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A COUNTEREXAMPLE TO A CONJECTURE OF BARR

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ABSTRACT. We discuss two versions of a conjecture attributed to M. Barr. The Harrison cohomology of a commutative algebra is known to coincide with the André/Quillen cohomology over a field of characteristic zero but not in prime characteristics. The conjecture is that a modified version of Harrison cohomology, taking into account torsion, always agrees with André/Quillen cohomology. We give a counterexample.

1. Definitions and Notation

Let A be a commutative algebra over a commutative ring k and let M be an A-bimodule. We denote $\otimes_k$ simply by $\otimes$. Consider the Hochschild cochain complex $(\mathrm{Hom}_k(A^{\otimes n}, M), \delta)$, where δ is the Hochschild coboundary,

$$\delta f(a_1 \otimes \cdots \otimes a_n) = a_1 f(a_2 \otimes \cdots \otimes a_n) + \sum_{i=1}^{n-1} (-1)^i f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) + (-1)^n f(a_1 \otimes \cdots \otimes a_{n-1}) a_n.$$

The symmetric group Σ_n acts on the left on $A^{\otimes n}$ by

$$\sigma(a_1 \otimes \ldots \otimes a_n) = (a_{\sigma^{-1}1} \otimes \ldots \otimes a_{\sigma^{-1}n}),$$

and we extend this linearly to an action of the group algebra $k\Sigma_n$.

Let $T^c(A)$ denote the cotensor algebra over A. That is, $(T^c(A))_n = A^{\otimes n}$, with graded commutative multiplication the signed shuffle, $\mu : A^{\otimes p} \otimes A^{\otimes q} \to A^{\otimes p+q}$,

$$\mu((a_1 \otimes \cdots \otimes a_p) \otimes (a_{p+1} \otimes \cdots \otimes a_{p+q})) = \sum (\mathrm{sgn}\, \sigma) \sigma(a_1 \otimes \cdots \otimes a_{p+q}),$$

where the sum is over (p, q)-shuffles σ in Σ_{p+q}.

Harrison cohomology, $\mathrm{Harr}^*(A; M)$, is the homology of the subcomplex of the Hochschild cochain complex consisting of those cochains which vanish on decomposable elements for the shuffle product. This commutative algebra cohomology theory is known to coincide with the André/Quillen theory when A is projective over k and k contains $\mathbb{Q}$ [4]. An example of André, described in [1], shows that the theories do not coincide in prime characteristics.

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2. The Counterexample

Note that the indecomposables of the cotensor algebra $T^c(A)$ may have torsion. For example, $\mu((a \otimes b) \otimes (a \otimes b)) = 2(a \otimes b \otimes a \otimes b)$, so that when k has characteristic zero a Harrison 4-cochain must vanish on $(a \otimes b \otimes a \otimes b)$, but not when k has characteristic 2. Let us denote by $HB^*(A; M)$ the homology of the subcomplex of all Hochschild cochains vanishing not only on shuffles but also on those elements some multiple of which is a shuffle. According to ([2] p232), Barr conjectures that this gives André/Quillen cohomology. We give a counterexample, by constructing a non-zero element in HB^5 of a polynomial algebra. (Of course, the higher André/Quillen cohomology groups of a polynomial algebra are zero.)

2.1. Proposition. *$HB^5(\mathbb{F}_2[x]; \mathbb{F}_2) \neq 0$, where $\mathbb{F}_2$ is a $\mathbb{F}_2[x]$-module via the trivial x-action.*

Proof. The idea is to give a Harrison 5-cocycle f as the coboundary of an explicitly defined Harrison 4-cochain g. So of course as a Hochschild or Harrison cycle f is cohomologous to zero. We then show that f satisfies Barr's condition, but is not the coboundary of any Barr 4-cochain.

Define an integral Hochschild 4-cochain G as follows,

$$G(1 \otimes x \otimes 1 \otimes x) = G(x \otimes 1 \otimes x \otimes 1) = G(x \otimes 1 \otimes 1 \otimes x) = G(1 \otimes x \otimes x \otimes 1) = 1,$$

G is zero on any other chain of the form $(x^{i_1} \otimes x^{i_2} \otimes x^{i_3} \otimes x^{i_4})$, and we extend linearly. Now let g be the reduction modulo 2 of G.

Let $f = \delta g$. Now $f \neq 0$, since for example $f(1 \otimes 1 \otimes x \otimes 1 \otimes x) = 1$.

