Theory and Applications of Categories, Vol. 19, No. 4, 2007, pp. 50–60.

$\mathcal{Q}$-MODULES ARE $\mathcal{Q}$-SUPLATTICES

ISAR STUBBE

ABSTRACT. It is well known that the internal suplattices in the topos of sheaves on a locale are precisely the modules on that locale. Using enriched category theory and a lemma on KZ doctrines we prove (the generalization of) this fact in the case of ordered sheaves on a small quantaloid. Comparing module-equivalence with sheaf-equivalence for quantaloids and using the notion of centre of a quantaloid, we refine a result of F. Borceux and E. Vitale.

1. Introduction

When studying topos theory one inevitably must study order theory too: if only because many advanced features of topos theory depend on order-theoretic arguments using the internal Heyting algebra structure of the subobject classifier in a topos, as C. J. Mikkelsen [1976] illustrates plainly. One of the results of [Mikkelsen, 1976] states that an ordered object in an elementary topos $\mathcal{E}$ is cocomplete, i.e. it is an internal suplattice, if and only if the "principal downset embedding" from that object to its powerobject has a left adjoint in $\mathsf{Ord}(\mathcal{E})$. In the case of a localic topos, it turns out that the internal suplattices in $\mathsf{Sh}(\Omega)$ are precisely the Ω-modules, and supmorphisms are just the module morphisms [Joyal and Tierney, 1984; Pitts, 1988].

Now consider quantaloids (i.e. Sup-enriched categories) as non-commutative, multi-typed generalization of locales. Using the theory of categories enriched in a quantaloid, and building further on results by B. Walters [1981] and F. Borceux and R. Cruciani [1998], I. Stubbe [2005b] proposed the notion of *ordered sheaf on a (small) quantaloid* $\mathcal{Q}$ (or $\mathcal{Q}$*-order* for short): one of several equivalent ways of describing a $\mathcal{Q}$-order is to say that it is a Cauchy complete category enriched in the split-idempotent completion of $\mathcal{Q}$. There is thus a locally ordered category $\mathsf{Ord}(\mathcal{Q})$ of $\mathcal{Q}$-orders and functors between them. If one puts $\mathcal{Q}$ to be the one-object suspension of a locale Ω, then $\mathsf{Ord}(\Omega)$ is equivalent to $\mathsf{Ord}(\mathsf{Sh}(\Omega))$. (And if one puts $\mathcal{Q}$ to be the one-object suspension of the Lawvere reals $[0,\infty]$, then $\mathsf{Ord}([0,\infty])$ is equivalent to the category of Cauchy complete generalized metric spaces.)

In this paper we shall explain how $\mathsf{Mod}(\mathcal{Q})$, the quantaloid of $\mathcal{Q}$-modules, is the category of Eilenberg-Moore algebras for the KZ doctrine on $\mathsf{Ord}(\mathcal{Q})$ that sends a $\mathcal{Q}$-order $\mathbb{A}$ to its free cocompletion $\mathcal{P}\mathbb{A}$. The proof of this fact is, altogether, quite straightforward:

Isar Stubbe is a Postdoctoral Fellow of the Research Foundation Flanders (FWO).
Received by the editors 2007-01-15 and, in revised form, 2007-07-15.
Transmitted by Stephen Lack. Published on 2007-07-25.
2000 Mathematics Subject Classification: 06F07, 18D05, 18D20.
Key words and phrases: Quantaloid, quantale, locale, ordered sheaf, module, centre, KZ doctrine.

a lot of the hard work – involving quantaloid-enriched categories – has already been done elsewhere [Stubbe, 2005a, 2005b, 2006], so we basically only need a lemma on KZ doctrines to put the pieces of the puzzle together. Applied to a locale Ω, and up to the equivalence of $\mathsf{Ord}(\Omega)$ with $\mathsf{Ord}(\mathsf{Sh}(\Omega))$, this KZ doctrine sends an ordered sheaf on Ω to (the sheaf of) its downclosed subobjects, so our more general theorem provides an independent proof of the fact that Ω-modules are precisely the internal cocomplete objects of $\mathsf{Ord}(\mathsf{Sh}(\Omega))$. This then explains the title of this paper: even for a small quantaloid $\mathcal{Q}$, "$\mathcal{Q}$-modules are $\mathcal{Q}$-suplattices"! We end the paper with a comment on the comparison of (small) quantaloids, their categories of ordered sheaves, their module categories, and their centres; thus we refine a result of F. Borceux and E. Vitale [1992].

In some sense, this paper may be considered a *prequel* to [Stubbe, 2007]: we can now rightly say that the latter paper treats those $\mathcal{Q}$-suplattices (in their guise of cocomplete $\mathcal{Q}$-enriched categories) that are totally continuous (or supercontinuous, as some say). It is hoped that this will lead to a better understanding and further development of "dynamic domains", i.e. "domains" in $\mathsf{Ord}(\mathcal{Q})$, so that applying general results to either Ω or $[0,\infty]$ then gives interesting results for "constructive domains" or "metric domains".

2. Preliminaries

Quantales and quantaloids. Let Sup denote the category of complete lattices and maps that preserve arbitrary suprema (*suplattices* and *supmorphisms*): it is symmetric monoidal closed for the usual tensor product. A *quantaloid* is a Sup-enriched category; it is *small* when it has a set of objects; and a one-object quantaloid (most often thought of as a monoid in Sup) is a *quantale*. A Sup-functor between quantaloids is a *homomorphism*; QUANT denotes the (illegitimate) category of quantaloids and their homomorphisms. A standard reference on quantaloids is [Rosenthal, 1996].

