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COMPUTING CROSSED MODULES INDUCED BY AN INCLUSION OF A NORMAL SUBGROUP, WITH APPLICATIONS TO HOMOTOPY 2-TYPES

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ABSTRACT. We obtain some explicit calculations of crossed Q-modules induced from a crossed module over a normal subgroup P of Q. By virtue of theorems of Brown and Higgins, this enables the computation of the homotopy 2-types and second homotopy modules of certain homotopy pushouts of maps of classifying spaces of discrete groups.

Introduction

A crossed module $\mathcal{M} = (\mu : M \to P)$ has a classifying space $B\mathcal{M}$ (see, for example, [4]) which is of the homotopy type of $B(P/M)$ if μ is the inclusion of the normal subgroup M of P. Consider a homotopy pushout X of the form

$$\begin{tikzcd} BP \arrow[r, "B\iota"] \arrow[d, "B\kappa"'] & BQ \arrow[d] \\ B\mathcal{M} \arrow[r] & X \end{tikzcd}$$

where $\iota : P \to Q$ is a morphism of groups and κ is the natural inclusion from the group P, regarded as a crossed module $1 \to P$, to the crossed module $\mathcal{M}$. It is shown in our previous paper [7], using results of Brown and Higgins in [3, 4], that the homotopy 2-type of X is determined by the induced crossed Q-module $\iota_*\mathcal{M}$. This explains the homotopical interest in calculating induced crossed modules. Such calculations include of course the calculation of a weaker invariant, namely the second homotopy module of X, which is in this case just the kernel of the boundary of $\iota_*\mathcal{M}$. Note also that results previous to [7] gave information on the homotopy type of X only if $\iota : P \to Q$ is also surjective (see [1] for 2-type in this case, and [6] for 3-type).

In all these cases, the key link between the topology and the algebra is provided by a higher dimensional Van Kampen Theorem. Proofs of these theorems require non traditional concepts, for example double groupoids, as in [3], or Loday's cat^n-groups, as in [6].

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The results even on the second homotopy modules seem unobtainable by more traditional methods, for example transversality and pictures, as described by Hog-Angeloni, Metzler, and Sieradski in [10].

Another interest of induced crossed modules is algebraic. Consider for example the inclusion crossed module $(\mu : M \to P)$ of a normal subgroup M of P, and suppose $\iota : P \to Q$ is an inclusion of a subgroup. Then the image of the boundary ∂ of the induced crossed module $(\partial : \iota_* M \to Q)$ is the normal closure $N^Q(\iota M)$ of ιM in Q. Thus the induced crossed module construction replaces this normal closure by a bigger group on which Q acts, and which has a universal property not usually enjoyed by $N^Q(\iota M)$. The algebraic significance of the kernel of ∂ has yet to be exposed.

The purpose of this paper is to give some new results on crossed modules induced by a morphism of groups $\iota : P \to Q$ in the case when ι is the inclusion of a normal subgroup. One of our main results (in section 1) determines $\iota_* M$, and so the kernel of $\iota_* M \to Q$, in the case P and M are normal in Q.

In section 2 we use the presentation of induced crossed modules given in [3, 7] to describe the crossed module induced by the normal inclusion ι in terms of the coproduct of crossed P-modules discussed in [12, 1]. This allows us to apply methods of Gilbert and Higgins in [9] to generalise the result of section 1, and to deduce a result on the index 2 case from the results on coproducts in [1].

For crossed modules, and modules, the action is a crucial part of the structure, and this is reflected in our Theorems and Examples.

The initial motivation of this set of papers was a conversation with Rafael Sivera in Zaragoza, in November 1993, which suggested the lack of explicit calculations of induced crossed modules. This led to discussions at Bangor on the use of computational group theory packages which culminated in a GAP [11] program for computing finite induced crossed modules. In some cases the form of the resulting calculations suggested some general results, such as those in [7].

A separate paper [8] in preparation discusses the algorithmic aspects of the GAP program and includes a table of explicit calculations. The work with GAP is being extended by the second author to a general package of calculations with crossed modules.

Reports on some of the results of this and the other papers were given by the authors at *Groups in Galway* in May, 1994, and by the first author at the *European Category Theory Meeting* at Tours, July, 1994 (see [2]).

Induced constructions may be thought of in terms of 'change of base'. For more background on related contexts, see [2].

1. Inducing from a normal subgroup P of Q

This section contains the following main result, which is proved by a direct verification of the universal property for an induced crossed module. We assume as known the definition of induced crossed modules given in [3, 7]. If $n \in M$, then the class of n in M^{ab} is written $[n]$. If R is a group, then $I(R)$ denotes the augmentation ideal of R. The augmentation ideal $I(Q/P)$ of a quotient group Q/P has basis $\{\bar{t}-1 \mid t \in T'\}$ where T is a transversal of P in Q, $T' = T \setminus \{1\}$ and $\bar{q}$ denotes the image of q in Q/P.