Next we check that g vanishes on shuffles (or, equivalently, G takes even values on shuffles). It is clearly sufficient to check for those shuffles which involve

$$(1 \otimes x \otimes 1 \otimes x),\ (x \otimes 1 \otimes x \otimes 1),\ (x \otimes 1 \otimes 1 \otimes x) \text{ or } (1 \otimes x \otimes x \otimes 1).$$

For example,

$$\begin{aligned} G(\mu((1) \otimes (x \otimes 1 \otimes x))) &= G(1 \otimes x \otimes 1 \otimes x) - G(x \otimes 1 \otimes 1 \otimes x) \\ &\quad + G(x \otimes 1 \otimes 1 \otimes x) - G(x \otimes 1 \otimes x \otimes 1) = 0. \end{aligned}$$

Similar calculations show that although G does not vanish on all these shuffles it does take even values on them. Hence g is a Harrison cochain and so $f = \delta g$ is a Harrison 5-cocycle.

Notice that as above $(1 \otimes x \otimes 1 \otimes x)$ has a multiple which is a shuffle and yet $g(1 \otimes x \otimes 1 \otimes x) \neq 0$, so g does not satisfy Barr's condition. In fact, if $f = \delta h$ then $1 = f(1 \otimes 1 \otimes x \otimes 1 \otimes x) = \delta h(1 \otimes 1 \otimes x \otimes 1 \otimes x) = h(1 \otimes x \otimes 1 \otimes x)$. So f is not the coboundary of any cochain satisfying Barr's condition.

It remains to check that f itself does satisfy Barr's condition. Note that f is the reduction modulo 2 of the integral cochain $F = \delta G$. Now one easily checks that the only chains of the form $(x^{i_1} \otimes x^{i_2} \otimes x^{i_3} \otimes x^{i_4} \otimes x^{i_5})$ on which F is non-zero are:

$$\begin{aligned} a_1 &= (1 \otimes 1 \otimes x \otimes 1 \otimes x), & a_4 &= (x \otimes 1 \otimes x \otimes 1 \otimes 1), \\ a_2 &= (1 \otimes x \otimes 1 \otimes 1 \otimes x), & a_5 &= (x \otimes 1 \otimes 1 \otimes x \otimes 1), \\ a_3 &= (1 \otimes 1 \otimes x \otimes x \otimes 1), & a_6 &= (1 \otimes x \otimes x \otimes 1 \otimes 1). \end{aligned}$$

We have $F(a_i) = 1$ for $i = 1, 2, 3$, $F(a_i) = -1$ for $i = 4, 5, 6$. We need only check for shuffle-submultiples involving these. We introduce notation:

$$\begin{aligned} b_1 &= (1 \otimes 1 \otimes 1 \otimes x \otimes x), & b_3 &= (x \otimes x \otimes 1 \otimes 1 \otimes 1), \\ b_2 &= (x \otimes 1 \otimes 1 \otimes 1 \otimes x), & b_4 &= (1 \otimes x \otimes 1 \otimes x \otimes 1). \end{aligned}$$

We show that f vanishes on any shuffle-submultiple of the form $\sum k_i a_i + \sum l_j b_j$, for $k_i, l_j \in \mathbb{F}_2$. The idea is that although F does not vanish on every shuffle-submultiple, it takes even values on such, so that f does vanish.

Consider all possible shuffles involving three 1's and two x's. There are a total of forty such shuffles, ten each of the $(1, 4)$, $(2, 3)$, $(3, 2)$ and $(4, 1)$ shuffles. Since the shuffle product is graded commutative it suffices to calculate only the twenty $(1, 4)$ and $(2, 3)$ shuffles. Calculating these directly one finds for the $(1, 4)$ shuffles:

$$\begin{aligned} &\mu((1) \otimes (1 \otimes 1 \otimes x \otimes x)) = b_1 - a_1 + a_3, & &\mu((1) \otimes (1 \otimes x \otimes 1 \otimes x)) = b_4, \\ &\mu((1) \otimes (x \otimes 1 \otimes 1 \otimes x)) = a_2 - b_2 + a_5, & &\mu((1) \otimes (1 \otimes x \otimes x \otimes 1)) = b_4, \\ &\mu((1) \otimes (x \otimes 1 \otimes x \otimes 1)) = b_4, & &\mu((1) \otimes (x \otimes x \otimes 1 \otimes 1)) = a_6 - a_4 + b_3, \\ &\mu((x) \otimes (1 \otimes 1 \otimes 1 \otimes x)) = b_2 - a_2 + a_1, & &\mu((x) \otimes (1 \otimes 1 \otimes x \otimes 1)) = a_5 - b_4 + a_1, \\ &\mu((x) \otimes (1 \otimes x \otimes 1 \otimes 1)) = a_4 - b_4 + a_2, & &\mu((x) \otimes (x \otimes 1 \otimes 1 \otimes 1)) = a_4 - a_5 + b_2. \end{aligned}$$