For a given quantaloid $\mathcal{Q}$ we write $\mathsf{Idm}(\mathcal{Q})$ for the new quantaloid whose objects are the idempotent arrows in $\mathcal{Q}$, and in which an arrow from an idempotent $e\colon A\longrightarrow A$ to an idempotent $f\colon B\longrightarrow B$ is a $\mathcal{Q}$-arrow $b\colon A\longrightarrow B$ satisfying $b\circ e=b=f\circ b$. Composition in $\mathsf{Idm}(\mathcal{Q})$ is done as in $\mathcal{Q}$, the identity in $\mathsf{Idm}(\mathcal{Q})$ on some idempotent $e\colon A\longrightarrow A$ is e itself, and the local order in $\mathsf{Idm}(\mathcal{Q})$ is that of $\mathcal{Q}$. (Note that $\mathsf{Idm}(\mathcal{Q})$ is small whenever $\mathcal{Q}$ is.) It is easy to verify that the quantaloid $\mathsf{Idm}(\mathcal{Q})$ is the universal split-idempotent completion of $\mathcal{Q}$ in QUANT, as the next lemma spells out.

2.1. Lemma. *If $\mathcal{R}$ is a quantaloid in which idempotents split, then, for any quantaloid $\mathcal{Q}$, the full embedding $i\colon\mathcal{Q}\longrightarrow\mathsf{Idm}(\mathcal{Q})\colon(f\colon A\longrightarrow B)\mapsto(f\colon 1_A\longrightarrow 1_B)$ determines an equivalence of quantaloids $-\circ i\colon\mathsf{QUANT}(\mathsf{Idm}(\mathcal{Q}),\mathcal{R})\longrightarrow\mathsf{QUANT}(\mathcal{Q},\mathcal{R})$.*

When $\mathcal{Q}$ is a small quantaloid, we use $\mathsf{Mod}(\mathcal{Q})$ as shorthand for $\mathsf{QUANT}(\mathcal{Q}^{\mathsf{op}},\mathsf{Sup})$: the objects of this (large) quantaloid are called the *modules on* $\mathcal{Q}$. Since idempotents split in Sup, it follows directly from 2.1 that $\mathsf{Mod}(\mathcal{Q})\simeq\mathsf{Mod}(\mathsf{Idm}(\mathcal{Q}))$.

Quantaloid-enriched categories. A quantaloid is a bicategory and therefore it may serve itself as base for enrichment. The theory of quantaloid-enriched categories, functors and distributors is surveyed in [Stubbe, 2005a] where also the appropriate references are given. To make this paper reasonably self-contained we shall go through some basic notions here; we follow the notations of *op. cit.* for easy cross reference.

To avoid size issues we work with a small quantaloid $\mathcal{Q}$. A *$\mathcal{Q}$-category* $\mathbb{A}$ consists of a set $\mathbb{A}_0$ of 'objects', a 'type' function $t\colon \mathbb{A}_0 \longrightarrow \mathcal{Q}_0$, and for any $a, a' \in \mathbb{A}_0$ a 'hom-arrow' $\mathbb{A}(a', a)\colon ta \longrightarrow ta'$ in $\mathcal{Q}$; these data are required to satisfy

$$\mathbb{A}(a'', a') \circ \mathbb{A}(a', a) \leq \mathbb{A}(a'', a) \quad \text{and} \quad 1_{ta} \leq \mathbb{A}(a, a)$$

for all $a, a', a'' \in \mathbb{A}_0$. A *functor* $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ between $\mathcal{Q}$-categories is a map $\mathbb{A}_0 \longrightarrow \mathbb{B}_0\colon a \mapsto Fa$ that satisfies

$$ta = t(Fa) \quad \text{and} \quad \mathbb{A}(a', a) \leq \mathbb{B}(Fa', Fa)$$

for all $a, a' \in \mathbb{A}_0$. For parallel functors $F, G\colon \mathbb{A} \rightrightarrows \mathbb{B}$ we put $F \leq G$ when $1_{ta} \leq \mathbb{B}(Fa, Ga)$ for every $a \in \mathbb{A}_0$. With the obvious composition and identities we obtain a locally ordered category $\mathsf{Cat}(\mathcal{Q})$ of $\mathcal{Q}$-categories and functors.

To give a *distributor* (or *module* or *profunctor*) $\Phi\colon \mathbb{A} \to\!\!\!\!\!\circ\,\, \mathbb{B}$ between $\mathcal{Q}$-categories is to specify for any $a \in \mathbb{A}_0$, $b \in \mathbb{B}_0$, an arrow $\Phi(b, a)\colon ta \longrightarrow tb$ in $\mathcal{Q}$, such that

$$\mathbb{B}(b, b') \circ \Phi(b', a) \leq \Phi(b, a) \quad \text{and} \quad \Phi(b, a') \circ \mathbb{A}(a', a) \leq \Phi(b, a)$$

for every $a, a' \in \mathbb{A}_0$, $b, b' \in \mathbb{B}_0$. Two distributors $\Phi\colon \mathbb{A} \to\!\!\!\!\!\circ\,\, \mathbb{B}$, $\Psi\colon \mathbb{B} \to\!\!\!\!\!\circ\,\, \mathbb{C}$ compose: we write $\Psi \otimes \Phi\colon \mathbb{A} \to\!\!\!\!\!\circ\,\, \mathbb{C}$ for the distributor with elements

$$\Big(\Psi \otimes \Phi\Big)(c, a) = \bigvee_{b \in \mathbb{B}_0} \Psi(c, b) \circ \Phi(b, a).$$

The identity distributor on a $\mathcal{Q}$-category $\mathbb{A}$ is $\mathbb{A}\colon \mathbb{A} \to\!\!\!\!\!\circ\,\, \mathbb{A}$ itself, i.e. the distributor with elements $\mathbb{A}(a', a)\colon ta \longrightarrow ta'$. We order parallel distributors $\Phi, \Phi'\colon \mathbb{A} \rightrightarrows \mathbb{B}$ by "elementwise comparison": we define $\Phi \leq \Phi'$ to mean that $\Phi(b, a) \leq \Phi'(b, a)$ for every $a \in \mathbb{A}_0$, $b \in \mathbb{B}_0$. It is easily seen that $\mathcal{Q}$-categories and distributors form a quantaloid $\mathsf{Dist}(\mathcal{Q})$.

