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ON FINITE INDUCED CROSSED MODULES, AND THE HOMOTOPY 2-TYPE OF MAPPING CONES

Dedicated to the memory of J. H. C. Whitehead

RONALD BROWN AND CHRISTOPHER D. WENSLEY

Transmitted by Lawrence Breen

ABSTRACT. Results on the finiteness of induced crossed modules are proved both algebraically and topologically. Using the Van Kampen type theorem for the fundamental crossed module, applications are given to the 2-types of mapping cones of classifying spaces of groups. Calculations of the cohomology classes of some finite crossed modules are given, using crossed complex methods.

Introduction

Crossed modules were introduced by J.H.C. Whitehead in [31]. They form a part of what can be seen as his programme of testing the idea of extending to higher dimensions the methods of combinatorial group theory of the 1930's, and of determining some of the extra structure that was necessary to model the geometry. Other papers of Whitehead of this era show this extension of combinatorial group theory tested in different directions.

In this case he was concerned with the algebraic properties satisfied by the boundary map

$$\partial : \pi_2(X, A) \to \pi_1(A)$$

of the second relative homotopy group, together with the standard action on it of the fundamental group $\pi_1(A)$. This is the *fundamental crossed module* $\Pi_2(X, A)$ of the pair (X, A). In order to determine the second homotopy group of a CW-complex, he formulated and proved the following theorem for this structure:

THEOREM W *Let* $X = A \cup \{e^2_\lambda\}$ *be obtained from the connected space* A *by attaching* 2*-cells. Then the second relative homotopy group* $\pi_2(X, A)$ *may be described as the free crossed* $\pi_1(A)$*-module on the* 2*-cells.*

The proof in [32] uses transversality and knot theory ideas from the previous papers [30, 31]. See [5] for an exposition of this proof. Several other proofs are available. The

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survey by Brown and Huebschmann [12], and the book edited by Hog-Angeloni, Metzler and Sieradski [20], give wider applications.

The paper of Mac Lane and Whitehead [26] uses Theorem W to show that the 2-dimension-al homotopy theory of pointed, connected CW-complexes is completely modelled by the theory of crossed modules. This is an extra argument for regarding crossed modules as 2-dimensional versions of groups.

One of our aims is the explicit calculation of examples of the crossed module

$$(\partial : \pi_2(A \cup \Gamma V, A) \to \pi_1(A)\ ) \tag{1}$$

of a mapping cone, when $\pi_1(V)$, $\pi_1(A)$ are finite. The key to this is the 2-dimensional Van Kampen Theorem (2-VKT) proved by Brown and Higgins in [9]. This implies a generalisation of Theorem W, namely that the crossed module (1) is induced from the identity crossed module $(1 : \pi_1(V) \to \pi_1(V))$ by the morphism $\pi_1(V) \to \pi_1(A)$.

Presentations of induced crossed modules are given in [9], and from these we prove a principal theorem (Theorem 2.1), that crossed modules induced from finite crossed modules by morphisms of finite groups are finite. We also use topological methods to prove a similar result for finite p-groups (Corollary 4.3). These results give a new range of finite crossed modules.

Sequels to this paper discuss crossed modules induced by a normal inclusion [15], and calculations obtained using a group theory package [16].

The origin of the 2-VKT was the idea of extending to higher dimensions the notion of the fundamental groupoid, as suggested in 1967 in [4]. This led to the discovery of the relationship of 2-dimensional groupoids to crossed modules, in work with Spencer [14]. This relationship reinforces the idea of 'higher dimensional group theory', and was essential for the proof of the 2-VKT for the fundamental crossed module [9]. In view of the results of Mac Lane and Whitehead [26], and of methods of classifying spaces of crossed modules by Loday [25] and Brown and Higgins [11] (see section 3), the 2-VKT allows for the explicit computation of some homotopy 2-types, in the form of the crossed modules which model them.

In some cases, the Postnikov invariant of these 2-types can be calculated, as the following example shows.

Corollary 5.5 Let C_n denote the cyclic group of order n, and let BC_n denote its classifying space. The second homotopy module of the mapping cone $X = BC_{n^2} \cup \Gamma BC_n$ is a particular cyclic C_n-module, A_n, say. The cohomology group $H^3(C_n, A_n)$ is a cyclic group of order n, and the first Postnikov invariant of X is a generator of this group.

The method used for the calculation of the cohomology class here is also of interest. It introduces a new small free crossed resolution of the cyclic group of order n in order to construct an explicit 3-cocycle corresponding to the above crossed module. This indicates a wider possibility of using crossed resolutions for explicit calculations. It is also related to Whitehead's use of what he called in [32] 'homotopy systems', and which are simply free crossed complexes.

The initial motivation of this paper was a conversation with Rafael Sivera in Zaragoza, in November, 1993, which suggested the lack of explicit calculations of induced crossed modules. This led to discussions at Bangor on the use of computational group theory packages which culminated in a GAP [29] program [16], and to the development of general theory.

1. Crossed modules and induced crossed modules

In this section, we recall the definition of induced crossed modules, and of results of [9] on presentations of induced crossed modules. We then give some basic examples of these.

Recall that a *crossed module* $\mathcal{M} = (\mu : M \to P)$ is a morphism of groups $\mu : M \to P$ together with an action $(m,p) \mapsto m^p$ of P on M satisfying the two axioms

- CM1) $\mu(m^p) = p^{-1}\mu(m)p$
- CM2) $m^{\mu n} = n^{-1}mn$

for all $m, n \in M$ and $p \in P$. We say that $\mathcal{M}$ is *finite* when M is finite.

