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REMARKS ON QUINTESSENTIAL AND PERSISTENT LOCALIZATIONS

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ABSTRACT. We define a localization $\mathcal{L}$ of a category $\mathcal{E}$ to be *quintessential* if the left adjoint to the inclusion functor is also right adjoint to it, and *persistent* if $\mathcal{L}$ is closed under subobjects in $\mathcal{E}$. We show that quintessential localizations of an arbitrary Cauchy-complete category correspond to idempotent natural endomorphisms of its identity functor, and that they are necessarily persistent. Our investigation of persistent localizations is largely restricted to the case when $\mathcal{E}$ is a topos: we show that persistence is equivalence to the closure of $\mathcal{L}$ under finite coproducts and quotients, and that it implies that $\mathcal{L}$ is coreflective as well as reflective, at least provided $\mathcal{E}$ admits a geometric morphism to a Boolean topos. However, we provide examples to show that the reflector and coreflector need not coincide.

Introduction

It is well known that a local operator (also called a Lawvere–Tierney topology) j on a topos $\mathcal{E}$ gives rise to two reflective subcategories of $\mathcal{E}$: the category $\mathbf{sh}_j(\mathcal{E})$ of j-sheaves, and the category $\mathbf{sep}_j(\mathcal{E})$ of j-separated objects. We shall denote the reflector $\mathcal{E} \to \mathbf{sh}_j(\mathcal{E})$ (that is, the left adjoint of the inclusion functor) by L (or L_j, if it is necessary to specify j), and the reflector $\mathcal{E} \to \mathbf{sep}_j(\mathcal{E})$ by M or M_j. It is also well known that L is a localization of $\mathcal{E}$ (that is, it preserves all finite limits), but M is not in general: it preserves finite products (since $\mathbf{sep}_j(\mathcal{E})$ is an exponential ideal in $\mathcal{E}$) and monomorphisms (since it is a subfunctor of L which preserves monomorphisms), but need not preserve equalizers.

One might ask: what can be said about j if M does preserve equalizers? If so, then by the 'little Giraud theorem' which says that the localizations of a topos $\mathcal{E}$ correspond bijectively to local operators on $\mathcal{E}$ (cf. [1], III 9.3.9) we know that $\mathbf{sep}_j(\mathcal{E})$ must coincide with $\mathbf{sh}_k(\mathcal{E})$ for some local operator k. But we can say more than this: since every k-separated object is a subobject of a k-sheaf, and since $\mathbf{sep}_j(\mathcal{E})$ is closed under subobjects in $\mathcal{E}$, k must have the property that its separated objects and sheaves coincide. Also, knowledge of $\mathbf{sep}_k(\mathcal{E})$ determines the local operator k uniquely (since the k-dense monomorphisms are precisely those mapped to epimorphisms in $\mathbf{sep}_k(\mathcal{E})$ by M_k), so we must in fact have $j = k$; that is, j itself has the property that all its separated objects are sheaves. Conversely, if j has this property, then M_j coincides with L_j, and so it does preserve finite limits.

Do there exist any examples of local operators with this curious property? One source

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of them is what we have chosen to call the *quintessential localizations* of a topos. We recall that a *localization* of an arbitrary category $\mathcal{E}$ is defined to be a reflective (full) subcategory $\mathcal{L}$ such that the reflector $L: \mathcal{E} \to \mathcal{L}$ preserves finite limits; it is called an *essential localization* [5] if L actually has a left as well as a right adjoint. We shall further call it *quintessential* if the left and right adjoints of L coincide (up to natural isomorphism): we devote the first section of this paper to studying the quintessential localizations of an arbitrary category $\mathcal{E}$. We shall show that, provided idempotents split in $\mathcal{E}$, its quintessential localizations correspond bijectively to idempotent natural endomorphisms of the identity functor on $\mathcal{E}$: in particular, they form a sub-semilattice of the lattice of all localizations of $\mathcal{E}$. Moreover, any quintessentially localizing subcategory is what we have chosen to call *persistently localizing*; that is, it is closed under subobjects in $\mathcal{E}$ (and under quotients).