Every functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ between $\mathcal{Q}$-categories represents an adjoint pair of distributors:

- the left adjoint $\mathbb{B}(-, F-)\colon \mathbb{A} \to\!\!\!\!\!\circ\,\, \mathbb{B}$ has elements $\mathbb{B}(b, Fa)\colon ta \longrightarrow tb$,
- the right adjoint $\mathbb{B}(F-, -)\colon \mathbb{B} \to\!\!\!\!\!\circ\,\, \mathbb{A}$ has elements $\mathbb{B}(Fa, b)\colon tb \longrightarrow ta$.

The assignment $F \mapsto \mathbb{B}(-, F-)$ is a faithful 2-functor from $\mathsf{Cat}(\mathcal{Q})$ to $\mathsf{Dist}(\mathcal{Q})$; it gives rise to a rich theory of $\mathcal{Q}$-categories. We shall briefly explain two notions that play an essential rôle in the current work: cocompleteness and Cauchy completeness.

Cocompleteness and modules. Given a distributor $\Phi\colon \mathbb{A} \to\!\!\!\!\!\circ\,\, \mathbb{B}$ and a functor $F\colon \mathbb{B} \longrightarrow \mathbb{C}$, a functor $K\colon \mathbb{A} \longrightarrow \mathbb{C}$ is the *Φ-weighted colimit of F* when it satisfies

$$\mathbb{C}(K-, -) = [\Phi, \mathbb{C}(F-, -)]$$

(and in that case it is essentially unique). The right hand side of this equation uses the adjunction between ordered sets

$$\begin{tikzcd}\mathsf{Dist}(\mathcal{Q})(\mathbb{C},\mathbb{A}) \arrow[r, bend left, "\Phi\otimes -"] \arrow[r, phantom, "\perp"] & \mathsf{Dist}(\mathcal{Q})(\mathbb{C},\mathbb{B}) \arrow[l, bend left, "{[\Phi,-]}"]\end{tikzcd}$$

which surely exists since $\mathsf{Dist}(\mathcal{Q})$ is a quantaloid. A functor is *cocontinuous* if it preserves all weighted colimits that happen to exist in its domain; and a $\mathcal{Q}$-category is *cocomplete* if it admits all weighted colimits. We write $\mathsf{Cocont}(\mathcal{Q})$ for the subcategory of $\mathsf{Cat}(\mathcal{Q})$ of cocomplete $\mathcal{Q}$-categories and cocontinuous functors. Much more can be found in [Stubbe, 2005a, Sections 5 and 6].

As stated in [Stubbe, 2006, 4.13] (but see also the references contained in that paper), $\mathsf{Mod}(\mathcal{Q})$ and $\mathsf{Cocont}(\mathcal{Q})$ are biequivalent locally ordered categories. Indeed, a $\mathcal{Q}$-module $M\colon \mathcal{Q}^{\mathsf{op}} \longrightarrow \mathsf{Sup}$ determines a $\mathcal{Q}$-category $\mathbb{A}_M$: as object set take $(\mathbb{A}_M)_0 = \uplus_{X\in\mathcal{Q}_0} MX$, then say that $tx = X$ precisely when $x \in MX$, and for $x \in MX$, $y \in MY$ let $\mathbb{A}_M(y,x) = \bigvee\{f\colon X \longrightarrow Y \mid Mf(y) \leq x\}$. A detailed analysis of why this $\mathbb{A}_M$ is cocomplete, and why every cocomplete $\mathcal{Q}$-category arises in this way, is precisely the subject of [Stubbe, 2006]; we shall not go into details here.

2.2. Corollary. *For a small quantaloid $\mathcal{Q}$,*

$$\mathsf{Cocont}(\mathcal{Q}) \simeq \mathsf{Mod}(\mathcal{Q}) \simeq \mathsf{Mod}(\mathsf{Idm}(\mathcal{Q})) \simeq \mathsf{Cocont}(\mathsf{Idm}(\mathcal{Q}))$$

are biequivalent locally ordered categories.

Cauchy completeness and orders. A $\mathcal{Q}$-category $\mathbb{C}$ is *Cauchy complete* if for any other $\mathcal{Q}$-category $\mathbb{A}$ the map

$$\mathsf{Cat}(\mathcal{Q})(\mathbb{A},\mathbb{C}) \longrightarrow \mathsf{Map}(\mathsf{Dist}(\mathcal{Q}))(\mathbb{A},\mathbb{C})\colon F \mapsto \mathbb{C}(-,F-)$$

is surjective, i.e. when any left adjoint distributor (also called *Cauchy distributor*) into $\mathbb{C}$ is represented by a functor. This is equivalent to the requirement that $\mathbb{C}$ admits any colimit weighted by a Cauchy distributor; and moreover such weighted colimits are *absolute* in the sense that they are preserved by any functor [Street, 1983]. We write $\mathsf{Cat}_{\mathsf{cc}}(\mathcal{Q})$ for the full subcategory of $\mathsf{Cat}(\mathcal{Q})$ whose objects are the Cauchy complete $\mathcal{Q}$-categories. For more details we refer to [Stubbe, 2005a, Section 7].

Now we have everything ready to state an important definition from [Stubbe, 2005b].

