Theory and Applications of Categories, Vol. 3, No. 1, 1997, pp. 1–23.

HIGHER DIMENSIONAL PEIFFER ELEMENTS IN SIMPLICIAL COMMUTATIVE ALGEBRAS

Z.ARVASI AND T.PORTER
Transmitted by Ronald Brown

ABSTRACT. Let E be a simplicial commutative algebra such that E_n is generated by degenerate elements. It is shown that in this case the $n^{\rm th}$ term of the Moore complex of E is generated by images of certain pairings from lower dimensions. This is then used to give a description of the boundaries in dimension $n-1$ for $n = 2$, 3, and 4.

Introduction

Simplicial commutative algebras occupy a place somewhere between homological algebra, homotopy theory, algebraic K-theory and algebraic geometry. In each sector they have played a significant part in developments over quite a lengthy period of time. Their own internal structure has however been studied relatively little. The present article is one of a series in which we will study the n-types of simplicial algebras and will apply the results in various, mainly homological, settings. The pleasing, and we believe significant, result of this study is that simplicial algebras lend themselves very easily to detailed general calculations of structural maps and thus to a determination of a remarkably rich amount of internal structure. These calculations can be done by hand in low dimensions, but it seems likely that more general computations should be possible using computer aided calculations.

R.Brown and J-L.Loday [5] noted that if the second dimension G_2 of a simplicial group G, is generated by the degenerate elements, that is, elements coming from lower dimensions, then the image of the second term NG_2 of the Moore complex (NG, ∂) of G by the differential, ∂, is $[\mathrm{Ker}d_1, \mathrm{Ker}d_0]$ where the square brackets denote the commutator subgroup. An easy argument then shows that this subgroup of NG_1 is generated by elements of the form $(s_0d_0(y)x(s_0d_0y^{-1}))(yx^{-1}y^{-1})$ and that it is thus exactly the Peiffer subgroup of NG_1, the vanishing of which is equivalent to $\partial_1 : NG_1 \to NG_0$ being a crossed module.

It is clear that one should be able to develop an analogous result for other algebraic structures and in the case of commutative algebras, it is not difficult to see, cf. Arvasi [2] and section 3 (below), that if $\mathbf{E}$ is a simplicial algebra in which the subalgebra, E_2, is generated by the degenerate elements then the corresponding image is the ideal $\mathrm{Ker}d_1\mathrm{Ker}d_0$ in

Received by the editors 3 October 1996 and, in revised form, 7 January 1997.
Published on 24 January 1997
1991 Mathematics Subject Classification : 18G30, 18G55, 16E99 .
Key words and phrases: Simplicial commutative algebra, boundaries, Moore complex .

NE_1 and that it is generated by the elements $(x - s_0d_0x)y$ which give the analogous Peiffer *ideal* in the theory of crossed modules of algebras, (cf. Porter [14]). The vanishing of these elements is important in the construction of the cotangent complex of Lichtenbaum and Schlessinger, [13], and the simplicial version of the cotangent complex of Quillen [15], André [1] and Illusie [12]. It is natural to hope for higher dimensional analogues of this result and for an analysis and interpretation of the structure of the resulting elements in $NE_n,\ n \geq 2$.

We generalise the complete result for commutative algebras to dimensions $2, 3$ and 4 and get partial results in higher dimensions. The methods we use are based on ideas of Conduché, [8] and techniques developed by Carrasco and Cegarra [7]. In detail, this gives the following:

Let $\mathbf{E}$ be a simplicial commutative algebra with Moore complex $\mathbf{NE}$ and for $n > 1$, let D_n be the ideal generated by the degenerate elements in dimension n. If $E_n = D_n$, then

$$\partial_n(NE_n) = \partial_n(I_n) \quad \text{for all } n > 1$$

where I_n is an ideal in E_n (generated by a fairly small set of elements which will be explicitly given later on).

