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A NOTE ON FREE REGULAR AND EXACT COMPLETIONS AND THEIR INFINITARY GENERALIZATIONS

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Transmitted by Michael Barr

ABSTRACT. Free regular and exact completions of categories with various ranks of weak limits are presented as subcategories of presheaf categories. Their universal properties can then be derived with standard techniques as used in duality theory.

Introduction

A category $\mathbf{A}$ with finite limits is *regular* (cf. [2]) if every morphism factors into a regular epimorphism followed by a monomorphism, with the regular epimorphisms being stable under pullback; it is *exact* if, in addition, equivalence relations are effective, that is, if every equivalence relation in $\mathbf{A}$ is a kernel pair. It was noted by Joyal that, in the definition of regular category, one may replace "regular epimorphism" by the weaker notion of "strong epimorphism" in the sense of Kelly [11].

In [4] Carboni and Magno presented a one-step construction of the free exact completion $\mathbf{C}_{\mathrm{ex}}$ of a category $\mathbf{C}$ with finite limits (=lex), in terms of so-called pseudo-equivalence relations. Recently, Carboni and Vitale [5] have constructed the free regular completion $\mathbf{C}_{\mathrm{reg}}$ of $\mathbf{C}$ with weak finite limits (= weakly lex), and the free exact completion $\mathbf{A}_{\mathrm{ex/reg}}$ for a regular category $\mathbf{A}$, so that $\mathbf{C}_{\mathrm{ex}}$ can be obtained as $(\mathbf{C}_{\mathrm{reg}})_{\mathrm{ex/reg}}$ for $\mathbf{C}$ with weak finite limits. The objects of $\mathbf{C}_{\mathrm{reg}}$ are given by finite sources $(f_i : X \to X_i)_{i\in I}$ of arrows in $\mathbf{C}$, and the morphisms are defined to be equivalence classes of suitably compatible $\mathbf{C}$-morphisms between the domains of the given sources. Also for $\mathbf{A}_{\mathrm{ex/reg}}$ the description of objects is simple, as $\mathbf{A}$-objects with a fixed equivalence relation, while morphisms are less easily described: they are given by relations between the underlying $\mathbf{A}$-objects satisfying certain compatibility conditions with the structure-preserving equivalence relations.

Quite a different approach to $\mathbf{C}_{\mathrm{ex}}$ for $\mathbf{C}$ weakly lex and $\mathbf{C}$ small was given by Hu [9] who generalized a result by Makkai [14]: the exact completion $\mathbf{C}_{\mathrm{ex}}$ is given by the category

$$\mathbf{C}^{*+} = \prod\mathrm{Filt}(\mathbf{C}^{*}, \mathbf{Set})$$

of product- and filtered-colimit-preserving set-valued functors on

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$$\mathbf{C}^* = \mathrm{Flat}(\mathbf{C}, \mathbf{Set}),$$

the category of flat functors on $\mathbf{C}$; note that $\mathbf{C}^*$ is a finitely accessible category since $\mathbf{C}$ is small, and that it has products since $\mathbf{C}$ is weakly lex. We show here that for any category $\mathbf{C}$ with finite limits, $\mathbf{C}_{\mathrm{reg}}$ and $\mathbf{C}_{\mathrm{ex}}$ are full subcategories of $\mathbf{C}^{*+}$ with some additional properties (see Remark 2.5).

In a talk in the Sydney Category Seminar in February 1995, Max Kelly proposed to make better use of the Yoneda embedding

$$y : \mathbf{C} \to (\mathbf{C}^{\mathrm{op}}, \mathbf{Set})$$

when constructing $\mathbf{C}_{\mathrm{reg}}$ and $\mathbf{C}_{\mathrm{ex}}$. Because of the correspondences

$$\frac{\dfrac{\dfrac{X \longrightarrow Y_i}{y(X) \longrightarrow y(Y_i)}}{y(X) \longrightarrow \prod_{i \in I} y(Y_i)}}{y(X) \xrightarrow{e} A \xrightarrow{m} \prod_{i \in I} y(Y_i)}$$

(with e being a regular epimorphism and m being a monomorphism) the Carboni-Vitale construction suggests to take as objects of $\mathbf{C}_{\mathrm{reg}}$ those $F \in (\mathbf{C}^{\mathrm{op}}, \mathbf{Set})$ which appear at the same time as quotients of representables and as subobjects of finite products of representables.

This paper outlines Kelly's construction of $\mathbf{C}_{\mathrm{reg}}$ and shows that $\mathbf{C}_{\mathrm{ex}}$ may be described conveniently within $(\mathbf{C}^{\mathrm{op}}, \mathbf{Set})$ as well, as Kelly had anticipated in his talk: take those $F \in (\mathbf{C}^{\mathrm{op}}, \mathbf{Set})$ which admit a regular epimorphism $e : y(X) \to F$ whose kernel pair K is again covered by a representable functor, so that there is a regular epimorphism $y(Y) \to K$. The proof that $\mathbf{C}_{\mathrm{ex}}$ constructed this way is indeed finitely complete remains a bit laborious, but we find it convenient that most proofs reduce to checking closedness properties of $\mathbf{C}_{\mathrm{reg}}$ and $\mathbf{C}_{\mathrm{ex}}$ within the familiar presheaf environment, which allows to present the intrinsic connection between both categories more directly.

We also stress the point that it takes no additional effort to prove all results for a category $\mathbf{C}$ with weak κ-limits, rather than just weak finite limits; here κ is a regular infinite cardinal or the symbol ∞. Recall that a (weak) κ-limit in $\mathbf{C}$ is a (weak) limit of a diagram $D : \mathbf{J} \to \mathbf{C}$ with $\#\mathbf{J} < \kappa$, which in case $\kappa = \infty$ simply means that $\mathbf{J}$ must be small. Then the notion of regularity and exactness must be "κ-fied" as follows: $\mathbf{C}$ is *κ-regular* (*κ-exact*) if $\mathbf{C}$ is regular (exact), has κ-limits, and if κ-products of regular epimorphisms are regular epimorphisms. Note that for $\kappa = \aleph_0$, the latter property comes for free:

$$f \times g = (f \times \mathrm{id}_D)(\mathrm{id}_A \times g) : A \times C \to B \times D$$

is the composition of a pullback of g with a pullback of f. Notice furthermore that κ-Barr-exactness in the sense of [13] and [9] implies κ-exactness; ∞-regular is called "completely regular" in [5].

A simple description of $\mathbf{C}_{\mathrm{ex}}$ with $\mathbf{C}$ co-accessible is also given at the end of the paper. We show that for a co-accessible category $\mathbf{C}$ with weak limits, the objects of $\mathbf{C}_{\mathrm{ex}}$ are exactly all those $F \in (\mathbf{C}^{\mathrm{op}}, \mathbf{Set})$ which admit a regular epimorphism into F with representable domain.

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1. Flat Functors

Let κ be an infinite regular cardinal or the symbol ∞. A category $\mathbf{C}$ is said to be *weakly κ-complete* if it has weak κ-limits, i.e., for any κ-diagram $D : I \to \mathbf{C}$ (so that the morphism set of I has cardinality less than κ which, in case $\kappa = \infty$, just means that I must be small), a weak limit of D exists in $\mathbf{C}$.