2.3. Definition. *For a small quantaloid $\mathcal{Q}$, we write $\mathsf{Ord}(\mathcal{Q})$ for the locally ordered category $\mathsf{Cat}_{\mathsf{cc}}(\mathsf{Idm}(\mathcal{Q}))$, and call its objects* ordered sheaves on $\mathcal{Q}$*, or simply* $\mathcal{Q}$-orders.

In fact, the definition of '$\mathcal{Q}$-order' in [Stubbe, 2005b, 5.1] is not quite this one: instead it is given in more "elementary" terms (avoiding the split-idempotent construction). But it is part of the investigations in that paper (more precisely in its Section 6) that what we give here as definition is indeed equivalent to what was given there; and for the purposes of the current paper this "structural" definition is best.

The notion of $\mathcal{Q}$-order has the merit of generalizing two – at first sight quite different – mathematical structures: On the one hand, taking $\mathcal{Q}$ to be the (one-object suspension of) the Lawvere reals $[0,\infty]$, $\mathsf{Ord}([0,\infty])$ is the category of Cauchy complete generalized metric spaces [Lawvere, 1973]. On the other hand, taking $\mathcal{Q}$ to be the (one-object suspension of) a locale Ω, $\mathsf{Ord}(\Omega)$ is the category of ordered objects in the topos $\mathsf{Sh}(\Omega)$ [Walters, 1981; Borceux and Cruciani, 1998]; obviously, this example inspired our terminology. For details we refer to [Stubbe, 2005b].

3. Monadicity of $\mathcal{Q}$-modules over $\mathcal{Q}$-orders

Recall from [Kock, 1995] that a *Kock–Zöberlein (KZ) doctrine* on a locally ordered 2-category $\mathcal{C}$ is a monad $(T\colon\mathcal{C}\longrightarrow\mathcal{C}, \eta\colon\mathsf{Id}_{\mathcal{C}}\Longrightarrow T, \mu\colon T\circ T\Longrightarrow T)$ for which $T(\eta_C)\leq\eta_{TC}$ for any $C\in\mathcal{C}$. This precisely means that "T-structures are adjoint to units". Further on we shall encounter an instance of the following lemma.

3.1. Lemma. *Consider locally ordered 2-categories and 2-functors as in*

$$\begin{tikzcd}
& \mathcal{B} \arrow[dr, hook, "W"] & \\
\mathcal{A} \arrow[ur, "V"] \arrow[rr, bend left, "U"] \arrow[rr, phantom, "\top"] & & \mathcal{C} \arrow[ll, bend left, "F"]
\end{tikzcd}$$

with W a local equivalence and $W\circ V=U$. Write $\eta\colon\mathsf{id}_{\mathcal{C}}\Longrightarrow U\circ F$ for the unit of the involved adjunction. Then

1. *$F\circ W\dashv V$ and its unit $\xi\colon\mathsf{Id}_{\mathcal{B}}\Longrightarrow V\circ(F\circ W)$ satisfies $\eta*\mathsf{id}_W=\mathsf{id}_V*\xi$, that is, $W(\xi_B)=\eta_{WB}$ for every $B\in\mathcal{B}$,*

and writing $T=U\circ F\colon\mathcal{C}\longrightarrow\mathcal{C}$ and $S=V\circ(F\circ W)\colon\mathcal{B}\longrightarrow\mathcal{B}$, these monads satisfy

2. *$T\circ W=W\circ S$,*
3. *if T is a KZ doctrine then*
   (a) *also S is a KZ doctrine,*
   (b) *$B\in\mathcal{B}$ is an S-algebra if and only if WB is a T-algebra,*
   (c) *for $A\in\mathcal{A}$, UA is a T-algebra if and only if VA is an S-algebra,*
   (d) *if $\mathcal{A}\simeq\mathcal{C}^T$ then $\mathcal{A}\simeq\mathcal{B}^S$.*

PROOF. To prove that $F \circ W \dashv V$, observe that for $B \in \mathcal{B}$ and $C \in \mathcal{C}$,

$$\begin{tikzcd}
\mathcal{B}(B, VC) \arrow[d, "\text{apply } W"] \\
\mathcal{A}(WB, WVC) \arrow[d, equal, "\text{use that } U = WV"] \\
\mathcal{A}(WB, UC) \arrow[d, "\text{use that } F \dashv U"] \\
\mathcal{C}(FWB, C)
\end{tikzcd}$$

are all equivalences (recall that W is supposed to be a local equivalence). Putting $C = FWB$ in the above, and tracing the element 1_{FWB} through the equivalences, results in $W(\xi_B) = \eta_{WB}$.

The second part of the lemma is trivial.

For the third part, suppose that $T(\eta_C) \leq \eta_{TC}$ for any $C \in \mathcal{C}$, then also

$$WS(\xi_B) = TW(\xi_B) = T(\eta_{WB}) \leq \eta_{TWB} = \eta_{WSB} = W(\xi_{SB})$$

for every $B \in \mathcal{B}$; but W is locally an equivalence, so $S(\xi_B) \leq \xi_{SB}$ as required to prove (a). Now, by the very nature of the algebras of KZ doctrines, $B \in \mathcal{B}$ is an S-algebra if and only if ξ_B is a right adjoint in $\mathcal{B}$, which is the same as $W(\xi_B) = \eta_{WB}$ being a right adjoint in $\mathcal{C}$ because W is locally an equivalence, and this in turn is just saying that WB is a T-algebra. This proves (b), and (c) readily follows by putting $B = VA$ for an $A \in \mathcal{A}$, and using that $W \circ V = U$; so (d) becomes obvious. ∎