If $n = 2, 3$ or 4, then the ideal of boundaries of the Moore complex of the simplicial algebra $\mathbf{E}$ can be shown to be of the form

$$\partial_n(NE_n) = \sum_{I,J} K_I K_J$$

for $\emptyset \neq I, J \subset [n-1] = \{0, 1, \ldots, n-1\}$ with $I \cup J = [n-1]$, where

$$K_I = \bigcap_{i \in I} \mathrm{Ker} d_i \quad \text{and} \quad K_J = \bigcap_{j \in J} \mathrm{Ker} d_j.$$

This gives internal criteria for the vanishing of the higher Peiffer elements which yield conditions for various crossed algebra structures on the Moore complex. In general however for $n > 4$, we can only prove

$$\sum_{I,J} K_I K_J \subseteq \partial_n(NE_n)$$

but suspect the opposite inclusion holds as well.

These results are quite technical, being internal to the theory of simplicial algebras themselves. It is known [3], [14] that simplicial algebras lead to crossed modules and crossed complexes of algebras, that free crossed modules are related to Koszul complex constructions and higher dimensional analogues have been proposed by Ellis [9] for use in homotopical and homological algebra. In a sequel to this paper it will be shown how technical results found here facilitate the study of these aspects of crossed higher dimensional algebra, in particular by examining a suitable way of defining free 'crossed algebras' of various types.

Acknowledgement

This work was partially supported by the Royal Society ESEP programme in conjunction with TÜBİTAK, the Scientific and Technical Research Council of Turkey.

1. Definitions and preliminaries

In what follows 'algebras' will be commutative algebras over an unspecified commutative ring, $\mathbf{k}$, but for convenience are not required to have a multiplicative identity.

A simplicial (commutative) algebra $\mathbf{E}$ consists of a family of algebras $\{E_n\}$ together with face and degeneracy maps $d_i = d_i^n : E_n \to E_{n-1}, \quad 0 \leq i \leq n, \quad (n \neq 0)$ and $s_i = s_i^n : E_n \to E_{n+1}, \quad 0 \leq i \leq n$, satisfying the usual simplicial identities given in André [1] or Illusie [12] for example. It can be completely described as a functor $\mathbf{E}$: $\Delta^{op} \to \mathbf{CommAlg}_k$ where Δ is the category of finite ordinals $[n] = \{0 < 1 < \cdots < n\}$ and increasing maps.

Quillen [15] and Illusie [12] both discuss the basic homotopical algebra of simplicial algebras and their application in deformation theory. André [1] gives a detailed examination of their construction and applies them to cohomology via the cotangent complex construction. Another essential reference from our point of view is Carrasco's thesis, [6], where many of the basic techniques used here were developed systematically for the first time and the notion of hypercrossed complex was defined.

The following notation and terminology is derived from [6] and the published version, [7], of the analogous group theoretic case.

For the ordered set $[n] = \{0 < 1 < \ldots < n\}$, let $\alpha_i^n : [n+1] \to [n]$ be the increasing surjective map given by

$$\alpha_i^n(j) = \begin{cases} j & \text{if } \; j \leq i \\ j-1 & \text{if } \; j > i. \end{cases}$$

Let $S(n, n-r)$ be the set of all monotone increasing surjective maps from $[n]$ to $[n-r]$. This can be generated from the various α_i^n by composition. The composition of these generating maps is subject to the following rule $\alpha_j \alpha_i = \alpha_{i-1}\alpha_j, \ j < i$. This implies that every element $\alpha \in S(n, n-r)$ has a unique expression as $\alpha = \alpha_{i_1} \circ \alpha_{i_2} \circ \ldots \circ \alpha_{i_r}$ with $0 \leq i_1 < i_2 < \ldots < i_r \leq n-1$, where the indices i_k are the elements of $[n]$ such that $\{i_1, \ldots, i_r\} = \{i : \alpha(i) = \alpha(i+1)\}$. We thus can identify $S(n, n-r)$ with the set $\{(i_r, \ldots, i_1) \ : \ 0 \leq i_1 < i_2 < \ldots < i_r \leq n-1\}$. In particular, the single element of $S(n,n)$, defined by the identity map on $[n]$, corresponds to the empty 0-tuple ( ) denoted by $\emptyset_n$. Similarly the only element of $S(n, 0)$ is $(n-1, n-2, \ldots, 0)$. For all $n \geq 0$, let