1.1. Definition. *([9]) Let $\mathbf{C}$ be a weakly κ-complete category, and $\mathbf{B}$ a category with κ-limits. A functor $F : \mathbf{C} \to \mathbf{B}$ is called* κ-flat, *if for any κ-diagram $G : I \to \mathbf{C}$, and for each weak limit cone $(f_i : D \to G(i))_{i\in I}$ on G, the morphism $k : F(D) \to limF \circ G$ with $F(f_i) = p_i \circ k$ for all $i \in I$, is a regular epi; here the morphisms p_i are limit projections.*

1.2. Remark. (i) For $\mathbf{C}$ small, and $\mathbf{B}$ the category $\mathbf{Set}$ of sets, as pointed out in [9], F is κ-flat iff it is a κ-filtered colimit of representable functors.
(ii) For $\mathbf{B}$ having (regular epi, mono)-factorization, we notice that F is κ-flat iff there is a cone $(f_i : D \to G(i))_{i\in I}$ on G with $F(f_i) = p_i \circ k$ as in 1.1, so that k is a regular epi.
(iii) $\aleph_0$-flat functors were called left covering functors in [5].

1.3. Proposition. *For any locally small category $\mathbf{C}$, let $y : \mathbf{C} \to (\mathbf{C}^{\mathrm{op}}, \mathbf{Set})$ be the Yoneda embedding. If $\mathbf{C}$ is weakly κ-complete, then y is κ-flat.*

Proof. Follows from 1.1 and the fact that a morphism $a : M \to N$ in $(\mathbf{C}^{\mathrm{op}}, \mathbf{Set})$ is regular epi iff $a_C : M(C) \to N(C)$ is surjective for each $C \in \mathbf{C}$. ∎

1.4. Proposition. *Let $F : \mathbf{C} \to \mathbf{B}$ be any functor. If $\mathbf{C}$ and $\mathbf{B}$ have κ-limits, then F is κ-flat iff it preserves κ-limits.*

Proof. One only needs to show that a κ-flat functor F preserves κ-limits, that is, equalizers and κ-products. Since similar proofs can be found in [7] and [5], we can omit the proof here. ∎

1.5. Proposition. *For any weakly κ-complete category $\mathbf{C}$, the category $\mathbf{C}^*$ (= $\kappa -$* Flat$(\mathbf{C}, \mathbf{Set})$*) of set-valued κ-flat functors has products. Moreover, for κ an infinite regular cardinal, $\mathbf{C}^*$ has κ-filtered colimits.*

PROOF. Let $(F_i)_{i\in I}$ be a small family of functors of $\mathbf{C}^*$. Since F_i is κ-flat, for any κ-diagram $G : J \to \mathbf{C}$, the morphism $k_i : F_i(D) \to \lim F_i \circ G$ is surjective for each $i \in I$. Therefore, the morphism $\prod k_i : \prod F_i(D) \to \prod \lim F_i \circ G$ is surjective. $\prod \lim F_i \circ G$ is isomorphic to $\lim \prod F_i \circ G$ in **Set**, so we have that $\mathbf{C}^*$ is closed under products in ($\mathbf{C}$, **Set**).

Suppose that κ is an infinite regular cardinal. Let $M : I \to \mathbf{C}^*$ be a κ-filtered diagram. Since $M(i)$ is κ-flat, the morphism $k_i : M(i)(D) \to \lim M(i) \circ G$ is surjective for each $i \in I$; here $G : J \to \mathbf{C}$ is a κ-diagram. For any $u \in \mathrm{colim}_J \lim_I M(i) \circ G$, there are $i \in I$ and $u_i \in \lim M(i) \circ G$ such that $u = f_i(u_i)$ with f_i the colimit injection. Denote $\mathrm{colim}_I \lim_J M(i) \circ G$ by Q, and consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{colim}\, M(i)(D) \arrow[r, "k"] & Q \\
M(i)(D) \arrow[u, "f_i(D)"] \arrow[r, "k_i"'] & \lim M(i) \circ G \arrow[u, "f_i"']
\end{tikzcd}$$

where k is a morphism induced by the family $\langle k_i \rangle_{i\in I}$. Each k_i is surjective, so there is $x \in M(i)(D)$ such that $k_i(x) = u_i$; we let $y = f_i(D)(x)$ and we have $k(y) = u$. Since κ-filtered colimits commute with κ-limits in **Set**, Q is isomorphic to $\lim_J \mathrm{colim}_I M(i) \circ G$. Therefore, $\mathbf{C}^*$ is closed under κ-filtered colimits in $(\mathbf{C}, \mathbf{Set})$. ∎

2. $\mathbf{C}_{\mathrm{reg}}$ and $\mathbf{C}_{\mathrm{ex}}$

2.1. DEFINITION. *Let $y : \mathbf{C} \to (\mathbf{C}^{\mathrm{op}}, \mathbf{Set})$ be the Yoneda embedding. A functor $F : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ is* weakly representable *if there is a regular epimorphism $y(C) \to F$ with C in $\mathbf{C}$; we also say that F is* regularly covered by *C or $y(C)$ in this case. We define extensions $\mathbf{C}_{\kappa-\mathrm{reg}}$ and $\mathbf{C}_{\kappa-\mathrm{ex}}$ of $\mathbf{C}$ as follows, for any weakly κ-complete category $\mathbf{C}$:*

(i) $\mathbf{C}_{\kappa-\mathrm{reg}}$ is the full subcategory of $(\mathbf{C}^{\mathrm{op}}, \mathbf{Set})$ whose objects are these weakly representable functors which are subfunctors of κ-products of representable functors.

(ii) $\mathbf{C}_{\kappa-\mathrm{ex}}$ is the full subcategory of $(\mathbf{C}^{\mathrm{op}}, \mathbf{Set})$ of functors F such that there is a regular epimorphism $e : y(C) \to F$ whose kernel pair $(m, n : G \to y(C))$ has the property that G again is weakly representable.

Since we keep κ fixed, we shall write $\mathbf{C}_{\mathrm{reg}}$, $\mathbf{C}_{\mathrm{ex}}$ for $\mathbf{C}_{\kappa-\mathrm{reg}}$, $\mathbf{C}_{\kappa-\mathrm{ex}}$, respectively.

2.2. REMARK. Given $B \in \mathbf{C}_{\mathrm{reg}}$, we have a mono $m : B \to \prod_{i\in I} y(C_i)$ with $\#I < \kappa$. Let the p_i's be the projections of the product $\prod y(C_i)$, and let $(u, v : A \to B)$ be the kernel pair of a regular epi $e : y(D) \to B$ with $D \in \mathbf{C}$. Then A is a limit of the family of $(p_i \circ m \circ e, p_i \circ m \circ e)_{i\in I}$. By 1.3, A can be regularly covered by $y(S)$, where S is a weak limit of the family $(p_i \circ m \circ e, p_i \circ m \circ e)$. Thus, $\mathbf{C}_{\mathrm{reg}}$ is a full subcategory of $\mathbf{C}_{\mathrm{ex}}$.

2.3. THEOREM. *If* $\mathbf{C}$ *is weakly* κ*-complete, then* $\mathbf{C}_{\mathrm{reg}}$ *is* κ*-regular, and* $\mathbf{C}_{\mathrm{ex}}$ *is* κ*-exact; moreover, the inclusions from* $\mathbf{C}_{\mathrm{reg}}$ *and* $\mathbf{C}_{\mathrm{ex}}$ *into* $(\mathbf{C}^{\mathrm{op}}, \mathbf{Set})$ *are* κ*-regular.*

PROOF. Step 1. $\mathbf{C}_{\mathrm{reg}}$ has κ-products.

Let $(F_i)_{i\in I}$ be a family of objects of $\mathbf{C}_{\mathrm{reg}}$ with $\#I < \kappa$. If, in $(\mathbf{C}^{\mathrm{op}}, \mathbf{Set})$, F_i is a subobject of the product $\prod_{j\in I_i} y(C_j)$, then $\prod F_i$ is a subobject of the product of all $\prod_{j\in I_i} y(C_j)$, with $i \in I$. If the morphisms $p_i : y(D_i) \to F_i$ are regular epis, also $\prod p_i : \prod y(D_i) \to \prod F_i$ is a regular epi. Let Q be a weak product of all D_i in $\mathbf{C}$; by 1.1, the unique arrow $y(Q) \to \prod y(D_i)$ is a regular epi. Consequently, we have a regular epi from $y(Q)$ onto $\prod F_i$. This shows that $\prod F_i$ is in $\mathbf{C}_{\mathrm{reg}}$.