The category $\mathcal{XM}$ of crossed modules has as objects all crossed modules. Morphisms in $\mathcal{XM}$ are pairs (g, f) forming commutative diagrams

$$\begin{tikzcd} M \arrow[r, "\mu"] \arrow[d, "g"'] & P \arrow[d, "f"] \\ N \arrow[r, "\nu"'] & Q \end{tikzcd}$$

in which the horizontal maps are crossed modules, and g, f preserve the action in the sense that for all $m \in M, p \in P$ we have $g(m^p) = (gm)^{fp}$. If P is a group, then the category $\mathcal{XM}/P$ of crossed P-modules is the subcategory of $\mathcal{XM}$ whose objects are the crossed P-modules and in which a *morphism* $(\mu : M \to P) \to (\nu : N \to P)$ of crossed P-modules is a morphism $g : M \to N$ of groups such that g preserves the action ($g(m^p) = (gm)^p$, for all $m \in M,\ p \in P$), and $\nu g = \mu$.

Standard algebraic examples of crossed modules are:
(i) an inclusion of a normal subgroup, with action given by conjugation;
(ii) an inner automorphism crossed module $(\chi : M \to \mathrm{Aut}\ M)$ in which χm is the automorphism $n \mapsto m^{-1}nm$;
(iii) a zero crossed module $(0 : M \to P)$ where M is a P-module;
(iv) an epimorphism $M \to P$ with kernel contained in the centre of M.

Examples of finite crossed modules may be found among those above, the induced crossed modules of this paper and its sequels [15, 16], and coproducts [6] and tensor products [13, 19] of finite crossed P-modules.

Further important examples of crossed modules are the free crossed modules, referred to in the Introduction, which are rarely finite. They arise algebraically in considering identities among relations [12, 20], which are non-abelian forms of syzygies.

We next define *pullback crossed modules*. Let $\iota : P \to Q$ be a morphism of groups and let $(\nu : N \to Q)$ be a crossed Q-module. Let $\nu' : \iota^*N \to P$ be the pullback of N by

ι, so that $\iota^*N = \{(p,n) \in P \times N \mid \iota p = \nu n\}$, and $\nu' : (p,n) \mapsto p$. Let P act on ι^*N by $(p_1,n)^p = (p^{-1}p_1p, n^{\iota p})$. The verification of the axiom CM1) is immediate, while CM2) is proved as follows:
Let $(p,n), (p_1,n_1) \in \iota^*N$. Then

$$\begin{aligned} (p,n)^{-1}(p_1,n_1)(p,n) &= (p^{-1}p_1p, n^{-1}n_1n) \\ &= (p^{-1}p_1p, n_1^{\nu n}) \\ &= (p^{-1}p_1p, n_1^{\iota p}) \\ &= (p_1,n_1)^{\nu'(p,n)}. \end{aligned}$$

1.1. Proposition. *The functor $\iota^* : \mathcal{XM}/Q \to \mathcal{XM}/P$ has a right adjoint ι_*.*

Proof. This follows from general considerations on Kan extensions. ∎

The universal property of induced crossed modules is the following. Let $(\mu : M \to P)$, $(\gamma : C \to Q)$ be crossed modules. In the diagram

$$\begin{tikzcd}
& M \arrow[r, "\mu"] \arrow[d, "\bar{\iota}"] \arrow[ddl, "f"'] & P \arrow[d, "\iota"] \\
& \iota_* M \arrow[r, "\delta"] \arrow[dl, dashed, "g"] & Q \\
C \arrow[urr, "\gamma"'] & &
\end{tikzcd}$$

the pair $(\bar{\iota},\ \iota)$ is a morphism of crossed modules such that for any crossed Q-module $(\gamma : C \to Q)$ and morphism of crossed modules $(f,\ \iota)$, there is a unique morphism $g : \iota_*M \to C$ of crossed Q-modules such that $g\bar{\iota} = f$.

It is a consequence of this universal property that if $M = P = F(R)$, the free group on a set R, and if $w : R \to Q$ is the restriction of ι to the set R, then $\iota_*F(R)$ is the free crossed module on w, in the sense of Whitehead [32] (see also [12, 20, 28]). Constructions of this free crossed module are given in these papers.

A presentation for induced crossed modules for a general morphism ι is given in Proposition 8 of [9]. We will need two more particular results. The first is Proposition 9 of that paper, and the second is a direct deduction from Proposition 10.

1.2. Proposition. *If $\iota : P \to Q$ is a surjection, and $(\mu : M \to P)$ is a crossed P-module, then $\iota_*M \cong M/[M,K]$, where $K = \mathrm{Ker}\ \iota$, and $[M,K]$ denotes the subgroup of M generated by all $m^{-1}m^k$ for all $m \in M,\ k \in K$.*

The following term and notation will be used frequently. Let P be a group and let T be a set. We define the *copower* $P \vec{*} T$ to be the free product of groups P_t, $t \in T$, each with elements (p,t), $p \in P$, and isomorphic to P under the map $(p,t) \mapsto p$. If Q is a group, then $P \vec{*} Q$ will denote the copower of P with the underlying set of the group Q.

1.3. Proposition. *If $\iota : P \to Q$ is an injection, and $(\mu : M \to P)$ is a crossed P-module, let T be a right transversal of ιP in Q. Let Q act on the copower $M \vec{*} T$ by the rule $(m,t)^q = (m^p, u)$, where $p \in P$, $u \in T$, and $tq = (\iota p)u$. Let $\delta : M \vec{*} T \to Q$ be defined by $(m,t) \mapsto t^{-1}(\iota\mu m)t$. Let S be a set of generators of M as a group, and let $S^P = \{x^p : x \in S,\ p \in P\}$. Then*

$$\iota_* M = (M \vec{*} T)/R$$

where R is the normal closure in $M \vec{*} T$ of the elements

$$\langle (r,t),(s,u)\rangle = (r,t)^{-1}(s,u)^{-1}(r,t)(s,u)^{\delta(r,t)} \quad (r,s \in S^P,\ t,u \in T).$$