$$S(n) = \bigcup_{0 \leq r \leq n} S(n, n-r).$$

We say that $\alpha = (i_r, \ldots, i_1) < \beta = (j_s, \ldots, j_1)$ in $S(n)$

$$\begin{array}{llll} \text{if} & i_1 = j_1, \ldots, i_k = j_k & \text{but} & i_{k+1} > j_{k+1} \ \ (k \geq 0) \text{ or} \\ \text{if} & i_1 = j_1, \ldots, i_r = j_r & \text{and} & r < s. \end{array}$$

This makes $S(n)$ an ordered set. For instance, the orders in $S(2)$ and in $S(3)$ are respectively:

$$\begin{aligned} S(2) &= \{\emptyset_2 < (1) < (0) < (1,0)\}; \\ S(3) &= \{\emptyset_3 < (2) < (1) < (2,1) < (0) < (2,0) < (1,0) < (2,1,0)\}. \end{aligned}$$

We also define $\alpha \cap \beta$ as the set of indices which belong to both α and β.

The Moore complex

The Moore complex $\mathbf{NE}$ of a simplicial algebra $\mathbf{E}$ is defined to be the differential graded module $(\mathbf{NE}, \partial)$ with

$$(\mathbf{NE})_n = \bigcap_{i=0}^{n-1} \mathrm{Ker} d_i$$

and with differential $\partial_n : NE_n \to NE_{n-1}$ induced from d_n by restriction.

The Moore complex has the advantage of being smaller than the simplicial algebra itself and being a differential graded module is of a better known form for manipulation. Its homology gives the homotopy groups of the simplicial algebra and thus in specific cases, e.g. a truncated free simplicial resolution of a commutative algebra, gives valuable higher dimensional information on syzygy-like elements.

The Moore complex, $\mathbf{NE}$, carries a hypercrossed complex structure (see Carrasco [6]) which allows the original $\mathbf{E}$ to be rebuilt. We recall briefly some of those aspects of this reconstruction that we will need later.

The Semidirect Decomposition of a Simplicial Algebra

The fundamental idea behind this can be found in Conduché [8]. A detailed investigation of it for the case of a simplicial group is given in Carrasco and Cegarra [7]. The algebra case of that structure is also given in Carrasco's thesis [6].

Given a split extension of algebras

$$0 \longrightarrow K \longrightarrow E \underset{s}{\overset{d}{\rightleftarrows}} R \longrightarrow 0$$

we write $E \cong K \rtimes s(R)$, the semidirect product of the ideal, K, with the image of R under the splitting s.

1.1. Proposition. *If* $\mathbf{E}$ *is a simplicial algebra, then for any* $n \geq 0$

$$\begin{aligned} E_n \cong\ & (\ldots(NE_n \rtimes s_{n-1}NE_{n-1}) \rtimes \ldots \rtimes s_{n-2}\ldots s_0 NE_1) \rtimes \\ & (\ldots(s_{n-2}NE_{n-1} \rtimes s_{n-1}s_{n-2}NE_{n-2}) \rtimes \ldots \rtimes s_{n-1}s_{n-2}\ldots s_0 NE_0). \end{aligned}$$

Proof. This is by repeated use of the following lemma. ∎

1.2. Lemma. *Let* $\mathbf{E}$ *be a simplicial algebra. Then* E_n *can be decomposed as a semidirect product:*

$$E_n \cong \mathrm{Ker} d_n^n \rtimes s_{n-1}^{n-1}(E_{n-1}).$$

Proof. The isomorphism can be defined as follows:

$$\begin{array}{rccl} \theta: & E_n & \longrightarrow & \mathrm{Ker} d_n^n \rtimes s_{n-1}^{n-1}(E_{n-1}) \\ & e & \longmapsto & (e - s_{n-1}d_n e, s_{n-1}d_n e). \end{array}$$

