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SYMMETRIC MONOIDAL CATEGORIES MODEL ALL CONNECTIVE SPECTRA

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ABSTRACT. The classical infinite loopspace machines in fact induce an equivalence of categories between a localization of the category of symmetric monoidal categories and the stable homotopy category of -1-connective spectra.

Introduction

Since the early seventies it has been known that the classifying spaces of small symmetric monoidal categories are infinite loop spaces, the zeroth space in a spectrum, a sequence of spaces $X_i, i \geq 0$ with given homotopy equivalences to the loops on the succeeding space $X_i \xrightarrow{\sim} \Omega X_{i+1}$. Indeed, many of the classical examples of infinite loop spaces were found as such classifying spaces (e.g. [Ma2], [Se]). These infinite loop spaces and spectra are of great interest to topologists. The homotopy category formed by inverting the weak equivalences of spectra is the stable homotopy category, much better behaved than but still closely related to the usual homotopy category of spaces (e.g., [Ad] III).

One has in fact classically a functor Spt from the category of small symmetric monoidal categories to the category of -1-connective spectra, those spectra X_i for which $\pi_k X_i = 0$ when $k < i$ ([Ma2], [Se], [Th2]). Moreover, any two such functors satisfying the condition that the zeroth space of $\mathrm{Spt}(\mathcal{S})$ is the "group completion" of the classifying space $B\mathcal{S}$ are naturally homotopy equivalent ([Ma4], [Th2]).

The aim of this article is to prove the new result (Thm. 5.1) that in fact Spt induces an equivalence of categories between the stable homotopy category of -1-connective spectra and the localization of the category of small symmetric monoidal categories by inverting those morphisms that Spt sends to weak homotopy equivalences. In particular, each -1-connective spectrum is weak equivalent to $\mathrm{Spt}(\mathcal{S})$ for some symmetric monoidal category $\mathcal{S}$.

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Thus the category of symmetric monoidal categories provides an alternate model for the -1-connective stable homotopy category, one which looks rather different from the classical model category of spectra. It is really "coordinate-free" in that there are no suspension coordinates at all, in contrast to May's coordinate-free spectra ([Ma3] II) which use all finite subspaces of an infinite vector space as coordinates, and thus are just free of choices of coordinates. The category of E_∞-spaces, spaces with an action of an E_∞ operad ([Ma2]), is similarly a model for - -1-connective spectra which is really coordinate free. But a symmetric monoidal category structure is much more rigid than a general E_∞-space structure, and as a consequence can be specified and manipulated much more readily. As convincing evidence for this claim, I refer to my talk at the Colloque en l'honneur de Michel Zisman at l'Université Paris VII in June 1993. There I used this alternate model of stable homotopy to give the first known construction of a smash product which is associative and commutative up to coherent natural isomorphism in the model category. This will be the subject of an article to appear.

The proof that the functor Spt induces an equivalence between a localization of the category of symmetric monoidal categories and the -1-connective stable homotopy category begins by constructing an inverse functor. The construction is made in several steps. First, there are known equivalences of homotopy categories induced by functors between the category of -1-connective spectra and the category of E_∞-spaces. Thus one reduces to finding an appropriate functor to the category of symmetric monoidal categories from that of E_∞-spaces. Any space X is weak homotopy equivalent to the classifying space of the category Null/X of weakly contractible spaces over X. When X is an E_∞-space the operad action on X induces operations on this category. For example, for each integer $n \geq 0$ there is an n-ary functor sending the objects $C_1 \to X, C_2 \to X, \ldots, C_n \to X$ to $E(n)\times C_1 \times C_2 \times \ldots \times C_n \longrightarrow E(n)\times \prod^n X \longrightarrow X$, where the last arrow $E(n)\times \prod^n X \longrightarrow X$ is given by the operad action. The internal composition functions of the operad induce certain natural transformations between composites of these operations on Null/X. Using Kelly's theory of clubs ([Ke3], [Ke4]) one finds that Null/X has been given the structure of a lax algebra for the club σ of symmetric monoidal categories. The next step is to replace this lax symmetric monoidal category by a symmetric monoidal one. There is a Godement resolution of a lax algebra by free lax algebras, yielding a simplicial lax algebra. By coherence theory, the latter is degree-wise stably homotopy equivalent to a simplicial free symmetric monoidal category. Taking a sort of homotopy colimit of the last simplicial object ([Th2]) yields the desired symmetric monoidal category.

The layout of the article is as follows: The first section is a review, starting with the definitions of non-unital symmetric monoidal categories, and of lax, strong and strict symmetric monoidal functors. I recall in 1.6 the basic properties of the classical functor from symmetric monoidal categories to spectra. Next in 1.7 comes a review of the definitions of op-lax functors into a 2-category and left op-lax natural transformations between such. The homotopy colimit, or op-lax colimit, of a diagram of symmetric monoidal functors is recalled in 1.8, and its good homotopy theoretic properties stated. The first section closes with a proof in 1.9 that all the variant categories of symmetric monoidal categories considered have equivalent homotopy categories. Section 2 begins with a review of lax algebras over a doctrine. A Godement type simplicial resolution of lax algebras by free lax algebras in given in 2.2. This resolution is shown to be left op-lax natural with respect to lax morphisms of lax algebras. In 2.3 I consider the special case of lax symmetric monoidal categories. The functor to Spectra is extended to these in 2.4. In 2.5 I use the Godement resolution and the homotopy colimit to show how to replace a lax symmetric monoidal category by a symmetric monoidal category. Section 3 reviews the notion of an E_∞-space and the equivalence of homotopy categories between E_∞-spaces and Spectra. Section 4 contains the construction of a lax symmetric monoidal category associated to an E_∞-space. Finally Section 5 finishes the proof by showing various round-trip functors made by joining the above pieces are linked to the identity by a chain of stable homotopy equivalences.

1 Symmetric monoidal categories and homotopy colimits

I will need to use the "homotopy colimit" of diagrams of symmetric monoidal categories. The based version of the homotopy colimit does not have good homotopy behavior except under stringent "non-degenerate basepoint conditions": essentially one would have to ask that the symmetric monoidal unit have no non-identity automorphisms and that every morphism to or from the unit be an isomorphism. Such a symmetric monoidal category is equivalent to the disjoint union of 0 with a possibly non-unital symmetric monoidal category, and finally all homotopy colimit results are easier to state if one works with variant categories of non-unital symmetric monoidal categories from the beginning. Thus:

1.1 Definition. By *unital symmetric monoidal category*, I mean a symmetric monoidal category in the classic sense, a category $\mathcal{S}$ provided with an object 0 and a functor $\oplus\colon \mathcal{S}\times\mathcal{S}\to\mathcal{S}$, together with natural isomorphisms of associativity, commutativity, and unitaricity for which certain simple diagrams are to commute. For the details see for example [McL] VII §1, §7.

1.2 Definition. A *symmetric monoidal category* is a category $\mathcal{S}$ together with a functor $\oplus\colon \mathcal{S}\times\mathcal{S}\to\mathcal{S}$ and natural isomorphisms:

$$\begin{aligned} \alpha\colon A\oplus(B\oplus C) &\xrightarrow{\sim} (A\oplus B)\oplus C \\ \gamma\colon A\oplus B &\xrightarrow{\sim} B\oplus A \end{aligned} \tag{1.2.1}$$

which are such that the following two pentagon and hexagon diagrams commute:

(1.2.2)

$$\begin{tikzcd}
A \oplus (B \oplus (C \oplus D))) \arrow[r, "\alpha"] \arrow[d, "1\oplus\alpha"'] & (A \oplus B) \oplus (C \oplus D) \arrow[r, "\alpha"] & ((A \oplus B) \oplus C) \oplus D \\
A \oplus ((B \oplus C) \oplus D) \arrow[rr, "\alpha"'] & & (A \oplus (B \oplus C)) \oplus D \arrow[u, "\alpha\oplus 1"']
\end{tikzcd}$$

(1.2.3)

$$\begin{tikzcd}
A \oplus (B \oplus C) \arrow[r, "\alpha"] \arrow[d, "1\oplus\gamma"'] & (A \oplus B) \oplus C \arrow[r, "\gamma"] & C \oplus (A \oplus B) \arrow[d, "\alpha"] \\
A \oplus (C \oplus B) \arrow[r, "\alpha"'] & (A \oplus C) \oplus B \arrow[r, "\gamma\oplus 1"'] & (C \oplus A) \oplus B
\end{tikzcd}$$

The usual results of coherence theory that "every diagram commutes" continue to hold for these non-unital symmetric monoidal categories; indeed the precise statements and the demonstrations become easier without the additional structure of the unitaricity natural isomorphisms ([Ep], [Ke1] 1.2).

To compare with the existing literature it will be useful to recall ([Ma2] 4.1 and [McLa] VII §1) that a *permutative category* is a unital symmetric monoidal category where the natural associativity isomorphism α is the identity. It follows from coherence theory that every unital symmetric monoidal category is equivalent to a permutative category. Indeed the equivalence is realized by strong unital symmetric monoidal functors and unital symmetric monoidal natural isomorphisms. See [Ma2] 4.2 or [Ke3] 1.2, 4.10, 4.8.

As to morphisms between symmetric monoidal categories, one needs to consider three kinds: the lax, the strong, and the strict symmetric monoidal functors. The usual definitions adapt to the the non-unital case easily.

1.3 Definition. For $\mathcal{S}$ and $\mathcal{T}$ symmetric monoidal categories, a *lax symmetric monoidal functor* from $\mathcal{S}$ to $\mathcal{T}$ consists of a functor $F\colon \mathcal{S} \to \mathcal{T}$ together with a natural transformation of functors from $\mathcal{S} \times \mathcal{S}$:

$$f\colon FA \oplus FB \to F(A \oplus B)$$

such that the following two diagrams commute:

(1.3.1)

$$\begin{tikzcd}
FA \oplus (FB \oplus FC) \arrow[r, "1\oplus f"] \arrow[d, "\alpha"'] & FA \oplus F(B \oplus C) \arrow[r, "f"] & F(A \oplus (B \oplus C)) \arrow[d, "F\alpha"] \\
(FA \oplus FB) \oplus FC \arrow[r, "f\oplus 1"'] & F(A \oplus B) \oplus FC \arrow[r, "f"'] & F((A \oplus B) \oplus C)
\end{tikzcd}$$

(1.3.2)

$$\begin{array}{ccc} FA \oplus FB & \xrightarrow{f} & F(A \oplus B) \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle F\gamma} \\ FB \oplus FA & \xrightarrow[f]{} & F(B \oplus A) \end{array}$$

One will often abusively denote the lax symmetric monoidal functor (F, f) as simply F, censoring the expression of the natural transformation f.

A *strong symmetric monoidal functor* is a lax symmetric monoidal functor such that the natural transformation f is in fact a natural isomorphism.

A *strict symmetric monoidal functor* is a strong symmetric monoidal functor such that the natural transformation f is the identity. Thus F strictly preserves the operation $\oplus$ and the natural isomorphisms α and γ.

The usual definition of a lax, strong, or strict unital symmetric monoidal functor between unital symmetric monoidal categories imposes the additional structure of a morphism $0 \to F0$ (respectively, an isomorphism, an identity) subject to a commutative diagram involving the unitaricity isomorphisms. See e.g. [Th2] (1.2).

1.4 DEFINITION. A *symmetric monoidal natural transformation* $\eta: F \to G$ between two lax symmetric monoidal functors $(F, f), (G, g): \mathcal{S} \to \mathcal{T}$ is a natural transformation η such that the following diagram commutes:

(1.4.2)

$$\begin{array}{ccc} FA \oplus FB & \xrightarrow{f} & F(A \oplus B) \\ {\scriptstyle\eta\oplus\eta}\downarrow & & \downarrow{\scriptstyle\eta} \\ GA \oplus GB & \xrightarrow[g]{} & G(A \oplus B) \end{array}$$

A symmetric monoidal natural transformation between two strict or two strong symmetric monoidal functors is a symmetric monoidal natural transformation between the underlying lax symmetric monoidal functors.

Note that such an η is automatically compatible with the associativity and commutativity isomorphisms α and γ by naturality of η, α, and γ. On the other hand, the definition of a unital symmetric monoidal natural transformation between unital symmetric monoidal functors imposes a new compatibility with the unitaricity isomorphisms, as in e.g. [Th2] (1.7). (Thus a symmetric monoidal natural transformation between two unital symmetrical monoidal functors need not be a *unital* symmetric monoidal natural transformation!)

1.5: Catalog of variant 2-categories of symmetric monoidal categories.

In order to model spectra, I want to consider only small symmetric monoidal categories, those for which the class of all morphisms is in fact a set. In order to be able to localize categories of such, I suppose Grothendieck's axiom of universes ([SGA4] I Appendice). This axiom is that each set is contained in a set U the elements of which form a model of set-theory such that the internal power sets in the model U are the true power sets. For each such universe U, one has several categories of U-small symmetric monoidal categories. A category is U-small if the class of all its morphisms is an element of U. The class of objects is then also an element of U. Any of the variant categories of U-small categories below will then itself be V-small for any universe containing U as an element. The localization of any of the variants by inverting any set of morphisms is then W-small for any universe W containing V.

The variants, differing in unitaricity and in degree of laxity of morphisms, are:

SymMon:
Objects: U-small symmetric monoidal categories
Morphisms: lax symmetric monoidal functors
2-cells: symmetric monoidal natural transformations

SymMonStrong:
Objects: U-small symmetric monoidal categories
Morphisms: strong symmetric monoidal functors
2-cells: symmetric monoidal natural transformations

SymMonStrict:
Objects: U-small symmetric monoidal categories
Morphisms: strict symmetric monoidal functors
2-cells: symmetric monoidal natural transformations

UniSymMon:
Objects: U-small unital symmetric monoidal categories
Morphisms: lax unital symmetric monoidal functors
2-cells: unital symmetric monoidal natural transformations

UniSymMonStrong:
Objects: U-small unital symmetric monoidal categories
Morphisms: strong unital symmetric monoidal functors
2-cells: unital symmetric monoidal natural transformations

UniSymMonStrict:
Objects: U-small unital symmetric monoidal categories
Morphisms: strict unital symmetric monoidal functors
2-cells: unital symmetric monoidal natural transformations

Of these, what the common man means by "the" category of small symmetric monoidal categories is UniSymMonStrong. To express the universal mapping property characterizing homotopy colimits, one needs instead to consider both SymMon and SymMonStrict. Adding the non-unital and unital analogs of all these produces the list of six 2-categories above.

