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ON THE SIZE OF CATEGORIES

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Transmitted by Michael Barr

ABSTRACT. The purpose is to give a simple proof that a category is equivalent to a small category if and only if both it and its presheaf category are locally small.

In one of his lectures (University of New South Wales, 1971) on Yoneda structures [SW], the second author conjectured that a category $\mathcal{A}$ is essentially small if and only if both $\mathcal{A}$ and the presheaf category $\mathcal{PA}$ are locally small. The first author was in the audience and at the end of the lecture suggested a proof of the conjecture using some of his own results. This was reported on page 352 of [SW] and used to motivate a definition of "small" in [St]; yet the proof was not published. The proof given in the present paper evolved via correspondence between the authors in 1976-77 while the second author was on sabbatical leave at Wesleyan University (Middletown, Connecticut) but has remained unpublished despite our expectation at various times that it would appear as an exercise in some textbook.

In 1979, a longer, but related, proof appeared in [F1]. We advised the author of this history and sent him our proof. This was reported in [F2], but our proof was still not published.

Now that there is actually an application [RW], we decided publication was in order. We have expressed the construction in a form we believe begs generalization to, for example, parametrized categories [SS]. Note throughout that "small" can mean "finite".

For an object A of a category $\mathcal{A}$, we let

$$\mathrm{Idem}(A) = \{e : A \longrightarrow A : ee = e\}$$

denote the set of idempotents on A. The category $\mathcal{A}$ has *small idempotency* [SS] when $\mathrm{Idem}(A)$ is a small set for all objects A. It is clear that every locally small category (that is, category with small homsets) has small idempotency. We shall see conversely that, if binary products exist, small idempotency implies local smallness.

We write $\mathcal{S}$ for the category of small sets and functions between them. We write $\mathcal{S}^{\mathcal{A}}$ for the category of functors $F : \mathcal{A} \longrightarrow \mathcal{S}$ and natural transformations between them. [We work with $\mathcal{S}^{\mathcal{A}}$ rather than the presheaf category to avoid contravariant functors.]

A *retraction pair* (m, r) at an object A in a category $\mathcal{A}$ consists of arrows $m : X \longrightarrow A$, $r : A \longrightarrow X$ with $rm = 1_X$. Two retraction pairs $(m, r), (n, s)$ at A are *equivalent* when there is an invertible arrow $h : X \longrightarrow Y$ such that $m = nh$ and $hr = s$. We write $\mathrm{Ret}(A)$

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for the set of equivalence classes $[m, r]$ of retraction pairs at A. There is a well-defined injective function

$$\mathrm{Ret}(A) \rightarrowtail \mathrm{Idem}(A)$$

taking the equivalence class $[m, r]$ to the idempotent $mr : A \longrightarrow A$. So $\mathrm{Ret}(A)$ is small if $\mathrm{Idem}(A)$ is. For our purposes here we could in fact assume $\mathcal{A}$ to admit splittings of all idempotents; so the above injective function would be bijective and we could avoid introducing Ret.

As foreshadowed, we can use this to show that *small idempotency is equivalent to local smallness when $\mathcal{A}$ has binary products.* For, we have an injective function

$$\mathcal{A}(A, B) \rightarrowtail \mathrm{Ret}(A \times B)$$

taking $f : A \longrightarrow B$ to the equivalence class $[m, p]$ where $pm = 1_A, qm = f$ and $p : A \times B \longrightarrow A, q : A \times B \longrightarrow B$ are the projections.

A *split monic* is an arrow $m : X \longrightarrow A$ with a left inverse. Two split monics $m : X \longrightarrow A$, $n : Y \longrightarrow A$ are *equivalent* when there is an invertible arrow $h : X \longrightarrow Y$ with $nh = m$. An equivalence class $[m : X \longrightarrow A]$ of split monics into A is called a *split subobject* of A. We write $S\mathrm{sub}(A)$ for the set of split subobjects of A. There is a well-defined surjective function

$$\mathrm{Ret}(A) \twoheadrightarrow S\mathrm{sub}(A)$$

taking the equivalence class $[m, r]$ to the split subobject $[m : X \longrightarrow A]$. So $S\mathrm{sub}(A)$ is small if $\mathrm{Ret}(A)$ is.

It is clear from the above that, if $\mathcal{A}$ has small idempotency then $S\mathrm{sub}(A)$ is a small set for all objects A of $\mathcal{A}$. In this case we define a functor

$$T : \mathcal{A} \longrightarrow \mathcal{S}$$

on objects by

$$TA = S\mathrm{sub}(A) + \{0\},$$

and, for $f : A \longrightarrow B$ in $\mathcal{A}$, the function $Tf : TA \longrightarrow TB$ is given by

$$(Tf)[m : X \longrightarrow A] = \begin{cases} [fm] & \text{when } fm \text{ has a left inverse} \\ 0 & \text{otherwise} \end{cases}$$

To see that T is a functor we use the fact that if gfm has a left inverse then so does fm.