In the rest of this section we let $\mathcal{Q}$ be a small quantaloid. It is a result from $\mathcal{Q}$-enriched category theory [Stubbe, 2005a, 6.11] that $\mathsf{Cocont}(\mathcal{Q})$, the locally ordered category of cocomplete $\mathcal{Q}$-categories and cocontinuous functors, is monadic over the locally ordered category $\mathsf{Cat}(\mathcal{Q})$ of all categories and functors: the forgetful functor $\mathsf{Cocont}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q})$ admits the presheaf construction as left adjoint,

$$\begin{tikzcd}
\mathsf{Cocont}(\mathcal{Q}) \arrow[r, bend right, "\mathcal{U}"'] \arrow[r, phantom, "\perp"] & \mathsf{Cat}(\mathcal{Q}). \arrow[l, bend right, "\mathcal{P}"']
\end{tikzcd} \tag{1}$$

The unit of the adjunction is given by the Yoneda embeddings $Y_{\mathbb{A}} : \mathbb{A} \longrightarrow \mathcal{P}\mathbb{A}$; and a $\mathcal{Q}$-category $\mathbb{A}$ is in $\mathsf{Cocont}(\mathcal{Q})$ if and only if $Y_{\mathbb{A}} : \mathbb{A} \longrightarrow \mathcal{P}\mathbb{A}$ admits a left adjoint in $\mathsf{Cat}(\mathcal{Q})$, which is then the structure map of the algebra $\mathbb{A}$. In short, the monad induced by (1) is a KZ-doctrine on $\mathsf{Cat}(\mathcal{Q})$.

Cauchy complete $\mathcal{Q}$-categories can be characterized as those $\mathcal{Q}$-categories that admit all absolute colimits [Stubbe, 2005a, 7.2]. Knowing this it is clear that the forgetful

$\mathsf{Cocont}(\mathcal{Q})\longrightarrow\mathsf{Cat}(\mathcal{Q})$ factors over the full embedding $\mathsf{Cat}_{\mathsf{cc}}(\mathcal{Q})\longrightarrow\mathsf{Cat}(\mathcal{Q})$ of Cauchy complete $\mathcal{Q}$-categories into all $\mathcal{Q}$-categories. Applying 3.1 to the adjunction in (1) we thus obtain that the forgetful $\mathsf{Cocont}(\mathcal{Q})\longrightarrow\mathsf{Cat}_{\mathsf{cc}}(\mathcal{Q})$ has a left adjoint, namely (the restriction of) the presheaf construction, and moreover $\mathsf{Cocont}(\mathcal{Q})$ is precisely the category of algebras for the induced KZ doctrine on $\mathsf{Cat}_{\mathsf{cc}}(\mathcal{Q})$.

We can apply all this to the quantaloid $\mathsf{Idm}(\mathcal{Q})$, and get the following result.

3.2. Proposition. *For any small quantaloid $\mathcal{Q}$, $\mathsf{Cocont}(\mathsf{Idm}(\mathcal{Q}))$ is the category of algebras for the "presheaf construction" KZ doctrine $\mathcal{P}\colon\mathsf{Cat}_{\mathsf{cc}}(\mathsf{Idm}(\mathcal{Q}))\longrightarrow\mathsf{Cat}_{\mathsf{cc}}(\mathsf{Idm}(\mathcal{Q}))$.*

In combination with the remarks on $\mathcal{Q}$-orders and $\mathcal{Q}$-modules in Section 2, we can now justify the title of the paper.

3.3. Theorem. *For a small quantaloid $\mathcal{Q}$, the diagram*

$$\begin{tikzcd}\mathsf{Mod}(\mathcal{Q})\simeq\mathsf{Cocont}(\mathsf{Idm}(\mathcal{Q})) \arrow[r, bend right, "\mathcal{U}"'] \arrow[r, phantom, "\perp"] & \mathsf{Cat}_{\mathsf{cc}}(\mathsf{Idm}(\mathcal{Q}))\simeq\mathsf{Ord}(\mathcal{Q}) \arrow[l, bend right, "\mathcal{P}"']\end{tikzcd}$$

exhibits the quantaloid $\mathsf{Mod}(\mathcal{Q})$ as (biequivalent to) the category of algebras for the "presheaf construction" KZ doctrine on $\mathsf{Ord}(\mathcal{Q})$.

As an example we shall point out how the preceding theorem is a precise generalization of the well-known fact that the internal suplattices in a localic topos $\mathsf{Sh}(\Omega)$ are exactly the Ω-modules [Joyal and Tierney, 1984; Pitts, 1988].

3.4. Example. Let Ω be a locale and $(F,\leq)$ and ordered object in $\mathsf{Sh}(\Omega)$. We can associate to this ordered sheaf a category $\mathbb{A}$ enriched in the quantaloid $\mathsf{Idm}(\Omega)$ (the split-idempotent completion of the monoid $(\Omega,\wedge,1)$) as follows:

- objects: $\mathbb{A}_0=\coprod_{v\in\Omega}F(v)$, with types $tx=v\Leftrightarrow x\in F(v)$,
- hom-arrows: for $x,y\in\mathbb{A}_0$, $\mathbb{A}(y,x)=\bigvee\{w\leq tx\wedge ty\mid y_{|w}\leq_w x_{|w}\}$.

That is to say, we can read off that

$$\mathbb{A}(y,x)=\text{“the greatest level at which } y\leq x \text{ in } F\text{”}.$$

With a slight adaptation of the arguments in [Walters, 1981; Borceux and Cruciani, 1998] one can prove that this construction extends to a (bi)equivalence of locally ordered categories $\mathsf{Ord}(\mathsf{Sh}(\Omega))\simeq\mathsf{Cat}_{\mathsf{cc}}(\mathsf{Idm}(\Omega))$; the details are in [Stubbe, 2005b].