In section 2, we specialize to the case when $\mathcal{E}$ is a topos, and consider local operators on $\mathcal{E}$ which correspond to persistently localizing subcategories. We provide a large number of equivalent characterizations of such local operators; we also show that, provided $\mathcal{E}$ is definable over a Boolean topos, every such localizing subcategory is coreflective as well as reflective in $\mathcal{E}$. However, in general the left and right adjoints of the inclusion fail to coincide, as we show by an example. We also consider the condition, for a local operator j, that every j-dense monomorphism should be split monic: we show that this implies that the localization is persistent, but it neither implies nor is implied by the assertion that it is quintessential.

1. Quintessential Localizations

Throughout this section, we assume that $\mathcal{E}$ is a Cauchy-complete (or Karoubian) category, that is one in which every idempotent endomorphism splits. We recall that, in any category, the monoid of natural endomorphisms of the identity functor is commutative: if α and β are such endomorphisms, then the commutativity of

$$\begin{tikzcd} A \arrow[r, "\alpha_A"] \arrow[d, "\beta_A"] & A \arrow[d, "\beta_A"] \\ A \arrow[r, "\alpha_A"] & A \end{tikzcd}$$

for each A is simply an instance of the naturality of α (or of β). It follows that the idempotent natural endomorphisms of the identity form a submonoid of this monoid (and indeeed a semilattice) under composition.

For future reference, we also note

1.1. LEMMA. *Suppose $\mathcal{E}$ has finite products, and let α be a natural endomorphism of $1_{\mathcal{E}}$. Then, given any internal algebraic structure carried by an object A of $\mathcal{E}$, α_A is a homomorphism for that structure. In particular, if $\mathcal{E}$ is a topos and α is idempotent, then α_Ω is a local operator on $\mathcal{E}$.*

PROOF. The naturality of α with respect to the product projections $A^n \to A$ tells us that α_{A^n} is simply $(\alpha_A)^n$; then, if $\omega\colon A^n \to A$ is an n-ary operation on A, naturality with respect to ω tells us that α_A is a homomorphism for it. The second assertion follows since two of the three conditions in the definition of a local operator $j : \Omega \to \Omega$ say simply that j is a meet-semilattice endomorphism of Ω, and the third is the idempotency of j. ∎

We now revert to consideration of an arbitrary Cauchy-complete category $\mathcal{E}$.

1.2. LEMMA. *Let ϵ be an idempotent natural endomorphism of the identity functor on $\mathcal{E}$. Then the full subcategory of objects A for which $\epsilon_A = 1_A$ is both reflective and coreflective in $\mathcal{E}$, and its reflector and coreflector coincide (up to natural isomorphism).*

PROOF. For each A, let us choose a splitting

$$A \overset{\alpha_A}{\twoheadrightarrow} LA \overset{\beta_A}{\rightarrowtail} A$$

of ϵ_A: then the naturality of ϵ makes L into a functor $\mathcal{E} \to \mathcal{E}$, in such a way that α and β become natural transformations. Moreover, the naturality of ϵ also ensures the commutativity of

$$\begin{tikzcd}
A \arrow[r, two heads, "\alpha_A"] \arrow[d, "\epsilon_A"] & LA \arrow[r, tail, "\beta_A"] \arrow[d, "\epsilon_{LA}"] & A \arrow[d, "\epsilon_A"] \\
A \arrow[r, two heads, "\alpha_A"] & LA \arrow[r, tail, "\beta_A"] & A
\end{tikzcd}$$

and either cell of this diagram forces ϵ_{LA} to be the identity; that is, the image of L is contained in the subcategory ($\mathcal{L}$, say) described in the statement of the Lemma. Moreover, α_A is an isomorphism iff $A \in \mathrm{ob}\,\mathcal{L}$, from which it follows easily that L is left adjoint to the inclusion $\mathcal{L} \to \mathcal{E}$ (with α as the unit of the adjunction); similarly, L is right adjoint to the inclusion, with β as the counit of this adjunction. ∎