Proof. Let $N = M \vec{*} T$. Proposition 10 of [9] yields that $\iota_* M$ is the quotient of N by the subgroup $\langle N,N\rangle$ generated by $\langle n,n_1\rangle = n^{-1}n_1^{-1}nn_1^{\delta n}, n,n_1 \in N$, and which is called in [12] the Peiffer subgroup of N. Now N is generated by the set $(S^P,T) = \{(s^p,t) : s \in S, p \in P, t \in T\}$, and this set is Q-invariant since $(s^p,t)^q = (s^{pp'},u)$ where $u \in T, p' \in P$ satisfy $tq = (\iota p')u$. It follows from Proposition 3 of [12] that $\langle N,N\rangle$ is the normal closure of the set $\langle (S^P,T),(S^P,T)\rangle$ of basic Peiffer commutators. ∎

1.4. Example. *The dihedral crossed module.* We show how this works out in the following case, which exhibits a number of typical features. We let Q be the dihedral group D_n with presentation $\langle x,y : x^n = y^2 = xyxy = 1\rangle$, and let $M = P$ be the cyclic subgroup C_2 of order 2 generated by y. Let $C_n = \{0,1,2,\ldots,n-1\}$ be the cyclic group of order n. A right transversal T of C_2 in D_n is given by the elements x^i, $i \in C_n$. Hence $\iota_* C_2$ has a presentation with generators $a_i = (y,x^i)$, $i \in C_n$, and relations given by $a_i^2 = 1, i \in C_n$, together with the Peiffer relations. Now $\delta a_i = x^{-i}yx^i = yx^{2i}$. Further the action is given by $(a_i)^x = a_{i+1}$, $(a_i)^y = a_{n-i}$. Hence $(a_i)^{\delta a_j} = a_{2j-i}$, so that the Peiffer relations are $a_j a_i a_j = a_{2j-i}$. It is well known that we now have a presentation of the dihedral group D_n, from which we recover the standard presentation $\langle u,v : u^n = v^2 = uvuv = 1\rangle$ by setting $u = a_0a_1$, $v = a_0$, so that $u^i = a_0a_i$. Then

$$\delta u = x^2,\ \delta v = y,$$

so that y acts on $\iota_* C_2$ by conjugation by v. However x acts by

$$u^x = u,\ v^x = vu.$$

Note that this is consistent with the crossed module axiom CM2) since

$$v^{x^2} = (vu)^x = vuu = u^{-1}vu.$$

We call $\mathcal{D}_n = (\delta : D_n \to D_n)$ the *dihedral crossed module*. It follows from these formulae that δ is an isomorphism if n is odd, and has kernel and cokernel isomorphic to C_2 if n is even. In particular, if n is even, then by results of section 3, $\pi_2(BD_n \cup \Gamma BC_2)$ can be regarded as having one non-trivial element represented by $u^{n/2}$.

1.5. Corollary. *Assume $\iota : P \to Q$ is injective. If M has a presentation as a group with g generators and r relations, the set of generators of M is P-invariant, and $n = [Q : \iota\mu(M)]$, then ι_*M has a presentation with gn generators and $rn + g^2n(n-1)$ relations.*

Another corollary determines induced crossed modules under some abelian conditions. This result has useful applications. If M is an abelian group, or P-module, and T is a set, we define the *copower* of M with T, written $M \vec{\oplus} T$, to be the sum of copies of M, one for each element of T.

1.6. Corollary. *Let $(\mu : M \to P)$ be a crossed P-module and $\iota : P \to Q$ a monomorphism of groups such that M is abelian and $\iota\mu(M)$ is normal in Q. Then ι_*M is abelian and as a Q-module is just the induced Q-module in the usual sense.*

Proof. We use the result and notation of Proposition 1.3. Note that if $u, t \in T$ and $r \in S$ then $u\delta(r,t) = ut^{-1}(\iota\mu r)t = (\iota\mu m)ut^{-1}t = (\iota\mu m)u$ for some $m \in M$, by the normality condition. The Peiffer commutator given in Proposition 1.3 can therefore be rewritten as

$$(r,t)^{-1}(s,u)^{-1}(r,t)(s,u)^{\delta(r,t)} = (r^{-1},t)(s,u)^{-1}(r,t)(s^m,u).$$

Since M is abelian, $s^m = s$. Thus the basic Peiffer commutators reduce to ordinary commutators. Hence ι_*M is the copower $M \vec{\oplus} T$, and this, with the given action, is the usual presentation of the induced Q-module. ∎

1.7. Example. Let $M = P = Q$ be the infinite cyclic group C_∞, let μ be the identity, and let ι be multiplication by 2. Then $\iota_*M \cong C_\infty \times C_\infty$, and the action of a generator of Q on ι_*M is to switch the two copies of C_∞. This result could also be deduced from well known results on free crossed modules. However, our results show that we get a similar conclusion simply by replacing each C_∞ in the above by a finite cyclic group C_{2n}, and this fact is new.

2. On the finiteness of induced crossed modules

In this section we give an algebraic proof that a crossed module induced from a finite crossed module by a morphism with finite cokernel is also finite. In a later section we will prove a slightly less general result, but by topological methods which will also yield results on the preservation of the Serre class of a crossed module under the inducing process.

2.1. Theorem. *Let $(\mu : M \to P)$ be a crossed module and let $\iota : P \to Q$ be a morphism of groups. Suppose that M and the index of $\iota(P)$ in Q are finite. Then the induced crossed module $(\delta : \iota_*M \to Q)$ is finite.*

Proof. Factor the morphism $\iota : P \to Q$ as $\tau\sigma$ where τ is injective and σ is surjective. Then ι_*M is isomorphic to $\tau_*\sigma_*M$. It is immediate from Proposition 1.2 that if M is finite then so also is σ_*M. So it is enough to assume that ι is injective, and in fact we assume it is an inclusion.