We shall now explain that, under the identification of $(F,\leq)$ in $\mathsf{Sh}(\Omega)$ with $\mathbb{A}$, there is a *bijective correspondence between downsets of F and presheaves on $\mathbb{A}$*; in particular do *principal downsets correspond with representable presheaves.*

A *downset S of $(F,\leq)$* is an $S\in\Omega^F$ (i.e. an $S\subseteq F_u\subseteq F$ for some $u\in\Omega$)[1] such that

$$(y\leq x)\wedge(x\in S)\Rightarrow(y\in S), \tag{2}$$

[1]We write F_u for the "truncation of F at u" [Borceux, 1994, vol. 3, 5.2.3]: it is the sheaf defined by $F_u(v)=F(v)$ whenever $v\leq u$ and otherwise $F_u(v)=\emptyset$.

this definition being written in the internal logic of $\mathsf{Sh}(\Omega)$. On the other hand, a *presheaf* on the $\mathsf{Idm}(\Omega)$-enriched category $\mathbb{A}$ is by definition a distributor $\phi\colon *_u \xrightarrow{\circ} \mathbb{A}$ for some $u \in \mathsf{Idm}(\Omega)$; equivalently, such is a map $\phi\colon \mathbb{A}_0 \longrightarrow \Omega$ such that for all $x, y \in \mathbb{A}_0$, $\phi(x) \leq u \wedge tx$ and

$$\mathbb{A}(y,x) \wedge \phi(x) \leq \phi(y). \tag{3}$$

The similarity between the formulas in (2) and (3) suggests that a downset S of $(F, \leq)$ is related with a presheaf ϕ on $\mathbb{A}$ by the clause

$$\phi(x) = \text{"the greatest level at which } x \in S\text{"}.$$

Here is how this can be made precise: Given a downset $S \subseteq F_u \subseteq F$ with its characteristic map $\varphi\colon F \longrightarrow \Omega$, consider the family of its components $\varphi_v\colon F(v) \longrightarrow \Omega(v)$ (indexed by $v \in \Omega$), extend their codomains in the obvious way to the whole of Ω and call these new maps $\phi_v\colon F(v) \longrightarrow \Omega$. The coproduct $\phi = \coprod_{v \in \Omega} \phi_v\colon \mathbb{A}_0 \longrightarrow \Omega$ satisfies, for $x \in \mathbb{A}_0$,

$$\phi(x) = \bigvee \{v \leq tx \mid x_{|v} \in S(v)\}$$

so that quite obviously $\phi(x) \leq tx \wedge u$, and moreover (3) holds because it is just a rephrasing of (2). Hence ϕ gives the elements of a presheaf $\phi\colon *_u \xrightarrow{\circ} \mathbb{A}$. Conversely, given a presheaf $\phi\colon *_u \xrightarrow{\circ} \mathbb{A}$ we decompose the map $\phi\colon \mathbb{A}_0 \longrightarrow \Omega$ into a family of maps $\phi_v\colon F(v) \longrightarrow \Omega\colon x \mapsto \phi(x)$ indexed by $v \in \Omega$. Since $\phi(x) \leq tx \wedge u$ we can restrict the codomains of each of these maps to obtain a new family

$$\big(\varphi_v\colon F(v) \longrightarrow \Omega(v)\colon x \mapsto \phi(x)\big)_{v \in \Omega}.$$

This family is natural in v: Let $w \leq v$ and take any $x \in F(v)$. Then $w = \mathbb{A}(x_{|w}, x)$ and therefore $w \wedge \varphi_v(x) = \mathbb{A}(x_{|w}, x) \wedge \phi(x) \leq \phi(x_{|w}) = \varphi_w(x_{|w})$ by (3). But also $w = \mathbb{A}(x, x_{|w})$ and so, again by (3), $\varphi_w(x_{|w}) = \phi(x_{|w}) = w \wedge \phi(x_{|w}) = \mathbb{A}(x, x_{|w}) \wedge \phi(x_{|w}) \leq \phi(x) = \varphi_v(x)$. Thus indeed $w \wedge \varphi_v(x) = \varphi_w(x_{|w})$. Now we let $S \in \Omega^F$ be the $S \subseteq F$ with characteristic map $\varphi\colon F \longrightarrow \Omega$: then actually $S \subseteq F_u \subseteq F$ because $\phi(x) \leq u$, and moreover S is a downset because (2) follows from (3). The constructions $S \mapsto \phi$ and $\phi \mapsto S$ are inverse to each other under the identification of the ordered sheaf $(F, \leq)$ with the enriched category $\mathbb{A}$.

In particular, the *principal downset S_x of F at $x \in F$* is the $S_x \in \Omega^F$ such that

$$(y \leq x) \Leftrightarrow (y \in S_x).$$

(Clearly such an S_x is always a downset.) The corresponding presheaf $\phi_x\colon *_u \xrightarrow{\circ} \mathbb{A}$ must thus satisfy

$$\mathbb{A}(y,x) = \phi_x(y),$$

that is to say, it is the *representable presheaf* $\mathbb{A}(-,x)$.

Now we can understand why *an ordered sheaf $(F, \leq)$ is an internal suplattice in* $\mathsf{Sh}(\Omega)$ *if and only if the associated* $\mathsf{Idm}(\Omega)$*-category $\mathbb{A}$ is cocomplete*: $(F, \leq)$ is an internal suplattice in $\mathsf{Sh}(\Omega)$ if and only if the "principal downset inclusion" $F \longrightarrow \Omega^F$ has a left adjoint

[Mikkelsen, 1976; Johnstone, 2002, B2.3.9]. But this is constructively equivalent with the existence of a left adjoint to its factorization over the (object of) downsets of F. By the above we know that this is the case if and only if the Yoneda embedding $Y_{\mathbb{A}}\colon \mathbb{A} \mapsto \mathcal{P}\mathbb{A}$ has a left adjoint, which in turn means precisely that $\mathbb{A}$ is cocomplete.