One has the obvious diagram of forgetful 2-functors:

(1.5.1)

$$\begin{tikzcd}
\mathrm{UniSymMonStrict} \arrow[r] \arrow[d] & \mathrm{UniSymMonStrong} \arrow[r] \arrow[d] & \mathrm{UniSymMon} \arrow[d] \\
\mathrm{SymMonStrict} \arrow[r] & \mathrm{SymMonStrong} \arrow[r] & \mathrm{SymMon}
\end{tikzcd}$$

There is also a 2-functor from the bottom to the top of each column in (1.5.1). On objects it sends the non-unital symmetric monoidal category $\mathcal{S}$ to the coproduct of categories $\mathcal{S} \coprod \underline{0}$, where $\underline{0}$ is the category with one object 0 and only the identity morphism. The symmetric monoidal structure on the coproduct is determined by saying that the inclusion of $\mathcal{S}$ is a strict symmetric monoidal functor and that there are natural identities of functors on the coproduct

$$\mathrm{Id} \oplus 0 = \mathrm{Id} = 0 \oplus \mathrm{Id}$$

which we take as the unitaricity isomorphisms. Thus 0 becomes a strict unit. This construction on objects extends to a 2-functor in the obvious way.

1.6: The functor $\mathrm{Spt}\colon \mathrm{SymMon} \to \mathrm{Spectra}$.

A slight elaboration ([Th2] Appendix) of either May's or Segal's infinite loop space machines gives a functor into the category of spectra:

$$\mathrm{Spt}\colon \mathrm{SymMon} \to \mathrm{Spectra}$$

Moreover, symmetric monoidal natural transformations canonically induce homotopies of maps of spectra ([Th2] 2.9).

Let $B\colon \mathrm{Cat} \to \mathrm{Top}$ denote the classifying space functor. By [Th2] 2.2 there is a natural transformation of functors from SymMon to Top

$$\iota\colon B\mathcal{S} \to \mathrm{Spt}(\mathcal{S})_0 \tag{1.6.1}$$

where $\mathrm{Spt}(\mathcal{S})_0$ is the underlying zeroth space of the spectrum $\mathrm{Spt}(\mathcal{S})$. When $\mathcal{S}$ admits the structure of a unital symmetric monoidal category, this ι is a group-completion. Equivalently, it induces an isomorphism on homology groups after inverting the action of the monoid $\pi_0 B\mathcal{S}$:

$$(\pi_0)^{-1} H_*(B\mathcal{S};\mathbb{Z}) \xrightarrow{\simeq} H_*(\mathrm{Spt}(\mathcal{S})_0;\mathbb{Z}) \tag{1.6.2}$$

When $\mathcal{S}$ is not unital, $\mathrm{Spt}(\mathcal{S})_0$ is a group-completion of the unital $\mathcal{S} \coprod \underline{0}$ since by [Th2] 2.1 the inclusion of $\mathcal{S}$ into the this symmetric monoidal category induces a weak homotopy equivalence of spectra:

$$\mathrm{Spt}(\mathcal{S}) \xrightarrow{\sim} \mathrm{Spt}(\mathcal{S} \coprod \underline{0}) \tag{1.6.3}$$

(This last assertion ultimately reduces to the observation that any monoid M has group-completion isomorphic to that of the monoid $M \coprod 0$ formed by forgetting there was already a unit and freely adding a new one 0. For the group completion process forces the identification of the new 0 with all other idempotent elements, and in particular with the old forgotten unit. For details of the reduction see [Th2] A.2.)

The proof of May's uniqueness theorem ([Ma4] Thm. 3) for functors defined from the category of permutative categories easily generalizes to show any two functors from SymMon to Spectra which satisfy the above-cited group-completion conditions of [Th2] 2.1 and 2.2 are connected by a chain of natural weak homotopy equivalences. (See [Th2] pp. 1603, 1646.) In particular, the two functors become naturally isomorphic after composition with the functor from Spectra into the stable homotopy category. The same statements hold for any two functors with the group-completion properties defined on any of the six variant categories of symmetric monoidal categories considered in 1.5.

1.6.4 Definition. A lax symmetric monoidal functor $F: \mathcal{S} \to \mathcal{T}$ is said to be a *stable homotopy equivalence* if $\mathrm{Spt}(F)$ is a weak homotopy equivalence of spectra. A morphism in any of the variant categories of symmetric monoidal categories listed in 1.5 is said to be a stable homotopy equivalence if the underlying lax symmetric monoidal functor is such.

1.6.5. If the lax symmetric monoidal functor F induces a homotopy equivalence of classifying spaces $B\mathcal{S} \xrightarrow{\sim} B\mathcal{T}$, then F is a stable homotopy equivalence. This follows from the group completion property of 1.6.1. For a map of -1-connective spectra is a stable homotopy equivalence if and only if the induced map on the zeroth spaces is a weak homotopy equivalence. (Indeed, the stable homotopy groups of the spectrum are 0 in negative degrees by -1-connectivity, and in non-negative degrees are isomorphic to the homotopy groups of the zeroth space.) And by the Whitehead theorem (for H-spaces not necessarily simply-connected) this condition is in turn equivalent to the map of zeroth spaces inducing an isomorphism on homology with $\mathbb{Z}$ coefficients.

More generally, this shows $\mathrm{Spt}(F)$ is a stable homotopy equivalence if and only if BF induces an isomorphism on the localizations of the homology groups by inverting the action of the monoid π_0:

$$\pi_0^{-1} H_*(BF): (\pi_0 B\mathcal{S})^{-1} H_*(B\mathcal{S};\mathbb{Z}) \xrightarrow{\sim} (\pi_0 B\mathcal{T})^{-1} H_*(B\mathcal{T};\mathbb{Z})$$

1.7: Left op-lax natural transformations.

I will need the notions of op-lax functors and left op-lax natural transformations between them. Theses concepts are ultimately derived from Benabou's work on bicategories and Grothendieck's theory of pseudofunctors. I follow the terminology of Street [St] suitably dualized from lax to op-lax and generalized from Cat to an arbitrary 2-category.

Let $\underline{L}$ be a category and $\underline{\underline{K}}$ a 2-category.

1.7.1 Definition. An *op-lax functor* $\Phi\colon \underline{L} \to \underline{\underline{K}}$ consists of functions assigning to each object L of $\underline{L}$ an object ΦL of $\underline{\underline{K}}$; to each morphism $\ell\colon L \to L'$ of $\underline{L}$ a morphism $\Phi\ell\colon \Phi L \to \Phi L'$ of $\underline{\underline{K}}$; to each composable pair of morphisms $L \xrightarrow{\ell} L' \xrightarrow{\ell'} L''$ in $\underline{L}$ a 2-cell in $\underline{\underline{K}}$ $\varphi_{\ell',\ell}\colon \Phi(\ell'\ell) \Rightarrow \Phi(\ell')\Phi(\ell)$; and to each object L of $\underline{L}$ a 2-cell in $\underline{\underline{K}}$ $\varphi_L\colon \Phi(1_L) \Rightarrow 1_{\Phi L}$. These are to satisfy the following three identities of 2-cells:

For each $L_0 \xrightarrow{\ell_1} L_1 \xrightarrow{\ell_2} L_2 \xrightarrow{\ell_3} L_3$ in $\underline{L}$

(1.7.1.1)

$$\begin{tikzcd}
\Phi L_0 \arrow[r, "\Phi(\ell_3\ell_2\ell_1)"] \arrow[d, "\Phi\ell_1"'] \arrow[dr, phantom, "\Downarrow\varphi", near start] & \Phi L_3 \\
\Phi L_1 \arrow[r, "\Phi\ell_2"'] \arrow[ur, "\Phi(\ell_3\ell_2)" description] \arrow[ur, phantom, "\varphi\Downarrow", near end, shift right=4] & \Phi L_2 \arrow[u, "\Phi\ell_3"']
\end{tikzcd}
=
\begin{tikzcd}
\Phi L_0 \arrow[r, "\Phi(\ell_3\ell_2\ell_1)"] \arrow[d, "\Phi\ell_1"'] \arrow[dr, "\Phi(\ell_2\ell_1)" description] \arrow[ur, phantom, "\Downarrow\varphi", shift left=4] & \Phi L_3 \\
\Phi L_1 \arrow[r, "\Phi\ell_2"'] \arrow[ur, phantom, "\varphi\Downarrow", shift right=4] & \Phi L_2 \arrow[u, "\Phi\ell_3"']
\end{tikzcd}$$

For each morphism $\ell\colon L \to L'$ in $\underline{L}$

(1.7.1.2)

$$\begin{tikzcd}
\Phi L \arrow[rr, "\Phi\ell"] \arrow[dr, "\Phi 1"] \arrow[dr, bend right=40, "1"'] \arrow[dr, phantom, bend right=20, "\Leftarrow\varphi"] & \arrow[d, phantom, "\Downarrow\varphi"] & \Phi L' \\
& \Phi L \arrow[ur, "\Phi\ell"'] &
\end{tikzcd}
= \mathrm{id} =
\begin{tikzcd}
\Phi L \arrow[rr, "\Phi\ell"] \arrow[dr, "\Phi\ell"'] & \arrow[d, phantom, "\varphi\Downarrow"] & \Phi L' \\
& \Phi L \arrow[ur, "\Phi 1"'] \arrow[ur, bend right=40, "1"'] \arrow[ur, phantom, bend right=20, "\Leftarrow\varphi"] &
\end{tikzcd}$$

A functor may be considered as an op-lax functor with identity structure 2-cells φ. A *pseudo-functor* is an op-lax functor where the structure 2-cells are isomorphisms. The definition of a *lax functor* is obtained by reversing the direction of the structure 2-cells in 1.7.1.

1.7.2 Definition. A *left op-lax natural transformation* between two op-lax functors $\Phi, \Psi\colon \underline{L} \to \underline{\underline{K}}$ is a function assigning to each object L of $\underline{L}$ a morphism in $\underline{\underline{K}}$, $\eta_L\colon \Phi L \to$

ΨL; and to each morphism $\ell: L \to L'$ a 2-cell in $\underline{\underline{K}}$, $\eta_\ell: \Psi\ell \circ \eta_L \Rightarrow \eta_{L'} \circ \Phi\ell$ These are to satisfy the following two identities of 2-cells:

For each $L_0 \xrightarrow{\ell_1} L_1 \xrightarrow{\ell_2} L_2$ in $\underline{L}$:

(1.7.2.1)

$$
\begin{tikzcd}
 & \Phi L_0 \arrow[r, "\eta_{L_0}"] \arrow[dl, "\Phi\ell_1"'] \arrow[dd, "\Phi\ell_2\ell_1"] \arrow[ddr, phantom, "\Downarrow\eta_{\ell_2\ell_1}"] & \Psi L_0 \arrow[dd, "\Psi\ell_2\ell_1"] \\
\Phi L_1 \arrow[dr, "\Phi\ell_2"'] \arrow[r, phantom, "\overset{\Longleftarrow}{\varphi}"] & {} & \\
 & \Phi L_2 \arrow[r, "\eta_{L_2}"'] & \Psi L_2
\end{tikzcd}
=
\begin{tikzcd}
 & \Phi L_0 \arrow[rr, "\eta_{L_0}"] \arrow[dl, "\Phi\ell_1"'] \arrow[d, phantom, "\Downarrow\eta_{\ell_1}"] & & \Psi L_0 \arrow[dl, "\Psi\ell_1"'] \arrow[dd, "\Psi(\ell_2\ell_1)"] \\
\Phi L_1 \arrow[rr, "\eta_{L_1}"] \arrow[dr, "\Phi\ell_2"'] & {} \arrow[d, phantom, "\Downarrow\eta_{\ell_2}"] & \Psi L_1 \arrow[dr, "\Psi\ell_2"] \arrow[r, phantom, "\overset{\Longleftarrow}{\psi}"] & {} \\
 & \Phi L_2 \arrow[rr, "\eta_{L_2}"'] & & \Psi L_2
\end{tikzcd}
$$

And for each object L of $\underline{L}$:

(1.7.2.2)

$$
\begin{tikzcd}
\Phi L \arrow[r, "\eta_L"] \arrow[d, bend right, "1"'] \arrow[d, bend left, "\Phi 1"] \arrow[d, phantom, "\overset{\Leftarrow}{\varphi}"] \arrow[dr, phantom, "\Downarrow\eta_1"] & \Psi L \arrow[d, "\Psi 1"] \\
\Phi L \arrow[r, "\eta_L"'] & \Psi L
\end{tikzcd}
=
\begin{tikzcd}
\Phi L \arrow[r, "\eta_L"] \arrow[d, "1"'] & \Psi L \arrow[d, bend right, "1"'] \arrow[d, bend left, "\Psi 1"] \arrow[d, phantom, "\overset{\Leftarrow}{\psi}"] \\
\Phi L \arrow[r, "\eta_L"'] & \Psi L
\end{tikzcd}
$$

Any natural transformation between two functors may be considered a left op-lax natural transformation with identity structure 2-cells η_ℓ. There is a notion of right op-lax natural transformation obtained by reversing the direction of the structure 2-cells. For a left lax natural transformation between two lax functors, the 2-cells η_ℓ go in the same sense as for a left op-lax natural transformation, but of course the structure two cells of Φ and Ψ go in the opposite sense. The conditions to impose on the 2-cells of a left lax natural transformation are analogous to 2.7.2.1 and 2.7.2.2. (See [St] §1.)

