_\rho\;} \mathfrak{LBun}^k_{-(\rho+k^*\rho)}(M) \tag{114}$$

converting a Jandl structure on the bundle gerbe $\mathcal{G}$ into a k-equivariant line bundle over M. It becomes obvious that the existence of a Jandl structure with involution k on the trivial bundle gerbe $\mathcal{I}_\rho$ constraints the 2-form ρ: as the curvature of a line bundle, the 2-form $-(\rho + k^*\rho)$ has to be closed and to have integer periods.

Let us now explain how Jandl structures enter in the definition of holonomy around unoriented surfaces, and how we can take further advantage of the 2-categorial formalism. We have learned before that to incorporate surfaces with boundary we had to do two steps: we first specified additional structure, a D-brane of the bundle gerbe $\mathcal{G}$, and then specified which maps $\phi : \Sigma \to M$ are compatible with this additional structure: those who send the boundary of Σ into the support of the D-brane. To discuss unoriented surfaces (without boundary), we also do these two steps: the additional structure we choose here is a Jandl structure $\mathcal{J} = (k, \mathcal{A}, \varphi)$ on the bundle gerbe $\mathcal{G}$. To describe the space of maps we want to consider, we have to introduce the following geometric structures [SSW05]:

- For any (unoriented) closed surface Σ there is an oriented two-fold covering $\mathrm{pr} : \hat{\Sigma} \to \Sigma$. It is unique up to orientation-preserving diffeomorphisms and it is connected if and only if Σ is not orientable. It has a canonical, orientation-reversing involution σ, which permutes the sheets and preserves the fibres. We call this two-fold covering the orientation covering of Σ.

- A fundamental domain of Σ in $\hat{\Sigma}$ is a submanifold F of $\hat{\Sigma}$ with (possibly only piecewise smooth) boundary, such that

  $$F \cup \sigma(F) = \hat{\Sigma} \quad \text{and} \quad F \cap \sigma(F) = \partial F. \tag{115}$$

  A key observation is that the involution σ restricts to an orientation-*preserving* involution on $\partial F \subset \hat{\Sigma}$. Accordingly, the quotient $\overline{\partial F}$ is an oriented closed 1-dimensional submanifold of Σ.

Now, given a closed surface Σ, we consider maps $\hat{\phi} : \hat{\Sigma} \to M$ from the orientation covering $\hat{\Sigma}$ to M, which are equivariant with respect to the two involutions on $\hat{\Sigma}$ and M, i.e. the diagram

$$\begin{tikzcd} \hat{\Sigma} \arrow[r, "\hat{\phi}"] \arrow[d, "\sigma"'] & M \arrow[d, "k"] \\ \hat{\Sigma} \arrow[r, "\hat{\phi}"'] & M \end{tikzcd}$$

has to be commutative.

3.12. Definition. *Let $\mathcal{J}$ be a Jandl structure on a bundle gerbe $\mathcal{G}$ over M, and let $\hat{\phi} : \hat{\Sigma} \to M$ be an equivariant smooth map. For a trivialization*

$$\mathcal{T} \;:\; \hat{\phi}^*\mathcal{G} \longrightarrow \mathcal{I}_\rho$$

let $\hat{R}$ be the σ-equivariant line bundle over $\hat{\Sigma}$ determined by the functor

$$\mathrm{Bun}^\rho_\sigma \circ J_{\mathcal{T}^{-1}} : \mathfrak{Jdl}(\hat{\phi}^*\mathcal{G}, \sigma) \longrightarrow \mathfrak{LBun}^\sigma_{-(\rho+\sigma^*\rho)}(\hat{\Sigma}) \tag{116}$$

from (114). In turn, $\hat{R}$ defines a line bundle R over Σ. Choose any fundamental domain F of Σ. Then, the holonomy of $\mathcal{G}$ with Jandl structure $\mathcal{J}$ around $\hat{\phi}$ is defined as

$$\mathrm{hol}_{\mathcal{G},\mathcal{J}}(\hat{\phi}) := \exp\left(\mathrm{i}\int_F \rho\right) \cdot \mathrm{hol}_R(\overline{\partial F}).$$