1.1. Theorem. *Let $M \subseteq P$ be normal subgroups of Q, so that Q acts on P and M by conjugation. Let $\mu : M \to P$, $\iota : P \to Q$ be the inclusions and let $\mathcal{M}$ denote the crossed module $(\mu : M \to P)$ with the conjugation action. Then the induced crossed Q-module $\iota_*\mathcal{M}$ is isomorphic as a crossed Q-module to*

$$(\zeta : M \times (M^{\mathrm{ab}} \otimes I(Q/P)) \to Q)$$

where for $m, n \in M$, $x \in I(Q/P)$:
(i) $\zeta(m, [n] \otimes x) = m \in Q$;
(ii) the action of Q is given by

$$(m, [n] \otimes x)^q = (m^q, [m^q] \otimes (\bar{q}-1) + [n^q] \otimes x\bar{q}).$$

The universal map $i : M \to M \times (M^{\mathrm{ab}} \otimes I(Q/P))$ is given by $m \mapsto (m, 0)$, and if (β, ι) is a morphism from $\mathcal{M}$ to the crossed module $\mathcal{C} = (\chi : C \to Q)$, then the morphism

$$\begin{tikzcd} & & C \arrow[dd, "\chi"] \\ M \arrow[urr, "\beta"] \arrow[r, "i"'] \arrow[d, "\mu"'] & Z \arrow[ur, dotted, "\phi"] \arrow[d, "\zeta"'] & \\ P \arrow[r, "\iota"'] & Q & \end{tikzcd} \tag{1}$$

$\phi : M \times (M^{\mathrm{ab}} \otimes I(Q/P)) \to C$ induced by β is, for $m, n \in M$, $q \in Q$, given by

$$\phi(m, [n] \otimes (\bar{q}-1)) = (\beta m)\,(\beta n)^{-1}\,\left(\beta\left(n^{q^{-1}}\right)\right)^{q}. \tag{2}$$

The following corollary is immediate.

1.2. Corollary. *The homotopy 2-type of*

$$X = BQ \cup_{BP} B(P/M)$$

is determined by the crossed Q-module $(\zeta : M \times (M^{\mathrm{ab}} \otimes I(Q/P)) \to Q)$ above. In particular, the second homotopy module of X is isomorphic to the Q/M-module $M^{\mathrm{ab}} \otimes I(Q/P)$.

Proof of Theorem 1.1. Let $Z = M \times (M^{\mathrm{ab}} \otimes I(Q/P))$. The proof that $\mathcal{Z} = (\zeta : Z \to Q)$ with the given action is indeed a crossed module is straightforward and is omitted.

Clearly we have a morphism of crossed modules $(i, \iota) : \mathcal{M} \to \mathcal{Z}$. We verify that this morphism satisfies the universal property of the induced crossed module.

Consider diagram (1). We prove below:

1.3. If $\phi : Z \to C$ is a morphism $\mathcal{Z} \to \mathcal{C}$ of crossed Q-modules such that $\phi i = \beta$, then ϕ is given the formula *(2)*.

We next prove that this formula does define a morphism of crossed Q-modules.

Let $q \in Q$. We define a function

$$\gamma_q : M \to C, \qquad m \mapsto (\beta m)^{-1}\left(\beta\left(m^{q^{-1}}\right)\right)^q .$$

We prove in turn:

1.4. $\gamma_q(M)$ is contained in the centre of C.

1.5. γ_q is a morphism, which factors through M^{ab}.

1.6. The morphisms γ_q depend only on the classes $\bar{q}$ of q in Q/P, and so define a morphism of groups $\gamma : M^{\mathrm{ab}} \otimes I(Q/P) \to C$, $[m] \otimes (\bar{q}-1) \mapsto \gamma_q(m)$.