1.7.3 Definition. A *modification* $s: \eta \Rightarrow \nu$ between two left op-lax natural transformations is a function assigning to each object L of $\underline{L}$ a 2-cell $s_L: \eta_L \Rightarrow \nu_L$ of $\underline{\underline{K}}$. These are to satisfy the 2-cell identity that for each morphism $\ell: L \to L'$ in $\underline{L}$:

(1.7.3.1)

$$
\begin{tikzcd}[column sep=large, row sep=huge]
\Phi L \arrow[r, bend left, "\eta_L"] \arrow[r, bend right, "\nu_L"'] \arrow[r, phantom, "\Downarrow s_L"] \arrow[d, "\Phi\ell"'] \arrow[dr, phantom, "\Downarrow \nu_\ell"] & \Psi L \arrow[d, "\Psi\ell"] & & \Phi L \arrow[r, "\eta_L"] \arrow[d, "\Phi\ell"'] \arrow[dr, phantom, "\Downarrow \eta_\ell"] & \Psi L \arrow[d, "\Psi\ell"] \\
\Phi L' \arrow[r, "\nu_{L'}"'] & \Psi L' \arrow[r, phantom, "="] & & \Phi L' \arrow[r, bend left, "\eta_{L'}"] \arrow[r, bend right, "\nu_{L'}"'] \arrow[r, phantom, "\Downarrow s_{L'}"] & \Psi L'
\end{tikzcd}
$$

1.7.4 Notation. Let $\underline{L}$ be a small category and $\underline{\underline{K}}$ a 2-category. Denote by

$$\mathrm{Cat}(\underline{L}, \underline{\underline{K}})$$

the 2-category of functors from $\underline{L}$ to $\underline{\underline{K}}$. Its 1-cells are natural transformations and its 2-cells are modifications. Denote by

$$\text{op-}\overset{\leftarrow}{\mathrm{Lax}}(\underline{L}, \underline{\underline{K}})$$

the 2-category of op-lax functors from $\underline{L}$ to $\underline{\underline{K}}$. Its objects are op-lax functors, the morphisms are left op-lax natural transformations, and the 2-cells are modifications. Denote by

$$\overset{\leftarrow}{\mathrm{Fun}}(\underline{L}, \underline{\underline{K}})$$

the sub 2-category whose objects are functors, whose morphisms are left op-lax natural transformations, and whose 2-cells are modifications.

1.8: Homotopy colimits.

I recall from [Th2] the homotopy colimit of a diagram of symmetric monoidal categories. This is a sort of op-lax colimit which turns out to have good properties with respect to stable homotopy theory. More precisely:

The homotopy colimit is a 2-functor

$$\mathrm{hocolim}_L \colon \overset{\leftarrow}{\mathrm{Fun}}(\underline{L}, \mathrm{SymMon}) \to \mathrm{SymMonStrict} \tag{1.8.1}$$

which is left 2-adjoint to the composite of the forgetful functor SymMonStrict $\to$ SymMon and the 2-functor SymMon $\to \overset{\leftarrow}{\mathrm{Fun}}(\underline{L}, \mathrm{SymMon})$ sending a symmetric monoidal category $\mathcal{S}$ to the constant functor from $\underline{L}$ sending each L to $\mathcal{S}$. Thus there is a natural adjunction isomorphism of categories for each functor $\Phi \colon \underline{L} \to \mathrm{SymMon}$ and each symmetric monoidal category $\mathcal{S}$ in SymMonStrict:

$$(\overset{\leftarrow}{\mathrm{Fun}}(\underline{L}, \mathrm{SymMon}))(\Phi, \mathcal{S}) \cong \mathrm{SymMonStrict}(\mathrm{hocolim}_L \Phi, \mathcal{S}) \tag{1.8.2}$$

This adjunction isomorphism is equivalent to the universal mapping property stated in [Th2] Prop. 3.21, as follows by a straightforward calculation on expanding out the various definitions and using ([Th2] (2.9)).

I call this op-lax colimit a homotopy colimit because of its relation to the homotopy colimit of diagrams of spectra. Recall ([BK] XII, [Th2] §3) the latter homotopy colimit is a functor:

$$\mathrm{hocolim}_L\colon \mathrm{Cat}(\underline{L}, \mathrm{Spectra}) \to \mathrm{Spectra}$$

which sends natural stable homotopy equivalences to stable homotopy equivalences equivalences. It induces a total derived functor of colim_L on the homotopy categories. Among its other good properties is a natural spectral sequence for the stable homotopy groups, whose E^2 term is expressed in terms of homology of the category $\underline{L}$:

$$E^2_{p,q} = H_p(\underline{L}; \pi_q\Phi) \Rightarrow \pi_{p+q}\mathrm{hocolim}_L\Phi \tag{1.8.3}$$

But by [Th2] Thm. 4.1, there is a natural stable homotopy equivalence between functors from $\mathrm{Cat}(\underline{L}, \mathrm{SymMon})$ to the category of Spectra,

$$\mathrm{hocolim}_L\mathrm{Spt}(\Phi) \xrightarrow{\sim} \mathrm{Spt}(\mathrm{hocolim}_L\Phi) \tag{1.8.4}$$

In particular, there is a spectral sequence natural in $\Phi \in \overleftarrow{\mathrm{Fun}}(\underline{L}, \mathrm{SymMon})$

$$E^2_{p,q} = H_p(\underline{L}; \pi_q\mathrm{Spt}(\Phi)) \Rightarrow \pi_{p+q}\mathrm{Spt}(\mathrm{hocolim}_L\Phi) \tag{1.8.5}$$

Since $\pi_q\mathrm{Spt} \cong 0$ for $q \leq -1$, the spectral sequence 1.8.5 lives in the first quadrant and converges strongly (e.g. [Th2] 3.14). Since a map of spectra is a stable homotopy equivalence if and only if it induces an isomorphism on stable homotopy groups π_*, this spectral sequence directly gives good homotopy-theoretic control of the symmetric monoidal category $\mathrm{hocolim}_L\Phi$. (The extended naturality of the spectral sequence (1.8.5) for left op-lax natural transformations is proved using rectification of op-lax functors as in the last paragraph of [Th2] §4, cf. [Th1] 3.3)

1.9: Comparison of variant homotopy categories.

As a first application of the homotopy colimit, I will now proceed to show all the variant categories of 1.5 all have equivalent localizations on inverting the stable homotopy equivalences.

As explained after the diagram 1.5.1, for each of the vertical forgetful functors in this diagram there is a functor going in the opposite direction which freely adds a unit 0. There are natural transformations between the identity functors and the composites of these vertical forgetful and free-unit functors. The components of these natural transformations are the inclusion $\iota: \mathcal{S} \to \mathcal{S} \coprod \underline{0}$ for $\mathcal{S}$ non-unital and the map sending 0 to the old unit $\rho: \mathcal{S} \coprod \underline{0} \to \mathcal{S}$ for $\mathcal{S}$ unital. By 1.6.3 $\mathrm{Spt}(\iota)$ is a stable homotopy equivalence. Since $\mathrm{Spt}(\rho)$ is a right inverse to an instance of $\mathrm{Spt}(\iota)$, it is also a stable homotopy equivalence. Thus these natural transformations become natural isomorphisms in the localizations, and the localizations of the unital and non-unital variants in each column of 1.5.1 are equivalent.

It remains only to see that the forgetful functors between the non-unital variants in the bottom row of 1.5.1 induce equivalences of localizations. But one has another functor in the opposite direction given by:

$$\mathrm{SymMon} \cong \mathrm{Cat}(\underline{0}, \mathrm{SymMon}) \to \overleftarrow{\mathrm{Fun}}(\underline{0}, \mathrm{SymMon}) \to \mathrm{SymMonStrict} \tag{1.9.1}$$

where the last functor is the $\mathrm{hocolim}_0$ of 1.8. Let this composite functor be denoted by $\widehat{(\quad)}$. Using the universal mapping property (1.8.2) of $\mathrm{hocolim}_0$, one gets a natural transformation of functors on SymMonStrict with components $\widehat{\mathcal{S}} = \mathrm{hocolim}_0 \mathcal{S} \to \mathcal{S}$. By 1.8.8 these components are stable homotopy equivalences. Moreover, this is still a natural stable homotopy equivalence after pre- or post-composing $\widehat{(\quad)}$ with any of the forgetful functors in the bottom row of 1.5.1. Thus these composites with $\widehat{(\quad)}$ induce inverses to the forgetful functors after localization. This has proved the following reassuring principle:

1.9.2 Lemma. *The forgetful functors in diagram 1.5.1 all induce equivalences of categories between the localizations of these variant categories by inverting the stable homotopy equivalences.*

2 Lax symmetric monoidal categories

In this section, I will study the category of lax algebras in Cat over the doctrine of symmetric monoidal categories, that is, the category of lax symmetric monoidal categories. Following Kelly, I give generators and relations for a club whose strict algebras are the lax symmetric monoidal categories. The functor $\mathrm{Spt}(\quad)$ of 1.6 extends to a functor on these. A Godement construction shows that each lax symmetric monoidal category admits a simplicial resolution by (strict) symmetric monoidal categories. Using this and the homotopy colimit along Δ^{op} I define a functor from the category of lax symmetric monoidal categories to that of symmetric monoidal categories which commutes up to natural stable homotopy equivalence with $\mathrm{Spt}(\quad)$.

2.1: Clubs and doctrines.

2.1.1. I will suppose the reader is familiar with Kelly's theory of clubs, an efficient means of describing algebraic structures imposed on categories. A club prescribes certain n-ary functors, which are to be operations on a category, and natural transformations between them. These may be considered as generated by iterated substitution and composition of a smaller basic set of operations and transformations. The club structure encodes this substitution process. I will consider only clubs over the skeletal category of finite sets with permutations as morphisms. That is, each operation in a club will have a finite arity n, and natural transformations between operations are allowed to specify a required permutation of the order of inputs between the source and the target operations. The "type" functor from the club to the category of finite sets specifies the arity of the operations and the permutations associated to the natural transformations. The reader may consult [Ke4] §10 and §1 for a quick review of club theory. See also [Ke2], [Ke1].

2.1.2. Denote by σ the club for symmetric monoidal categories. The underlying category of this club has as objects a T_n of type $n \in \mathbb{N}$ for each way to build up an n-ary operation by iterated substitution of a binary operation $\oplus$ into instances of itself. (This includes an empty set of substitutions, which yields the identity functor 1 as a 1-ary operation.) For each $n \geq 1$ the type functor is an equivalence of categories between the subcategory of objects of type n and the symmetric group of order n. Thus any two objects of the same type are isomorphic, and the group of automorphisms of any object of type n is the symmetric group Σ_n. All this follows from standard coherence theory, given that in this paper symmetric monoidal categories are not assumed to have units.

In order to construct the Godement resolution needed to pass from lax symmetric monoidal categories to symmetric monoidal ones, I will need to consider the doctrine associated to a club. The reader may consult [KS] §3.5 for a review of doctrines. I recall the definitions for its convenience.

2.1.3 Definitions. A *doctrine* in a 2-category $\underline{\underline{K}}$ is an endo 2-functor D together with 2-natural transformations $j\colon \mathrm{Id} \to D$ and $\mu\colon DD \to D$ satisfying the defining identities:

$$\mu \circ D\mu = \mu \circ \mu D \qquad \mu \circ Dj = \mathrm{id} = \mu \circ jD \tag{2.1.3.1}$$

A D-*algebra* is an object A of $\underline{\underline{K}}$ together with a morphism $a\colon DA \to A$ ("the action") satisfying the defining identities:

$$a \circ Da = a \circ \mu A\colon DDA \to A \qquad \mathrm{id}_A = a \circ j \tag{2.1.3.2}$$

A *lax morphism* between two D-algebras (A, a) and (B, b) consists of a morphism in $\underline{\underline{K}}$, $f\colon A \to B$ together with a 2-cell $\overline{f}$:

(2.1.3.3)

$$\begin{tikzcd}
DA \arrow[r, "Df"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\overline{f}"] & DB \arrow[d, "b"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

which is to satisfy the following two identities of 2-cells:

(2.1.3.4)

$$
\begin{tikzcd}
DDA \arrow[r, "DDf"] \arrow[d, "\mu A"'] & DDB \arrow[d, "\mu B"] \\
DA \arrow[r, "Df"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\overline{f}"] & DB \arrow[d, "b"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
=
\begin{tikzcd}
DDA \arrow[r, "DDf"] \arrow[d, "Da"'] \arrow[dr, phantom, "\Downarrow D\overline{f}"] & DDB \arrow[d, "Db"] \\
DA \arrow[r, "Df"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\overline{f}"] & DB \arrow[d, "b"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
$$

(2.1.3.5)

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "jA"'] & B \arrow[d, "jB"] \\
DA \arrow[r, "Df"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\overline{f}"] & DB \arrow[d, "b"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "1"'] & B \arrow[d, "1"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
$$

A *strong morphism* between D-algebras is a lax morphism such that the 2-cell $\overline{f}$ is an isomorphism. A *strict morphism* of D-algebras is a lax morphism such that $\overline{f}$ is the identity.

A *D-2-cell* between two (lax, strong, or strict) morphisms of D-algebras $f, g\colon A \to B$ is a 2-cell of $\underline{\underline{K}}$ $s\colon f \Rightarrow g$ such that one has the identity of 2-cells:

$$\overline{g} \circ Ds = s \circ \overline{f} \tag{2.1.3.6}$$

Denote by D-Alg the 2-category whose objects are D-algebras, whose 1-cells are *lax* morphisms of D-algebras, and whose 2-cells are D-2-cells. The variant sub 2-categories whose 1-cells are the strong or strict morphisms of D-algebras are denoted D-AlgStrong and D-AlgStrict respectively.

2.1.4. The doctrine $D = \kappa\circ$ on the 2-category Cat associated to a club κ sends a category $\underline{C}$ to the underlying category of the free κ algebra on $\underline{C}$. From [Ke1] §2 and [Ke2] § 2.3 or [Ke4] §10, one derives the following description. Denote by $\kappa(n)$ the category whose objects are (T, τ) where T is an object of type n in the club, and $\tau \in \Sigma_n$. A morphism $(T, \tau) \to (T', \tau')$ consists of a morphism $u: T \to T'$ in κ whose type ν satisfies $\nu\tau = \tau'$. $\kappa(n)$ is thus roughly speaking the category of operations induced by elements of the club and permutations of inputs. Then the doctrine corresponding to κ is:

$$\kappa\underline{C} = \coprod_{n\geq 0} \kappa(n) \ltimes_{\Sigma_n} \underline{C}^n \tag{2.1.4.1}$$

Here Σ_n acts on the n-fold product $\underline{C}^n$ by permuting the factors, and on $\kappa(n)$ by the free right action $(T, \tau) \times \sigma \mapsto (T, \tau\sigma)$.

The natural transformation $\mu: \kappa \circ \kappa\underline{C} \to \kappa\underline{C}$ is given as part of the club structure and encodes the substitution of operations into other operations. It is induced by a collection of functors:

$$\kappa(n) \times \kappa(j_1) \times \cdots \times \kappa(j_n) \to \kappa(j_1 + \cdots + j_n) \tag{2.1.4.2}$$

The natural transformation $j: \underline{C} \to \kappa\underline{C}$ is induced by the inclusion

$$1 \in \kappa(1) \tag{2.1.4.3}$$

of the distinguished object $(1, 1)$ corresponding to the identity operation.