Step 2. $\mathbf{C}_{\mathrm{reg}}$ has equalizers.

Let $(u, v : M \to N)$ be a pair of arrows in $\mathbf{C}_{\mathrm{reg}}$, and $m : P \to M$ be an equalizer of u and v in $(\mathbf{C}^{\mathrm{op}}, \mathbf{Set})$. P is a subobject of M, and M is a subobject of a κ-product of representable functors, so P is a subobject of the product. To show that P can be regularly covered by a representable functor, let $s : y(C) \to M$ and $t : y(D) \to N$ be regular epis. Let $m' : P' \to y(C)$ be an equalizer of $u \circ s$ and $v \circ s$. We then have a unique arrow $w : P' \to P$ making the following diagram a pullback:

$$\begin{tikzcd}
P' \arrow[r, "m'"] \arrow[d, "w"'] & y(C) \arrow[d, "s"] \\
P \arrow[r, "m"'] & M
\end{tikzcd}$$

where w is a regular. Also, P' can be regularly covered by a representable functor. Indeed, let $k : N \to \prod_{i\in I} y(B_i)$ be mono, and $q_i : \prod y(B_i) \to y(B_i)$ be the product projections. Then P' is a joint equalizer of the family $(q_i \circ u \circ s, q_i \circ v \circ s : y(C) \to y(B_i))_{i\in I}$. Since C and B_i are in $\mathbf{C}$ and since y is full and faithful, we can write $y(u_i) = q_i \circ u \circ s$ and $y(v_i) = q_i \circ v \circ s$. By 1.2, P' is regularly covered by $y(W)$; here W is a weak joint equalizer of the family $(u_i, v_i : C \to B_i)_{i\in I}$.

This completes the proof that $\mathbf{C}_{\mathrm{reg}}$ has κ-limits.

Step 3. $\mathbf{C}_{\mathrm{ex}}$ has κ-products.

Let $(F_i)_{i\in I}$ be in $\mathbf{C}_{\mathrm{ex}}$, with $\#I < \kappa$. There are regular epis $p_i : y(D_i) \to F_i$ such that the kernel pair M_i of p_i

$$M_i \underset{g_i}{\overset{f_i}{\rightrightarrows}} y(D_i) \xrightarrow{p_i} F_i$$

can be regularly covered by some $y(B_i)$. Given the regular epis $q_i : y(B_i) \to M_i$ (for all i), then also $\prod q_i : \prod y(B_i) \to \prod M_i$ is a regular epi. So $\prod M_i$ is regularly covered by $y(W)$; here W is a weak product of the family of B_i. M_i is a subobject of the product $y(D_i) \times y(D_i)$, so $\prod M_i$ is a subobject of the product of all $y(D_i) \times y(D_i)$. This shows that $\prod M_i$ is in $\mathbf{C}_{\mathrm{reg}}$. Let $s : y(Z) \to \prod y(D_i)$ be a regular epi; here Z is a weak product of D_i.

Thus, we have the regular epi $\prod p_i \circ s : y(Z) \to \prod F_i$. Forming pullbacks, we obtain the following diagram:

$$\begin{tikzcd}
H \arrow[r, "b'"] \arrow[d, "a'"'] & G' \arrow[r, "g"] \arrow[d, "a"] & y(Z) \arrow[d, "s"] \\
G \arrow[r, "b"'] \arrow[d, "f"'] & \prod M_i \arrow[r, "\prod f_i"'] \arrow[d, "\prod g_i"] & \prod y(D_i) \arrow[d, "\prod p_i"] \\
y(Z) \arrow[r, tail, "s"'] & \prod y(D_i) \arrow[r, "\prod p_i"'] & \prod F_i
\end{tikzcd}$$

Note that $y(Z)$, $\prod M_i$ and $\prod y(D_i)$ are in $\mathbf{C}_{\mathrm{reg}}$. Hence, also H is in $\mathbf{C}_{\mathrm{reg}}$ and can be regularly covered by a representable functor. But H is the domain of the kernel pair of the regular epi $\prod p_i \circ s$. This shows that $\prod F_i$ is in $\mathbf{C}_{\mathrm{ex}}$.

Step 4. $\mathbf{C}_{\mathrm{ex}}$ has pullbacks.

Case 1. First we deal with the case of a kernel pair

$$M \overset{f}{\underset{g}{\rightrightarrows}} y(D) \xrightarrow{p} G$$

of a regular epi $p : y(C) \to G$ with $G \in \mathbf{C}_{\mathrm{ex}}$ such that M is regularly covered by a representable functor. Also, M is a subobject of the product $y(C) \times y(C)$, so $M \in \mathbf{C}_{\mathrm{reg}}$. From Remark 2.2, we have that M is in $\mathbf{C}_{\mathrm{ex}}$.

Case 2. For arrows $s : y(A) \to G$ and $t : y(B) \to G$ with $G \in \mathbf{C}_{\mathrm{ex}}$, we can write $s = p \circ s'$ and $t = p \circ t'$ for the regular epimorphism p of Case 1. We form the following diagram of pullbacks:

$$\begin{tikzcd}
H \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & y(B) \arrow[d, "t'"] \\
N \arrow[r] \arrow[d] & G' \arrow[r] \arrow[d] & y(C) \arrow[d, "p"] \\
y(A) \arrow[r, "s'"] & y(C) \arrow[r, "p"] & G
\end{tikzcd}$$

Since G', $y(A)$ and $y(B)$ are in $\mathbf{C}_{\mathrm{reg}}$, so is H. Thus, H is in $\mathbf{C}_{\mathrm{ex}}$, i.e., the pullback of s and t is in $\mathbf{C}_{\mathrm{ex}}$.

Case 3. For any arrows $s : F \to G$ and $t : F' \to G$ of $\mathbf{C}_{\mathrm{ex}}$, let $p : y(A) \to F$ and $q : y(B) \to F'$ be regular epis. We form the following diagram of pullbacks:

$$\begin{tikzcd}
H \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & y(B) \arrow[d, "q"] \\
N \arrow[r] \arrow[d] & G' \arrow[r] \arrow[d] & F' \arrow[d, "t"] \\
y(A) \arrow[r, "p"] & F \arrow[r, "s"] & G
\end{tikzcd}$$

As in Case 2, H is in $\mathbf{C}_{\mathrm{reg}}$. Also, G' is regularly covered by H as p and q are regular epis. Say that $e : H \to G'$ is the regular epi given by the diagram, and $m : G' \to F \times F'$ is the mono determined by the pullback of s and t. Note that $F \times F'$ is in $\mathbf{C}_{\mathrm{ex}}$. We form the kernel pair of $m \circ e$:

$$H' \underset{v}{\overset{u}{\rightrightarrows}} H \xrightarrow{m \circ e} F \times F'$$

That H' can be regularly covered by a representable functor follows from the same property for H. Also, H' is a subobject of $H \times H$, so $H' \in \mathbf{C}_{\mathrm{reg}}$. This shows that the kernel pair of e is in $\mathbf{C}_{\mathrm{reg}}$. Using the same argument as in Step 3, we can show that G' is in $\mathbf{C}_{\mathrm{ex}}$. This completes the proof that $\mathbf{C}_{\mathrm{ex}}$ has pullbacks.

Step 5. Every kernel pair of any morphism in $\mathbf{C}_{\mathrm{reg}}$ has a coequalizer.