By 3.3 we thus get an independent proof of the fact that the internal suplattices in $\mathsf{Sh}(\Omega)$ are precisely the modules on Ω: $\mathsf{Sup}(\mathsf{Sh}(\Omega)) \simeq \mathsf{Mod}(\Omega)$.

4. Module equivalence compared with sheaf equivalence

For any quantaloid $\mathcal{Q}$, let $\mathcal{Z}(\mathcal{Q})$ be shorthand for $\mathsf{QUANT}(\mathcal{Q},\mathcal{Q})(\mathsf{Id}_{\mathcal{Q}},\mathsf{Id}_{\mathcal{Q}})$ and call it the *centre of* $\mathcal{Q}$. This is by definition a commutative quantale: that $\mathcal{Z}(\mathcal{Q})$ is a quantale, is because it is an endo-hom-object of the quantaloid $\mathsf{QUANT}(\mathcal{Q},\mathcal{Q})$; that it is moreover commutative, is because $\mathsf{QUANT}(\mathcal{Q},\mathcal{Q})$ is monoidal with $\mathsf{Id}_{\mathcal{Q}}$ the unit object for the tensor (which is composition). Unraveling the definition, an element $\alpha \in \mathcal{Z}(\mathcal{Q})$ is a family of endo-arrows

$$\Big(\begin{tikzcd} A \arrow[loop above, "\alpha_A"] \end{tikzcd} \;\Big|\; A \in \mathcal{Q}_0 \Big)$$

such that for every $f\colon A \longrightarrow B$ in $\mathcal{Q}$, $\alpha_B \circ f = f \circ \alpha_A$. Inspired by [Bass, 1968, p. 56] it is then straightforward to prove the following proposition. (Since I believe that this is a "folk theorem" – and moreover the case for quantales is already mentioned in [Borceux and Vitale, 1992] – I shall only sketch the proof.)

4.1. Proposition. *For any quantaloid $\mathcal{Q}$, $\mathcal{Z}(\mathcal{Q}) \cong \mathcal{Z}(\mathsf{Mod}(\mathcal{Q}))$. Therefore Morita-equivalent quantaloids have isomorphic centres.*

Sketch of proof: Given a natural transformation $\alpha\colon \mathsf{Id}_{\mathcal{Q}} \longrightarrow \mathsf{Id}_{\mathcal{Q}}$, build the natural transformation $\widehat{\alpha}\colon \mathsf{Id}_{\mathsf{Mod}(\mathcal{Q})} \longrightarrow \mathsf{Id}_{\mathsf{Mod}(\mathcal{Q})}$ whose component at $M \in \mathsf{Mod}(\mathcal{Q})$ is the natural transformation $\widehat{\alpha}_M\colon M \longrightarrow M$, whose component at $A \in \mathcal{Q}$ is the Sup-arrow

$$\widehat{\alpha}_M^A = M(\alpha_A)\colon M(A) \longrightarrow M(A).$$

Conversely, given a natural transformation $\beta\colon \mathsf{Id}_{\mathsf{Mod}(\mathcal{Q})} \longrightarrow \mathsf{Id}_{\mathsf{Mod}(\mathcal{Q})}$, build the natural transformation $\overline{\beta}\colon \mathsf{Id}_{\mathcal{Q}} \longrightarrow \mathsf{Id}_{\mathcal{Q}}$ whose component at $A \in \mathcal{Q}$ is the $\mathcal{Q}$-morphism

$$\overline{\beta}_A = \beta^A_{\mathcal{Q}(A,-)}(1_A)\colon A \longrightarrow A.$$

The mappings $\mathcal{Z}(\mathcal{Q}) \longrightarrow \mathcal{Z}(\mathsf{Mod}(\mathcal{Q}))\colon \alpha \mapsto \widehat{\alpha}$ and $\mathcal{Z}(\mathsf{Mod}(\mathcal{Q})) \longrightarrow \mathcal{Z}(\mathcal{Q})\colon \beta \mapsto \overline{\beta}$ thus defined are quantale homomorphisms which are each other's inverse. ∎

The following is now an easy consequence.

4.2. Proposition. *For small quantaloids $\mathcal{Q}$ and $\mathcal{Q}'$,*

$$\mathcal{Q} \simeq \mathcal{Q}' \Longrightarrow \mathsf{Ord}(\mathcal{Q}) \simeq \mathsf{Ord}(\mathcal{Q}') \Longrightarrow \mathsf{Mod}(\mathcal{Q}) \simeq \mathsf{Mod}(\mathcal{Q}') \Longrightarrow \mathcal{Z}(\mathcal{Q}) \cong \mathcal{Z}(\mathcal{Q}').$$

Proof. The first implication holds because "equivalent bases give equivalent enriched structures". The second implication is due to the monadicity explained in 3.3. For the third implication, see 4.1. ∎

It is an interesting problem to study the converse implications in the above proposition, for they do not hold in general. However, since a quantale is commutative if and only if it equals its centre, we do have the following special case which is a refinement of the conclusion of [Borceux and Vitale, 1992].

4.3. Corollary. *For commutative quantales $\mathcal{Q}$ and $\mathcal{Q}'$,*

$$\mathcal{Q} \simeq \mathcal{Q}' \iff \mathsf{Ord}(\mathcal{Q}) \simeq \mathsf{Ord}(\mathcal{Q}') \iff \mathsf{Mod}(\mathcal{Q}) \simeq \mathsf{Mod}(\mathcal{Q}').$$

A *locale* Ω is in particular a commutative quantale, so the above applies. Moreover, and this in strong contrast with the case of quantaloids or even quantales, besides the category $\mathsf{Ord}(\Omega)$ of ordered sheaves and its subcategory $\mathsf{Mod}(\Omega)$ of modules (i.e. cocompletely ordered sheaves) on Ω, we may now also consider the category $\mathsf{Sh}(\Omega)$ of all sheaves. But a locale Ω is (isomorphic to) the locale of subobjects of the terminal object in $\mathsf{Sh}(\Omega)$ (see [Borceux, 1994, vol. 3, 2.2.16] for example), thus we may end with the following.