Assigning to each object and morphism in the club an operation and natural transformation of operations on $\underline{C}$ of the appropriate arity corresponds to giving a morphism $c: \kappa\underline{C} \to \underline{C}$. If the assignment is compatible with substitution of operations and the identity, then c is the structure of an algebra for the doctrine κ.

2.1.5. In the case of the club σ of 2.1.2, the structure of a σ-algebra is exactly the structure of a symmetric monoidal category. The lax, strong, and strict morphisms of σ-algebras correspond respectively to lax, strong, and strict symmetric monoidal functors. The coherence theorem for symmetric monoidal categories noted in 2.1.2 is equivalent to the fact that for each $n \geq 1$ there is a unique isomorphism between any two objects of $\sigma(n)$.

2.2: Godement enriched and lax.

Recall that Δ^{op}, the category such that functors $\Delta^{op} \to \underline{C}$ are the simplicial objects of $\underline{C}$, is the opposite category of the skeletal category Δ of finite non-empty totally ordered sets $\underline{n} = \{0 < 1 < \cdots < n\}$ and monotone increasing maps. Following [Th2] 1.1 let Δ^+

be the category with objects $\underline{n} = \{-1 < 0 < 1 < \cdots < n\}$ for $n \geq -1$ and morphisms the monotone increasing maps sending -1 to -1. There is a standard inclusion functor from Δ to Δ^+ sending $\underline{n}$ to $\underline{n}$. A functor $X: \Delta^{+op} \to \underline{C}$ is a simplicial object $X.$ together with an augmentation $\epsilon = d_0: X_0 \to X_{-1}$ and a system of extra degeneracies $s_{-1}: X_n \to X_{n+1}$ for $n \geq -1$. These must satisfy certain relations extending the usual simplicial identities. That is, one requires on X_n that:

$$
\begin{aligned}
d_i d_j &= d_{j-1} d_i && \text{if } 0 \leq i < j \leq n \\
s_i s_j &= s_{j+1} s_i && \text{if } --1 \leq i \leq j \leq n \\
d_i s_j &= s_{j-1} d_i && \text{if } 0 \leq i < j \leq n \\
d_i s_j &= \mathrm{id} && \text{if } -1 \leq j \leq n \text{ and } 0 \leq i \in \{j, j+1\} \\
d_i s_j &= s_j d_{i-1} && \text{if } --1 \leq j \text{ and } j+1 < i \leq n+1
\end{aligned}
\tag{2.2.1}
$$

The extra degeneracies imply that the augmentation $X. \to X_{-1}$ is a simplicial homotopy equivalence, even that there is a simplicial homotopy on $X.$ that is a deformation retraction to the constant simplicial object on X_{-1}. (Cf. e.g. [Ma1] §9)

2.2.2 Proposition. *Let D be a doctrine on the 2-category $\underline{\underline{K}}$. Recall the notations of 1.7.5 and of §2 above.*

Then there is a natural 2-functor, the Godement resolution,

$$R_*: D\text{-}Alg \longrightarrow \overleftarrow{Fun}(\Delta^{+op}, \underline{\underline{K}})$$

such that $R_n = D^{n+1}$ for $n \geq 0$ while R_{-1} is the functor sending a D-algebra to its underlying object in $\underline{\underline{K}}$.

After restriction of the values in $\overleftarrow{Fun}$ from Δ^{+op} to the standard subcategory Δ^{op}, R_* lifts canonically to a 2-functor

$$R_* \mid \Delta^{op}: D\text{-}Alg \longrightarrow \overleftarrow{Fun}(\Delta^{op}, D\text{-}AlgStrict)$$

Considering $R_{-1} = Id: D\text{-}Alg \to D\text{-}Alg$ as taking values in constant simplicial D-algebras, the augmentation induces a 2-natural transformation ${d_0}^{*+1}: R_* \mid \Delta^{op} \Rightarrow R_{-1}$ of functors into $\overleftarrow{Fun}(\Delta^{op}, D\text{-}Alg)$

The restriction of R to D-AlgStrict factors through the sub 2-category of $\overleftarrow{Fun}$ whose 1-cells are (strict) natural transformations.

$$R_*: D\text{-}AlgStrict \longrightarrow Cat(\Delta^{+op}, \underline{\underline{K}})$$

Thus for a D-algebra A, $R_* A$ is a simplicial free D-algebra with a D-algebra augmentation to A, the augmentation being a simplicial homotopy equivalence after forgetting the D-algebra structure. The whole thing is strictly natural for strict morphisms of D-algebras, and is left op-lax natural for lax morphisms of D-algebras. If the 2-category

$\underline{\underline{K}}$ has only identity 2-cells, the doctrine D is just a monad and all reduces to the classical Godement resolution [Go] Appendice or to May's reformulation as a two-sided bar construction $B_*(D, D, \mathrm{Id})$ in [Ma1] §9.

PROOF. For (A, a) a D-algebra, define the functor $R_*(A, a)$ from Δ^{+op} by giving values on on objects and on the generating morphisms d_i and s_i of Δ^{+op}. On objects, $R_nA = D^{n+1}A$ for $n \geq -1$. For the morphisms:

$$d_i: R_nA \to R_{n-1}A = \begin{cases} D^i\mu D^{n-i-1}: D^{n+1}A \to D^nA & \text{for } 0 \leq i < n \\ D^na: D^{n+1}A \to D^nA & \text{for } i = n \end{cases} \tag{2.2.2.1}$$

$$s_i: R_nA \to R_{n+1}A = D^{i+1}jD^{n-i}: D^{n+1}A \to D^{n+2}A \text{ for } -1 \leq i \leq n \tag{2.2.2.2}$$

Naturality and the defining identities 2.1.3 for the structure maps μ, j, and a of doctrines and algebras give that these d_i and s_i satisfy the extended simplicial identities (2.2.1) and so define a functor on Δ^{+op}. All d_i and the s_i other than s_{-1} are strict maps of D-algebras.

Given a morphism of D-algebras $f: A \to B$, let $R_nf = D^{n+1}f$. Note that for a strict algebra morphism f this formula defines a natural transformation $R_*f: R_*A \to R_*B$ of functors on Δ^{+op}. For such an f strictly commutes with μ, j, and a. For a D-2-cell between two morphisms $s: f \Rightarrow g$ let $R_ns = D^{n+1}s: D^{n+1}f \Rightarrow D^{n+1}g$. Routine verification now yields the results of the last paragraph of Proposition 2.2.2.

The essential point remaining is to define the structure of a left op-lax natural transformation (1.7.2) $R_*f: R_*A \to R_*B$ for each lax morphism of D-algebras $(f, \overline{f}): A \to B$. The structure 1-cells of this R_*f are the $R_nf = D^{n+1}$ above. The structure 2-cell (2.2.2.3)

$$\begin{tikzcd} D^{n+1}A \arrow[r, "D^{n+1}f"] \arrow[d, "\varphi^*"'] \arrow[dr, phantom, "\Downarrow\overline{Rf}_\varphi"] & D^{n+1}B \arrow[d, "\varphi^*"] \\ D^{q+1}A \arrow[r, "D^{q+1}f"'] & D^{q+1}B \end{tikzcd}$$

associated to a morphism $\varphi: \underline{q} \to \underline{n}$ in Δ^+ is derived from the structure 2-cells of the lax algebra map as follows:

If $\varphi: \{-1 < 0 < \cdots < q\} \to \{-1 < \cdots < n\}$ preserves the maximal element, $\varphi(q) = n$, then φ in Δ^{+op} can be written as a composite of s_i and those d_i for which i is not maximal. Thus φ^* can be written as a composite of $D^i\mu D^k$ and D^ijD^k, without using any action maps D^ia. In this case the diagram of 1-cells in (2.2.2.3) commutes and one takes $\overline{Rf}_\varphi$ to be the identity 2-cell. In the other case where φ does not preserve the maximal element, it factors uniquely in Δ^+ as:

$$\varphi = \{-1 < \cdots < q\} \xrightarrow{\iota} \{-1 < \cdots < q < q+1\} \xrightarrow{\psi} \{-1 < \cdots < p\}$$

where ι is the inclusion of the initial segment and ψ is determined by $\psi\iota = \varphi$ with $\psi(q+1) = n$. Thus ψ preserves the maximal element. As for ι, $\iota^* = d_q^* = D^{q+1}a$. One sets the structure 2-cell $\overline{Rf}_\varphi$ to be that induced by the structure 2-cell of the lax algebra morphism:

(2.2.2.4)

$$\begin{tikzcd} D^{n+1}A \arrow[r, "D^{n+1}f"] \arrow[d, "\psi^*"'] & D^{n+1}B \arrow[d, "\psi^*"] \\ D^{q+2}A \arrow[r, "D^{q+2}f"] \arrow[d, "D^{q+1}a"'] \arrow[dr, phantom, "\Downarrow D^{q+1}\overline{f}"] & D^{q+2}B \arrow[d, "D^{q+1}b"] \\ D^{q+1}A \arrow[r, "D^{q+1}f"'] & D^{q+1}B \end{tikzcd}$$

That these $\overline{Rf}_\varphi$ satisfy the defining identities 1.7.2.1 and 1.7.2.2 for an op-lax natural transformation follows by routine case by case analysis of these specifications and the identities 2.1.3.5 and 2.1.3.4 of lax morphisms of algebras.

Finally, for s a D-2-cell, R_*s is a modification of left op-lax natural transformations. The required identity 1.7.3.1 results from 2.1.3.6. ∎

2.2.3 Addendum. *Preserving the hypotheses and notations of 2.2.2, suppose that* $E\colon \underline{\underline{K}} \to \underline{\underline{Q}}$ *is a 2-functor which is a right D-algebra. That is, suppose there is a 2-natural transformation* $e\colon ED \to E$ *such that*

$$e\mu = e \circ eD\colon EDD \to E$$

and

$$1 = e \circ Ej\colon E \to ED \to E$$

.

Then there is a 2-functor

$$ED_*\colon D\text{-}Alg \longrightarrow \overleftarrow{Fun}(\Delta^{op}, \underline{\underline{Q}})$$

specified as follows:

Given (A, a) a D-algebra, the simplicial object $ED_*A\colon \Delta^{op} \to \underline{\underline{Q}}$ is defined by:

$$n \mapsto ED^nA$$

$$s_i = ED^i j D^{n-i}: ED^n A \to ED^{n+1} A$$

$$d_i: ED^n A \to ED^{n-1} A = \begin{cases} eD^{n-1} & \textit{for } i = 0 \\ ED^{i-1}\mu D^{n-i-1} & \textit{for } 0 < i < n \\ ED^{n-1}a & \textit{for } i = n \end{cases}$$

Given $(f, \overline{f}): (A, a) \to (B, b)$ a lax morphism of D-algebras, the associated left op-lax natural transformation $ED_*A \to ED_*B$ has structure 1-cells $ED^n f: ED^n A \to ED^n B$ and structure 2-cells given by the obvious generalization of 2.2.2.4 on replacing all $D^{n+1}A$ by $ED^n A$, etc.

Given $s: f \Rightarrow g$ a modification of lax morphisms of D-algebras, the associated modification of left op-lax natural transformations has component 2-cells

$$ED^n s: ED^n f \Rightarrow ED^n g$$

Verification of all these points results from calculations which are trivial modifications of those required for the verification of 2.2.2.

2.3: LaxSymMon.

2.3.1 Definition. The club $\tilde{\sigma}$ for lax symmetric monoidal categories is the club over the category of finite ordinals and permutations with the following presentation:

The objects of $\tilde{\sigma}$ are generated under substitution by a $\overline{T}$ for each object T of the club for symmetric monoidal categories σ. The type of the generator $\overline{T}_n$ equals the type of T_n. No relations are imposed on the objects. (In particular, if 1 denotes the objects in σ and $\tilde{\sigma}$ corresponding to the identity operation, one does *not* have $\overline{1} = 1$.)

The morphisms of $\tilde{\sigma}$ are generated by:

i) A morphism

$$\overline{u}: \overline{T} \to \overline{R}$$

for each morphism $u: T \to S$ in σ. The type of $\overline{u}$ is the permutation which is the type of u.

ii) A morphism

$$\tilde{a}_{T[S_1, S_2, \cdots, S_n]}: \overline{T}(\overline{S}_1, \cdots, \overline{S}_n) \to \overline{T(S_1, \cdots, S_n)}$$

of type the identity permutation for each object T of σ and each n-tuple of objects $S_1, \cdots, S_n$ of σ. Here $n \in \mathbb{N}$ is the type of T.

iii) A morphism

$$\hat{a}: 1 \to \overline{1}$$

of type 1.

The relations imposed on these generators are a)-e) below:

(2.3.1.a)

$$\overline{u} \circ \overline{v} = \overline{uv}$$
$$\overline{\mathrm{id}_T} = \mathrm{id}_{\overline{T}}$$

(2.3.1.b)

$$\begin{tikzcd} \overline{T}(1,\cdots,1) \arrow[r, "\overline{T}(\hat{a},\cdots,\hat{a})"] \arrow[dr, "\mathrm{id}"'] & \overline{T}(\overline{1},\cdots,\overline{1}) \arrow[d, "\tilde{a}_{T[1,\cdots,1]}"] \\ & \overline{T} \end{tikzcd}$$

(2.3.1.c)

$$\begin{tikzcd} 1[\overline{S}] \arrow[r, "\hat{a}[\overline{S}]"] \arrow[dr, "\mathrm{id}"'] & \overline{1}[\overline{S}] \arrow[d, "\tilde{a}_{1[S]}"] \\ & \overline{S} \end{tikzcd}$$

(2.3.1.d)

$$\begin{tikzcd} \overline{T}(\overline{S}_1(\overline{R}_{11},\cdots),\cdots,\overline{S}_n(\overline{R}_{n1},\cdots)) \arrow[r, "\tilde{a}"] \arrow[d, "\overline{T}(\tilde{a},\cdots,\tilde{a})"'] & \overline{T(S_1,\cdots,S_n)}(\overline{R}_{11},\cdots,\overline{R}_{nq}) \arrow[d, "\tilde{a}"] \\ \overline{T}(\overline{S_1(R_{11},\cdots)},\cdots,\overline{S_n(R_{n1},\cdots)}) \arrow[r, "\tilde{a}"'] & \overline{T(S_1(R_{11},\cdots),\cdots,S_n(\cdots,R_{nq}))} \end{tikzcd}$$

(2.3.1.e)

$$\begin{tikzcd} \overline{T}(\overline{S}_1,\cdots,\overline{S}_n) \arrow[r, "\tilde{a}_{T[S_1,\cdots,S_n]}"] \arrow[d, "\overline{u}(\overline{v}_1,\cdots,\overline{v}_n)"'] & \overline{T(S_1,\cdots,S_n)} \arrow[d, "\overline{u(v_1,\cdots,v_n)}"] \\ \overline{T'}(\overline{S'}_1,\cdots,\overline{S'}_n) \arrow[r, "\tilde{a}_{T'[S'_1,\cdots,S'_n]}"'] & \overline{T'(S'_1,\cdots,S'_n)} \end{tikzcd}$$

for each ordered set of morphisms in σ consisting of a $u: T \to T'$ and of $v_i: S_i \to S'_{\varphi(i)}$ where φ is the type of u.