1.7. The function ϕ defined in the theorem satisfies $\phi i = \beta$ and is a well defined morphism of crossed modules.

Proof of 1.3. Let $\phi : Z \to C$ be a morphism of crossed Q-modules such that $\phi i = \beta$. Let $m, n \in M, q \in Q$. Then

$$\begin{aligned}
\phi(1, [n] \otimes (\bar{q}-1)) &= \phi((n^{-1}, 0)(n, [n] \otimes (\bar{q}-1))) \\
&= \beta\left(n^{-1}\right)\phi\left(\left(n^{q^{-1}}, 0\right)^q\right) \\
&= \beta(n)^{-1}\left(\phi\left(n^{q^{-1}}, 0\right)\right)^q \\
&= \beta(n)^{-1}\left(\beta\left(n^{q^{-1}}\right)\right)^q \\
&= \gamma_q(n).
\end{aligned}$$

The result follows since $(m, [n] \otimes (\bar{q}-1)) = (m, 0)(1, [n] \otimes (\bar{q}-1))$. ∎

Proof of 1.4 This follows from the facts that if $m \in M$, then $\chi\gamma_q(m) = 1$, and that $\mathcal{C}$ is a crossed module. ∎

Proof of 1.5. Let $m, n \in M$. Then

$$\begin{aligned}
\gamma_q(mn) &= (\beta(mn))^{-1}\left(\beta\left((mn)^{q^{-1}}\right)\right)^q \\
&= (\beta n)^{-1}(\beta m)^{-1}\left(\beta\left(m^{q^{-1}}\right)\right)^q\left(\beta\left(n^{q^{-1}}\right)\right)^q \\
&= (\beta n)^{-1}(\gamma_q m)\left(\beta\left(n^{q^{-1}}\right)\right)^q \\
&= (\gamma_q m)(\beta n)^{-1}\left(\beta\left(n^{q^{-1}}\right)\right)^q \\
&= \gamma_q(m)\gamma_q(n).
\end{aligned}$$

This proves that γ_q is a morphism of groups. By 1.4, $\gamma_q(m)\gamma_q(n) = \gamma_q(n)\gamma_q(m)$, and so γ_q factors through M^{ab}. ∎

Proof of 1.6. The first part follows from the fact that β is a P-morphism. The second follows from the fact that the elements $\bar{q}-1, \bar{q} \in Q/P$, form a basis of $I(Q/P)$. ∎

PROOF OF 1.7. The function ϕ is clearly a well-defined morphism of groups since it is of the form $\phi(m,u) = (\beta m)(\gamma u)$, where β, γ are morphisms of groups and γu belongs to the centre of C. Further, $\phi i = \beta$, and $\chi\phi = \zeta$ since $\chi\gamma$ is trivial.

Next we prove that ϕ preserves the action. This is the crucial part of the argument. Recall that $\gamma([n] \otimes (\bar{q} - 1)) = \gamma_q(n) = (\beta n)^{-1}\left(\beta\left(n^{q^{-1}}\right)\right)^q$.

Let $m, n \in M,\ r, q \in Q$. Then

$$
\begin{aligned}
& \phi\left((m, [n] \otimes (\bar{r} - 1))^q\right) \\
= \ & \phi(m^q, [m^q] \otimes (\bar{q} - 1) + [n^q] \otimes (\bar{r} - 1)\bar{q}) \\
= \ & (\beta(m^q))\gamma([m^q] \otimes (\bar{q} - 1) + [n^q] \otimes ((\bar{r}\bar{q} - 1) - (\bar{q} - 1))) \\
= \ & (\beta(m^q))\,(\beta(m^q))^{-1}\,(\beta m)^q\,(\beta(n^q))^{-1}\left(\beta\left(n^{q(q^{-1}r^{-1})}\right)\right)^{rq}(\gamma_q(n^q))^{-1} \\
= \ & (\beta m)^q(\gamma_q(n^q))^{-1}\,(\beta(n^q))^{-1}\left(\beta\left(n^{r^{-1}}\right)\right)^{rq} \\
= \ & (\beta m)^q\,((\beta n)^q)^{-1}\left(\beta\left(n^{r^{-1}}\right)\right)^{rq} \\
= \ & (\phi(m, [n] \otimes (\bar{r} - 1)))^q.
\end{aligned}
$$

■

This completes the proof of the theorem. ■

An intuitive explanation of this result is that the part $(\beta n)^{-1}\left(\beta\left(n^{q^{-1}}\right)\right)^q$ measures the deviation of β from being a Q-morphism.

1.8. COROLLARY. *In particular, if the index $[Q : P]$ is finite, and $\mathcal{P}$ is the crossed module $1 : P \to P$, then $\iota_*\mathcal{P}$ is isomorphic to the crossed module $(\mathrm{pr}_1 : P \times (P^{\mathrm{ab}})^{[Q:P]-1} \to Q)$ with action as above.*

1.9. REMARK. It might be imagined from this that the Postnikov invariant of this crossed module is trivial, since one could argue that the projection

$$\mathrm{pr}_2 : P \times P^{\mathrm{ab}} \otimes I(Q/P) \to P^{\mathrm{ab}} \otimes I(Q/P)$$

should give a morphism from $\iota_* P$ to the crossed module $0 : P^{\mathrm{ab}} \otimes I(Q/P) \to Q/P$, which represents 0 in the cohomology group $H^3(Q/P, P^{\mathrm{ab}} \otimes I(Q/P))$ (see [7]). However, the projection pr_2 is a P-morphism, but is not in general a Q-morphism, as the above results show. In fact, in the next Theorem we give a precise description of the Postnikov invariant of $\iota_* P$ when Q/P is cyclic of order n. This generalises the result for the case $P = C_n,\ Q = C_{n^2}$ in Theorem 5.4 of [7].