Let T be a right transversal of ιP in Q. Let $Y = M \vec{*} T$ be the copower of M and T, and let $\delta : Y \to Q$ and the action of Q on Y be as in Proposition 1.3. The equations $tq = (\iota p)u$ which determine this action in fact provide a function

$$(\xi, \eta) : T \times Q \to P \times T, \ \ (t, q) \mapsto (p, u).$$

A basic Peiffer relation is then of the form

$$(m, t)(n, v) = (n, v)(m^{\xi(t, v^{-1}(\iota\mu m)v)}, \eta(t, v^{-1}(\iota\mu m)v)) = (n, v)(m^p, u) \tag{2}$$

where $m, n \in M$, $t, u \in T$ and $q = v^{-1}(\iota\mu m)v$.

We now assume that the finite set T has l elements and has been given the total order $t_1 < t_2 < \cdots < t_l$. An element of Y may be represented as a word

$$(m_1, u_1)(m_2, u_2) \ldots (m_e, u_e). \tag{3}$$

Such a word is said to be *reduced* when $u_i \neq u_{i+1}$, $1 \leq i < e$, and to be *ordered* if $u_1 < u_2 < \cdots < u_e$ in the given order on T. This yeilds a partial ordering of $M \vec{*} T$ where $(m_i, u_i) \leq (m_j, u_j)$ whenever $u_i \leq u_j$.

A *twist* uses the Peiffer relation (2) to replace a reduced word $w = w_1(m, t)(n, v)w_2$, with $v < t$, by $w' = w_1(n, v)(m^p, u)w_2$. If the resulting word is not reduced, multiplication in M_v and M_u may be used to reduce it. In order to show that any word may be ordered by a finite sequence of twists and reductions, we define an integer weight function on the set W_n of non-empty words of length at most n by

$$\Omega_n : W_n \to \mathbb{Z}^+, \ \ (m_1, t_{j_1})(m_2, t_{j_2}) \ldots (m_e, t_{j_e}) \mapsto l^e \sum_{i=1}^{e} l^{n-i} j_i.$$

It is easy to see that $\Omega_n(w') < \Omega_n(w)$ when $w \to w'$ is a reduction. Similarly, for a twist

$$w = w_1(m_i, t_{j_i})(m_{i+1}, t_{j_{i+1}})w_2 \ \to \ w' = w_1(m_{i+1}, t_{j_{i+1}})(n, t_k)w_2$$

the weight reduction is

$$\Omega_n(w) - \Omega_n(w') = l^{n+e-i-1}(\, l(j_i - j_{i+1}) + j_{i+1} - j_k \,) \geq l^{n+e-i-1},$$

so the process terminates in a finite number of moves.

This ordering process is a special case of a purely combinatorial sorting algorithm discussed in [17].

We now specify an algorithm for converting a reduced word to an ordered word. Various algorithms are possible, some more efficient than others, but we are not interested in efficiency here. We call a reduced word k-ordered if the subword consisting of the first k elements is ordered and the remaining elements are greater than these. Every reduced word is at least 0-ordered. Given a k-ordered, reduced word, find the rightmost minimal element to the right of the k-th position. Move this element one place to the left with a twist, and reduce if necessary. The resulting word may only be j-ordered, with $j < k$, but its weight will be less than that of the original word. Repeat until an ordered word is obtained.

Let $Z = M_{t_1} \times M_{t_2} \times \ldots \times M_{t_l}$ be the product of the sets $M_{t_i} = M \times \{t_i\}$. Then the algorithm yeilds a function $\phi : Y \to Z$ such that the quotient morphism $Y \to \iota_* M$ factors through ϕ. Since Z is finite, it follows that $\iota_* M$ is finite. ∎

2.2. Remark. In this last proof, it is in general not possible to give a group structure on the set Z such that the quotient morphism $Y \to \iota_* M$ factors through a morphism to Z. For example, in the dihedral crossed module of example 1.4, with $n = 3$, the set Z will have 8 elements, and so has no group structure admitting a morphism onto D_3. This explains why the above method does not give an algebraic proof of Corollary 4.3, which gives conditions for $\iota_* M$ to be a finite p-group. However, in [15], we will give an algebraic proof for the case P is normal in Q.

3. Topological applications

As explained in the Introduction, the fundamental crossed module functor Π_2 assigns a crossed module $(\partial : \pi_2(X, A) \to \pi_1(A))$ to any base pointed pair of spaces (X, A). We will use the following consequence of Theorem C of [9], which is a 2-dimensional Van Kampen type theorem for this functor.

3.1. Theorem. ([9], Theorem D) *Let (B, V) be a cofibred pair of spaces, let $f : V \to A$ be a map, and let $X = A \cup_f B$. Suppose that A, B, V are path-connected, and the pair (B, V) is 1-connected. Then the pair (X, A) is 1-connected and the diagram*

$$\begin{array}{ccc} \pi_2(B,V) & \xrightarrow{\delta} & \pi_1(V) \\ {\scriptstyle \epsilon}\downarrow & & \downarrow{\scriptstyle \lambda} \\ \pi_2(X,A) & \xrightarrow[\delta']{} & \pi_1(A) \end{array}$$

presents $\pi_2(X, A)$ as the crossed $\pi_1(A)$-module $\lambda_*(\pi_2(B, V))$ induced from the crossed $\pi_1(V)$-module $\pi_2(B, V)$ by the group morphism $\lambda : \pi_1(V) \to \pi_1(A)$ induced by f.

As pointed out earlier, in the case P is a free group on a set R, and μ is the identity, then the induced crossed module $\iota_* P$ is the free crossed Q-module on the function $\iota | R : R \to Q$. Thus Theorem 3.1 implies Whitehead's Theorem W of the Introduction. A considerable amount of work has been done on this case, because of the connections with identities among relations, and methods such as transversality theory and "pictures" have proved successful ([12, 28]), particularly in the homotopy theory of 2-dimensional complexes [20]. However, the only route so far available to the wider geometric applications of induced crossed modules is Theorem 3.1. We also note that this Theorem includes the relative Hurewicz Theorem in this dimension, on putting $A = \Gamma V$, and $f : V \to \Gamma V$ the inclusion.