Thus every idempotent natural endomorphism of $1_{\mathcal{E}}$ gives rise to a quintessential localization of $\mathcal{E}$. Conversely, suppose $\mathcal{L}$ is a quintessential localization of $\mathcal{E}$, with reflector/coreflector $L\colon \mathcal{E} \to \mathcal{L}$. Then, for any A, we have morphisms

$$A \xrightarrow{\alpha_A} LA \xrightarrow{\beta_A} A$$

which are respectively the unit of ($L \dashv$ inclusion) and the counit of (inclusion $\dashv L$). The composite $\epsilon_A = \beta_A\alpha_A$ is clearly the A-component of a natural endomorphism of $1_{\mathcal{E}}$. The composite $\theta_A = \alpha_A\beta_A$ need not be the identity; but, from the commutativity of

$$\begin{tikzcd}
LA \arrow[r, "L\alpha_A"] \arrow[d, "\beta_A"] & LLA \arrow[d, "\beta_{LA}"] \\
A \arrow[r, "\alpha_A"] & LA
\end{tikzcd}$$

and the idempotency of both adjunctions, we see that it is the A-component of a natural automorphism of the functor L. And the composite $\beta'_A = \beta_A(\theta_A)^{-1}$ has the same universal property as β_A; so, if we replace β by β', we may reduce to the case where ϵ is idempotent. We have thus established the main result of this section:

1.3. Theorem. *For any Cauchy-complete category $\mathcal{E}$, there is a bijection between the quintessential localizations of $\mathcal{E}$ and the idempotent natural endomorphisms of the identity functor on $\mathcal{E}$. Moreover, the quintessentially localizing subcategories of $\mathcal{E}$ form a semilattice under the operation of intersection.*

Proof. The only part which requires further comment is the last assertion. But if ϵ and δ are two idempotent natural endomorphisms of $1_{\mathcal{E}}$, it is clear that we have $\epsilon_A\delta_A = 1_A$ iff ϵ_A is epic and δ_A is monic, iff $\epsilon_A = \delta_A = 1_A$. So the semilattice operation of composition on idempotent endomorphisms of $1_{\mathcal{E}}$ becomes the operation of intersection on the corresponding localizing subcategories of $\mathcal{E}$. ∎

The second assertion of 1.3 should be compared with the results of [5] that, although the essentially localizaing subcategories of a suitably well-behaved category form a complete lattice, meets in this lattice (even binary ones) need not be simply intersections. In contrast, although the quintessentially localizing subcategories are always closed under finite intersections, they need not be closed under infinite intersections or even form a complete lattice, as the following example shows.

1.4. Example. Let $\mathcal{E}$ be the category of (left) M-sets, where M is a monoid. Given any element m of the centre of M, the mapping $a \mapsto m \cdot a$ defines an endomorphism μ_A of an arbitrary M-set A, which is clearly natural with respect to arbitrary M-equivariant maps. Conversely, if μ is any natural endomorphism of $1_{\mathcal{E}}$, let us define $m = \mu_M(1)$, where M acts on itself by left translations (and 1 denotes the identity element of M); then naturality tells us that we have $\mu_A(a) = m \cdot a$ for any element a of an arbitrary M-set A, and the M-equivariance of this map (in the case $A = M$) tells us that m must lie in the centre of M. Thus the monoid of natural endomorphisms of $1_{\mathcal{E}}$ is isomorphic to the centre of M; in particular, if M is commutative, it is isomorphic to M itself.

Specializing further to the case when M is a semilattice, we see that in this case every endomorphism of $1_{\mathcal{E}}$ is idempotent, and so corresponds to a quintessential localization of $\mathcal{E}$; moreover, the semilattice of quintessential localizations of $\mathcal{E}$ is isomorphic to M. So any semilattice can occur as a semilattice of quintessential localizations (indeed, as the semilattice of quintessential localizations of a topos).

Before leaving this section, we note two further consequences of Theorem 1.3.

1.5. Corollary. *Let $\mathcal{E}$ be a Cauchy-complete category, and $\mathcal{L}$ a quintessentially localizing subcategory of $\mathcal{E}$. Then $\mathcal{L}$ is closed under arbitrary subobjects and quotient objects in $\mathcal{E}$.*

Proof. By 1.3, we can identify $\mathcal{L}$ with the full subcategory of objects A such that $\epsilon_A = 1_A$, for some idempotent natural endomorphism ϵ of $1_{\mathcal{E}}$. But if this holds for A, and $m\colon A' \to A$ is monic, then we have $m\epsilon_{A'} = \epsilon_A m = m$ and hence $\epsilon_{A'} = 1_{A'}$, so A' is also in $\mathcal{L}$. The argument for quotients is dual. ∎