∎

The bracketting and the order of terms in this multiple semidirect product are generated by the sequence:

$$\begin{array}{rcl} E_1 & \cong & NE_1 \rtimes s_0 NE_0 \\ E_2 & \cong & (NE_2 \rtimes s_1 NE_1) \rtimes (s_0 NE_1 \rtimes s_1 s_0 NE_0) \\ E_3 & \cong & ((NE_3 \rtimes s_2 NE_2) \rtimes (s_1 NE_2 \rtimes s_2 s_1 NE_1))\rtimes \\ & & \qquad ((s_0 NE_2 \rtimes s_2 s_0 NE_1) \rtimes (s_1 s_0 NE_1 \rtimes s_2 s_1 s_0 NE_0)). \end{array}$$

and

$$\begin{array}{rcl} E_4 & \cong & (((NE_4 \rtimes s_3 NE_3) \rtimes (s_2 NE_3 \rtimes s_3 s_2 NE_2))\rtimes \\ & & \quad ((s_1 NE_3 \rtimes s_3 s_1 NE_2) \rtimes (s_2 s_1 NE_2 \rtimes s_3 s_2 s_1 NE_1)))\rtimes \\ & & \qquad s_0(\text{decomposition of } E_3). \end{array}$$

Note that the term corresponding to $\alpha = (i_r, \ldots, i_1) \in S(n)$ is

$$s_\alpha(NE_{n-\#\alpha}) = s_{i_r \ldots i_1}(NE_{n-\#\alpha}) = s_{i_r} \ldots s_{i_1}(NE_{n-\#\alpha}),$$

where $\#\alpha = r$. Hence any element $x \in E_n$ can be written in the form

$$x = y + \sum_{\alpha \in S(n)} s_\alpha(x_\alpha) \qquad \text{with } y \in NE_n \text{ and } x_\alpha \in NE_{n-\#\alpha}.$$

Crossed Modules of Commutative Algebras

Recall from [14] the notion of a crossed module of commutative algebras. Let $\mathbf{k}$ be a fixed commutative ring and let R be a $\mathbf{k}$-algebra with identity. *A crossed module of commutative algebras*, (C, R, ∂), is an R-algebra C, together with an action of R on C and an R-algebra morphism

$$\partial : C \longrightarrow R,$$

such that for all $c, c' \in C, r \in R$,

$$\text{CM1)} \quad \partial(r \cdot c) = r\partial c \qquad \text{CM2)} \quad \partial c \cdot c' = cc'.$$

The second condition (CM2) is called the *Peiffer identity.*

A standard example of a crossed module is any ideal I in R giving an inclusion map $I \to R$, which is a crossed module. Conversely, given any crossed module $\partial : C \to R$, the image $I = \partial C$ of C is an ideal in R.

2. Hypercrossed Complex Pairings and Boundaries in the Moore Complex

The following lemma is noted by Carrasco [6].

2.1. Lemma. *For a simplicial algebra* $\mathbf{E}$*, if* $0 \leq r \leq n$ *let* $\overline{NE}_n^{(r)} = \bigcap_{i \neq r} \mathrm{Ker} d_i$ *then the mapping*

$$\varphi : NE_n \longrightarrow \overline{NE}_n^{(r)}$$

in E_n*, given by*

$$\varphi(e) = e - \sum_{k=0}^{n-r-1} (-1)^{k+1} s_{r+k} d_n e,$$

is a bijection. ∎

This easily implies:

2.2. Lemma. *Given a simplicial algebra* $\mathbf{E}$*, then we have the following*

$$d_n(NE_n) = d_r(\overline{NE}_n^{(r)}).$$

∎

2.3. Proposition. *Let* $\mathbf{E}$ *be a simplicial algebra, then for* $n \geq 2$ *and nonempty* $I, J \subseteq [n-1]$ *with* $I \cup J = [n-1]$

$$(\bigcap_{i \in I} \mathrm{Ker} d_i)(\bigcap_{j \in J} \mathrm{Ker} d_j) \subseteq \partial_n NE_n.$$

Proof. For any $J \subset [n-1], J \neq \emptyset$, let r be the smallest element of J. If $r = 0$, then replace J by I and restart and if $0 \in I \cap J$, then redefine r to be the smallest nonzero element of J. Otherwise continue.