1.10. THEOREM. *Let P be a normal subgroup of Q such that P/Q is isomorphic to C_n, the cyclic group of order n. Let t be an element of Q which maps to the generator $\bar{t}$ of C_n under the quotient map. Then the first Postnikov invariant k^3 of the mapping cone $X = BQ \cup \Gamma BP$ of the inclusion $BP \to BQ$ lies in a third cohomology group*

$$H^3(C_n, P^{\mathrm{ab}} \otimes I(C_n))$$

This group is isomorphic to

$$P^{\mathrm{ab}} \otimes C_n,$$

and under this isomorphism the element k^3 is taken to the element

$$[t^n] \otimes \bar{t}.$$

PROOF. We have to determine the cohomology class represented by the crossed module

$$\xi : P \times P^{\mathrm{ab}} \otimes I(C_n) \to Q.$$

Let $A = P^{\mathrm{ab}} \otimes I(C_n)$. As in [7] for the case $Q = C_{n^2},\ P = C_n$, we consider the diagram

$$\begin{tikzcd}
\mathbb{Z}[C_n] \arrow[r, "\delta_4"] \arrow[d, "0"'] & \mathbb{Z}[C_n] \arrow[r, "\delta_3"] \arrow[d, "f_3"'] & \mathbb{Z}[C_n] \arrow[r, "\delta_2"] \arrow[d, "f_2"] & C_\infty \arrow[r] \arrow[d, "f_1"] & C_n \arrow[d, "1"] \\
0 \arrow[r] & A \arrow[r, "i"'] & P \times A \arrow[r, "\nu_n"'] & Q \arrow[r] & C_n .
\end{tikzcd}$$

Here the top row is the begining of a free crossed resolution of C_n. The free C_n-modules $\mathbb{Z}[C_n]$ have generators y_4, y_3, y_2 respectively, C_∞ has generator y_1 and $\delta_2(y_2) = y_1^n,\ \delta_3(y_3) = y_2.(\bar{t} - 1)$ (here C_∞ operates on each $\mathbb{Z}[C_n]$ via the morphism to C_n); $\delta_4(y_4) = y_3.(1+\bar{t}+\bar{t}^2+\cdots+\bar{t}^{n-1})$. Further, we define $f_1(y_1) = t,\ f_2(y_2) = (t^n, 0),\ f_3(y_3) = [t^n] \otimes (\bar{t} - 1)$, and $i(a) = (1, a),\ a \in A$. Thus the diagram gives a morphism of crossed complexes, and the cohomology class of the cocycle f_3 is the Postnikov invariant of the crossed module.

As in [7], Theorem 5.4, since $\mathbb{Z}[C_n]$ is a free C_n-module on one generator, the cohomology group $H^3(C_n, A)$ is isomorphic to the homology group of the sequence

$$A \xleftarrow{\delta_4^*} A \xleftarrow{\delta_3^*} A$$

where δ_4^* is multiplication by $1 + \bar{t} + \bar{t}^2 + \cdots + \bar{t}^{n-1}$ and δ_3^* is multiplication by $\bar{t} - 1$. It follows that $\delta_4^* = 0$, and it is easy to check that $I(C_n)/I(C_n)(\bar{t} - 1)$ is a cyclic group of order n generated by $\bar{t} - 1$. The cocycle f_3 determines the element $f_3(y_3) = [t^n] \otimes (\bar{t} - 1)$ of A, and the result follows. ∎

1.11. REMARK. The reason for the success of this last determination is that we have a convenient small free crossed resolution of the cyclic group C_n.

2. Coproducts of crossed P-modules

We refer to [7] for further background information that we require on crossed modules.

Let $\mathcal{XM}/P$ be the category of crossed modules over the group P. It is well known that arbitrary coproducts exist in this category. They may be constructed in the following way, which is given in essence, but not with this terminology, in [12].