1.6. REMARK. If $\mathcal{E}$ is a topos, and ϵ is an idempotent endomorphism of $1_{\mathcal{E}}$, then ϵ_Ω is a local operator, as we saw in 1.1. It is easy to see that a monomorphism $A' \rightarrowtail A$ is dense for this local operator iff it contains the image of ϵ_A. Since ϵ_A and 1_A are equalized by the image of ϵ_A, it follows that an ϵ_Ω-separated object A must satisfy $\epsilon_A = 1_A$; but conversely any object satisfying this condition is an ϵ_Ω-sheaf. Thus $\mathbf{sh}_{\epsilon_\Omega}(\mathcal{E})$ is exactly the quintessentially localizing subcategory corresponding to ϵ. In particular, we note that an idempotent natural endomorphism of the identity functor on a topos is uniquely determined by its component at the single object Ω. (It would be interesting to know whether this statement remains true with the word 'idempotent' deleted.)

2. Persistent Local Operators

We now specialize to the case when $\mathcal{E}$ is a topos, and consider localizations of $\mathcal{E}$ which have the property that the localizing subcategory is closed under subobjects: as we observed in the Introduction, this is equivalent to saying that every separated object for the corresponding local operator j is a sheaf, and to saying that the separated reflector M_j preserves finite limits. For want of a better name, we shall call localizations (or local operators) with this property *persistent*; our next result collects together a large number of characterizations of persistent local operators, including the three just mentioned.

2.1. THEOREM. *For a local operator j on a topos $\mathcal{E}$, the following conditions are equivalent:*

(i) *Every j-separated object is a j-sheaf.*

(ii) $\mathbf{sh}_j(\mathcal{E})$ *is closed under subobjects in $\mathcal{E}$.*

(iii) *For every A, the unit map $A \to LA$ is epic.*

(iv) $\mathbf{sh}_j(\mathcal{E})$ *is closed under finite coproducts and quotients in $\mathcal{E}$.*

(v) $\mathbf{sep}_j(\mathcal{E})$ *is closed under finite coproducts and quotients in $\mathcal{E}$.*

(vi) *M_j preserves finite limits.*

(vii) *M_j preserves equalizers.*

(viii) $\mathbf{sep}_j(\mathcal{E})$ *is a topos.*

(ix) $\mathbf{sep}_j(\mathcal{E})$ *is a pretopos.*

(x) $\mathbf{sep}_j(\mathcal{E})$ *is balanced (that is, every morphism which is both monic and epic in* $\mathbf{sep}_j(\mathcal{E})$ *is an isomorphism).*

PROOF. (i) $\Leftrightarrow$ (ii) since, for any j, the j-separated objects are exactly the $\mathcal{E}$-subobjects of j-sheaves ([3], 3.29).

(ii) $\Leftrightarrow$ (iii) is a standard result on reflective subcategories (cf. [1], I 3.6.2).

(ii) $\Rightarrow$ (iv): First, the initial object 0 of $\mathcal{E}$ is a j-sheaf, because it is a subobject of 1. Next, if A and B are sheaves, their coproduct in $\mathbf{sh}_j(\mathcal{E})$ contains two subobjects isomorphic to A and B, whose intersection is 0, since the inclusion $\mathbf{sh}_j(\mathcal{E}) \to \mathcal{E}$ preserves monomorphisms and pullbacks. But this implies that it contains the coproduct $A \amalg B$ in $\mathcal{E}$ ([2], 1.621), and hence that the latter is a sheaf.

Now suppose $f\colon A \twoheadrightarrow B$ is an epimorphism in $\mathcal{E}$, where A is a j-sheaf. Let $R \rightrightarrows A$ be the kernel-pair of f; then R is a subobject of $A \times A$, so by (ii) it is a sheaf. Since the associated sheaf functor preserves finite limits, $R \rightrightarrows A$ is also the kernel-pair of $A \to B \to LB$; but the latter is epic by (iii). Since any epimorphism in $\mathcal{E}$ is the coequalizer of its kernel-pair ([3], 1.53), it follows that $B \cong LB$.