We will apply this Theorem 3.1 to the *classifying space of a crossed module*, as defined by Loday in [25] or Brown and Higgins in [11]. This classifying space is a functor B

assigning to a crossed module $\mathcal{M} = (\mu : M \to P)$ a pointed CW-space $B\mathcal{M}$ with the following properties:

3.2. *The homotopy groups of the classifying space of the crossed module $\mathcal{M} = (\mu : M \to P)$ are given by*

$$\pi_i(B\mathcal{M}) \cong \begin{cases} Coker\,\mu & for\ i = 1 \\ Ker\,\mu & for\ i = 2 \\ 0 & for\ i > 2. \end{cases}$$

3.3. *The classifying space $B(\iota : 1 \to P)$ is the usual classifying space BP of the group P, and BP is a subcomplex of $B\mathcal{M}$. Further, there is a natural isomorphism of crossed modules*

$$\Pi_2(B\mathcal{M}, BP) \cong \mathcal{M}.$$

3.4. *If X is a reduced CW-complex with 1-skeleton X^1, then there is a map*

$$X \to B(\Pi_2(X, X^1))$$

inducing an isomorphism of π_1 and π_2.

It is in these senses that it is reasonable to say, as in the Introduction, that crossed modules model all pointed homotopy 2-types.

We now give two direct applications of Theorem 3.1.

3.5. Corollary. *Let $\mathcal{M} = (\mu : M \to P)$ be a crossed module, and let $\iota : P \to Q$ be a morphism of groups. Let $\beta : BP \to B\mathcal{M}$ be the inclusion. Consider the pushout*

$$\begin{tikzcd} BP \arrow[r, "\beta"] \arrow[d, "B\iota"'] & B\mathcal{M} \arrow[d] \\ BQ \arrow[r, "\beta'"'] & X. \end{tikzcd} \tag{4}$$

Then the fundamental crossed module of the pair (X, BQ) is isomorphic to the induced crossed module $(\delta : \iota_* M \to Q)$, and this crossed module determines the 2-type of X.

Proof. The first statement is immediate from Theorem 3.1. The final statement follows from results of [11], since the morphism $Q \to \pi_1(X)$ is surjective. ∎

3.6. Remark. An interesting special case of the last corollary is when $\mathcal{M}$ is an inclusion of a normal subgroup, since then $B\mathcal{M}$ is of the homotopy type of $B(P/M)$. So we have determined the 2-type of a homotopy pushout

$$\begin{tikzcd} BP \arrow[r, "Bp"] \arrow[d, "B\iota"'] & BR \arrow[d] \\ BQ \arrow[r, "p'"'] & X \end{tikzcd}$$

in which $p : P \to R$ is surjective.

We write ΓV for the cone on a space V.

3.7. COROLLARY. *Let $\iota : P \to Q$ be a morphism of groups. Then the fundamental crossed module $\Pi_2(BQ \cup_{B\iota} \Gamma BP, BQ)$ is isomorphic to the induced crossed module $(\delta : \iota_* P \to Q)$.*

4. Finiteness theorems by topological methods

The aim of this section is to show that the property of being a finite p-group is preserved by the process of induced crossed modules. We use topological methods.

An outline of the method is as follows. Suppose that Q is a finite p-group. To prove that $\iota_* M$ is a finite p-group, it is enough to prove that Ker $(\iota_* M \to Q)$ is a finite p-group. But this kernel is the second homotopy group of the space X of the pushout (4), and so is also isomorphic to the second homology group of the universal cover $\widetilde{X}$ of X. In order to apply the homology Mayer-Vietoris sequence to this universal cover, we need to show that it may be represented as a pushout, and we need information on the homology of the spaces determining this pushout. So we start with the necessary information on covering spaces.

We work in the convenient category $\mathcal{TOP}$ of weakly Hausdorff k-spaces [24]. Let $\alpha : \widetilde{X} \to X$ be a map of spaces. In the examples we will use, α will be a covering map. Then α induces a functor

$$\alpha^* : \mathcal{TOP}/X \to \mathcal{TOP}/\widetilde{X}.$$

It is known that α has a right adjoint and so preserves colimits [2, 1, 24].

For regular spaces, the pullback of a covering space in the above category is again a covering space. These results enable us to identify a covering space of an adjunction space as an adjunction space obtained from the induced covering spaces.

If further, α is a covering map, and X is a CW-complex, then $\widetilde{X}$ may be given the structure of a CW-complex [27].

We also need a special case of the basic facts on the path components and fundamental group of induced covering maps ([7, 8, 27]). Given the following pullback

$$\begin{tikzcd} \widehat{A} \arrow[r] \arrow[d, "\alpha'"'] & \widetilde{X} \arrow[d, "\alpha"] \\ A \arrow[r, "f"'] & X \end{tikzcd}$$

and points $a \in A$, $\tilde{x} \in \widetilde{X}$ such that $fa = \alpha\tilde{x}$, in which α is a universal covering map and $X, A, \widetilde{X}$ are path connected, then there is a sequence

$$1 \to \pi_1(\widehat{A}, (a, \tilde{x})) \to \pi_1(A, a) \xrightarrow{f_*} \pi_1(X, fa) \to \pi_0(\widehat{A}) \to 1. \qquad (5)$$

This sequence is exact in the sense of sequences arising from fibrations of groupoids [7], which involves an operation of the fundamental group $\pi_1(X, fa)$ on the set $\pi_0(\widehat{A})$ of path components of $\widehat{A}$. It follows that the fundamental group of $\widehat{A}$ is isomorphic to Ker f_*, and that $\pi_0(\widehat{A})$ is bijective with the set of cosets $(\pi_1(X, fa))/(f_*\pi_1(A, a))$. It is also clear that the covering $\widehat{A} \to A$ is regular and that all the components of $\widehat{A}$ are homeomorphic.