2.3.2. Comparison with [Ke3] 4.10, 4.1-4.2 shows that this club $\tilde{\sigma}$ is indeed the club whose doctrine has as strict algebras the lax algebras over the doctrine of symmetric monoidal categories.

There is a map of clubs

(2.3.2.1) $$s: \tilde{\sigma} \to \sigma$$

sending the object $\overline{T}$ to T, the morphism $\overline{u}$ to u, and the morphisms $\hat{a}$ and $\tilde{a}_{T[S_1,\cdots,S_n]}$ to identity morphisms. This map of clubs induces a morphism of the corresponding doctrines.

By [Ke3] 4.8 Thm. 4.1 there is also a lax doctrine map going in the opposite direction:

$$(h, \tilde{h}, \hat{h}) \colon \sigma \to \tilde{\sigma} \tag{2.3.2.2}$$

As in [Ke3] 4.10 h is induced by the functor between the underlying categories of the clubs which sends the object T to $\overline{T}$ and the morphism $u\colon T \to S$ to $\overline{u}$. This functor is not a (strict) morphism of clubs since it does not strictly preserve the operational substitution which is part of the club structure.

The following facts are immediately deduced from the "cheap" coherence result of [Ke3] 1.4 and the description of σ above in 2.1. For each object U of $\tilde{\sigma}$ there is an object V in σ, and a morphism in $\tilde{\sigma}$ $U \to \overline{V} = hV$. Given U, any V for which there exists a morphism $U \to \overline{V}$ is isomorphic to sU in σ. Two morphisms $U \to \overline{V}$ in $\tilde{\sigma}$ are equal if and only if their images under s are equal, that is to say, if and only if they have the same permutation as type.

Rephrasing this, we get that there is a 2-cell of lax maps of doctrines $\tau\colon 1 \Rightarrow hs$ such that $s\tau = 1$ and $\tau h = 1$. In particular for any category $\underline{C}$ the functor $h\colon \sigma\underline{C} \to \tilde{\sigma}\underline{C}$ is right adjoint to $s\colon \tilde{\sigma}\underline{C} \to \sigma\underline{C}$.

Note that these results of coherence theory together with 2.1.4, 2.1.5, and the elements of the homotopy theory of categories ([Qu] §1) imply that one has the following homotopy equivalences of classifying spaces, naturally in any category $\underline{C}$:

$$B(\tilde{\sigma}\underline{C}) \xrightarrow[s]{\sim} B(\sigma\underline{C}) \xrightarrow[\text{type}]{\sim} \coprod_{n=1}^{\infty} E\Sigma_n \times_{\Sigma_n} B(\underline{C})^n \tag{2.3.2.3}$$

2.3.3 Definition. The 2-category LaxSymMonStrict has as objects the U-small algebras for the club $\tilde{\sigma}$, i.e. the (U-small) lax symmetrical monoidal categories. The 1-cells are the *strict* morphisms of $\tilde{\sigma}$-algebras, those functors which strictly preserve the operations $\overline{T}$ and the natural transformations $\overline{u}$, $\tilde{a}$, and $\hat{a}$. The 2-cells are the modifications of 1-cells (2.1.3).

There are variant 2-categories LaxSymMonStrong and LaxSymMon whose 1-cells are respectively the strong and the lax morphisms of $\tilde{\sigma}$-algebras (2.1.3) and whose 2-cells are the modifications of such.

To preserve compatibility with the more common terminology of §1, I will use the term "lax symmetric monoidal functor" only for a 1-cell in the category SymMon of 1.5.1. A 1-cell in LaxSymMonStrict will be a "strict functor between lax symmetric monoidal categories".

There is a forgetful 2-functor

$$\mathrm{SymMon} \to \mathrm{LaxSymMon} \tag{2.3.4}$$

induced by the map of clubs $s: \tilde{\sigma} \to \sigma$.

2.4: Spt( ) on LaxSymMon.

The functor $\mathrm{Spt}: \mathrm{SymMon} \to \mathrm{Spectra}$ of 1.6 extends to a functor on LaxSymMon. Indeed the construction in [Th2] Appendix generalizes easily. The only differences arise because now for $\mathcal{A}$ a lax symmetric monoidal category which has a strict unit 0, one step of the construction gives a lax functor (1.7.1, or [St]) $A: \Gamma^{op} \to \mathrm{Cat}$ instead of a pseudofunctor. To construct this, choose for each finite ordinal $n \geq 1$ one of the isomorphic objects T_n of type n in the club σ. Let T_0 denote the strict unit of $\mathcal{A}$. The lax functor A sends p to $\prod^p \mathcal{A}$. For each morphism $\varphi: p \to q$ in the category of finite based sets Γ^{op}, let $A(\varphi): \prod^p \mathcal{A} \to \prod^q \mathcal{A}$ be the functor sending $(A_1, \cdots, A_p)$ to

$$(\overline{T}_{\varphi^{-1}(1)}(A_i; i \in \varphi^{-1}(1)), \cdots, \overline{T}_{\varphi^{-1}(q)}(A_i; i \in \varphi^{-1}(q)))$$

Here in each $\overline{T}_{\varphi^{-1}(j)}(A_i; i \in \varphi^{-1}(j))$ one orders the A_i by increasing order of the indexes i. The structure natural transformations of the lax functor $A(\varphi)A(\psi) \Rightarrow A(\varphi\psi)$ and $\mathrm{id} \Rightarrow A(\mathrm{id})$ have components induced by the unique tuple of morphisms in the club $\tilde{\sigma}$ that have the appropriate type to universally define such a natural transformation. The essential ingredients are the $\tilde{a}$ and $\hat{a}$ of 2.3.1. The identities required by the lax functor axioms hold by the coherence result mentioned in 2.3.2. In the case where $\mathcal{A}$ is a strict symmetric monoidal category, this lax functor is in fact a pseudofunctor, that associated to $\mathcal{A}$ in [Th2] Appendix. A lax morphism of lax symmetric monoidal categories induces a left lax natural transformation of lax functors $\Gamma^{op} \to \mathrm{Cat}$ by trivial generalization of the formulae in [Th2]. To the lax functor A one now applies Street's "second construction" of [St] to convert this into a strict functor from Γ^{op} to Cat which sends p to a category related to $\mathcal{A}^p$ by an adjoint pair of functors. From this it follows that its classifying space is homotopy equivalent to $\prod^p B(\mathcal{A})$ ([Qu] §1 Prop.2 Cor.1). Thus on applying the classifying space functor $B: \mathrm{Cat} \to \mathrm{Top}$ to this, one has a "special Γ space " in the sense of Segal [Se], to which his infinite loop space machine associates a spectrum. By [St] Thm. 2, Street's second construction is in fact a functor

$$\overleftarrow{\mathrm{Lax}}(\Gamma^{op}, \mathrm{Cat}) \to \mathrm{Cat}(\Gamma^{op}, \mathrm{Cat})$$

and thus a left lax natural transformation yields a map of "special Γ spaces".

The rest of the argument, including generalization to the non-unital case, now proceeds exactly as in [Th2].

In the case where $\mathcal{A}$ is a strict symmetric monoidal category the construction coincides with that given in [Th2], except that instead of considering a pseudofunctor on Γ^{op} as an op-lax functor and applying Street's first construction for op-lax functors, one considers it as a lax-functor and applies the second construction. The proof given in [Th2] shows that this variant also yields a functor Spt: SymMon $\to$ Spectra satisfying conditions 2.1 and 2.2 of [Th2], and thus ([Th2] p. 1603) is linked by a chain of natural stable homotopy equivalences to the original version Spt( ).

2.4.1. By an argument identical to that in [Th2], one obtains the analogs of the group completion results (1.6.1), (1.6.2). Similarly as in 1.6.5, these imply that if $F\colon \mathcal{S} \to \mathcal{T}$ is a 1-cell in LaxSymMon such that BF is a homotopy equivalence of spaces, then Spt(F) is a stable homotopy equivalence of spectra.

2.5 Proposition. *There is a 2-functor:*

$$S\colon LaxSymMon \to SymMon \tag{2.5.1}$$

and a chain of natural stable homotopy equivalences of spectra:

$$Spt(S\mathcal{S}) \xleftarrow{\sim} hocolim_{\Delta^{op}} Spt(\tilde{R}_* \mathcal{S}) \xrightarrow{\sim} Spt(\mathcal{S}) \tag{2.5.2}$$

It follows that S preserves stable homotopy equivalences between lax symmetric monoidal categories.

Denote by $I\colon SymMon \to LaxSymMon$ the inclusion functor. There is a 2-natural transformation to the identity functor:

$$\eta\colon S \circ I \to 1_{\mathrm{SymMon}} \tag{2.5.3}$$

such that $Spt(\eta)$ is a stable homotopy equivalence of spectra.

If U is a Grothendieck universe inspection of the construction in 2.6 shows that the functor S preserves U-smallness.

Proof. Recall that LaxSymMon is the 2-category of algebras for the doctrine $\tilde{\sigma}$.

The strict map of doctrines $s\colon \tilde{\sigma} \to \sigma$ of 2.3.2 induces the structure of a right $\tilde{\sigma}$-algebra on the doctrine for symmetric monoidal categories σ. The action map is given by: $\mu \circ \sigma s\colon \sigma\tilde{\sigma} \to \sigma\sigma \to \sigma$. Thus the addendum 2.2.3 to the Godement resolution yields a 2-functor $\sigma(\tilde{\sigma})_*$ from LaxSymMon to $\overleftarrow{\mathrm{Fun}}(\Delta^{op}, \mathrm{SymMonStrict})$ sending $\mathcal{S}$ to the simplicial symmetric monoidal category $n \mapsto \sigma\tilde{\sigma}^n\mathcal{S}$

Let S: LaxSymMon $\to$ SymMon be the composite of this 2-functor with the homotopy colimit functor of 1.8

$$\mathrm{hocolim}_{\Delta^{op}}\colon \overleftarrow{\mathrm{Fun}}(\Delta^{op}, \mathrm{SymMon}) \to \mathrm{SymMonStrict} \subseteq \mathrm{SymMon}$$

To construct the chain of stable homotopy equivalences of spectra 2.5.2, let $\tilde{R}_*$ be the Godement resolution functor 2.2.2 $n \mapsto \tilde{\sigma}\tilde{\sigma}^n$. Then $s\tilde{\sigma}^n\colon \tilde{\sigma}\tilde{\sigma}^n \to \sigma\tilde{\sigma}^n$ gives a simplicial map $\tilde{R} \to \sigma\tilde{\sigma}^*$ which is a stable homotopy equivalence in each degree $n \in \Delta^{op}$ by (2.3.2.3). By 1.8.4 and 1.8.3 this map induces a natural stable homotopy equivalence:

(2.5.4)
$$\mathrm{hocolim}_{\Delta^{op}}\mathrm{Spt}\tilde{R}_*\mathcal{S} \xrightarrow{\sim} \mathrm{hocolim}_{\Delta^{op}}\mathrm{Spt}\sigma\tilde{\sigma}^*\mathcal{S} \xrightarrow{\sim} \mathrm{Spt}(\mathrm{hocolim}_{\Delta^{op}}\sigma\tilde{\sigma}^*\mathcal{S}) = \mathrm{Spt}S\mathcal{S}$$

On the other hand, since $\tilde{R}_*\mathcal{S}$ is a resolution of $\mathcal{S}$, the augmentation map $\tilde{R}_*\mathcal{S} \to \mathcal{S}$ induces a stable homotopy equivalence $\mathrm{hocolim}_{\Delta^{op}}\mathrm{Spt}\tilde{R}_*\mathcal{S} \xrightarrow{\sim} \mathrm{Spt}\mathcal{S}$. Indeed, by a standard reasoning on -1-connective spectra using the group completion property 2.4.1 (cf. [Th2] 4.5) it suffices to show that the map induced by the augmentation from the geometric realization of the simplicial classifying space $\|n \mapsto B\tilde{\sigma}^{n+1}\mathcal{S}\| \to B\mathcal{S}$ is a weak homotopy equivalence of spaces. But this map is such since the resolution simplicially deformation retracts to $\mathcal{S}$ because of the extra degeneracies s_{-1} (2.2). (cf. e.g. [Ma1] 9.8). This completes the construction of the chain of stable homotopy equivalences (2.5.2).

It remains to construct the natural stable homotopy equivalence $\eta: S \circ I \to \mathrm{Id}$. But there is a 2-natural transformation

$$\sigma(\tilde{\sigma})_* \circ I \Rightarrow R_*: \mathrm{SymMon} \to \overleftarrow{\mathrm{Fun}}(\Delta^{op}, \mathrm{SymMon}) \tag{2.5.5}$$

to the Godement resolution 2.2.2 for σ-algebras. The component at $\mathcal{S}$ is the natural transformation of functors on Δ^{op} whose component at $n \in \Delta^{op}$ is $\sigma s^n\mathcal{S}: \sigma\tilde{\sigma}^n\mathcal{S} \to \sigma^{n+1}\mathcal{S}$. The 2-natural transformation (2.5.5) induces a map of homotopy colimits

$$SI = \mathrm{hocolim}_{\Delta^{op}}(n \mapsto \sigma\tilde{\sigma}^n\mathcal{S}) \to \mathrm{hocolim}_{\Delta^{op}}(n \mapsto \sigma^{n+1}\mathcal{S}) \tag{2.5.6}$$

By the universal mapping property of homotopy colimits 1.8.2, the augmentation of the Godement resolution for σ-algebras $\sigma_*\mathcal{S} \to \mathcal{S}$ yields a natural 1-cell

$$\mathrm{hocolim}_{\Delta^{op}}(n \mapsto \sigma^{n+1}\mathcal{S}) \to \mathcal{S} \tag{2.5.7}$$

The 2-natural transformation $\eta: SI(\mathcal{S}) \to \mathcal{S}$ is the composite of 2.5.6 and 2.5.7. To show it is a stable homotopy equivalence, it suffices to show 2.5.6 and 2.5.7 are such.