For the $(2, 3)$ shuffles we have:

$$\begin{aligned} &\mu((x \otimes x) \otimes (1 \otimes 1 \otimes 1)) = b_3 - a_4 + a_5 - b_2 + a_6 - b_4 + a_2 + a_3 - a_1 + b_1, \\ &\mu((1 \otimes x) \otimes (1 \otimes 1 \otimes x)) = a_2 + a_1, \\ &\mu((1 \otimes x) \otimes (1 \otimes x \otimes 1)) = 2b_4, \\ &\mu((1 \otimes x) \otimes (x \otimes 1 \otimes 1)) = b_4 - a_2 + a_4 + b_2, \\ &\mu((x \otimes 1) \otimes (1 \otimes 1 \otimes x)) = b_2 - a_5 + b_4 + a_1, \\ &\mu((x \otimes 1) \otimes (1 \otimes x \otimes 1)) = 2b_4, \\ &\mu((x \otimes 1) \otimes (x \otimes 1 \otimes 1)) = a_4 + a_5, \\ &\mu((1 \otimes 1) \otimes (x \otimes x \otimes 1)) = a_3 - b_4 + a_5 + b_3, \\ &\mu((1 \otimes 1) \otimes (x \otimes 1 \otimes x)) = a_1 - b_4 + b_2 + a_4, \\ &\mu((1 \otimes 1) \otimes (1 \otimes x \otimes x)) = b_1 + a_2 - b_4 + a_6. \end{aligned}$$

Now it is easily checked that any Z-linear combination of these can be expressed in the following form.

$$\begin{aligned} s = z_1(a_1 + a_2) + z_2(a_1 + a_5) + z_3(a_2 + a_4) + z_4(b_1 + a_2 + a_3) + z_5(b_1 + a_2 + a_6) \\ + z_6(b_2 + a_1 + a_4) + z_7(b_3 + a_3 + a_5) + z_8(b_3 + a_2 + a_6) + z_9 b_4, \end{aligned}$$

where $z_i \in \mathbb{Z}$. Note that $F(s) = 2(z_1 + z_4)$.

Suppose that s has the coefficient of each a_i and b_j divisible by q. Then we need to show that f vanishes on the shuffle-submultiple s/q, that is $F(s/q)$ is even, or equivalently, $F(s)$ is divisible by $2q$. Now adding the coefficients of a_2 and b_2 and subtracting the coefficients of a_4 and a_6 gives $z_1 + z_4$. So $z_1 + z_4$ is divisible by q. Thus $F(s) = 2(z_1 + z_4)$ is divisible by $2q$ as required.

Hence, f does vanish on any element some multiple of which is a shuffle. So $0 \neq [f] \in HB^5(\mathbb{F}_2[x]; \mathbb{F}_2)$. ■

Another version of the conjecture is given by considering the Hochschild chain complex with the shuffle product as a divided power algebra, and factoring out the divided powers. (See ([2] p232) and ([3] p271).) The above is also a counterexample to the conjecture that the cohomology of this complex then gives André/Quillen cohomology. Firstly, since $1 \otimes x \otimes 1 \otimes x$ is a divided power we see that f is not the coboundary of any cochain vanishing on divided powers. Secondly, the arguments above show that f itself does vanish on all divided powers.

References

[1] M. Barr. *Harrison homology, Hochschild homology and triples.* Journal of Algebra, 8, pp314–323, 1968.

[2] M. Gerstenhaber and S.D. Schack. *A Hodge-type decomposition for commutative algebra cohomology.* Journal of Pure and Applied Algebra, 48, pp229–247, 1987.

[3] M. Gerstenhaber and S.D. Schack. *The shuffle bialgebra and the cohomology of commutative algebras.* Journal of Pure and Applied Algebra, 70, pp263–272, 1991.

[4] D. Quillen. *On the (co)-homology of commutative rings.* Proc. Symp. Pure Math., 17, pp65–87, 1970.

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