In fact, let $(u, v : M \to N)$ be the kernel pair of $f : N \to G$ in $\mathbf{C}_{\mathrm{reg}}$. Take the coequalizer of u and v:

$$M \underset{v}{\overset{u}{\rightrightarrows}} N \xrightarrow{g} G'$$

Then G' is a subobject of G. Thus, G' is in $\mathbf{C}_{\mathrm{reg}}$.

Step 6. Every equivalence relation in $\mathbf{C}_{\mathrm{ex}}$ has a coequalizer.

Let $(u, v : M \to N)$ be an equivalence relation of objects in $\mathbf{C}_{\mathrm{ex}}$, and $g : N \to G$ be the coequalizer of u and v in $(\mathbf{C}^{\mathrm{op}}, \mathbf{Set})$; we show that G is in $\mathbf{C}_{\mathrm{ex}}$, as follows. Given the regular epi $s : y(A) \to N$, we form the following diagram of pullbacks:

$$
\begin{tikzcd}
H \arrow[r] \arrow[d] & M'' \arrow[r] \arrow[d] & y(A) \arrow[d, "s"] \\
M' \arrow[r] \arrow[d] & M \arrow[r, "u"] \arrow[d, "v"] & N \arrow[d, "g"] \\
y(A) \arrow[r, "s"] & N \arrow[r, "g"] & G
\end{tikzcd}
$$

Since $y(A)$, N and M are in $\mathbf{C}_{\mathrm{ex}}$, so is H. Thus, the kernel pair H of the regular epi $g \circ s$ is regularly covered by a representable functor. This shows that G is in $\mathbf{C}_{\mathrm{ex}}$.

Finally, that regular epis in $\mathbf{C}_{\mathrm{reg}}$ and $\mathbf{C}_{\mathrm{ex}}$ are stable under pullback and under κ-products follows immediately from the corresponding properties of $(\mathbf{C}^{\mathrm{op}}, \mathbf{Set})$. ∎

2.4. Corollary. *Let $k : \mathbf{C} \to \mathbf{C}_{\mathrm{reg}}$ and $l : \mathbf{C} \to \mathbf{C}_{\mathrm{ex}}$ be the restricted Yoneda embeddings. If $\mathbf{C}$ is weakly κ-complete, then*

(i) k and l are κ-flat;

(ii) for every $C \in \mathbf{C}$, $k(C)$ and $l(C)$ are regularly projective in $\mathbf{C}_{\mathrm{reg}}$ and $\mathbf{C}_{\mathrm{ex}}$, respectively;

(iii) for each $A \in \mathbf{C}_{\mathrm{ex}}$, there are $C \in \mathbf{C}$ and a regular epi $l(C) \to A$ in $\mathbf{C}_{\mathrm{ex}}$. Moreover, A is in $\mathbf{C}_{\mathrm{reg}}$ iff A is a subobject of a κ-product of objects in $\mathbf{C}$.

Proof. By 1.3 and 2.3. ∎

2.5. Remark. (i) In the next section we only use the properties of 2.4 to show the universal properties of $\mathbf{C}_{\mathrm{reg}}$ and $\mathbf{C}_{\mathrm{ex}}$. Consequently, these are necessary and sufficient conditions describing the free regular and exact completions of $\mathbf{C}$.
(ii) For any weakly κ-complete category, $\mathbf{C}^*$ of 1.5 has products. Let

$$\prod(\mathbf{C}^*, \mathbf{Set})$$

be the category of set-valued functors preserving products. Then $\prod(\mathbf{C}^*, \mathbf{Set})$ is ∞-exact, since products commute with regular epimorphisms and limits in $\mathbf{Set}$. Consider the evaluation functor

$$e_{\mathbf{C}} : \mathbf{C} \to \prod(\mathbf{C}^*, \mathbf{Set})$$

It is clear that $e_{\mathbf{C}}$ is full and faithful. From 2.3 and (i) above, $\mathbf{C}_{\mathrm{ex}}$ is equivalent to the full subcategory of $\prod(\mathbf{C}^*, \mathbf{Set})$ whose objects F are covered by a regular epimorphism $e_{\mathbf{C}}(C) \to F$ whose kernel pairs have the same property again. Likewise, we can describe $\mathbf{C}_{\mathrm{reg}}$ as a full subcategory of $\prod(\mathbf{C}^*, \mathbf{Set})$.

(iii) For κ an infinite regular cardinal, let

$$\mathbf{C}^{*+} = \prod \mathrm{Filt}_\kappa(\mathbf{C}^*, \mathbf{Set})$$

be the category of set-valued functors preserving products and κ-filtered colimits. $\mathbf{C}^{*+}$ is κ-exact, as κ-filtered colimits commute with regular epimorphisms and κ-limits in **Set**. Therefore, $\mathbf{C}_{\mathrm{reg}}$ and $\mathbf{C}_{\mathrm{ex}}$ can be described as full subcategories of $\mathbf{C}^{*+}$ as in (ii).

3. Universal properties of $\mathbf{C}_{\mathrm{reg}}$ and $\mathbf{C}_{\mathrm{ex}}$

3.1. PROPOSITION. *Let* $\mathbf{C}$ *be a weakly* κ*-complete category. With* k *and* l *as in 2.4 one has:*

(i) If $\mathbf{B}$ *is* κ*-regular, then every* κ*-flat functor* $F : \mathbf{C} \to \mathbf{B}$ *has a left Kan extension* $F!$ *along* k*, and* $F!$ *preserves regular epimorphisms.*

(ii) If $\mathbf{B}$ *is* κ*-exact, then every* κ*-flat functor* $F : \mathbf{C} \to \mathbf{B}$ *has a left Kan extension* $F!$ *along* l*, and* $F!$ *preserves regular epimorphisms.*

PROOF. We only give the proof of part (ii), as (i) can be done in the same manner. For convenience, we assume that l is the inclusion functor.

The proof here follows the same argumentation as in 3.5 of [9]. For the existence of $F!$, by the dual of Theorem X.3.1. in [12], it suffices to show that the composite $F \circ P : l/C' \to \mathbf{C} \to \mathbf{B}$ has a colimit in $\mathbf{B}$ for each $C' \in \mathbf{C}_{\mathrm{ex}}$, where P is the projection $\langle C, C \to C' \rangle \mapsto C$. Since $C' \in \mathbf{C}_{\mathrm{ex}}$, we have a regular epimorphism $e : A \to C'$ with A in $\mathbf{C}$. Let

$$D \underset{v'}{\overset{u'}{\rightrightarrows}} A$$

be the kernel pair of e; so e is the coequalizer of (u', v'), and there is a regular epimorphism $d : S \to D$ with $S \in \mathbf{C}$. Then e is a coequalizer of the morphisms $(u' \circ d, v' \circ d)$. Denote $u' \circ d$ by u, and $v' \circ d$ by v. Define a category $\mathbf{E}$ whose only non-trivial arrows are

$$e \underset{v}{\overset{u}{\rightrightarrows}} e \circ u$$

Let $i : \mathbf{E} \to l/C'$ be the inclusion functor. One then has:

3.2. LEMMA. *i is final.*

PROOF. Firstly, f/i is non-empty, for any $f : C \to C'$ with $C \in \mathbf{C}$. Indeed, by the projectivity of C, there is a morphism $w : C \to A$ such that $f = e \circ w$.

To show that f/i is connected, let m, n be any two morphisms in f/i. Then we only need to consider the following three cases.

Case 1: $m, n : f \to e$, i.e., $e \circ m = e \circ n = f$. Since $\langle u', v' \rangle$ is the kernel pair of e, there is a unique morphism $k' : C \to D$ such that $m = u' \circ k'$ and $n = v' \circ k'$. By the projectivity of C , we obtain a morphism $k : C \to S$ with $k' = d \circ k$. Thus, $m = u \circ k$ and $n = v \circ k$, i.e., $u : k \to m$ and $v : k \to n$; here $k : f \to e \circ u$ is in f/i.