- From the homotopy equivalences (2.3.2.3) it follows that $s: \tilde{\sigma}\underline{C} \to \sigma\underline{C}$ induces a homotopy equivalence on classifying spaces of categories, and that both functors $\tilde{\sigma}$ and σ preserve homotopy equivalences of categories. Thus in each degree $n \in \Delta^{op}$ the component of (2.5.5) is a homotopy equivalence of categories. Then it is a stable homotopy equivalence for each n and so by 1.8 the induced map 2.5.6 of homotopy colimits is a stable homotopy equivalence.

As for the map 2.5.7, to show it is a stable homotopy equivalence it suffices by 1.8.4 to show that the augmentation induces a stable homtopy equivalence

$$\mathrm{hocolim}_{\Delta^{op}}\mathrm{Spt}R_*\mathcal{S} \xrightarrow{\sim} \mathrm{Spt}\mathcal{S}$$

But as above this holds since $R_*\mathcal{S}$ is a resolution of $\mathcal{S}$. ∎

3 From Spectra to E_∞-spaces

In this section, I will recall May's notion of E_∞-operad and his result on the equivalence of the homotopy category of -1-connective spectra with a localization of the category of E_∞-spaces. This is an essential link in the chain of equivalences connecting the former with the localization of SymMon. I begin by recalling some definitions from [Ma1], since one will need to have the details available for §4.

3.1 Definition. An *operad* in the category of compactly generated Hausdorff spaces consists of the following data:

i) For each integer $n \geq 0$, a space $E(n)$ and a right action of the symmetric group Σ_n on $E(n)$.
ii) A point $1 \in E(1)$
iii) For each $n \geq 1$ and each sequence $j_1, j_2, \cdots, j_n$ of non-negative integers, a continuous function:

$$\gamma\colon E(n) \times \big(E(j_1) \times E(j_2) \times \cdots \times E(j_n)\big) \to E(j_1 + j_2 + \cdots + j_n) \tag{3.1.1}$$

These are to satisfy the following conditions:

a) The space $E(0) = *$ is a single point.
b) The distinguished point $1 \in E(1)$ is an identity for the composition law γ. That is, for any $f \in E(n)$ and $g \in E(j)$ one has:

$$\gamma(f; 1, \cdots, 1) = f$$
$$\gamma(1; g) = g$$

c) The composition law is compatible with the action of the symmetric group. That is, for any $\sigma \in \Sigma_n$, any sequence of $\sigma_k \in \Sigma_{j_k}$ for $k = 1, \cdots, n$ and any $f \in E(n), g_k \in E(j_k)$ one has:

$$\gamma(f\sigma; g_1, \cdots, g_n) = \gamma(f; g_{\sigma^{-1}(1)}, \cdots g_{\sigma^{-1}(n)})\sigma(j_1, \cdots, j_n)$$
$$\gamma(f; g_1\sigma_1, \cdots, g_n\sigma_n) = \gamma(f; g_1, \cdots, g_n)(\sigma_1 \amalg \cdots \amalg \sigma_n)$$

Here $\sigma(j_1, \cdots, j_n)$ denotes the permutation in $\Sigma_{j_1+\cdots+j_n}$ that permutes the n blocks of j_k successive integers according to $\sigma \in \Sigma_n$, leaving the order within each block fixed. $\sigma_1 \amalg \cdots \amalg \sigma_n$ is the permutation leaving the n blocks invariant and which restricts to σ_k on the k^{th} block.
d) The composition law is associative. That is, given $f \in E(n)$; $g_i \in E(j_i)$ for $i = 1, 2, \cdots, n$; and $h_{ik} \in E(\ell_k)$ for $k = 1, 2, \cdots, j_i$; one has:

$$\gamma(f; \gamma(g_i; h_{ik})) = \gamma(\gamma(f; g_i); h_{ik})$$

3.2 Definition. An E_∞-*operad* is an operad such that for each n the space $E(n)$ is homotopy equivalent to a point and Σ_n acts freely on $E(n)$.

3.3 Definition. An E-*space* for E an operad is a based space X together with continuous functions for each non-negative integer n:

$$\alpha_n \colon E(n) \times \prod^n X \to X$$

such that this action α is based, unital, and respects the permutations and the composition law of the operad. More precisely, one requires that:

a) $\alpha_0(*)$ is the basepoint of X.
b) $\alpha_1(1; x) = x$ for the distinguished point $1 \in E(1)$.
c)
$$\alpha_n(f\sigma; x_1, x_2, \cdots, x_n) = \alpha_n(f; x_{\sigma^{-1}(1)}, \cdots, x_{\sigma^{-1}(n)})$$
for any $f \in E(n)$ and $\sigma \in \Sigma_n$.
d)
$$\alpha_{j_1+\cdots+j_n}(\gamma(f; g_1, \cdots, g_n); x_{11}, \cdots, x_{1j_1}, x_{21}, \cdots, x_{nj_n}) = \\ \alpha_n(f; \alpha_{j_1}(g_1; x_{11}, \cdots, x_{1j_1}), \cdots, \alpha_{j_n}(g_n; x_{n1}, \cdots, x_{nj_n}))$$
for any $f \in E(n)$, and any sequence $g_i \in E(j_i)$ for $i = 1, \dots, n$

A morphism of E-spaces is a continuous function $\varphi \colon X \to Y$ such that for all n the following diagram commutes:

(3.4)
$$\begin{tikzcd} E(n) \times \prod^n X \arrow[r, "\mathrm{id} \times \prod^n \varphi"] \arrow[d, "\alpha_n"'] & E(n) \times \prod^n Y \arrow[d, "\alpha_n"'] \\ X \arrow[r, "\varphi"'] & Y \end{tikzcd}$$

A morphism of E-spaces is a weak homotopy equivalence if it is a weak homotopy equivalence on the underlying spaces.

These definitions are due to May ([Ma1] 1.1, 3.5, 1.4) inspired by earlier work of Adams, Beck, Boardman, MacLane, Stasheff, and Vogt.

One fixes an E_∞ operad E in a Grothendieck universe U and considers the category of U-small E-spaces. For any two choices of such E, there are functors between the two categories such that the composites each way are linked to the identity by a chain of natural homotopy equivalences of E-spaces ([MaT] 1.8, §4, §5; cf. [Ma1] §13 up to 13.1). Thus these categories are essentially interchangeable. I will therefore conform to the standard abuse and speak of any of them as *the* category of E_∞-spaces.

May's approach to infinite loop space theory produces an functor Spt' from the category of E_∞-spaces to the category of -1-connective spectra ([Ma2]). In fact, he shows this functor induces an equivalence of a localization of the category of E_∞-spaces and the stable homotopy category of - -1-connective spectra. The inverse functor is given by imposing a natural action of an E_∞ operad on the zeroth space of a spectrum.

3.5 Proposition (May). *The functor Spt': E_∞-spaces $\to$ Spectra induces an equivalence from the localization of E_∞-spaces by inverting all maps that Spt' sends to stable homotopy equivalences, to the full subcategory of the stable homotopy category consisting of -1-connective spectra. The inverse equivalence is induced by the zeroth space functor Ω^∞.*

This is just a combination and slight reinterpretation of parts of the statements ([Ma2] Thm. 2.3, Cor. 2.4, Thm. 3.2), to which I refer the reader for the proof.

3.5.1. The morphisms of E_∞-spaces $\varphi\colon X \to Y$ that Spt' sends to stable homotopy equivalences are precisely those which induce isomorphisms on homology after localizing by inverting the action of the abelian monoid π_0 of the E_∞-spaces:

$$\varphi_*\colon \pi_0(X)^{-1}H_*(X;\mathbb{Z}) \xrightarrow{\sim} \pi_0(Y)^{-1}H_*(Y;\mathbb{Z})$$

This follows from the group-completion theorem ([Ma2] 2.3iv, 1.3, [Se] §4).

3.6. Let κ be a club such that the categories $\kappa(n)$ of 2.1.4 have contractible classifying spaces $B\kappa(n)$. Suppose further that $\kappa(0)$ is isomorphic to the category with one morphism. By coherence theory the club for *strict unital* symmetric monoidal categories is such a club. Then setting $E(n) = B\kappa(n)$ for $n \geq 0$ yields an E_∞-operad. For the free action of Σ_n on $\kappa(n)$ induces a free right Σ_n action on $B\kappa(n)$. The distinguished object $1 \in \kappa(1)$ induces a distinguished point $1 \in B\kappa(1)$. Since the classifying space functor B preserves finite products, the composition law of the club expressed in form (2.1.4.2) induces a composition law γ on the $B\kappa(n)$ for $n \geq 1$. The conditions 3.1.a-d for an operad hold as a consequence of the similar conditions satisfied by a club.

Moreover, an action of such a club κ on a category $\underline{C}$ induces an action of the operad $B\kappa$ on $B\underline{C}$. For the action map of the club

$$\coprod_{n\geq 0} \kappa(n) \ltimes_{\Sigma_n} \underline{C}^n \cong \kappa\underline{C} \to \underline{C} \tag{3.6.1}$$

induces an action of the operad on applying B.

This procedure is natural for strict morphisms of κ actions. In particular, taking κ to be the club for unital symmetric monoidal categories, this procedure gives a functor

$$B\colon \text{UniSymMonStrict} \to E_\infty\text{-spaces} \tag{3.6.2}$$

The May functor [Ma2] from UniSymMonStrict to Spectra is the composition of this functor B and the May machine Spt'. As noted in 1.6, this functor is linked by a chain of natural stable homotopy equivalences to the restriction from SymMon to UniSymMonStrict of our functor Spt.

4 From E_∞-spaces to lax symmetric monoidal categories

In this section, I construct a functor from E_∞-spaces to LaxSymMonStrict that will be the essential constituent of an inverse up to natural stable homotopy equivalence to the functor Spt of §2. The strategy of the construction is first to show that the category of contractible spaces over a space X has a classifying space weak homotopy equivalent to X, and then to use the action of the E_∞-operad on X to produce a lax symmetric monoidal structure on this category.

4.1. I begin by recalling some well-known facts from the theory of simplicial sets. Denote by $\Delta[\quad]: \Delta \to \mathrm{Top}$ the functor sending p to the standard topological p-simplex. Recall that the singular functor

$$\mathrm{Sing}(\quad): \mathrm{Top} \to \Delta^{op}\text{-Sets}$$

sends a space X to the simplicial set which in degree p is the set $\mathrm{Top}(\Delta[p], X)$. There is a natural weak homotopy equivalence from the geometric realization of the singular complex

$$|\mathrm{Sing}(X)| \xrightarrow{\sim} X \tag{4.1.1}$$

Denote by

$$\Delta/X \tag{4.1.2}$$

the category whose objects are the singular simplices of X, $c: \Delta[p] \to X$ and whose morphisms from c to $c': \Delta[q] \to X$ are morphisms $\varphi: p \to q$ in Δ such that the following diagram commutes in Top:
(4.1.3)

$$\begin{tikzcd} \Delta[p] \arrow[dd, "{\Delta[\varphi]}"'] \arrow[dr, "c"] & \\ & X \\ \Delta[q] \arrow[ur, "c'"'] & \end{tikzcd}$$

This construction yields a functor from Top to Cat.

4.2 Proposition (Quillen). *There is a natural weak homotopy equivalence from the classifying space of the category Δ/X to X:*

$$B\Delta/X \xrightarrow{\sim} X \tag{4.2.1}$$

Proof. Given the weak homotopy equivalence of 4.1.1, it suffices to find a natural weak equivalence of simplicial sets from the nerve of Δ/X to $\mathrm{Sing}(X)$. For the geometric realization of this weak equivalence is then a weak equivalence of spaces and can then be composed with the map of 4.1.1 to yield the equivalence of 4.2.1.

By definition, a p-simplex of the nerve $N\Delta/X$ is a sequence of p composable morphisms in Δ/X:

(4.2.2)

$$\begin{tikzcd} \Delta[k_0] \arrow[r, "\Delta[\varphi_1]"] \arrow[drrr, "c_0"'] & \Delta[k_1] \arrow[r, "\Delta[\varphi_2]"] \arrow[drr] & \cdots \arrow[r, "\Delta[\varphi_p]"] & \Delta[k_p] \arrow[d, "c_p"] \\ & & & X \end{tikzcd}$$

One sends this to the p-simplex of $\mathrm{Sing}(X)$ specified as follows. The sequence of morphisms φ_i determine a map in Δ $\psi: \{0, 1, \cdots, p\} \to \{0, 1, \cdots, k_p\}$ in the category of finite ordinals and monotone maps Δ by $\psi(i) = \varphi_p\varphi_{p-1}\cdots\varphi_{i+1}(k_i)$ for $i = 0, 1, \cdots, p$. (For $i = p$ one considers the composition of an empty set of φ to be the identity, so $\psi(p) = k_p$.) The desired p-simplex of $\mathrm{Sing}(X)$ is the composite:

$$c_p\Delta[\psi]: \Delta[p] \to \Delta[k_p] \to X$$

One easily checks this defines a map of simplicial sets:

$$N\Delta/X \to \mathrm{Sing}(X) \tag{4.2.3}$$

It remains to show this map is a weak homotopy equivalence.