(iv) $\Rightarrow$ (ii): Let $f\colon A \rightarrowtail B$ be a monomorphism in $\mathcal{E}$, where B is a j-sheaf. Form its cokernel-pair $B \rightrightarrows C$; then C is a sheaf since it is a quotient of $B \amalg B$. But f is the equalizer of $B \rightrightarrows C$ ([3], 1.21); and $\mathbf{sh}_j(\mathcal{E})$ is closed under equalizers. So A is a sheaf.

(iv) $\Rightarrow$ (v) is immediate, given that (iv) implies (i).

(v) $\Rightarrow$ (x): (v) implies that the inclusion $\mathbf{sep}_j(\mathcal{E}) \to \mathcal{E}$ preserves cokernel-pairs and hence epimorphisms; but it always preserves monomorphisms. So $\mathbf{sep}_j(\mathcal{E})$ inherits balancedness from $\mathcal{E}$.

(i) $\Rightarrow$ (vi) is immediate since L_j preserves finite limits.

(vi) $\Leftrightarrow$ (vii) is immediate since M_j always preserves finite products.

(vi) $\Rightarrow$ (viii) is immediate from 'Giraud's little theorem' (cf. the Introduction).

(viii) $\Rightarrow$ (ix) is immediate from [2], 1.95.

(ix) $\Rightarrow$ (x) is immediate from [2], 1.652.

(x) $\Rightarrow$ (i): By [7], 43.6 and 45.8, $\mathbf{sep}_j(\mathcal{E})$ is a quasitopos, and its coarse objects are the j-sheaves. But in a balanced quasitopos every object is coarse. ■

2.2. REMARK. We note in particular that any localizing subcategory of a topos which is closed under subobjects is also closed under quotients, by condition (iv) of 2.1. However, the two halves of condition (iv) are independent. Any proper closed subtopos (cf. [3], 3.53) provides an example of a localizing subcategory which is closed under quotients but not under coproducts; and the subtopos of the Sierpiński topos $[\mathbf{2}, \mathbf{Set}]$ consisting of those objects $(f\colon A_0 \to A_1)$ for which f is an isomorphism is closed under coproducts (indeed, it is coreflective as well as reflective in $[\mathbf{2}, \mathbf{Set}]$) but not under quotients.

As we saw in 1.5, any quintessential localization of a topos is persistent. We shall see eventually that not every persistent localization is quintessential (or even essential); however, as one might guess from condition (iv) of 2.1, there is a strong tendency for the inclusion $\mathbf{sh}_j(\mathcal{E}) \to \mathcal{E}$ to have a right adjoint in this case. We have not been able to find a completely 'elementary' proof of this fact (that is, one that works entirely within the topos $\mathcal{E}$); but we have a proof using the ideas of indexed category theory, provided $\mathcal{E}$ can be indexed over another topos in such a way that the 'indexing objects' are all j-sheaves. In fact this condition is necessary as well as sufficient:

2.3. Proposition. *Let j be a persistent local operator on a topos $\mathcal{E}$. Then the inclusion $\mathbf{sh}_j(\mathcal{E}) \to \mathcal{E}$ has a right adjoint iff there exists a geometric morphism $p\colon \mathcal{E} \to \mathcal{S}$ such that p^*I is a j-sheaf for every object I of $\mathcal{S}$.*

Proof. The condition is necessary, since if the inclusion has a right adjoint then it is itself the inverse image of a geometric morphism with the required property.

Conversely, suppose the condition holds. Given an object B of $\mathcal{E}$, we shall write j_B for the local operator $j \times 1_B\colon \Omega \times B \to \Omega \times B$ on $\mathcal{E}/B$; it is easy to see that this operator satisfies the conditions of 2.1 if j does. Also, if B itself is a j-sheaf (for example, if $B = p^*I$ for some I), then the category of j_B-sheaves is simply the slice category $\mathbf{sh}_j(\mathcal{E})/B$.