Let $\mathcal{M}$ be the crossed module $(\mu : M \to P)$ and let $\iota : P \to Q$ be a morphism of groups. Let

$$X = BQ \cup_{B\iota} B\mathcal{M}$$

as in diagram (4). Let $\alpha : \widetilde{X} \to X$ be the universal covering map, and let $\widehat{BQ}$, $\widehat{B\mathcal{M}}$, $\widehat{BP}$ be the pullbacks of $\widetilde{X}$ under the maps $BQ \to X$, $B\mathcal{M} \to X$, $BP \to X$. Then we may write

$$\widetilde{X} \cong \widehat{BQ} \cup_{\widehat{B\iota}} \widehat{B\mathcal{M}}, \tag{6}$$

by the results of section 3.

From the exact sequence (5) we obtain the following exact sequences, in which $\pi_1 X \cong Q *_P (P/\mu M)$:

$$\begin{array}{cccccccccc} 1 \to & \pi_1(\widehat{BQ}) & \to & Q & \to & Q *_P (P/\mu M) & \to & 1, & \\ 1 \to & \pi_1(\widehat{B\mathcal{M}}) & \to & P/\mu M & \to & Q *_P (P/\mu M) & \to & \pi_0 \widehat{B\mathcal{M}} & \to 1, \\ 1 \to & \pi_1(\widehat{BP}) & \to & P & \to & Q *_P (P/\mu M) & \to & \pi_0 \widehat{BP} & \to 1. \end{array}$$

4.1. Proposition. *Under the above situation, let the groups $\pi_1(\widehat{BP})$, $\pi_1(\widehat{B\mathcal{M}})$, $\pi_1(\widehat{BQ})$ be denoted by P', M', Q' respectively, and let $B\mathcal{M}'$ denote a component of $\widehat{B\mathcal{M}}$. Then there is an exact sequence*

$$H_2(P') \vec{\oplus} \pi_0 \widehat{BP} \to (H_2(B\mathcal{M}') \vec{\oplus} \pi_0 \widehat{B\mathcal{M}}) \oplus H_2(Q') \to \pi_2(X) \to$$

$$\to H_1(P') \vec{\oplus} \pi_0 \widehat{BP} \to (H_1(M') \vec{\oplus} \pi_0 \widehat{B\mathcal{M}}) \oplus H_1(Q') \to 0.$$

Proof. This is immediate from the Mayer-Vietoris sequence for the pushout (6) and the fact that $H_2(\widetilde{X}) \cong \pi_2(X)$. ∎

4.2. Corollary. *If $\iota : P \to Q$ is the inclusion of a normal subgroup, and $X = BQ \cup_{B\iota} \Gamma BP$, then $\pi_2(X)$ is isomorphic to $H_1(P) \otimes I(Q/P)$, where $I(G)$ denotes the augmentation ideal of a group G.*

This result agrees with Corollary 2.5 of [15], in which the induced crossed module itself is computed, in the case P is normal in Q, *via* the use of coproducts of crossed P-modules.

4.3. Corollary. *Let $\mathcal{M} = (\mu : M \to P)$ be a crossed module and let $\iota : P \to Q$ be a morphism of groups. If M, P and Q are finite p-groups, then so also is $\iota_* M$.*

PROOF. It is standard that the (reduced) homology groups of a finite p-group are finite p-groups. The same applies to the reduced homology of the classifying space of a crossed module of finite p-groups. The latter may be proved using the spectral sequence of a covering, and Serre $\mathcal{C}$ theory, as in Chapters IX and X of [21]. In the present case, we need information only on $H_2(B\mathcal{M})$, and some of its connected covering spaces, and this may be deduced from the exact sequence due to Hopf

$$H_3K \to H_3G \to (\pi_2K) \otimes_{\mathbb{Z}G} \mathbb{Z} \to H_2K \to H_2G \to 0$$

for any connected space K with fundamental group G (see for example Exercise 6 on p.175 of [3]). Proposition 4.1 shows that Ker $(\iota_*(M) \to Q) \cong \pi_2(X)$ is a finite p-group. Since Q is a finite p-group, it follows that ι_*M is a finite p-group. ∎

Note that these methods extend also to results on the Serre class of an induced crossed module, which we leave the reader to formulate.

5. Cohomology classes

Recall [22, 3] that if G is a group and A is a G-module, then elements of $H^3(G, A)$ may be represented by equivalence classes of *crossed sequences*

$$0 \to A \to M \xrightarrow{\mu} P \to G \to 1, \tag{7}$$

namely exact sequences as above such that $(\mu : M \to P)$ is a crossed module. The equivalence relation between such crossed sequences is generated by the *basic equivalences*, namely the existence of a commutative diagram of morphisms of groups as follows

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "1"] & M \arrow[r, "\mu"] \arrow[d, "f"] & P \arrow[r] \arrow[d, "g"] & G \arrow[r] \arrow[d, "1"] & 1 \\
0 \arrow[r] & A \arrow[r] & M' \arrow[r, "\mu'"'] & P' \arrow[r] & G \arrow[r] & 1
\end{tikzcd}$$

such that (f, g) is a morphism of crossed modules. Such a diagram is called a *morphism* of crossed sequences.

The zero cohomology class is represented by the crossed sequence

$$0 \to A \xrightarrow{1} A \xrightarrow{0} G \xrightarrow{1} G \to 1,$$

which we sometimes abbreviate to

$$A \xrightarrow{0} G.$$

In a similar spirit, we say that a crossed module $(\mu : M \to P)$ represents a cohomology class, namely an element of $H^3(\mathrm{Coker}\ \mu, \mathrm{Ker}\ \mu)$.