We shall now introduce a function θ from the objects of $\mathcal{A}$ to the endomorphisms of T. We show that θ may be viewed as a function, not from the objects but, from the isomorphism classes of objects, and when so viewed, is an injection. The definition of the function

$$\theta : \mathrm{obj}\mathcal{A} \longrightarrow \mathcal{S}^{\mathcal{A}}(T, T)$$

is as follows. For each $K \in \mathrm{obj}\mathcal{A}$, the natural transformation $\theta K : T \longrightarrow T$ has component

$$(\theta K)_A : TA \longrightarrow TA$$

at A given by

$$(\theta K)_A[m : X \longrightarrow A] = \begin{cases} [m] & \text{when } X \cong K \\ 0 & \text{otherwise} \end{cases}$$

Clearly $K \cong L$ implies $\theta K = \theta L$; but the converse is true too. For, if $\theta K = \theta L$ then

$$(\theta K)_L[1_L : L \longrightarrow L] = (\theta L)_L[1_L : L \longrightarrow L] = [1_L : L \longrightarrow L] \neq 0;$$

so, from the definition of $(\theta K)_L$, we have $L \cong K$. It is clear that $(\theta K)(\theta K) = \theta K$, so we have an injective function

$$\mathrm{obj}\mathcal{A}/\cong \quad \longrightarrow \quad \mathrm{Idem}(T, T)$$

induced by θ. We have proved:

Theorem. *If $\mathcal{A}$ and $\mathcal{S}^{\mathcal{A}}$ have small idempotency then the set of isomorphism classes of objects of $\mathcal{A}$ is small.*

Corollary. *A category $\mathcal{A}$ is equivalent to a small category if and only if $\mathcal{A}$ and $\mathcal{S}^{\mathcal{A}}$ are locally small.*

Question. Suppose $\mathcal{A}$ is a locally small site such that the category $Sh\mathcal{A}$ of sheaves on $\mathcal{A}$ is locally small. Does it follow that $Sh\mathcal{A}$ is a Grothendieck topos? We do not know. Notice that, if the Grothendieck topology on $\mathcal{A}$ is such that every presheaf is a sheaf, then $\mathcal{A}$ is equivalent to a small category by the above Corollary; so $Sh\mathcal{A}$ is a Grothendieck topos in that case.

Remarks. Lest our functor $T : \mathcal{A} \longrightarrow \mathcal{S}$ seem mysterious, we provide two constructions of T which may help the reader. The first construction demands more of the ambient set theory, but makes T more transparent. The second is choice free and invokes no large sets.

(1) Suppose $\mathcal{A}$ is locally small. Each representable functor $H_A = \mathcal{A}(A, -) : \mathcal{A} \longrightarrow \mathcal{S}$ has a unique maximal subfunctor $M_A : \mathcal{A} \longrightarrow \mathcal{S}$; an element $f \in H_A B = \mathcal{A}(A, B)$ is in $M_A B$ if and only if it is **not** split monic. Let K_A denote the result of smashing M_A to a point; that is, M_A is defined by the pushout

$$\begin{tikzcd} M_A \arrow[r] \arrow[d] & H_A \arrow[d] \\ \mathbf{1} \arrow[r] & K_A \end{tikzcd}$$

in $\mathcal{S}^{\mathcal{A}}$ where $\mathbf{1}$ is the terminal object (which, of course, is the functor constantly valued at the one-point set also denoted by $\mathbf{1}$). The functor K_A can be regarded as landing in the category $\mathcal{S}_*$ of pointed small sets wherein $\mathbf{1}$ is both terminal and initial (= coterminal). Note that K_A takes to $\mathbf{1}$ precisely those objects for which A is not a retract. Let Λ denote a set of representatives of the isomorphism classes of objects of the category $\mathcal{A}$. Put

$$S = \sum_{A \in \Lambda} K_A$$

where the summation denotes the coproduct of pointed sets. In fact, S takes values in the category $\mathcal{S}_*$ of *small* pointed sets since, for all $B \in \mathcal{A}$, only a small set of terms in the coproduct are non-trivial (because $K_A B$ is non-trivial if and only if B is a retract of A). It is clear that S has at least as many endomorphisms as the coproduct has terms; so we have reproved the Corollary above.

Moreover, the functor T defined earlier is a quotient of S. For each object A, note that the group $\mathrm{Aut}(A)$ of automorphisms of A acts on the right of K_A (since it acts by composition on H_A and M_A). Let J_A denote the "orbit space" (that is, $J_A B = K_A B/\mathrm{Aut}(A)$ for all B). Then there is a natural isomorphism

$$T \cong \sum_{A \in \Lambda} J_A$$

of pointed-set-valued functors.

(2) Now we turn to the second construction of T assuming $\mathcal{A}$ has small idempotency. There is a preorder on each set $\mathrm{Idem}(A)$ motivated by the natural order on the "images" of the idempotents; the preorder is defined by: $e \leq e'$ when $e = e'e$. Let $e \approx e'$ be the equivalence relation generated; so $e \approx e'$ means $e = e'e$ and $ee' = e'$. Let UA be the set of equivalence classes together with the empty set (so, as a set, UA is the free partial order with first element on the preorder $\mathrm{Idem}(A)$). For each $f : A \longrightarrow B$, define the function $Uf : UA \longrightarrow UB$ by, for all elements $E \in UA$,

$$(Uf)(E) = \{feu : u \in \mathcal{A}(B, A), uf \in E, e \in E\}$$

It can be verified that $U : \mathcal{A} \longrightarrow \mathcal{S}$ is a functor. For each object X of $\mathcal{A}$, let $\phi(X)$ be the endomorphism of U whose component $\phi(X)_A$ at A takes $E \in UA$ to the element

$$\phi(X)_A(E) = \{xy : yx = 1_X, xy \in E\}$$

of UB. This ϕ induces an injective function

$$\mathrm{obj}\mathcal{A}/\cong \quad \longrightarrow \quad \mathrm{End}(U)$$

Our functor T is isomorphic to a subfunctor of U, namely, the smallest subfunctor containing the images of all the natural transformations $\phi(X), X \in \mathcal{A}$. In other words, T is isomorphic to the subfunctor of U obtained by discarding the idempotents which do not split.

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