Letting $e_0 \in \bigcap_{j \in J} \mathrm{Ker} d_j$ and $e_1 \in \bigcap_{i \in I} \mathrm{Ker} d_i$, one obtains

$$d_i(s_{r-1} e_0 s_r e_1) = 0 \text{ for } i \neq r$$

and hence $s_{r-1} e_0 s_r e_1 \in \overline{NE}_n^{(r)}$. It follows that

$$e_0 e_1 = d_r(s_{r-1} e_0 s_r e_1) \in d_r(\overline{NE}_n^{(r)}) = d_n NE_n \quad \text{by the previous lemma,}$$

and this implies

$$(\bigcap_{i \in I} \mathrm{Ker} d_i)(\bigcap_{j \in J} \mathrm{Ker} d_j) \subseteq \partial_n NE_n.$$

∎

Writing the abbreviations,

$$K_I = \bigcap_{i \in I} \mathrm{Ker} d_i \quad \text{and} \quad K_J = \bigcap_{j \in J} \mathrm{Ker} d_j$$

then 2.3 implies:

2.4. Theorem. *For any simplicial algebra* $\mathbf{E}$ *and for* $n \geq 2$

$$\sum_{I,J} K_I K_J \subseteq \partial_n NE_n$$

for $\emptyset \neq I, J \subset [n-1]$ *and* $I \cup J = [n-1]$. ■

Truncated Simplicial Algebras and n-type Simplicial Algebras.

By a *n-truncated simplicial algebra of order n* or *n-type simplicial algebra*, we mean a simplicial algebra $\mathbf{E}'$ obtained by killing dimensions of order $> n$ in the Moore complex $\mathbf{NE}$ of some simplicial algebra, $\mathbf{E}$.

2.5. Corollary. *Let* $\mathbf{E}$ *be a simplicial algebra and let* $\mathbf{E}'$ *be the corresponding n-type simplicial algebra, so we have a canonical morphism* $\mathbf{E} \longrightarrow \mathbf{E}'$. *Then* $\mathbf{E}'$ *satisfies the following property:*

For all nonempty sets of indices $(I \neq J)$ $I, J \subset [n-1]$ *with* $I \cup J = [n-1]$,

$$\left(\bigcap_{j \in J} \mathrm{Ker} d_j^{n-1}\right)\left(\bigcap_{i \in I} \mathrm{Ker} d_i^{n-1}\right) = 0.$$

Proof. Since $\partial_n NE'_n = 0$, this follows from proposition 2.3. ■

Hypercrossed complex pairings

We recall from Carrasco [6] the construction of a family of $\mathbf{k}$-linear morphisms. We define a set $P(n)$ consisting of pairs of elements (α, β) from $S(n)$ with $\alpha \cap \beta = \emptyset$, where $\alpha = (i_r, \ldots, i_1), \beta = (j_s, \ldots, j_1) \in S(n)$. The $\mathbf{k}$-linear morphisms that we will need,

$$\{C_{\alpha,\beta} : NE_{n-\#\alpha} \otimes NE_{n-\#\beta} \longrightarrow NE_n : (\alpha, \beta) \in P(n), \quad n \geq 0\}$$

are given as composites by the diagrams

$$\begin{tikzcd} NE_{n-\#\alpha} \otimes NE_{n-\#\beta} \arrow[r, "C_{\alpha,\beta}"] \arrow[d, "s_\alpha \otimes s_\beta"] & NE_n \\ E_n \otimes E_n \arrow[r, "\mu"] & E_n \arrow[u, "p"] \end{tikzcd}$$

where $\otimes$ is the tensor product of $\mathbf{k}$-modules,

$$s_\alpha = s_{i_r} \ldots s_{i_1} : NE_{n-\#\alpha} \longrightarrow E_n \ , \quad s_\beta = s_{j_s} \ldots s_{j_1} : NE_{n-\#\beta} \longrightarrow E_n,$$