This could be deduced from trivial modifications to the argument given in [Il] VII §3 for the weaker formulation of the proposition given there. I prefer an alternate proof:

Δ/X is isomorphic to the opposite category of the Grothendieck construction ([Th1] 1.1) $\Delta^{op}\int \mathrm{Sing}(X)$ on $\mathrm{Sing}(X)$ considered as a functor $\Delta^{op} \to \mathrm{Sets} \subseteq \mathrm{Cat}$. Thus one may conclude by dualizing the homotopy colimit theorem of [Th1] 1.2, that there is a natural weak homotopy equivalence of simplicial sets:

$$\mathrm{hocolim}_{\Delta^{op}}(p \mapsto \mathrm{Sing}_p(X)) \xrightarrow{\sim} И\big(\Delta^{op}\int(p \mapsto \mathrm{Sing}_p(X))\big) \cong N(\Delta/X) \tag{4.2.4}$$

Here $И$ is the version of the nerve functor used in [BK] XI 2.1 and [Th1], which has the opposite orientation to the nerve functor N of [Qu] §1 used in this paper (cf. 4.2.2). The two are related by:

$$NC = ИC^{op}$$

By [BK] XII 3.4, there is a natural weak homotopy equivalence from the homotopy colimit of a $F: \Delta^{op} \to \Delta^{op}\mathrm{Sets}$ to the diagonal simplicial set of F considered as a bisimplicial set. Applied to the simplicial set $\mathrm{Sing}(X)$ considered as a bisimplicial set constant in one direction, this gives a weak equivalence:

$$\text{hocolim}_{\Delta^{op}} \text{Sing}(X) \xrightarrow{\sim} \text{Diag}\big(p, q \mapsto \text{Sing}_p(X)\big) \cong \text{Sing}(X) \tag{4.2.5}$$

Explicit formulas for the equivalences 4.2.4 and 4.2.5 may be deduced from [Th1] 1.2.1 and [BK] XI 2.6, XII 3.4. A routine calculation then yields that these equivalences fit into a commutative triangle (4.2.6) with the map (4.2.3), which is therefore also an equivalence as required.

(4.2.6)

$$\begin{tikzcd}
& \text{hocolim}_{\Delta^{op}} N\text{Sing}(X) \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & \\
N\Delta/X \arrow[rr] & & \text{Sing}(X)
\end{tikzcd}$$

(In verifying the commutativity it is important to note that [BK] XII 3.4 is incorrect in stating that "obviously" $\Delta^{op}\backslash p = \Delta/p$. What is correct is that $\Delta^{op}\backslash p = (\Delta/p)^{op}$. When this correction is fed through [BK] XII 3.4, the result is that the description given above of 4.2.2 is right, whereas the erroneous formulae in [BK] XII 3.4 and XI 2.6 would lead one to expect a description using minima 0 instead of the maxima k_i) ∎

4.3 Notation. For X a topological space, let Null$/X$ be the category whose objects are maps of spaces $c: C \to X$ where C is weak homotopy equivalent to a point. A morphism $(C, c) \to (C', c')$ is a map of spaces $\gamma: C \to C'$ such that $c = c'\gamma$.

4.4 Lemma. *The obvious inclusion of categories $\iota: \Delta/X \to Null/X$ induces a natural weak homotopy equivalence of classifying spaces:*

$$B\Delta/X \xrightarrow{\sim} B\,Null/X \tag{4.4.1}$$

Proof. By Quillen's Thm. A ([Qu] §1), it suffices to show for each object $c: C \to X$ that the comma category $\iota/(C, c)$ is contractible. But this comma category is just Δ/C. Thus Prop. 4.2 gives that its classifying space is weakly equivalent to C, hence contractible as required. ∎

It follows immediately from 4.4 and 4.2 that the functors B: Cat $\to$ Top and Null/( ): Top $\to$ Cat induce inverse equivalences of localized categories on inverting the weak homotopy equivalences. (Modulo the usual precautions to take the categories of U-small objects.)

4.5 Proposition. *The functor Null/( ): Top $\rightarrow$ Cat lifts to a functor E_∞-spaces $\rightarrow$ LaxSymMonStrict.*

Proof. This claim is that one can construct a natural lax symmetric monoidal structure on Null/X from an action of an E_∞-operad $\{E(n)\}$ on X. By the presentation of the club $\tilde{\sigma}$ for lax symmetric monoidal categories given in 2.3.1, and the description of a club action in terms of its presentation ([Ke4] 10.2–10.7), this amounts to giving the following data:

4.5.1

i) For each object T of type n in the club σ for symmetric monoidal categories, i.e. for each n-ary operation built up by iterated substitution of a binary operation into itself, a functor:

$$\overline{T}: \prod^{n} \mathrm{Null}/X \rightarrow \mathrm{Null}/X$$

ii) For each morphism $u: T \rightarrow S$ in the club σ, a natural transformation with components:

$$\overline{u}: \overline{T}(c_1, \cdots c_n) \rightarrow \overline{S}(c_{\upsilon^{-1}(1)}, \cdots, c_{\upsilon^{-1}(n)})$$

Here υ is the permutation which is the type of u.

iii) A natural transformation

$$\hat{a}: 1 \rightarrow \overline{1}$$

iv) For each T in σ of type n, and each n-tuple S_i of objects in σ, a natural transformation of type the identity

$$\tilde{a}_{T[S_1, S_2, \cdots, S_n]}: \overline{T}(\overline{S}_1, \cdots, \overline{S}_n) \rightarrow \overline{T(S_1, \cdots, S_n)}$$

such that the diagrams 2.3.1.a,b,c,d, and e commute.

I construct all this as follows. Note that the category Top is a symmetric monoidal category under product of spaces. Thus for each T of type n in the club σ, and each n-tuple of spaces $Y_1, \cdots, Y_n$, one has the product space $T(Y_1, \cdots, Y_n)$, usually denoted $Y_1 \times \cdots \times Y_n$ with the choice of parentheses censored.

$\overline{T}$ will be the functor sending an n-tuple of objects $c_i: C_i \rightarrow X$ in Null/X to the contractible space over X given by the product of the C_i and $E(n)$:

(4.5.2)

$$E(n) \times T(C_1, \cdots, C_n) \xrightarrow{1 \times T(c_1, \cdots, c_n)} E(n) \times T(X, \cdots, X) \xrightarrow{\alpha_n} X$$

The natural transformations of ii) $\overline{u}: \overline{T} \rightarrow \overline{S}$ for $u: T \rightarrow S$ in σ will have as components the maps induced by the symmetric monoidal structure of Top:

(4.5.3)
$$1 \times u\colon E(n) \times T(C_1, \cdots, C_n) \to E(n) \times S(C_{v^{-1}(1)}, \cdots, C_{v^{-1}(n)})$$

This map is compatible with the structure maps to X by naturality of u and the compatibility (3.3.c) of the operad action α with the symmetric group action on the $E(n)$. Thus it is a map in Null$/X$.

The natural transformation of iii), $\hat{a}1\colon \to \overline{1}$ will have components:

(4.5.4)
$$C = 1 \times C \to E(1) \times C = \overline{1}(C)$$

induced by the inclusion of the operad's distinguished point $1 \in E(1)$.

This is a map over X by 3.3.b.

The natural transformation of iv), $\tilde{a}_{T[S_1,\cdots,S_n]}$ will have as components the maps induced by the composition law of the operad γ:

(4.5.5)
$$\begin{array}{c}
E(n) \times T\big(E(k_1) \times S_1(C_{11}, \cdots, C_{1k_1}), \cdots, E(k_n) \times S_n(C_{n1}, \cdots, C_{nk_n})\big) \\
\cong \Big\downarrow \\
E(n) \times E(k_1) \times \cdots \times E(k_n) \times T(S_1, \cdots, S_n)\big(C_{11}, \cdots, C_{nk_n}\big) \\
\gamma \times \mathrm{id} \Big\downarrow \\
E(k_1 + \cdots + k_n) \times T(S_1, \cdots, S_n)\big(C_{11}, \cdots, C_{nk_n}\big)
\end{array}$$

This is a map over X by 3.3.d.

These data satisfy the conditions imposed because of the conditions satisfied by the structure of an operad and of and operad action. Thus condition 2.3.1.a holds by naturality, conditions 2.3.1.b and 2.3.1.c by 3.1.b, 2.3.1.d by 3.1.d, and 2.3.1.e by naturality with 3.1.c and 3.3.c.

Finally, the construction is functorial, that is, it takes maps of operad actions to strict morphisms between lax symmetric monoidal categories. ∎

4.6 Lemma. *The functor Null/( ): E_∞-spaces $\to$ LaxSymMonStrict of 4.5 preserves stable homotopy equivalences.*

Proof. By group completion (1.6.5, 2.4.1, 3.5.1) it suffices to show that if $f\colon X \to Y$ is a map of E_∞-spaces which induces an isomorphism on homology localized by the action of the monoid π_0:

(4.6.1)
$$f_*\colon \pi_0(X)^{-1}H_*(X;\mathbb{Z}) \xrightarrow{\sim} \pi_0(Y)^{-1}H_*(Y;\mathbb{Z})$$

then is also an isomorphism the map:

$$B(\mathrm{Null}/f)_*\colon \pi_0(B\mathrm{Null}/X)^{-1}H_*(B\mathrm{Null}/X;\mathbb{Z}) \xrightarrow{\sim} \pi_0(B\mathrm{Null}/Y)^{-1}H_*(B\mathrm{Null}/Y;\mathbb{Z})$$

But by 4.4 and 4.2, there is a chain of natural homotopy equivalences between the E_∞-space Z and $B\mathrm{Null}/Z$, inducing an isomorphism on $H_*(\quad;\mathbb{Z})$ and on the set $\pi_0(\quad)$. Thus it suffices to show this isomorphism respects the actions of of π_0 on the homology groups, and so induces an isomorphism of the localizations. For $z \in Z$ considered as a representative of a class in $\pi_0 Z$, and for any choice of m in the contractible $E(2)$, the action on $H_*(Z;\mathbb{Z})$ is the map on homology induced by the homotopy class of the endomorphism $Z \to Z$ given by

$$Z \cong m \times z \times Z \subseteq E(2) \times Z \times Z \xrightarrow{\alpha_2} Z \tag{4.6.2}$$

By naturality of the chain of homotopy equivalences, this map on homology agrees with the endomorphism of $H_*(B\mathrm{Null}/Z;\mathbb{Z})$ induced by $Z \to Z$. This is the map on homology induced by the endofunctor of Null/Z sending $C \to Z$ to

$$C \cong m \times z \times C \subseteq E(2) \times Z \times Z \xrightarrow{\alpha_2} Z \tag{4.6.3}$$

On the other hand, the action of $z \in \pi_0 Z \cong \pi_0 B\mathrm{Null}/Z$ for the lax symmetric monoidal structure is the map on homology induced by the endofunctor sending $C \to Z$ to

$$\overline{T}(z \to Z, C \to Z) = E(2) \times z \times C \to E(2) \times Z \times Z \xrightarrow{\alpha_2} Z \tag{4.6.4}$$

for any choice of T in the contractible $\sigma(2)$. But the two endofunctors are homotopic, since they are linked by the natural transformation with components

$$C = m \times z \times C \hookrightarrow E(2) \times z \times C \tag{4.6.5}$$

This shows the actions on homology by an element of π_0 are compatible under the isomorphism induced by the chain of 4.4 and 4.2. Similarly, the isomorphism of $\pi_0 Z$ with $\pi_0 B\mathrm{Null}/Z$ respects the translation action of π_0 on itself, and so is an isomorphism of monoids. Thus the two π_0 actions are isomorphic.

∎

5 The main theorem

Recall that one has fixed a Grothendieck universe U and by convention considers only symmetric monoidal categories and spectra which are U-small.

5.1 THEOREM. *Let $Spectra_{\geq 0}$ be the category of -1-connective spectra, and SymMon the category of symmetric monoidal categories (1.5). Then the functor (1.6)*

$$Spt\colon SymMon \to Spectra_{\geq 0}$$

induces an equivalence between their homotopy categories formed by inverting the stable homotopy equivalences.

The inverse equivalence is induced by the functor:

$$\text{(5.1.1)} \qquad \mathit{Spectra}_{\geq 0} \to E_\infty\text{-}\mathit{spaces} \to \mathit{LaxSymMon} \to \mathit{SymMon}$$

which is the composite of the zeroth space functor $\Omega^\infty\colon \mathit{Spectra}_{\geq 0} \to E_\infty$*-spaces of 3.5, the functor* $\mathit{Null}/(\quad)\colon E_\infty$*-spaces* $\to$ *LaxSymMonStrict of 4.5, and the functor* $S\colon \mathit{LaxSymMon} \to \mathit{SymMon}$ *of 2.5.*

Proof. The functors of 2.5, 3.5, and 4.5 all preserve stable homotopy equivalences. Thus the composite functor does induce a functor on the homotopy categories. It remains to show the two functors are inverse on the homotopy categories.

One already knows by 4.5 that $\mathrm{Spectra}_{\geq 0}$ and E_∞-spaces are linked by functors Ω^∞ and Spt' inducing inverse equivalences of homotopy categories. Similarly, by 1.9.2, the inclusions between all the variants of SymMon listed in 1.5.1 induce equivalences of homotopy categories. Moreover one knows (3.6.2) that the restriction of Spt to UniSymMonStrict is linked by a chain of natural stable homotopy equivalences to the composite of a lift of the classifying space functor B: UniSymMonStrict $\to E_\infty$-spaces and the May machine functor Spt': E_∞-spaces $\to \mathrm{Spectra}_{\geq 0}$.

In light of this, to prove the two functors of the theorem are inverse to each other on the homotopy category, it suffices to show:

a) The composite functor

$$\mathrm{UniSymMonStrict} \xrightarrow{B} E_\infty\text{-spaces} \xrightarrow{S\mathrm{Null}/(\quad)} \mathrm{SymMon}$$

is linked to the inclusion functor by natural stable homotopy equivalences.

b) The composite functor

$$\mathrm{Spectra}_{\geq 0} \to \mathrm{SymMon} \to \mathrm{Spectra}_{\geq 0}$$

is linked to the identity functor by natural stable homotopy equivalences.

These will be proved in the course of this section. The statement a) will be proven by direct construction of the link, and b) will result from the uniqueness theorem for infinite loop space machines of [MaT]. ∎

5.2: proof of 5.1.a).

In order to prove 5.1.a) it suffices to construct a chain of functors and natural stable homotopy equivalences linking the composite

$$\text{(5.2.1)} \qquad \mathrm{UniSymMonStrict} \xrightarrow{B} E_\infty\text{-spaces} \xrightarrow{\mathrm{Null}/(\quad)} \mathrm{LaxSymMon}$$

to the inclusion I of UniSymMonStrict into LaxSymMon. For then on applying the functor S: LaxSymMon $\to$ SymMon of 2.5 to this link one obtains a link of $S\mathrm{Null}/B(\quad)$

to SI, which in turn is linked to the inclusion of UniSymMonStrict in SymMon by the natural stable homotopy equivalence η of 2.5.