Case 2: $m, n : f \to e \circ u$, i.e., $e \circ u' \circ d \circ m = e \circ v' \circ d \circ n = f$. Since $\langle u', v'\rangle$ is the kernel pair of e, there is a unique morphism $k' : C \to D$ such that $u' \circ d \circ m = u' \circ k'$ and $v' \circ d \circ n = v' \circ k'$. By the projectivity of C, we have a morphism $k : C \to S$ with $k = d \circ k'$. We conclude that $u \circ m = u \circ k$ and $v \circ n = v \circ k$. Thus, we have four morphisms $u : m \to u \circ k$, $u : k \to u \circ k$, $v : n \to v \circ k$, and $v : k \to v \circ k$ joining m and n.

Case 3: $m : f \to e$ and $n : f \to e \circ u$. By the projectivity of C, there is a morphism $m' : f \to e \circ u$ such that $m = u \circ m'$, because u is regular epi. Thus, we have morphisms $m', n : f \to e \circ u$. That f/i is connected now follows from Case 2. This completes the proof of that f/i is connected.

We continue with the proof of 3.1. Since i is final, according to Theorem IX.3.1 in [12], to prove that $F!$ exists, we only need to show that the pair of morphisms $(F(u), F(v))$ has a coequalizer in $\mathbf{B}$.

Let (p, q) be the product projections of $F(A) \times F(A)$, and let $\epsilon : F(S) \to F(A) \times F(A)$ be the unique morphism so that $F(u) = p \circ \epsilon$ and $F(v) = q \circ \epsilon$. Since $\mathbf{B}$ is κ–exact, ϵ has a factorization $\epsilon = y \circ x$ with $y : Q \to F(A) \times F(A)$ mono and $x : F(S) \to Q$ regular epi, for some $Q \in \mathbf{B}$.

3.3. Lemma. *y is an equivalence relation on $F(A)$.*

Proof. (i) (Reflexivity) The diagonal $\Delta : F(A) \to F(A) \times F(A)$ factors through y.

Let $k' : A \to D$ be the morphism so that $\mathrm{id}_A = u' \circ k' = v' \circ k'$. Since S is projective, one obtains a morphism $k : A \to S$ with $k' = d \circ k$, hence $\mathrm{id}_A = u \circ k = v \circ k$. It follows that

$$\mathrm{id}_{F(A)} = F(u) \circ F(k) = p \circ y \circ x \circ F(k) = q \circ y \circ x \circ F(k).$$

Consequently, $\Delta = y \circ (x \circ F(k))$.

(ii)(Symmetry) There exists a morphism $t : Q \to Q$ such that $p \circ y = q \circ y \circ t$ and $q \circ y = p \circ y \circ t$. Let (π_1, π_2) be the product projections of $A \times A$, and let $m : D \to A \times A$ be the induced morphism of (u', v'). Since the kernel pair of a morphism always yields an equivalence relation, there exists $n : D \to D$ such that $\pi_1 \circ m = \pi_2 \circ m \circ n$ and $\pi_2 \circ m = \pi_1 \circ m \circ n$. Since $d : S \to D$ is regular epi, by the projectivity of S, there is a morphism $n' : S \to S$ such that $n \circ d = d \circ n'$. Thus, we have $\pi_1 \circ m \circ d \circ n' = \pi_2 \circ m \circ d$ and $\pi_2 \circ m \circ d \circ n' = \pi_1 \circ m \circ d$, i.e., $u \circ n' = v$ and $v \circ n' = u$. Applying F to the above equalities, we obtain $F(u) \circ F(n') = F(v)$ and $F(v) \circ F(n') = F(u)$. Hence

$$p \circ y \circ x \circ F(n') = q \circ y \circ x, \text{ and } q \circ y \circ x \circ F(n') = p \circ y \circ x.$$

Let (f, g) be the kernel pair of x. Then,

$$p \circ y \circ x \circ F(n') \circ f = q \circ y \circ x \circ f = q \circ y \circ x \circ g = p \circ y \circ x \circ F(n') \circ g.$$

Similarly,

$$q \circ y \circ x \circ F(n') \circ f = q \circ y \circ x \circ F(n') \circ g.$$

Consequently, $y \circ x \circ F(n') \circ f = y \circ x \circ F(n') \circ g$. But y is monic, so $x \circ F(n') \circ f = x \circ F(n') \circ g$. Since (f, g) is the coequalizer of x, there is a unique morphism $t : Q \to Q$ such that $t \circ x = x \circ F(n')$. It is easily seen that t is the required morphism.

(iii) (Transitivity) For the pullback diagram of $p \circ y$ and $q \circ y$

$$\begin{tikzcd} Q \arrow[r, "p \circ y"] & F(A) \\ P \arrow[u, "b"] \arrow[r, "a"'] & Q \arrow[u, "q \circ y"'] \end{tikzcd}$$

the morphism $\delta = \langle (p \circ y) \circ a, (q \circ y) \circ b \rangle : P \to F(A) \times F(A)$ factors through y.

Let $(z, w : U \to F(S))$ be the pullback of $F(u)$ and $F(v)$. There is a unique morphism $\alpha : U \to P$ such that $x \circ z = b \circ \alpha$ and $x \circ w = a \circ \alpha$. Then, α is a regular epi. In fact, let $b' : U_1 \to F(S)$ and $x_1 : U_1 \to P$ be the pullback of x and b, and let $x_2 : U_2 \to P$ and $a' : U_2 \to F(S)$ be the pullback of a and x. Since x is regular epi, so are x_1 and x_2. If x_2' and x_1' is the pullback of x_1 and x_2, then $b' \circ x_2'$ and $a' \circ x_1'$ is the pullback of $F(u)$ and $F(v)$. Therefore, $\alpha = x_1 \circ x_2' = x_2 \circ x_1'$. That α is regular epi follows from the fact that x_1 and x_2' are regular epis.

Since F is κ-flat, there are two morphisms $s, t : V \to S$ in $\mathbf{C}$ with $u \circ s = v \circ t$ such that $F(s) = z \circ \beta$ and $f(t) = w \circ \beta$ for some regular epi $\beta : F(V) \to U$ in $\mathbf{B}$. Thus, $\alpha \circ \beta : F(V) \to P$ is a regular epi in $\mathbf{B}$.

Note that (u', v') is an equivalence relation on A. Let

$$\begin{tikzcd} D \arrow[r, "u'"] & A \\ T \arrow[u, "c_1"] \arrow[r, "c_2"'] & D \arrow[u, "v'"'] \end{tikzcd}$$

be the pullback diagram of u' and v'. Thus, the morphism $c = \langle u' \circ c_2, v' \circ c_1 \rangle : T \to A \times A$ factors as $c = m \circ r'$ for some $r' : T \to D$; here m is as in the proof for symmetry. Since $u \circ s = v \circ t$, i.e., $u' \circ d \circ s = v' \circ d \circ t$, there exists a unique morphism $d' : V \to T$ such that $d \circ s = c_1 \circ d'$ and $d \circ t = c_2 \circ d'$. By the projectivity of V, we have $r : V \to S$ with $d \circ r = r' \circ d'$. Since

$$u' \circ d \circ t = u' \circ c_2 \circ d' = \pi_1 \circ c \circ d' = \pi_1 \circ m \circ r' \circ d' = u' \circ d \circ r,$$

one obtains $u \circ t = u \circ r$. Similarly, $v \circ s = v \circ r$. Applying F to the above equalities, one gets $F(u) \circ F(t) = F(u) \circ F(r)$ and $F(v) \circ F(s) = F(v) \circ F(r)$. Since $F(u) = p \circ y \circ x$ and $F(v) = q \circ y \circ x$, it follows that

$$p \circ y \circ x \circ F(r) = p \circ y \circ x \circ F(t) = p \circ y \circ x \circ w \circ \beta = p \circ y \circ a \circ \alpha \circ \beta$$

and $q \circ y \circ x \circ F(r) = q \circ y \circ b \circ \alpha \circ \beta$. Let (f', g') be the kernel pair of $\alpha \circ \beta$. Then

$$p \circ y \circ x \circ F(r) \circ f' = p \circ y \circ x \circ F(r) \circ g',$$
$$q \circ y \circ x \circ F(r) \circ f' = q \circ y \circ x \circ F(r) \circ g',$$

and consequently $y \circ x \circ F(r) \circ f' = y \circ x \circ F(r) \circ g'$, which implies and $x \circ F(r) \circ f' = x \circ F(r) \circ g'$ as y is monic. Since $\alpha \circ \beta$ is the coequalizer of f' and g', there is a unique morphism $\eta : P \to Q$ such that $\eta \circ \alpha \circ \beta = x \circ F(r)$. Thus, $p \circ y \circ a = p \circ y \circ \eta$ and $q \circ y \circ b = q \circ y \circ \eta$. Consequently, $\delta = y \circ \eta$.