5.1. Example. Let C_{n^2} denote the cyclic group of order n^2, written multiplicatively, with generator u. Let $\gamma_n : C_{n^2} \to C_{n^2}$ be given by $u \mapsto u^n$. This defines a crossed module, with trivial operations. This crossed module represents the trivial cohomology class in $H^3(C_n, C_n)$, in view of the morphism of crossed sequences

$$\begin{tikzcd}
0 \arrow[r] & C_n \arrow[r, "1"] \arrow[d, "1"] & C_n \arrow[r, "0"] \arrow[d, "\lambda"] & C_n \arrow[r, "1"] \arrow[d, "\lambda"] & C_n \arrow[r] \arrow[d, "1"] & 0 \\
0 \arrow[r] & C_n \arrow[r] & C_{n^2} \arrow[r, "\gamma_n"'] & C_{n^2} \arrow[r] & C_n \arrow[r] & 0
\end{tikzcd}$$

where, if t is the generator of the top C_n, then $\lambda(t) = u^n$.

5.2. Example. We show that the dihedral crossed module $\mathcal{D}_n$ of Example 1.4 represents the trivial cohomology class. This is clear for n odd, since then δ is an isomorphism. For n even, we simply construct a morphism of crossed sequences as in the following diagram

$$\begin{tikzcd}
0 \arrow[r] & C_2 \arrow[r, "1"] \arrow[d, "\cong"] & C_2 \arrow[r, "0"] \arrow[d, "f_2"] & C_2 \arrow[r, "1"] \arrow[d, "f_1"] & C_2 \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & C_2 \arrow[r] & D_n \arrow[r, "\delta"'] & D_n \arrow[r] & C_2 \arrow[r] & 0
\end{tikzcd}$$

where if t denotes the non trivial element of C_2 then $f_1(t) = x, f_2(t) = u^{n/2}$. Just for interest, we leave it to the reader to prove that there is no morphism in the other direction between these crossed sequences.

A crossed module $\mathcal{M} = (\mu : M \to P)$ determines a cohomology class

$$k_{\mathcal{M}} \in H^3(\text{Coker } \mu, \text{Ker } \mu).$$

If X is a connected, pointed CW-complex with 1-skeleton X^1, then the class

$$k_X^3 \in H^3(\pi_1 X, \pi_2 X)$$

of the crossed module $\Pi_2(X, X^1)$ is called the *first Postnikov invariant* of X. This class is also represented by $\Pi_2(X, A)$ for any connected subcomplex A of X such that (X, A) is 1-connected and $\pi_2(A) = 0$. It may be quite difficult to determine this Postnikov invariant from a presentation of this last crossed module, and even the meaning of the word "determine" in this case is not so clear. There are practical advantages in working directly with the crossed module, since it is an algebraic object, and so it, or families of such objects, may be manipulated in many convenient and useful ways. Thus the advantages of crossed modules over the corresponding 3-cocycles are analogous to some of the advantages of homology groups over Betti numbers and torsion coefficients.

However, in work with crossed modules, and in applications to homotopy theory, information on the corresponding cohomology classes, such as their non-triviality, or their order, is also of interest. The aim of this section is to give background to such a determination, and to give two example of finite crossed modules representing non-trivial elements of the corresponding cohomology groups.

The following general problem remains. If G, A are finite, where A is a G-module, how can one characterise the subset of $H^3(G, A)$ of elements represented by finite crossed modules? This subset is a subgroup, since the addition may be defined by a sum of crossed sequences, of the Baer type. (An exposition of this is given by Danas in [18].) It might always be the whole group.

The natural context in which to show how a crossed sequence gives rise to a 3-cocycle is not the traditional chain complexes with operators but that of crossed complexes [22]. We explain how this works here. For more information on the relations between crossed complexes and the traditional chain complexes with operators, see [10].

Recall that a free crossed resolution of the group G is a free aspherical crossed complex F_* together with an epimorphism $\phi : F_1 \to G$ with kernel $\delta_2(F_2)$.

5.3. Example. The cyclic group C_n of order n is written multiplicatively, with generator t. We give for it a free crossed resolution F_* as follows. Set $F_1 = C_\infty$, with generator w, and for $r \geq 2$, set $F_r = (C_\infty)^n$. Here for $r \geq 2$, F_r is regarded as the free C_n-module on one generator w_0, and we set $w_i = (w_0)^{t^i}$. The morphism $\phi : C_\infty \to C_n$ sends w to t, and the operation of F_1 on F_r for $r \geq 2$ is via ϕ. The boundaries are given by

1. $\delta_2(w_i) = w^n$,
2. for r odd, $\delta_r(w_i) = w_i w_{i+1}^{-1}$,
3. for r even and greater than 2, $\delta_r(w_i) = w_0 w_1 \dots w_{n-1}$.

Previous calculations show that δ_2 is the free crossed C_∞-module on the element $w^n \in C_\infty$. Thus F_* is a free crossed complex. It is easily checked to be aspherical, and so is, with ϕ, a crossed resolution of C_n.