Let T be an indexing set and let $\{\mathcal{M}_t = (\mu_t : M_t \to P) \mid t \in T\}$ be a family of crossed P-modules. Let Y be the free product of the groups M_t, $t \in T$. Let $\partial' : Y \to P$ be defined by the morphisms μ_t. The operation of P on the M_t extends to an operation of P on Y, so that $(\partial' : Y \to P)$ becomes a precrossed P-module. The standard functor from precrossed modules to crossed modules, obtained by factoring out the Peiffer subgroup [5, 10], is left adjoint to the inclusion of crossed modules into precrossed modules, and so takes coproducts into coproducts. Applying this to $(\partial' : Y \to P)$ gives the coproduct $(\partial : \bigcirc_{t\in T} M_t \to P)$ in the category $\mathcal{XM}/P$, determined by the canonical morphisms of crossed P-modules

$$i_u : M_u \to Y \to \bigcirc_{t\in T} M_t,$$

where the first morphism is the inclusion to the coproduct of groups, and the second is the quotient morphism. As is standard for coproducts in any category, the coproduct in $\mathcal{XM}/P$ is associative and commutative up to natural isomorphisms.

We now assume that P is a normal subgroup of Q, and show in Theorem 2.2 that the coproduct of crossed P-modules may be used to give a presentation of induced crossed P-modules analogous to known presentations of induced modules.

Suppose first given a crossed P-module $\mathcal{M} = (\mu : M \to P)$. Let α be an automorphism of P. The proof that the following definition does give a morphism of crossed modules is left to the reader.

2.1. Definition. The crossed module $\mathcal{M}_\alpha = (\mu_\alpha : M_\alpha \to P)$ *associated to* an automorphism α and an isomorphism $(k_\alpha, \alpha) : \mathcal{M} \to \mathcal{M}_\alpha$,

$$\begin{tikzcd} M \arrow[r, "k_\alpha"] \arrow[d, "\mu"'] & M_\alpha \arrow[d, "\mu_\alpha"] \\ P \arrow[r, "\alpha"'] & P. \end{tikzcd}$$

are defined as follows. The group M_α is just $M \times \{\alpha\}$ and $k_\alpha m = (m, \alpha)$, $m \in M$. The morphism μ_α is given by $(m, \alpha) \mapsto \alpha\mu m$. The action of P is given by $(m, \alpha)^p = (m^{\alpha^{-1}p}, \alpha)$.

We shall apply this construction to the case $\alpha = \alpha_t : p \mapsto t^{-1}pt$, for some $t \in Q$, and we write $\mathcal{M}_{\alpha_t}$ as $\mathcal{M}_t = (\mu_t : M_t \to P)$ where $\mu_t(m, t) = t^{-1}(\mu m)t$.

Given a set T of elements of Q, we write

$$\mathcal{M} \,\vec{\circ}\, T = (\partial : M \,\vec{\circ}\, T \to P)$$

for the coproduct crossed P-module $\bigcirc_{t\in T}\mathcal{M}_t$, and

$$i_t : \mathcal{M}_t \to \mathcal{M} \,\vec{\circ}\, T, \quad t \in T$$

for the canonical morphisms of crossed P-modules defining the coproduct.

2.2. Theorem. *Let $\mathcal{M} = (\mu : M \to P)$ be a crossed P-module, and let $\iota : P \to Q$ be an inclusion of a normal subgroup. Let T be a right transversal of P in Q. For $t \in T$, let $\mathcal{M}_t = (\mu_t : M_t \to P)$ be the crossed P-module in which the elements of M_t are (m,t), $m \in M$ with*

$$\mu_t(m,t) = t^{-1}(\mu m)t, \ \ (m,t)^p = (m^{tpt^{-1}}, t).$$

Then there is a unique action of Q on $M \,\vec{\circ}\, T$ which satisfies

$$(i_t(m,t))^q = i_u(m^p, u), \tag{3}$$

for $q \in Q$, $p \in P$, $t, u \in T$, such that $tq = pu$. This action makes $\mathcal{M}\,\vec{\circ}_\iota\, T = (\iota\partial : M \,\vec{\circ}\, T \to Q)$ a crossed Q-module and the morphism $(i_1, \iota) : \mathcal{M} \to \mathcal{M}\,\vec{\circ}_\iota\, T$ has the universal property of the induced crossed Q-module $\iota_*\mathcal{M} = (\iota\partial : \iota_* M \to Q)$ as shown in the diagram

$$\begin{tikzcd}
 & & C \arrow[dl, "\chi"] \\
M \arrow[r, "i_1"'] \arrow[urr, "\beta"] \arrow[d, "\mu"'] & M \,\vec{\circ}\, T \arrow[ur, dotted, "\phi"] \arrow[d, "\iota\partial"'] & \\
P \arrow[r, "\iota"'] & Q &
\end{tikzcd}$$

Further, given a morphism $(\beta, \iota) : \mathcal{M} \to \mathcal{C} = (\chi : C \to Q)$, the induced morphism $\phi : \mathcal{M}\,\vec{\circ}_\iota\, T \to \mathcal{C}$ is given by

$$\phi(i_t(m,t)) = (\beta m)^t. \tag{4}$$

Proof. The construction of the induced crossed module given in [3] and used in [7] is to form the precrossed module

$$\partial' : Y \to Q$$

where Y is the free product $*_{t \in T} M_t$, where the M_t are copies of M, with elements $(m,t), m \in M$ and action as above. The new aspect of the current situation is that the part $\mu_t : M_t \to P$ of ∂' is also a crossed P-module.