Definition 3.12 is a generalization of Definition 3.4 of holonomy around an oriented surface: for an orientable surface Σ and *any* choice of an orientation, they coincide [SSW05]. To show that Definition 3.12 does not depend on the choice of the trivialization $\mathcal{T}$, we combine all the collected tools. Let $\mathcal{T}' : \hat{\phi}^*\mathcal{G} \to \mathcal{I}_{\rho'}$ be any other trivialization. We consider the 1-isomorphism

$$\mathcal{B} \;:=\; \mathcal{T} \circ \mathcal{T}'^{-1} \;:\; \mathcal{I}_{\rho'} \longrightarrow \mathcal{I}_\rho \tag{117}$$

and the corresponding line bundle $T := \mathrm{Bun}(\mathcal{B})$. To compare the two σ-equivariant line bundles $\hat{R}$ and $\hat{R}'$ corresponding to the two trivializations, we first compare the Jandl structures $J_{\mathcal{T}^{-1}}(\mathcal{J})$ on $\mathcal{I}_\rho$ and $J_{\mathcal{T}'^{-1}}(\mathcal{J})$ on $\mathcal{I}_{\rho'}$. By Proposition 3.11 a), b) and c), there exists an isomorphism

$$J_{\mathcal{T}'^{-1}}(\mathcal{J}) \cong J_{\mathcal{B}}(J_{\mathcal{T}^{-1}}(\mathcal{J})) \tag{118}$$

of Jandl structures on $\mathcal{I}_\rho$. By definition of the functor $J_{\mathcal{B}}$, this isomorphism is a 2-isomorphism

$$\mathcal{A}' \cong \mathcal{B}^* \circ \mathcal{A} \circ \sigma^*\mathcal{B}, \tag{119}$$

where $\mathcal{A}$ is the 1-morphism of $J_{\mathcal{T}^{-1}}(\mathcal{J})$ and $\mathcal{A}'$ is the 1-morphism of $J_{\mathcal{T}'^{-1}}(\mathcal{J})$. Now we apply the functor Bun and obtain an isomorphism

$$\hat{R}' \cong T \otimes \hat{R} \otimes \sigma^* T \tag{120}$$

of σ-equivariant line bundles over $\hat{\Sigma}$, where $\hat{Q} := \sigma^* T \otimes T$ has the canonical σ-equivariant structure by exchanging the tensor factors. Thus, we have isomorphic line bundles

$$R' \cong R \otimes Q \tag{121}$$

over Σ. Notice that the holonomy of the line bundle Q is

$$\mathrm{hol}_Q(\overline{\partial F}) = \mathrm{hol}_T(\partial F) = \exp\left(\mathrm{i} \int_F \rho - \rho'\right) \tag{122}$$

This shows

$$\begin{aligned} \exp\left(\mathrm{i} \int_F \rho'\right) \cdot \mathrm{hol}_{R'}(\overline{\partial F}) &= \exp\left(\mathrm{i} \int_F \rho'\right) \cdot \mathrm{hol}_Q(\overline{\partial F}) \cdot \mathrm{hol}_R(\overline{\partial F}) \\ &= \exp\left(\mathrm{i} \int_F \rho\right) \cdot \mathrm{hol}_R(\overline{\partial F}) \end{aligned} \tag{123}$$

so that Definition 3.12 does not depend on the choice of the trivialization. In [SSW05] we have deduced from the equation $\mathrm{curv}(\hat{R}) = -(\rho + \sigma^*\rho)$ that it is also independent of the choice of the fundamental domain.

Unoriented surface holonomy, defined in terms of Jandl structures on bundle gerbes, provides a candidate for the Wess-Zumino term in two-dimensional conformal field theory for unoriented worldsheets, as they appear in type I string theories. Following the examples of $M = SU(2)$ and $M = SO(3)$ we give in [SSW05], we reproduce results known from other approaches. This indicates, that a bundle gerbe with Jandl structure, together with a metric, is the background field for unoriented WZW models. In this setup, Proposition 3.11 assures, that – just like for oriented WZW models – only the isomorphism class of the bundle gerbe is relevant.

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