I will need to define a functor

$$\underline{\text{Null}}/(\quad): \text{UniSymMonStrict} \to \text{LaxSymMon} \tag{5.2.2}$$

which will be the analog for categories of Null/( ) for spaces. First, I will construct the underlying endofunctor of Cat and show there is a natural homotopy equivalence η from the identity to this endofunctor. For $\mathcal{A}$ a category, let $\underline{\text{Null}}/\mathcal{A}$ be the category whose objects are (C, c) where C is a U-small category such that BC is contractible and $c: C \to \mathcal{A}$ is a functor. A morphism in $\underline{\text{Null}}/\mathcal{A}$ from (C, c) to (C', c') is a functor $\gamma: C \to C'$ such that $c = c'\gamma$. Let $\underline{\text{Term}}/\mathcal{A}$ be the full subcategory of those (C, c) such that C has a terminal object. For each such C, chose a terminal t_C out of the isomorphism class of terminal objects. Then one may construct a functor $\rho': \underline{\text{Term}}/\mathcal{A} \to \mathcal{A}$ sending the object (C, c) to the image of the terminal object, $c(t_C)$. ρ' sends the morphism $(C, c) \to (C', c')$ to c' of the unique morphism in C' from the image of t_C to the terminal object $t_{C'}$. This functor ρ' is not strictly natural in $\mathcal{A}$ because of the choice of terminal object. Consider now the full subcategory $\underline{\Delta}/\mathcal{A}$ whose objects are those (C, c) for which C is the total order $\{0 < 1 < \cdots < n\}$ for some n. Since here the terminal object n is unique, the restriction of ρ' gives a functor $\rho: \underline{\Delta}/\mathcal{A} \to \mathcal{A}$, is strictly natural in $\mathcal{A}$. Moreover, given any object $A \in \mathcal{A}$, the comma category ρ/A has a terminal object $((\{0\}, c)1: c(0) = A)$, and so is contractible. By Quillen's Theorem A ([Qu] §1) it follows that ρ is a homotopy equivalence of categories. This statement is the analog of Prop. 4.2. As in the proof of Lemma 4.4, the comma categories $\iota/(C, c)$ of the inclusion $\iota: \underline{\Delta}/\mathcal{A} \to \underline{\text{Null}}/\mathcal{A}$ are isomorphic to $\underline{\Delta}/C$ and hence are homotopy equivalent to the contractible C. Then by Quillen's Theorem A the inclusion ι is a homotopy equivalence. Similarly, the inclusion of $\underline{\Delta}/\mathcal{A}$ into $\underline{\text{Term}}/\mathcal{A}$ is a homotopy equivalence. Thus $\underline{\text{Null}}/\mathcal{A}$ is linked by a chain of homotopy equivalences of categories to $\mathcal{A}$. In fact the following natural functor

$$\eta: \mathcal{A} \to \underline{\text{Null}}/\mathcal{A} \tag{5.2.3}$$

is a homotopy equivalence. This η sends an object A to the canonical forgetful functor from the comma category $C = \mathcal{A}/A \to \mathcal{A}$. η sends a morphism $A/ \to A'$ to the canonical map of comma categories $\mathcal{A}/A \to \mathcal{A}/A'$. As each $\mathcal{A}/A$ has a terminal object, η is the composite of a functor η' into $\underline{\text{Term}}/\mathcal{A}$ and the homotopy equivalence given by the inclusion of the latter in Null$/\mathcal{A}$. But $\rho'\eta' = 1$, and since ρ' is a homotopy equivalence, so is η' and η. This completes the proof of the chain of homotopy equivalences of categories for a general category $\mathcal{A}$.

Now return to the case where $\mathcal{A}$ runs over UniSymMonStrict. Let κ be the club for strict unital symmetric monoidal categories. Then as in 3.6, the natural action of κ on the unital symmetric monoidal $\mathcal{A}$ is expressed by action maps (3.6.1). The action of the E_∞-operad $E(n) = B\kappa(n)$ on $B\mathcal{A}$ is given by applying the classifying space functor B to the categorical action (3.6.1). Then replacing everywhere in the construction 4.5

of the lax symmetric monoidal structure on Null/X the operad $E(n) = B\kappa(n)$ by the categories $\kappa(n)$ of 2.1.4, and Top by Cat, one finds a construction of a lax symmetric monoidal category structure on $\underline{\mathrm{Null}}/\mathcal{A}$ strictly natural for $\mathcal{A}$ in UniSymMonStrict.

There is a natural strict morphism of lax symmetric monoidal categories:

$$B\colon \underline{\mathrm{Null}}/\mathcal{A} \to \mathrm{Null}/B\mathcal{A} \tag{5.2.4}$$

This B sends the object $(C, c\colon C \to \mathcal{A})$ to $(BC, Bc\colon BC \to B\mathcal{A})$. Moreover, this B induces a homotopy equivalence on the underlying categories. For since $B(\{0 < \cdots < n\}) = \Delta[n]$, B restricts to a functor $\underline{\Delta}/\mathcal{A} \to \Delta/B\mathcal{A}$, and one has a commutative ladder of classifying spaces whose sides are given by the homotopy equivalences of 4.2 and 4.4, and the maps induced by their above categorical analogs:

(5.2.5)

$$\begin{tikzcd}
B\underline{\mathrm{Null}}/\mathcal{A} \arrow[d, "BB"'] & B\underline{\Delta}/\mathcal{A} \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d] & B\mathcal{A} \arrow[d, "1"] \\
B\mathrm{Null}/B\mathcal{A} & B\Delta/B\mathcal{A} \arrow[l, "\sim"'] \arrow[r, "\sim"] & B\mathcal{A}
\end{tikzcd}$$

This diagram shows that up to homotopy equivalence, the B of (5.2.4) is indentified to the identity map of $\mathcal{A}$. A fortiori is a fortiori a natural stable homotopy equivalence.

For $\mathcal{A}$ in UniSymMonStrict the natural homotopy equivalence $\eta\colon \mathcal{A} \to \underline{\mathrm{Null}}/\mathcal{A}$ of 5.2.3 is a lax morphism of lax symmetric monoidal categories. To give η such a structure, it suffices considering the presentation of the club for lax symmetric monoidal categories 2.3.1 and the description of lax morphisms in terms of presentations ([Ke3] 4.3 [Ke4] 10.2-10.7) to give for each object T of type n in the club σ for symmetric monoidal categories, a natural transformation

$$\eta_T\colon \overline{T}(\eta A_1, \cdots \eta A_n) \to \eta(T(A_1, \cdots A_n)) \tag{5.2.6}$$

in $\underline{\mathrm{Null}}/\mathcal{A}$ satisfying the following compatibilities with the natural transformations which are part of the lax symmetric monoidal structure.

(5.2.7.a) For each morphism $u\colon T \to R$ in σ of type $\nu \in \Sigma_n$, the following diagram commutes:

$$\begin{tikzcd}
\overline{T}(\eta A_1, \cdots, \eta A_n) \arrow[r, "\eta_T"] \arrow[d, "\overline{u}(\eta A)"'] & \eta(T(A_1, \cdots, A_n)) \arrow[d, "\eta(u(A))"] \\
\overline{R}(\eta A_{\nu^{-1}1}, \cdots, \eta A_{\nu^{-1}n}) \arrow[r, "\eta_R"'] & \eta(R(A_{\nu^{-1}1}, \cdots, A_{\nu^{-1}n}))
\end{tikzcd}$$

(5.2.7.b) For each natural transformation $\tilde{a}_{T[R_1,\cdots R_n]}$ as in 4.5.1.iv), the following diagram commutes:

$$\begin{tikzcd}
\overline{T}(\overline{R}_i)(\eta A_j) \arrow[r, "\overline{T}(\eta_{R_i}(A_{ij}))"] \arrow[d, "\tilde{a}_{T[R_i]}(A_j)"'] & \overline{T}(\eta(R_i(A_{ij})) \arrow[r, "\eta_T(R_i(A_{ij})"] & \eta(T(R_i(A_{ij}))) \arrow[d, "="] \\
\overline{T(R_i)}(\eta A_j) \arrow[rr, "\eta_{T(R_i)}(A_j)"'] & & \eta(T(R_i(A_{ij})))
\end{tikzcd}$$

(5.2.7.c) For the natural transformation $\hat{a}: 1 \to \overline{1}$ of 4.5.1.iii), the following diagram commutes:

$$\begin{tikzcd}
1[\eta(A)] \arrow[r, "="] \arrow[d, "\hat{a}"'] & \eta(1[A]) \arrow[d, "\eta(1_A)"] \\
\overline{1}[\eta(A)] \arrow[r, "\eta_1"'] & \eta(\overline{1}(A))
\end{tikzcd}$$

But for T of type n, $\overline{T}(\eta A_1, \cdots \eta A_n)$ is the contractible category over $\mathcal{A}$

$$\sigma(n) \times \mathcal{A}/A_1 \times \cdots \mathcal{A}/A_n \to \sigma(n) \times \mathcal{A} \times \cdots \mathcal{A} \to \mathcal{A}$$

Since T is a terminal object in $\sigma(n)$ and A_i is terminal in $\mathcal{A}/A_i$, the above functor factors canonically through the comma category $\mathcal{A}/T(A_1, \cdots, A_n) = \eta(T(A_1, \cdots, A_n))$. Then letting η_T be this factorization yields the required natural transformation. It is routine to check the properties (5.2.7) hold.

Thus the η of 5.2.3 is a natural stable homotopy equivalence between functors SymMonStrict $\to$ LaxSymMon. Composing this with the natural stable homotopy equivalence of 5.2.4 yields a natural stable homotopy equivalence from the inclusion functor I to Null$/B(\quad)$, as required to complete the proof of 5.1.a.

5.3: proof of 5.1.b.

It remains to link the functor Spectra$_{\geq 0}$ $\to$ SymMon $\to$ Spectra$_{\geq 0}$ to the identity functor by a chain of natural stable homotopy equivalences.

After composing with the zeroth space functor $(\quad)_0$: Spectra $\to$ Top, one has the chain of homotopy equivalences of spaces, all naturally in $X \in$ Spectra:

$$(5.3.1) \qquad \mathrm{Spt}(\mathrm{Null}/X_0)_0 \xleftarrow{\sim} B(\mathrm{Null}/X_0) \xleftarrow{\sim} B(\Delta/X_0) \xrightarrow{\sim} X_0$$

The second and third maps of 5.3.1 are the natural homotopy equivalences of 4.2 and 4.4. The first map is the canonical group completion map 1.6.1, which is a homotopy equivalence in this case. For as in the proof of 4.6, $\pi_0 B(\mathrm{Null}/X_0)$ is isomorphic as a monoid to $\pi_0 X_0$ and so is already a group.

It would be quite difficult to specify by hand a chain of E_∞-homotopy equivalences between the two ends of 5.3.1. Instead, I will circumvent the need to do this by the following Lemma. Its proof will complete the proof of 5.1.b and hence of the Theorem 5.1. It is essentially an avatar of the May-Thomason uniqueness theorem for infinite loop space machines [MaT].

5.3.2 LEMMA. *Let F: $Spectra_{\geq 0} \to Spectra_{\geq 0}$ be an endofunctor of the category of -1-connective spectra. Suppose there exists a chain of functors G_i: $Spectra_{\geq 0} \to Top$ and of natural homotopy equivalences of spaces:*

$$(F)_0 = G_0 \xleftarrow{\sim} G_1 \xrightarrow{\sim} \cdots \xleftarrow{\sim} G_{n-1} \xrightarrow{\sim} G_n = (\quad)_0 \tag{5.3.2.1}$$

Then there is a chain of endofunctors of $Spectra_{\geq 0}$ and natural stable homotopy equivalences linking F to Id.

PROOF. Recall (1.6.5) that a map of -1-connective spectra is a stable homotopy equivalence if and only if it induces a homotopy equivalence on the zeroth space. From this and the chain of homotopy equivalences of zeroth spaces $\cdots G_i \cdots$ one sees that F preserves stable homotopy equivalences. Also all the G_i send products of spectra to products of spaces, at least up to homotopy.

Consider the category of "special Γ-spaces " of Segal [Se], the category of functors from the category of finite based sets Γ^{op} to Top that take wedges to products up to homotopy. Segal's infinite loop space machine Sg is a functor from the category of special Γ-spaces to Spectra$_{\geq 0}$. By [Se] §3 there is a functor is the opposite direction, with the two composites linked to the identity by stable homotopy equivalences. - From this it follows that it suffices to link FSg to Sg by a chain of natural stable homotopy equivalences of functors from special Γ-spaces to Spectra$_{\geq 0}$.

But this can be shown to hold by a slight modification of the proof of [MaT] 2.5. In more detail, one has a functor given by smash product of finite based sets:

$$\Gamma^{op} \times \Gamma^{op} \to \Gamma^{op}$$

If $A: \Gamma^{op} \to \mathrm{Top}$ is a special Γ-space, then the induced $A: \Gamma^{op} \times \Gamma^{op} \to \mathrm{Top}$ is such that for each $p \quad q \mapsto A(pq)$ is a special Γ-space. Indeed, as p varies it is a special Γ-(special Γ-space), so $\Gamma^{op} \to \mathrm{Spectra}_{\geq 0}$ sending p to $Sg(q \mapsto A(pq))$ is homotopy equivalent to the p-fold product of the spectrum $Sg(q \mapsto A(1q))$. Applying the chain (5.3.2.1) of product-preserving natural homotopy equivalences of spaces to the spectra $Sg(q \mapsto A(pq))$ yields a chain of homotopy equivalences of special Γ-spaces linking $p \mapsto (FSg(q \mapsto A(pq)))_0$ to $p \mapsto (Sg(q \mapsto A(pq)))_0$. The Segal machine gives a group completion map natural in the Γ-space $A(p*)$: $A(p1) \to (Sg(q \mapsto A(pq)))_0$. Thus it induces a homotopy equivalence of associated spectra $Sg(p \mapsto A(p)) \xrightarrow{\sim} Sg(p \mapsto (Sg(q \mapsto A(pq)))_0)$. Applying $Sg(p \mapsto \quad)$

to the chain of homotopy equivalences of special Γ-spaces gives a chain of natural stable homotopy equivalences from the latter spectrum to $Sg(p \mapsto (FSg(q \mapsto A(pq)))_0)$. By the "over and across theorem" [MaT] 3.9 for bispectra associated to special Γ-spectra, there is a natural chain of stable homotopy equivalences linking $Sg(p \mapsto (FSg(q \mapsto A(pq)))_0)$ to $FSg(q \mapsto A(1q)) = FSg(A)$. Combining the above chains give the required chain of stable homotopy equivalences linking Sg to FSg. ∎

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