Now we see that both the induced crossed Q-module and the coproduct crossed P-module are obtained by factoring Y by the Peiffer subgroup, which is the same whether Y is considered as a precrossed P-module $Y \to P$ or as a precrossed Q-module $Y \to Q$. This proves the theorem. ∎

We remark that the result of Theorem 2.2 is analogous to descriptions of induced modules, except that here we have replaced the direct sum which is used in the module case by the coproduct of crossed modules. Corresponding descriptions in the non-normal case look to be considerably harder.

As a consequence of the theorem we obtain:

2.3. Proposition. *If M is a finite p-group and P is normal and of finite index in Q, then the induced crossed module $\iota_* M$ is a finite p-group.*

Proof. The coproduct of two crossed P-modules is shown in [1] to be obtained as a quotient of their semidirect product, so that the coproduct of two, and hence of a finite number, of finite crossed P-modules is finite. ■

Note that a similar result is proved in [7] by topological methods, without the normality condition, but assuming that Q also is a finite p-group.

We can now apply a result of Gilbert and Higgins [9] to obtain a description of an induced crossed module in more general circumstances than in section 1. We are careful about giving when possible the Q-action for this crossed module, since this is of course a key element of the structure.

If a group M acts on a group N, then the quotient of N by the action of M is written N_M; it is the quotient of N by the 'displacement subgroup' generated by the elements $n^{-1}n^m$ for all $n \in N,\ m \in M$.

2.4. Theorem. *Suppose that $\mathcal{M} = (\mu : M \to P)$ is a crossed module and that the restriction $\mu' : M \to \mu M$ of μ has a section $\sigma : \mu M \to M$. Let $\iota : P \to Q$ be the inclusion of a normal subgroup. Suppose that for all $q \in Q$, $q^{-1}(\mu M)q \subseteq \mu M$. Let T be a transversal of P in Q and let $T' = T \setminus \{1\}$. Then the group $\iota_* M$ of the induced crossed module $\iota_*\mathcal{M}$ is isomorphic to the group $M \times \bigoplus_{t\in T'}(M_t)_M$ and this yields by transference of actions an isomorphism of $\iota_*\mathcal{M}$ to a crossed module of the form*

$$\mathcal{X} = (\xi = \iota\mu\,\mathrm{pr}_1 : M \times \bigoplus_{t\in T'}(M_t)_M \to Q).$$

If, further, the section σ is P-equivariant, then the action of Q in $\mathcal{X}$ is given as follows, where $m \in M,\ r \in P,\ t, v \in T,\ q = rv \in Q$, and $[m, v]$ denotes the class of (m, v) in $(M_v)_M$:

(i)

$$(m,0)^q = \begin{cases} (m^q, 0) & \text{if } v = 1, \\ (\sigma((\mu m)^q), [m^r, v]) & \text{if } v \neq 1; \end{cases}$$

(ii) if $tq = pu,\ t \in T',\ p \in P,\ u \in T$, then

$$(1,[m,t])^q = \begin{cases} (1, [m^p, t]) & \text{if } v = 1, \\ (\sigma(\mu m^p)^{-1} m^p, -[\sigma((\mu m^p)^{v^{-1}}), v]) & \text{if } v \neq 1,\ u = 1, \\ (1, -[\sigma((\mu m^p)^{uv^{-1}}), v] + [m^p, u]) & \text{if } v \neq 1,\ u \neq 1. \end{cases}$$

Given a morphism $(\beta, \iota) : \mathcal{M} \to \mathcal{C} = (\chi : C \to Q)$, the induced morphism $\phi : M \times \bigoplus_{t\in T'}(M_t)_M \to C$ is given by

$$\phi(m, 0) = \beta m, \quad \phi(m, [n, v]) = (\beta m)\, \beta(\sigma((\mu n)^v))^{-1}\, (\beta n)^v.$$

Proof. We identify M and M_1, so that $i = i_1 : (m, 1) \mapsto (m, 0)$.