$p : E_n \to NE_n$ is defined by composite projections $p = p_{n-1} \dots p_0$ with

$$p_j = 1 - s_j d_j \qquad \text{with} \quad j = 0, 1, \dots n-1$$

and where $\mu : E_n \otimes E_n \to E_n$ denotes multiplication. Thus

$$\begin{aligned} C_{\alpha,\beta}(x_\alpha \otimes y_\beta) &= p\mu(s_\alpha \otimes s_\beta)(x_\alpha \otimes y_\beta) \\ &= p(s_\alpha(x_\alpha)s_\beta(y_\beta)) \\ &= (1 - s_{n-1}d_{n-1}) \dots (1 - s_0 d_0)(s_\alpha(x_\alpha)s_\beta(y_\beta)). \end{aligned}$$

We now define the ideal I_n to be that generated by all elements of the form

$$C_{\alpha,\beta}(x_\alpha \otimes y_\beta)$$

where $x_\alpha \in NE_{n-\#\alpha}$ and $y_\beta \in NE_{n-\#\beta}$ and for all $(\alpha, \beta) \in P(n)$.

Example. For $n = 2$, suppose $\alpha = (1)$, $\beta = (0)$ and $x, y \in NE_1 = \mathrm{Ker} d_0$. It follows that

$$\begin{aligned} C_{(1)(0)}(x \otimes y) &= p_1 p_0(s_1 x s_0 y) \\ &= s_1 x s_0 y - s_1 x s_1 y \\ &= s_1 x(s_0 y - s_1 y) \end{aligned}$$

and these give the generator elements of the ideal I_2.

For $n = 3$, the linear morphisms are the following

$$\begin{array}{lll} C_{(1,0)(2)}, & C_{(2,0)(1)}, & C_{(2,1)(0)}, \\ C_{(2)(0)}, & C_{(2)(1)}, & C_{(1)(0)}. \end{array}$$

For all $x \in NE_1$, $y \in NE_2$, the corresponding generators of I_3 are:

$$\begin{aligned} C_{(1,0)(2)}(x \otimes y) &= (s_1 s_0 x - s_2 s_0 x)s_2 y, \\ C_{(2,0)(1)}(x \otimes y) &= (s_2 s_0 x - s_2 s_1 x)(s_1 y - s_2 y), \\ C_{(2,1)(0)}(x \otimes y) &= s_2 s_1 x(s_0 y - s_1 y + s_2 y); \end{aligned}$$

whilst for all $x, \ y \in NE_2$,

$$\begin{aligned} C_{(1)(0)}(x \otimes y) &= s_1 x(s_0 y - s_1 y) + s_2(xy), \\ C_{(2)(0)}(x \otimes y) &= (s_2 x)(s_0 y), \\ C_{(2)(1)}(x \otimes y) &= s_2 x(s_1 y - s_2 y). \end{aligned}$$

In the following we analyse various types of elements in I_n and show that sums of them give elements that we want in giving an alternative description of $\partial_n NE_n$ in certain cases.

2.6. Proposition. *Let* $\mathbf{E}$ *be a simplicial algebra and* $n > 0$, *and* D_n *the ideal in* E_n *generated by degenerate elements. We suppose* $E_n = D_n$, *and let* I_n *be the ideal generated by elements of the form*

$$C_{\alpha,\beta}(x_\alpha \otimes y_\beta) \qquad \text{with } (\alpha, \beta) \in P(n)$$

where $x_\alpha \in NE_{n-\#\alpha}$, $y_\beta \in NE_{n-\#\beta}$. *Then*

$$\partial_n(NE_n) = \partial_n(I_n).$$

We defer the proof until we have some technical lemmas out of the way

2.7. LEMMA. *Given* $x_\alpha \in NE_{n-\#\alpha}$, $y_\beta \in NE_{n-\#\beta}$ *with* $\alpha = (i_r, \ldots, i_1)$, $\beta = (j_s, \ldots, j_1) \in S(n)$. *If* $\alpha \cap \beta = \emptyset$ *with* $\alpha < \beta$ *and* $u = s_\alpha(x_\alpha)s_\beta(y_\beta)$, *then*