Given an object A of $\mathcal{E}$, let $I = p_*(\Omega^A)$, and let $A' \rightarrowtail A \times p^*I$ be the generic I-indexed family of subobjects of A (i.e. the pullback of the membership relation $\in_A \rightarrowtail A \times \Omega^A$ along the counit map). As an $\mathcal{S}$-indexed subcategory of $\mathcal{E}$, $\mathbf{sh}_j(\mathcal{E})$ is definable: that is, given any object $f\colon B \to p^*I$ of $\mathcal{E}/p^*I$, we can find a subobject $I' \rightarrowtail I$ such that a morphism $x\colon K \to I$ factors through $I' \rightarrowtail I$ iff the pullback of f along $p^*(x)$ is a j_{p^*K}-sheaf in $\mathcal{E}/p^*K$. (To see this, note that f is a j_{p^*I}-sheaf iff it is j_{p^*I}-separated, iff the inclusion of the diagonal $B \rightarrowtail B \times_{p^*I} B$ in its j_{p^*I}-closure is an isomorphism; but isomorphisms are definable in $\mathcal{E}$, since it is locally small as an $\mathcal{S}$-indexed category.)

Applying this to A', we obtain a subobject $I' \rightarrowtail I$ and a subobject $A'' \rightarrowtail A \times p^*I'$ in $\mathcal{E}/p^*I'$, such that $(A'' \to p^*I')$ is a $j_{p^*I'}$-sheaf (and hence, since p^*I' is a j-sheaf, A'' itself is a j-sheaf). Let $RA \rightarrowtail A$ be the image of the composite $A'' \rightarrowtail A \times p^*I' \to A$; then RA is a j-sheaf by 2.1(iv). Moreover, any subobject of A which is a j-sheaf is obtainable as the pullback of A'' along a suitable morphism $1 \to p^*I'$, so RA contains all such subobjects; in other words, it is the largest subobject of A which is a sheaf. Again using the fact that epimorphic images of sheaves are sheaves, it follows that any morphism from a sheaf to A factors (uniquely) through $RA \rightarrowtail A$; so R defines a functor $\mathcal{E} \to \mathbf{sh}_j(\mathcal{E})$, right adjoint to the inclusion. ∎

2.4. Corollary. *Suppose there exists a geometric morphism $p\colon \mathcal{E} \to \mathcal{S}$ where $\mathcal{S}$ is a Boolean topos. Then, for every persistent local operator j on $\mathcal{E}$, the inclusion $\mathbf{sh}_j(\mathcal{E}) \to \mathcal{E}$ has a right adjoint.*

Proof. If $\mathcal{S}$ is Boolean, then every object of the form p^*I is decidable (i.e. has complemented diagonal) and hence separated for any dense local operator (i.e. any local operator for which 0 is a sheaf). But any persistent local operator is dense, since 0 is a subobject of the sheaf 1; and its separated objects are all sheaves. So the condition of 2.3 is satisfied. ∎

The assumption of Booleanness in 2.4 could be weakened to the condition (QD) studied in [4], since quotients of j-sheaves are j-sheaves. There are known examples of toposes which do not admit any geometric morphism to a Boolean topos or even to one satisfying (QD) (see [6]); but we do not know any example of a persistent local operator for which the inclusion $\mathbf{sh}_j(\mathcal{E}) \to \mathcal{E}$ fails to have a right adjoint.

In the situation of 2.3, we write $\alpha_A\colon A \twoheadrightarrow LA$ for the unit of the reflection, and $\beta_A\colon RA \rightarrowtail A$ for the counit of the coreflection. The composite $\alpha_A\beta_A$ is monic, since we can also factor it as

$$RA \xrightarrow{\alpha_{RA}} LRA \xrightarrow{L\beta_A} LA \ :$$

α_{RA} is an isomorphism since RA is a sheaf, and $L\beta_A$ is monic since L preserves monomorphisms. However, it is not an isomorphism in general, so we do not obtain a quintessential localization. Before giving a counterexample, we note a further property of local operators satisfying the conditions of 2.1:

2.5. Lemma. *In any topos $\mathcal{E}$, the class of persistent local operators is closed under any joins which exist in the lattice of local operators.*

Proof. We recall that joins of local operators correspond to intersections of the corresponding sheaf subcategories, by [3], exercise 3.9. So the result is immediate from condition (ii) (or from condition (iv)) of 2.1. ■

We saw that the quintessentially localizing subcategories of $\mathcal{E}$ form a semilattice under (finitary) intersection; but this semilattice need not be complete, by 1.4 (and even if it is, the infinite meets in it need not correspond to infinite intersections of localizing subcategories). This is the key to providing examples of persistent localizations which are not quintessential.