Let A be a G-module. Let $C(G, A, 3)$ denote the crossed complex C which is G in dimension 1, A in dimension 3, with the given action of G on A, and which is 0 elsewhere, as in the following diagram

$$\cdots \longrightarrow 0 \longrightarrow A \longrightarrow 0 \longrightarrow G.$$

Let (F_*, ϕ) be a free crossed resolution of G. It follows from the discussions in [10, 11] that a 3-cocycle of G with coefficients in A can be represented as a morphism of crossed complexes $f : F_* \to C(G, A, 3)$ over ϕ. This cocycle is a coboundary if there is an operator morphism $l : F_2 \to A$ over $\phi : F_1 \to G$ such that $l\delta_3 = f_3$.

$$\begin{tikzcd}
F_4 \arrow[r, "\delta_4"] \arrow[d] & F_3 \arrow[r, "\delta_3"] \arrow[d, "f_3"'] & F_2 \arrow[r, "\delta_2"] \arrow[d] \arrow[dl, "l"'] & F_1 \arrow[d, "\phi"] \\
0 \arrow[r] & A \arrow[r] & 0 \arrow[r] & G
\end{tikzcd}$$

To construct a 3-cocycle on F_* from the crossed sequence (7), first construct a morphism of crossed complexes as in the diagram

$$\begin{tikzcd} F_4 \arrow[r] \arrow[d] & F_3 \arrow[r] \arrow[d, "f_3"] & F_2 \arrow[r] \arrow[d, "f_2"] & F_1 \arrow[r, "\phi"] \arrow[d, "f_1"] & G \arrow[d, "1"] \\ 0 \arrow[r] & A \arrow[r] & M \arrow[r, "\mu"'] & P \arrow[r, "\psi"'] & G \end{tikzcd} \tag{8}$$

using the freeness of F_* and the exactness of the bottom row. Then compose this with the morphism of crossed sequences

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "1"] & M \arrow[r, "\mu"] \arrow[d] & P \arrow[r, "\psi"] \arrow[d, "\psi"] & G \arrow[d, "1"] \\ 0 \arrow[r] & A \arrow[r] & 0 \arrow[r] & G \arrow[r, "1"'] & G \end{tikzcd}$$

Hence it is reasonable to say that the morphism f_3 of diagram (8) is a 3-cocycle corresponding to the crossed sequence.

We now use these methods in an example.

5.4. THEOREM. *Let $n \geq 2$, and let $\iota : C_n \to C_{n^2}$ denote the injection sending a generator t of C_n to u^n, where u denotes a generator of C_{n^2}. Let A_n denote the C_n-module which is the kernel of the induced crossed module $\mathcal{N} = (\partial : \iota_* C_n \to C_{n^2})$. Then $H^3(C_n, A_n)$ is cyclic of order n and has as generator the class of this induced crossed module.*

PROOF. By Corollary 1.6 the abelian group $\iota_* C_n$ is the product $V = (C_n)^n$. As a C_n-module it is cyclic, with generator v, say. Write $v_i = v^{t^i}$, $i = 0, 1, \ldots, n-1$. Then each v_i is a generator of a C_n factor of V. The kernel A_n of ∂ is a cyclic C_n-module on the generator $a = v_0 v_1^{-1}$. Write $a_i = a^{t^i} = v_i v_{i+1}^{-1}$. As an abelian group, A_n has generators $a_0, a_1, \ldots, a_{n-1}$ with relations $a_i^n = 1$, $a_0 a_1 \ldots a_{n-1} = 1$.

We define a morphism f_* from F_* to the crossed sequence containing $\mathcal{N}$ as in diagram (9), where

1. f_1 maps w to u,
2. f_2 maps the module generator w_0 of F_2 to $v = v_0$.
3. f_3 maps the module generator w_0 of F_3 to a_0.

$$\begin{tikzcd} (C_\infty)^n \arrow[r, "\delta_4"] \arrow[d, "0"'] & (C_\infty)^n \arrow[r, "\delta_3"] \arrow[d, "f_3"'] & (C_\infty)^n \arrow[r, "\delta_2"] \arrow[d, "f_2"] \arrow[dl, "l"'] & C_\infty \arrow[r] \arrow[d, "f_1"] & C_n \arrow[d, "1"] \\ 0 \arrow[r] & A_n \arrow[r] & (C_n)^n \arrow[r, "\partial"'] & C_{n^2} \arrow[r] & C_n \end{tikzcd} \tag{9}$$

The operator morphisms f_r over f_1 are defined completely by these conditions.

The group of operator morphisms $g : (C_\infty)^n \to A_n$ over f_1 may be identified with A_n under $g \mapsto g(w_0)$. Under this identification, the boundaries δ_4, δ_3 are transformed respectively to 0 and to $a_i \mapsto a_i(a_i^t)^{-1}$. So the 3-dimensional cohomology group is the group A_n with a_i identified with $a_{i+1}, i = 0, \ldots, n-1$. This cohomology group is therefore isomorphic to C_n, and a generator is the class of the above cocycle f_3. ∎

5.5. Corollary. *The mapping cone $X = BC_{n^2} \cup_{B\iota} \Gamma BC_n$ satisfies $\pi_1 X = C_n$, and $\pi_2 X$ is the C_n-module A_n of Theorem 5.4. The first Postnikov invariant of X is a generator of the cohomology group $H^3(\pi_1 X, \pi_2 X)$, which is a cyclic group of order n.*

The following is another example of a determination of a non-trivial cohomology class by a crossed module. The method of proof is similar to that of Theorem 5.4, and is left to the reader.

5.6. Example. Let n be even. Let C_n' denote the C_n-module which is C_n as an abelian group but in which the generator t of the group C_n acts on the generator t' of C_n' by sending it to its inverse. Then $H^3(C_n, C_n') \cong C_2$ and a generator of this group is represented by the crossed module $(\nu_n : C_n \times C_n \to C_{n^2})$, with generators t_0, t_1, u say, and where $\nu_n t_0 = \nu_n t_1 = u^n$. Here $u \in C_{n^2}$ operates by switching t_0, t_1. It is not clear if this crossed module can be an induced crossed module for $n > 2$. However, $n = 2$ gives the case $n = 2$ of Theorem 5.4.

5.7. Remark. The crossed module $(\nu_2 : C_2 \times C_2 \to C_4)$ also appears as an example in [23, pp.332-333]. The proof given there that its corresponding cohomology class is non-trivial is obtained by relating this class to the obstruction to a certain kind of extension.

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School of Mathematics
University of Wales, Bangor
Gwynedd LL57 1UT, U.K.

Email: `r.brown@bangor.ac.uk` and `mas023@clvax.bangor.ac.uk`

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