(i) *if* $k \le j_1$, *then* $p_k(u) = u$,

(ii) *if* $k > j_s + 1$ *or* $k > i_r + 1$, *then* $p_k(u) = u$,

(iii) *if* $k \in \{i_1, \ldots, i_r, i_r + 1\}$ *and* $k = j_l + 1$ *for some* l, *then*

$$p_k(u) = s_\alpha(x_\alpha)s_\beta(y_\beta) - s_k(z_k),$$

for some $z_k \in E_{n-1}$,

(iv) *if* $k \in \{j_1, \ldots, j_s, j_s + 1\}$ *and* $k = i_m + 1$ *for some* m, *then*

$$p_k(u) = s_\alpha(x_\alpha)s_\beta(y_\beta) - s_k(z_k),$$

where $z_k \in E_{n-1}$ *and* $0 \le k \le n - 1$.

PROOF. Assuming $\alpha < \beta$ and $\alpha \cap \beta = \emptyset$ which implies $j_1 < i_1$. In the range $0 \le k \le j_1$,

$$\begin{aligned} p_k(u) &= s_\alpha(x_\alpha)s_\beta(y_\beta) - (s_k d_k s_\alpha x_\alpha)(s_k d_k s_\beta y_\beta) \\ &= s_\alpha(x_\alpha)s_\beta(y_\beta) - (s_k s_{i_r - 1} \ldots s_{i_1 - 1} d_k x_\alpha)(s_k d_k s_\beta y_\beta) \\ &= s_\alpha(x_\alpha)s_\beta(y_\beta) \qquad \text{since } d_k(x_\alpha) = 0. \end{aligned}$$

Similarly if $k > j_s + 1$, or if $k > i_r + 1$.

If $k \in \{i_1, \ldots, i_r, i_r + 1\}$ and $k = j_l + 1$ for some l, then

$$\begin{aligned} p_k(u) &= s_\alpha(x_\alpha)s_\beta(y_\beta) - s_k[d_k(s_\alpha(x_\alpha)s_\beta(y_\beta))] \\ &= s_\alpha(x_\alpha)s_\beta(y_\beta) - s_k(z_k) \end{aligned}$$

where $z_k = s_{\alpha'}(x_{\alpha'})s_{\beta'}(y_{\beta'}) \in E_{n-1}$ for new strings α', β' as is clear. The proof of (iv) is essentially the same so we will leave it out. ∎

2.8. LEMMA. *If* $\alpha \cap \beta = \emptyset$ *and* $\alpha < \beta$, *then*

$$p_{n-1} \ldots p_0(s_\alpha(x_\alpha)s_\beta(y_\beta)) = s_\alpha(x_\alpha)s_\beta(y_\beta) - \sum_{k=1}^{n-1} s_k(z_k)$$

where $z_k \in E_{n-1}$.

PROOF. We prove this by using induction on n. Write $u = s_\alpha(x_\alpha)s_\beta(y_\beta)$. For $n = 1$, it is clear to see that the equality is verified. We suppose that it is true for $n - 2$. It then follows that

$$\begin{aligned} p_{n-1} \ldots p_0(u) &= p_{n-1}(u - \sum_{k=1}^{n-2} s_k(z_k)) \\ &= p_{n-1}(u) - p_{n-1}(\sum_{k=1}^{n-2} s_k(z_k)) \end{aligned}$$

as p_{n-1} is a linear map. Next look at $p_{n-1}(u) = u - s_{n-1}(\underbrace{d_{n-1}u}_{z'}) = u - s_{n-1}(z')$ and

$$\begin{aligned} p_{n-1}(\sum_{k=1}^{n-2} s_k(z_k)) &= \sum_{k=1}^{n-2} s_k(z_k) - s_{n-1}(\underbrace{\sum_{k=1}^{n-2} d_{n-1}s_k(z_k)}_{z''}) \\ &= \sum_{k=1}^{n-2} s_k(z_k) - s_{n-1}(z''). \end{aligned}$$