Finally we can complete the proof of 3.1. Since $\mathbf{B}$ is κ-exact, every equivalence relation is effective, so $(p \circ y, q \circ y)$ has a coequalizer. Since $F(u) = p \circ y \circ x$ and $F(v) = q \circ y \circ x$, it follows that $(F(u), F(v))$ has a coequalizer as x is regular epi. This completes the proof of the existence of $F!$.

From the above proof, we see that $F!$ takes any regular epi with domain in $\mathbf{C}$ into a regular epi. Indeed, given a regular epi $e : P \to Q$ in $\mathbf{C}_{\text{ex}}$, we take a regular epi $d : C \to P$ with $C \in \mathbf{C}$. Since $F(e) \circ F(d)$ is a regular epi, so is $F(e)$. ■

For κ-regular categories $\mathbf{A}$ and $\mathbf{B}$, recall that a functor $F : \mathbf{A} \to \mathbf{B}$ is *κ-regular* if F preserves κ-limits and regular epimorphisms. We denote the category of κ-regular functors from $\mathbf{A}$ into $\mathbf{B}$ by κ-Reg$(\mathbf{A}, \mathbf{B})$. For $\mathbf{C}$ and $\mathbf{B}$ as in 2.1, κ-Flat$(\mathbf{C}, \mathbf{B})$ is the category of κ-flat functors from $\mathbf{C}$ into $\mathbf{B}$.

3.4. Theorem. *Let $\mathbf{C}$ be a locally small category with weak κ-limits.*

(a) $\mathbf{C}_{\text{reg}}$ has the following universal property which characterizes it as the free κ-regular completion of $\mathbf{C}$:

(i) For any κ-regular $\mathbf{B}$, the functor

$$\Sigma : \kappa\text{–Reg}(\mathbf{C}_{\text{reg}}, \mathbf{B}) \to \kappa\text{–Flat}(\mathbf{C}, \mathbf{B}), \;\; M \mapsto M \circ k$$

induced by k of 2.4 is an equivalence of categories.

(ii) The quasi-inverse of the equivalence Σ of (i) takes a κ-flat functor $F : \mathbf{C} \to \mathbf{B}$ to its left Kan extension $F!$ along k.

(b) $\mathbf{C}_{\text{ex}}$ has the following universal property which characterizes it as the free κ-exact completion of $\mathbf{C}$:

(i) For any κ-exact category $\mathbf{B}$, the functor

$$\Sigma : \kappa\text{–Reg}(\mathbf{C}_{\text{ex}}, \mathbf{B}) \to \kappa\text{–Flat}(\mathbf{C}, \mathbf{B}), \;\; M \mapsto M \circ l$$

induced by l is an equivalence of categories.

(ii) The quasi-inverse of the equivalence Σ of (i) takes a κ-flat functor $F : \mathbf{C} \to \mathbf{B}$ to its left Kan extension $F!$ along l.

Proof. We give a proof of part (b); the proof of (a) proceeds similarly.

The fullness and faithfulness of Σ follow from the properties of $\mathbf{C}_{\mathrm{ex}}$ described in 2.5. For details, see the proof of Proposition 5.8 in [8]. We now prove that Σ is essentially surjective on objects. Since Σ is full and faithful, by Corollary X.3.3 in [10], it suffices to show that for any κ-flat functor $F : \mathbf{C} \to \mathbf{B}$, F has a left Kan extension $F!$ of F along $e_{\mathbf{C}}$, and $F!$ is κ-regular. The existence of $F!$ was shown in Proposition 3.1. Since $F!$ preserves regular epimorphisms, it remains to be shown that $F!$ preserves κ-limits, and by 1.4, we only need to show that $F!$ is κ-flat. We proceed in several steps.

Step 1. $F!$ is flat w.r.t. κ-products. Indeed, let $(B_i)_{i \in I}$ be a family of objects in $\mathbf{C}_{\mathrm{ex}}$ with $\#I < \kappa$, and $p_i : C_i \to B_i$ be regular epis with C_i in $\mathbf{C}$. By 3.1, $F!(p_i)$ is regular epi for every $i \in I$. Since regular epis are stable under κ-products in $\mathbf{B}$, $\prod F!(p_i) : \prod F!(C_i) \to \prod F!(B_i)$ is a regular epi. Let W be a weak product of C_i in $\mathbf{C}$. The induced arrow $t : F(W) \to \prod F!(C_i)$ is a regular epi as F is κ-flat. There is a canonical arrow $s : F(\prod B_i) \to \prod F!(B_i)$, and the weak projections $e_i : W \to C_i$ of W in $\mathbf{C}$, when composed with p_i induce an $m : W \to \prod B_i$. Since $s \circ F!(m) = \prod F!(q_i) \circ t$, with $\prod F!(q_i) \circ t$ regular epi, also s must be a regular epimorphism.

Step 2. $F!$ is flat w.r.t. pullbacks. We proceed as in 2.3.

Case 1. Let $e : A \to B$ be a regular epi in $\mathbf{C}_{\mathrm{ex}}$ with $A \in \mathbf{C}$. Consider the kernel pair of $F!(e)$:

$$Q \underset{g}{\overset{f}{\rightrightarrows}} F!(A) \xrightarrow{F!(e)} F!(B)$$

Let $(u, v : D \to A)$ be the kernel pair of e in $\mathbf{C}_{\mathrm{ex}}$, and $a : S \to D$ be a regular epi with $S \in \mathbf{C}$. From the proof of 3.1, we can see that the unique arrow $x : F!(S) \to Q$ is a regular epi in $\mathbf{B}$. Let $b : F!(D) \to Q$ be the unique arrow so that $F!(u) = f \circ b$ and $F!(v) = g \circ b$. Then $x = b \circ F!(a)$, and this implies that b is a regular epi.