Form the crossed P-module

$$\mathcal{W} = (\nu : W = \bigcirc_{t\in T'} M_t \to P).$$

Then there is a coproduct crossed P-module

$$\mathcal{M} \circ \mathcal{W} = (M \circ W \to P).$$

By Theorem 2.2 there is an action of Q on $M \circ W$ giving an isomorphism of crossed Q-modules

$$\iota_* \mathcal{M} \cong \mathcal{M} \circ \mathcal{W}.$$

Let W_M be the quotient of W by the action of M on W via P. Then P acts on W_M and diagonally on $M \times W_M$ so that the morphism $\mu\, pr_1 : M \times W_M \to P$ gives a crossed P-module. Now $\nu(W) \subseteq \mu(M)$, and so Proposition 2.1 and Corollary 2.3 of [9] give an isomorphism of groups

$$\begin{array}{rcl} \alpha : M \circ W & \cong & M \times W_M \\ m \circ w & \mapsto & (m(\sigma\nu w), [w]) \end{array}$$

where $[w]$ denotes the class in W_M of $w \in W$. It is also observed in [9] that α preserves the P-action, if σ is P-equivariant.

We next observe that since $\mu_t M_t \subseteq \mu M$ for all $t \in T'$, we have

$$W_M \cong \bigoplus_{t \in T'} (M_t)_M.$$

The reason is that under these circumstances the Peiffer commutators

$$(m,t)^{-1}(m_1,t_1)^{-1}(m,t)(m_1,t_1)^{\partial'(m,t)}$$

which generate the Peiffer subgroup of $*_{t\in T'} M_t$ reduce to ordinary commutators.

In order to describe the action of Q in the case σ is P-equivariant we examine carefully the isomorphism of groups

$$\bigcirc_{t \in T} M_t \to M \times \bigoplus_{t \in T'} (M_t)_M.$$

This is determined by

$$i_t : (m,t) \mapsto \begin{cases} (m, 0) & \text{if} \quad t = 1, \\ (\sigma((\mu m)^t), [m,t]) & \text{if} \quad t \neq 1. \end{cases}$$

The formulae (i) and (ii) for the case $v = 1$ follow from the description of the action of P on M_t given by Definition 2.1. The remaining cases will be deduced from the formula for the action of Q given in Theorem 2.2, namely if $m \in M,\ t \in T,\ q \in Q$ then

$$(i_t(m,t))^q = \begin{cases} i_1(m^p, 1) \;=\; (m^p, 0), & \text{if} \quad tq = p \in P, \\ i_u(m^p, u) \;=\; (\sigma((\mu m^p)^u), [m^p, u]), & \text{if} \quad tq = pu,\ p \in P,\ u \in T'. \end{cases} \tag{5}$$

We first prove (i) for $v \neq 1$. We have since $q = rv,\ v \in T'$,

$$\begin{aligned}(m,0)^q &= (i_1(m,1))^{rv}\\ &= i_v(m^r, v)\\ &= (\sigma((\mu m^r)^v), [m^r, v]).\end{aligned}$$

To prove (ii) with $v \neq 1$, first note that

$$\begin{aligned}(1,[m,t]) &= (\sigma((\mu m)^t), 0)^{-1}\,(\sigma((\mu m)^t), [m,t])\\ &= (\sigma((\mu m)^t), 0)^{-1}\, i_t(m,t).\end{aligned}$$

But

$$\begin{aligned}(\sigma((\mu m)^t), 0)^q &= (\sigma((\mu(\sigma((\mu m)^t)))^q), [(\sigma((\mu m)^t))^r, v]) && \text{by (i)}\\ &= (\sigma((\mu m)^{tq}), [\sigma((\mu m)^{tr}), v]) && \text{since } \mu\sigma = 1,\end{aligned}$$

and, from equation (5),

$$(i_t(m,t))^q = \begin{cases} (m^p, 0) & \text{if } u = 1,\\ (\sigma((\mu m)^{tq}), [m^p, u]) & \text{if } u \neq 1.\end{cases}$$

It follows that

$$(1,[m,t])^q = \begin{cases} (\sigma(\mu m^p)^{-1} m^p, -[\sigma((\mu m^p)^{v^{-1}}), v]) & \text{if } u = 1,\\ (1, -[\sigma((\mu m^p)^{uv^{-1}}), v] + [m^p, u]) & \text{if } u \neq 1.\end{cases}$$

Finally, the formula for ϕ is obtained as follows:

$$\begin{aligned}\phi(m,[n,v]) &= \phi(m,0)\,\phi(\sigma((\mu n)^v), 0)^{-1}\,\phi(i_v(n,v))\\ &= (\beta m)\,(\beta(\sigma((\mu n)^v))^{-1})\,(\beta n)^v.\end{aligned}$$

∎

We now include an example for Theorem 2.4 showing the action in the case $v \neq 1,\ u = 1$.