4.4. Corollary. *For locales Ω and Ω',*

$$\Omega \simeq \Omega' \iff \mathsf{Sh}(\Omega) \simeq \mathsf{Sh}(\Omega') \iff \mathsf{Ord}(\Omega) \simeq \mathsf{Ord}(\Omega') \iff \mathsf{Mod}(\Omega) \simeq \mathsf{Mod}(\Omega').$$

References

[1] Hyman Bass, 1968, *Algebraic K-theory.* Mathematics Lecture Notes Series, W. A. Benjamin, New York.

[2] Francis Borceux, 1994, *Handbook of categorical algebra (3 volumes).* Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge.

[3] Francis Borceux and Rosanna Cruciani, 1998, Skew Ω-sets coincide with Ω-posets, *Cahiers Topol. Géom. Différ. Catég.* **39**, pp. 205–220.

[4] Francis Borceux and Enrico M. Vitale, 1992, A Morita theorem in topology, *Rend. Circ. Mat. Palermo (2) Suppl.* **29**, pp. 353–362.

[5] Aurelio Carboni and Ross Street, 1986, Order ideals in categories, *Pacific J. Math.* **124**, pp. 275–288.

[6] Peter T. Johnstone, 2002, *Sketches of an elephant: a topos theory compendium (2 volumes published, 3rd in preparation).* Oxford Logic Guides, The Clarendon Press Oxford University Press, New York.

[7] André Joyal and Myles Tierney, 1984, An extension of the Galois theory of Grothendieck, *Mem. Amer. Math. Soc.* **51**.

[8] Anders Kock, 1995, Monads for which structures are adjoint to units, *J. Pure Appl. Algebra* **104**, pp. 41–59.

[9] F. William Lawvere, 1973, Metric spaces, generalized logic and closed categories, *Rend. Sem. Mat. Fis. Milano* **43**, pp. 135–166. Republished in: *Reprints in Theory Appl. Categ.* **1**, pp. 1–37 (2002).

[10] Christian J. Mikkelsen, 1976, *Lattice theoretic and logical aspects of elementary topoi.* Various Publications Series No. 25, Matematisk Institut, Aarhus University, Aarhus.

[11] Andrew M. Pitts, 1988, Applications of sup-lattice enriched category theory to sheaf theory, *Proc. London Math. Soc.* **57**, pp. 433–480.

[12] Kimmo I. Rosenthal, 1996, *The theory of quantaloids.* Pitman Research Notes in Mathematics Series. Longman, Harlow.

[13] Ross H. Street, 1983, Absolute colimits in enriched categories, *Cahiers Topol. Géom. Différ. Catég.* **24**, pp. 377–379.

[14] Isar Stubbe, 2005a, Categorical structures enriched in a quantaloid: categories, distributors and functors, *Theory Appl. Categ.* **14**, pp. 1–45.

[15] Isar Stubbe, 2005b, Categorical structures enriched in a quantaloid: orders and ideals over a base quantaloid, *Appl. Categ. Structures* **13**, pp. 235–255.

[16] Isar Stubbe, 2006, Categorical structures enriched in a quantaloid: tensored and cotensored categories, *Theory Appl. Categ.* **16**, pp. 283–306.

[17] Isar Stubbe, 2007, Towards 'dynamic domains': totally continuous cocomplete Q-categories, *Theoret. Comput. Sci.* **373**, pp. 142–160.

[18] Robert F. C. Walters, 1981, Sheaves and Cauchy-complete categories, *Cahiers Topologie Géom. Différentielle* **22**, pp. 283–286.

Isar Stubbe
Department of Mathematics and Computer Science, University of Antwerp
Middelheimlaan 1, 2020 Antwerp, Belgium

Email: `isar.stubbe@ua.ac.be`

This article may be accessed at `http://www.tac.mta.ca/tac/` or by anonymous ftp at `ftp://ftp.tac.mta.ca/pub/tac/html/volumes/19/4/19-04.{dvi,ps,pdf}`

THEORY AND APPLICATIONS OF CATEGORIES (ISSN 1201-561X) will disseminate articles that significantly advance the study of categorical algebra or methods, or that make significant new contributions to mathematical science using categorical methods. The scope of the journal includes: all areas of pure category theory, including higher dimensional categories; applications of category theory to algebra, geometry and topology and other areas of mathematics; applications of category theory to computer science, physics and other mathematical sciences; contributions to scientific knowledge that make use of categorical methods.

Articles appearing in the journal have been carefully and critically refereed under the responsibility of members of the Editorial Board. Only papers judged to be both significant and excellent are accepted for publication.

Full text of the journal is freely available in .dvi, Postscript and PDF from the journal's server at `http://www.tac.mta.ca/tac/` and by ftp. It is archived electronically and in printed paper format.

SUBSCRIPTION INFORMATION. Individual subscribers receive abstracts of articles by e-mail as they are published. To subscribe, send e-mail to `tac@mta.ca` including a full name and postal address. For institutional subscription, send enquiries to the Managing Editor, Robert Rosebrugh, `rrosebrugh@mta.ca`.

INFORMATION FOR AUTHORS. The typesetting language of the journal is TEX, and LATEX2e strongly encouraged. Articles should be submitted by e-mail directly to a Transmitting Editor. Please obtain detailed information on submission format and style files at `http://www.tac.mta.ca/tac/`.