Case 2. Let e be the morphism of Case 1, and $f : C \to B$ be any arrow with $C \in \mathbf{C}$. Since C is regular projective, $f = e \circ g$ for some $g : C \to A$. We form the following diagram of pullbacks:

$$\begin{tikzcd}
Q' \arrow[r, "c'"] \arrow[d, "g'"'] & F!(C) \arrow[d, "F!(g)"] \\
Q \arrow[r, "c"] \arrow[d, "c"'] & F!(A) \arrow[d, "F!(e)"] \\
F!(C) \arrow[r, "F!(e)"] & F!(B)
\end{tikzcd}$$

As before, we let D be the kernel pair of e in $\mathbf{C}_{\text{ex}}$, and $b : F!(D) \to Q$ and $a : S \to D$ be regular epis in Case 1. We form a weak pullback S' of g and $u \circ a$ in $\mathbf{C}_{\text{ex}}$; here $F!(u) = f \circ b$. Thus, we have the following diagram of pullbacks:

$$\begin{tikzcd}
Q'' \arrow[r, "s"] \arrow[d, "F!(d)"'] & Q' \arrow[r, "c'"] \arrow[d, "g'"] & F!(C) \arrow[d, "F!(g)"] \\
F!(S) \arrow[r, "a \circ b"] & Q \arrow[r, "c"] & F!(A)
\end{tikzcd}$$

where s is a regular epi as $a \circ b$ is a regular epi. Let S' be a weak pullback of g and $u \circ a$ in $\mathbf{C}$. Since F is κ-flat, the unique arrow $p : F!(S') \to Q''$ is a regular epi. Thus, we have a regular epi $s \circ p : F!(S') \to Q'$. Let D' be the pullback of e and f in $\mathbf{C}_{\text{ex}}$, and $b' : F!(D') \to Q'$ be the unique arrow determined by the universal property of the pullback Q'. Then $s \circ p = b' \circ F!(s')$; here $s' : S' \to D'$ is the unique arrow determined by the universal property of the pullback D'. This shows that b' is a regular epi.

Case 3. Now we consider arrows $f : K \to B$ and $g : A \to B$ of $\mathbf{C}_{\text{ex}}$ with $K, A \in \mathbf{C}$. Let $e : C \to B$ be a regular epi with $C \in \mathbf{C}$. Then $f = e \circ f'$ and $g = e \circ g'$. We form the following pullback diagrams

$$\begin{tikzcd}
Q' \arrow[r, "m'"] \arrow[d] & F!(K) \arrow[d, "F!(g')"] \\
Q \arrow[r, "m"] \arrow[d, "e'"'] & F!(C) \arrow[d, "F!(e)"] \\
F!(A) \arrow[r, "F!(f)"] & F!(B)
\end{tikzcd}$$

Let $(u : S \to C, v : S \to A)$ be the pullback of e and f. From the argument of Case 2, the unique arrow $b : F!(S) \to Q$ is a regular epi. Let $a : D \to S$ be a regular epi with $D \in \mathbf{C}$. Then $b \circ F!(a) : F!(D) \to Q$ is a regular epi. We form the consecutive pullback diagrams

$$\begin{tikzcd}
Q'' \arrow[r, "b'"] \arrow[d] & Q' \arrow[r, "m'"] \arrow[d] & F!(K) \arrow[d, "F!(g')"] \\
F!(D) \arrow[r, "b \circ F!(a)"] & Q \arrow[r, "m"] & F!(C)
\end{tikzcd}$$

Let D' be a weak pullback of $u \circ a$ and g'. Since F is κ-flat, the unique arrow $b' : F!(D') \to Q''$ is a regular epi. Thus, $c \circ b' : F!(D') \to Q'$ is a regular epi. Consequently, the unique arrow $F!(S') \to Q'$ is a regular epi; here S' is the pullback of f and g.

Case 4. Next we consider arrows $f : A \to B$ and $g : K \to B$ with $A \in \mathbf{C}$. Let $e : C \to K$ be a regular epi with $C \in \mathbf{C}$. We form the pullback diagrams

$$\begin{tikzcd}
Q' \arrow[r, "e'"] \arrow[d] & Q \arrow[r, "g'"] \arrow[d] & F!(A) \arrow[d, "F!(f)"] \\
F!(C) \arrow[r, "F!(e)"] & F!(K) \arrow[r, "F!(g)"] & F!(B)
\end{tikzcd}$$

Let S' be the pullback of f and $g \circ e$. From the Case 3, the unique arrow $b : F!(S') \to Q'$ is a regular epi. Thus, we have a regular epi $e' \circ b : F!(S') \to Q$ as e' is a regular epi. Consequently, the unique arrow $a : F!(S) \to Q$ is a regular epi; here S be the pullback of f and g.

Case 5. Finally, for arbitrary arrows $f : A \to B$ and $g : K \to B$ of $\mathbf{C}_{\text{ex}}$, we can reduce the proof to the case just mentioned.

Step 3. $F!$ is flat w.r.t. equalizers.

Given $f, g : A \to B$ in $\mathbf{C}_{\text{ex}}$, let $m : Q \to F!(A)$ be the equalizer of $F!(f)$, $F!(g)$ in $\mathbf{B}$. With a regular epi $u : C \to A$ in $\mathbf{C}_{\text{ex}}$ with $C \in \mathbf{C}$ one forms the pullback

$$\begin{tikzcd} P \arrow[r, "n"] \arrow[d, "k"'] & F!(C) \arrow[d, "F!(u)"] \\ Q \arrow[r, "m"] & F!(A) \end{tikzcd}$$

so that n is the equalizer of $F!(f \circ u)$, $F!(g \circ u)$. The pullback

$$\begin{tikzcd} R \arrow[r, "s"] \arrow[d, "t"'] & F!(C) \arrow[d, "F!(g \circ u)"] \\ F!(C) \arrow[r, "F!(f \circ u)"] & F!(B) \end{tikzcd}$$

gives an arrow $q : P \to R$ which is an equalizer of s, t. There is a weak pullback of $f \circ u$, $g \circ u$, with projections $x, y : E \to C$ in $\mathbf{C}$, and the induced arrow $p : F(E) \to R$ is regular epi. For the pullback diagram

$$\begin{tikzcd} R' \arrow[r, "s'"] \arrow[d, "t'"'] & P \arrow[d, "q"] \\ F(E) \arrow[r, "p"] & R \end{tikzcd}$$

t' is easily recognized as the equalizer of $F(x)$, $F(y)$. But since F is flat, there is a weak equalizer Z of x and y such that the induced arrow $i : F(Z) \to R'$ is regular epi. As pullbacks of regular epis, also s' and k are regular epi. Consequently, $k \circ s' \circ i : F(Z) \to Q$ is a regular epi.

Step 4. $F!$ preserves equalizers. Looking at the kernel pair of a monomorphism f in $\mathbf{C}_{\text{ex}}$, one sees immediately that $F!(f)$ is mono as well, since $F!$ is flat w.r.t. pullbacks. But flatness w.r.t. equalizers and preservation of monos makes $F!$ preserve equalizers.

Step 5. $F!$ is κ-flat. As usual, one presents the limit of a κ-diagram in $\mathbf{C}_{\mathrm{ex}}$ as an equalizer of two arrows between κ-products. A routine diagram chase shows that flatness w.r.t. κ-products and preservation of equalizers yield the desired result. ∎

4. Simplifying the description of $\mathbf{C}_{\mathrm{ex}}$

We have seen that the objects of $\mathbf{C}_{\mathrm{ex}}$ consist of functors regularly covered by representable functors which satisfy the property: the kernel pair of any regular epi (see the proof of 2.3) from a representable functor into an object of $\mathbf{C}_{\mathrm{ex}}$ is regularly covered by a representable functor. The question is under which condition on $\mathbf{C}$ this latter condition can be omitted from the definition of the objects of $\mathbf{C}_{\mathrm{ex}}$.

First we consider a small category $\mathbf{C}$ and the evaluation functor

$$e_{\mathbf{C}} : \mathbf{C} \to (\kappa\text{–Flat}(\mathbf{C}), \mathbf{Set})$$

It was shown in [9] that κ-Flat($\mathbf{C}$) has products and κ-filtered colimits (as a κ-accessible category), and $\mathbf{C}_{\mathrm{ex}}$ is equivalent to the category of functors from κ-Flat($\mathbf{C}$) into $\mathbf{Set}$ which preserve products and κ-filtered colimits. The new embedding $e_{\mathbf{C}}$ gives that if a functor of $\mathbf{C} \to \mathbf{Set}$ can be regularly covered by a representable functor, then it satisfies the additional property mentioned above automatically.