2.5. Example. Let n be an odd integer and let $Q = D_{8n}$ be the dihedral group of order $8n$ generated by elements $\{t, y\}$ with relators $\{t^{4n}, y^2, (ty)^2\}$. Let $P = D_{4n}$ be generated by $\{x, y\}$, and let $\iota : P \to Q$ be the monomorphism given by $x \mapsto t^2,\ y \mapsto y$. Then let $M = C_{2n}$ be generated by $\{m\}$. Define $\mathcal{X} = (\mu : M \to P)$ where $\mu m = x^2,\ m^x = m$ and $m^y = m^{-1}$. This crossed module is isomorphic to a sub-crossed module of $(D_{4n} \to \mathrm{Aut}(D_{4n}))$ and has kernel $\{1, m^n\}$.

The image μM is the cyclic group of order n generated by x^2, and there is an equivariant section $\sigma : \mu M \to M,\ x^2 \mapsto m^{n+1}$ since $(x^2)^{(n+1)} = x^2$ and $\gcd(n+1, 2n) = 2$. Then

$Q = P \cup Pt$, $T = \{1, t\}$ is a transversal, M_t is generated by (m, t) and $\mu_t(m, t) = x^2$. The action of P on M_t is given by

$$(m,t)^x = (m,t), \quad (m,t)^y = (m^{-1},t).$$

Since M acts trivially on M_t,

$$\iota_* M \;\cong\; M \times M_t \;\cong\; C_{2n} \times C_{2n}.$$

Using the section σ given above, Q acts on $\iota_* M$ by

$$\begin{aligned}(m,0)^t &= (m^{n+1}, [m,t]),\\ (m,0)^y &= (m^{-1}, 0),\\ (1,[m,t])^t &= (m^n, (n-1)[m,t]),\\ (1,[m,t])^y &= (1, -[m,t]).\end{aligned}$$

We can obtain some information on the kernel of induced crossed modules in the case P is of index 2 in Q by using results of [1].

2.6. PROPOSITION. *Let $(\mu : M \to P)$ and $(\iota : P \to Q)$ be inclusions of normal subgroups. Suppose that P is of index 2 in Q, and $t \in Q \setminus P$. Then the kernel of the induced crossed module $(\partial : \iota_* M \to Q)$ is isomorphic to*

$$(M \cap t^{-1}Mt) \,/\, [M, t^{-1}Mt].$$

PROOF. By previous results, $\iota_* M$ is isomorphic to the coproduct crossed P-module $M \circ M_t$ with a further action of Q. The result now follows from Proposition 2.8 of [1]. ∎

We now give two homotopical applications of the last result.

2.7. EXAMPLE. Let $\iota : P = D_{4n} \to Q = D_{8n}$ be as in Example 2.5, and let $M = D_{2n}$ be the subgroup of P generated by $\{x^2, y\}$, so that $\iota M \lhd \iota P \lhd Q$ and $t^{-1}Mt$ is isomorphic to a second D_{2n} generated by $\{x^2, yx\}$. Then

$$M \cap t^{-1}Mt = [M, t^{-1}Mt]$$

(since $[y, yx] = x^2$), and both are isomorphic to C_n generated by $\{x^2\}$. It follows from Proposition 2.6 that if X is the homotopy pushout of the maps

$$BC_2 \longleftarrow BD_{4n} \longrightarrow BD_{8n},$$

where the lefthand map is induced by $D_{4n} \to D_{4n}/D_{2n} \cong C_2$, then $\pi_2(X) = 0$.

2.8. Example. Let M, N be normal subgroups of the group G, and let Q be the wreath product

$$Q = G \wr C_2 = (G \times G) \rtimes C_2.$$

Take $P = G \times G$, and consider the crossed module $(\partial : Z \to Q)$ induced from $M \times N \to P$ by the inclusion $P \to Q$. If t is the generator of C_2 which interchanges the two factors of $G \times G$, then $Q = P \cup Pt$ and $t^{-1}(M \times N)t = N \times M$. So

$$(M \times N) \cap t^{-1}(M \times N)t = (M \cap N) \times (N \cap M)$$

and

$$[M \times N, N \times M] = [M, N] \times [N, M].$$

It follows that if X is the homotopy pushout of

$$B(G/M) \times B(G/N) \leftarrow BG \times BG \rightarrow B(G \wr C_2),$$

then

$$\pi_2(X) \cong ((M \cap N)/[M, N])^2.$$

If $([m], [n])$ denotes the class of $(m, n) \in (M \cap N)^2$ in $\pi_2(X)$, the action of Q is determined by

$$([m], [n])^{(g,h)} = ([m^g], [n^h]), \ (g, h) \in P, \quad ([m], [n])^t = ([n], [m]).$$

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