2.6. Example. Let M be any (join-)semilattice containing a non-principal ideal I, and let $\mathcal{E}$ be the topos of M-sets. Let $\mathcal{L}$ be the full subcategory of $\mathcal{E}$ consisting of those objects on which every element of I acts as the identity. Since $\mathcal{L}$ is the intersection of the quintessentially localizing subcategories corresponding to the elements of I, it is persistent by 2.5. But it cannot itself be quintessentially localizing, since I has no largest element. If we form the quotient of M by the semilattice congruence $\{(m,n) \mid (\exists i \in I)(m \vee i = n \vee i)\}$, we obtain an M-set A which lies in $\mathcal{L}$, but on which no element of $M \setminus I$ acts by the identity map. This object is in fact the reflection in $\mathcal{L}$ of M itself; its coreflection may be identified with the sub-M-set of A consisting of those congruence classes which have a greatest member (equivalently, contain an upper bound for I).

Another condition on local operators which is closely related to those of 2.1 is described in the next lemma.

2.7. Lemma. *For a local operator j on a topos $\mathcal{E}$, the following conditions are equivalent, and imply those of 2.1:*

(i) *Every object of $\mathcal{E}$ is injective for the class of j-dense monomorphisms.*

(ii) *Every j-dense monomorphism is split.*

Proof. If (i) holds and $m \colon A' \rightarrowtail A$ is j-dense, then injectivity of A' yields a splitting for m. Conversely if (ii) holds and we are given

$$A \xleftarrow{\ m\ } A' \xrightarrow{\ f\ } B$$

with m dense, then any splitting r for m yields an extension fr of f along m. And it is immediate that (i) implies condition (i) of 2.1, since a j-sheaf is exactly a j-separated object which is injective for j-dense monomorphisms. ■

However, the conditions of 2.7 are not equivalent either to those of 2.1 or to those of 1.3.

2.8. Example. As in 2.6, let M be a semilattice, I an ideal of M, $\mathcal{E}$ the topos of M-sets and $\mathcal{L}$ the persistently localizing subcategory of those M-sets on which every element of I acts as the identity. In 2.6 we saw that $\mathcal{L}$ is quintessentially localizing iff I is principal; we now show that (the local operator corresponding to) $\mathcal{L}$ satisfies the conditions of 2.7 iff I is well-ordered (in the semilattice ordering on M). Since it is easy to give examples of principal ideals which are not well-ordered, and of well-ordered ideals which are not principal, this shows that there is no implication in either direction between these two conditions.

First suppose that I is well-ordered. It is easy to see that an inclusion $A' \subseteq A$ of M-sets is j-dense (for the local operator j corresponding to $\mathcal{L}$) iff, for every $a \in A$, there exists $i \in I$ such that $i \cdot a \in A'$. Since I is well-ordered, for every a there exists a *least* $i_a \in I$ with this property; we define $r(a) = i_a \cdot a$. Then clearly r is a retraction of A onto A'; we must show that it is M-equivariant. Let $a \in A$ and $m \in M$: then clearly $i_{m \cdot a} \leq i_a$, since $i_a \cdot (m \cdot a) = m \cdot (i_a \cdot a) \in A'$. But we also have $i_a \leq (i_{m \cdot a} \vee m)$, since $(i_{m \cdot a} \vee m) \cdot a = i_{m \cdot a} \cdot (m \cdot a) \in A'$. So $(i_a \vee m) = (i_{m \cdot a} \vee m)$, from which it follows that $m \cdot r(a) = r(m \cdot a)$.

Now suppose I is not well-ordered; let I' be a nonempty subset of it with no least member. Let $M' = \{m \in M \mid (\exists i \in I')(i \leq m)\}$; then it is clear that M' is a sub-M-set of M, and that the inclusion $M' \rightarrowtail M$ is j-dense. Suppose we have a retraction $r \colon M \to M'$. Then $r(0)$ is not the least element of I', so we can find $i \in I'$ with $i \not\geq r(0)$. Now we have

$$i \vee r(0) \neq i = r(i) = r(i \vee 0) ,$$

so r is not M-equivariant. Thus the inclusion $M' \rightarrowtail M$ is not split monic in $\mathcal{E}$.

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