If $\mathbf{C}$ is not small, this is no longer true, for the reason that κ-Flat($\mathbf{C}$) may no longer be the κ-filtered colimit completion of $\mathbf{C}^{\mathrm{op}}$. Consequently, one may have a functor $F : \kappa\text{-Flat}(\mathbf{C}) \to \mathbf{Set}$ which preserves κ-filtered colimits and products and satisfies the solution set condition, such that F is regularly covered by a representable functor on κ-Flat($\mathbf{C}$), but this representable fails to be of the form $e(C)$ although it preserves κ-filtered colimits. In what follows we give sufficient conditions on $\mathbf{C}$ for overcoming these difficulties.

As a first preparatory step we consider a category $\mathbf{A}$ with products. Since regular epis commute with products in $\mathbf{Set}$, a weakly representable functor $F : \mathbf{A} \to \mathbf{Set}$ preserves products. The following characterization of weakly representable functors is analgous to Theorem V.6.3 of [12].

4.1. Proposition. *Let $\mathbf{A}$ be a locally small category with products, and let $F : \mathbf{A} \to \mathbf{Set}$ be a functor. The following conditions are equivalent:*

(i) F is weakly representable.

(ii) F preserves products and satisfies the solution set condition on $1 \in \mathbf{Set}$.

Proof. If F is weakly representable, we already saw that F preserves products. To verify that F satisfies the solution set condition on 1, it is easy to show that $1/F$ has a weakly initial object: take a regular epimorphism $x : \mathbf{A}(A, -) \to F$ in $(\mathbf{A}, \mathbf{Set})$; then $x_B : \mathbf{A}(A, B) \to F(B)$ is surjective for any $B \in \mathbf{A}$. This says that $(A, x \in F(A))$ is a weakly initial object of $1/F$.

Conversely, let F preserve products and satisfy the solution set condition on $1 \in \mathbf{Set}$, so that there is a weakly initial family $(A_i, x_i \in F(A_i))_{i \in I}$ in $1/F$. Since F preserves products, we obtain a weakly initial object $(\prod A_i, x \in F(\prod A_i))$ with $x = \langle x_i \rangle_{i\in I}$ in $1/F$. By the Yoneda Lemma, such an x corresponds to a natural transformation $x : \mathbf{A}(A, -) \to F$ with $A = \prod A_i$, and this implies that $x_B : \mathbf{A}(A, B) \to F(B)$ is surjective for any $B \in \mathbf{A}$. We conclude that x is a regular epimorphism in $(\mathbf{A}, \mathbf{Set})$. ∎

Recall that an object A of a category $\mathbf{A}$ is said to be κ-presentable if the representable functor $\mathbf{A}(A, -)$ preserves κ-filtered colimits existing in $\mathbf{A}$. $\mathbf{A}$ is *κ-accessible* if: (i) $\mathbf{A}$ has κ-filtered colimits; (ii) there is a small subcategory $\mathbf{C}$ of $\mathbf{A}$ consisting of κ-presentable objects such that every object of $\mathbf{A}$ is a κ-filtered colimit of a diagram of objects in $\mathbf{C}$. The full subcategory of $\mathbf{A}$ whose objects are the κ-presentable ones is denoted by $\mathbf{A}_\kappa$. A category is accessible if it is κ-accessible for some κ. A functor between accessible categories is accessible if it preserves κ-filtered colimits for some κ ([1] and [14]).

Recall that every accessible functor $F : \mathbf{A} \to \mathbf{B}$ satisfies the solution set condition for any $B \in \mathbf{B}$ (cf. Proposition 6.1.2 of [14])), with the validity of the converse proposition being logically equivalent to the set-theroetic Vopěnka's Principle, see [15]. It is natural to ask whether preservation of products makes the accessibility of a functor equivalent to the solution set condition.

4.2. PROPOSITION. *Let $\mathbf{A}$ be an accessible category with products (called a weakly locally presentable category in the sense of [1]), and let $F : \mathbf{A} \to \mathbf{Set}$ be a functor preserving products. Then the following conditions are equivalent:*

(1) F is weakly representable.

(2) F is accesssible.

(3) F satisfies the solution set condition for any $X \in \mathbf{Set}$.

PROOF. If F is weakly representable, take a regular epimorphism $\mathbf{A}(A, -) \to F$ with $A \in \mathbf{A}_\kappa$ for some κ, then it is easy to see that F is κ-accessible. That (2) implies (3) follows from Proposition 6.1.2 of [14]. Proposition 4.1 shows that (3) implies (1). ∎

4.3. COROLLARY. *For an accessible category $\mathbf{A}$ with products, a product-preserving functor $F : \mathbf{A} \to \mathbf{B}$ into an accessible category $\mathbf{B}$ is accessible iff F satisfies the solution set condition for any $B \in \mathbf{B}$.*

PROOF. Only the "if" part still needs to be proved. For an arbitrary accessible category $\mathbf{B}$, each representable functor $\mathbf{B}(B, -)$ satisfies the solution set condition (since it is accessible). Hence $\mathbf{B}(B, -) \circ F$ satisfies the solution set condition and is therefore accessible (see Proposition 4.2), for any $B \in \mathbf{B}$. But since the representables collectively detect accessibility (see [14]), the proof is complete now. ∎

For a κ-accessible category $\mathbf{A}$ with products, as we mentioned before, the category $\prod \mathrm{Filt}_\kappa(\mathbf{A}, \mathbf{Set})$ is κ-exact. Furthermore, for any accessible category $\mathbf{A}$ with products, let

$$\prod \mathrm{Acc}(\mathbf{A}, \mathbf{Set})$$

be the category of set-valued accessible functors preserving products, i.e., of weakly representable functors. Then $\prod \mathrm{Acc}(\mathbf{A}, \mathbf{Set})$ is ∞-exact. A typical example of this kind of category is the opposite of the category of R-modules, for any commutative ring R with unity. It will follows Theorem 4.5 below that this category is the free ∞-exact completion of the opposite of the category of injective R-modules (for details, see [8]).

4.4. PROPOSITION. *For any accessible category* $\mathbf{A}$ *with products, the restricted Yoneda embedding*

$$y : \mathbf{A}^{\mathrm{op}} \to \prod \mathrm{Acc}(\mathbf{A}, \mathbf{Set})$$

has the following properties:
(i) y *is full and faithful;*
(ii) $y(A)$ *is regularly projective in* $\prod \mathrm{Acc}(\mathbf{A}, \mathbf{Set})$, *for any* $A \in \mathbf{A}$;
(iii) for any $F \in \prod \mathrm{Acc}(\mathbf{A}, \mathbf{Set})$, *there are* $A \in \mathbf{A}$ *and a regular epimorphism* $\mathbf{A}(A, -) \to F$ *in* $\prod \mathrm{Acc}(\mathbf{A}, \mathbf{Set})$;
(iv) y *is* ∞*-flat.*

PROOF. (i) is trivial. (ii) and (iii) follows from Proposition 4.3. (iv) follows from (i), (ii) and (iii). ∎

As proved in [1] and [9], an accessible category $\mathbf{A}$ has weak colimits iff it has products. Considering $\mathbf{C} = \mathbf{A}^{\mathrm{op}}$, we therefore obtain the following theorem:

4.5. THEOREM. *Let* $\mathbf{C}$ *be a co-accessible category with weak limits. Then the free* ∞*-exact completion of* $\mathbf{C}$ *is equivalent to the category* $\prod \mathrm{Acc}(\mathbf{C}^{\mathrm{op}}, \mathbf{Set})$.

4.6. COROLLARY. *Let* $\mathbf{C}$ *be a co-accessible category with weak limits, then the objects of the free* ∞*-exact completion of* $\mathbf{C}$ *are exactly the weakly representable functors from* $\mathbf{C}^{\mathrm{op}}$.

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