Thus

$$\begin{aligned} p_{n-1}\dots p_0(u) &= u - \sum_{k=1}^{n-2} s_k(z_k) + s_{n-1}(\underbrace{z'' - z'}_{z_{n-1}}) \\ &= u - \sum_{k=1}^{n-2} s_k(z_k) + s_{n-1}(z_{n-1}) \\ &= u - \sum_{k=1}^{n-1} s_k(z_k). \end{aligned}$$

as required. ∎

Note that: For $x, y \in NE_{n-1}$, it is easy to see that

$$p_{n-1}\dots p_0(s_{n-1}(x)s_{n-2}(y)) = s_{n-1}(x)(s_{n-2}y - s_{n-1}y)$$

and taking the image of this element by d_n gives

$$d_n[s_{n-1}(x)(s_{n-2}y - s_{n-1}y)] = x(s_{n-2}d_{n-1}y - y)$$

which gives a Peiffer type element of dimension n.

2.9. Lemma. *Let $x_\alpha \in NE_{n-\#\alpha}$, $y_\beta \in NE_{n-\#\beta}$ with $\alpha, \beta \in S(n)$, then*

$$s_\alpha(x_\alpha)s_\beta(y_\beta) = s_{\alpha\cap\beta}(z_{\alpha\cap\beta})$$

where $z_{\alpha\cap\beta}$ has the form $(s_{\alpha'}x_\alpha)(s_{\beta'}y_\beta)$ and $\alpha' \cap \beta' = \emptyset$.

Proof. If $\alpha \cap \beta = \emptyset$, then this is trivially true. Assume $\#(\alpha\cap\beta) = t$, with $t \in I\!N$. Take $\alpha = (i_r, \dots, i_1)$ and $\beta = (j_s, \dots, j_1)$ with $\alpha \cap \beta = (k_t, \dots, k_1)$,

$$s_\alpha(x_\alpha) = s_{i_r}\dots s_{k_t}\dots s_{i_1}(x_\alpha) \quad \text{and} \quad s_\beta(y_\beta) = s_{j_s}\dots s_{k_t}\dots s_{j_1}(y_\beta).$$

Using repeatedly the simplicial axiom $s_e s_d = s_d s_{e-1}$ for $d < e$ until obtaining that $s_{k_t}\dots s_{k_1}$ is at the beginning of the string, one gets the following

$$s_\alpha(x_\alpha) = s_{k_t\dots k_1}(s_{\alpha'}x_\alpha) \quad \text{and} \quad s_\beta(y_\beta) = s_{k_t\dots k_1}(s_{\beta'}y_\beta).$$

Multiplying these expressions together gives

$$\begin{aligned} s_\alpha(x_\alpha)s_\beta(y_\beta) &= s_{k_t}\dots s_{k_1}(s_{\alpha'}x_\alpha)s_{k_t}\dots s_{k_1}(s_{\beta'}y_\beta) \\ &= s_{k_t\dots k_1}((s_{\alpha'}x_\alpha)(s_{\beta'}y_\beta)) \\ &= s_{\alpha\cap\beta}(z_{\alpha\cap\beta}), \end{aligned}$$

where $z_{\alpha\cap\beta} = (s_{\alpha'}x_\alpha)(s_{\beta'}y_\beta) \in E_{n-\#(\alpha\cap\beta)}$ and where $\alpha\backslash(\alpha\cap\beta) = \alpha'$, $\beta\backslash(\alpha\cap\beta) = \beta'$. Hence $\alpha' \cap \beta' = \emptyset$. Moreover $\alpha' < \alpha$ and $\beta' < \beta$ as $\#\alpha' < \#\alpha$ and $\#\beta' < \#\beta$. ∎

PROOF. (of Proposition 2.6) From proposition 1.3, E_n is isomorphic to

$$NE_n \rtimes s_{n-1}NE_{n-1} \rtimes s_{n-2}NE_{n-1} \rtimes \ldots \rtimes s_{n-1}s_{n-2}\ldots s_0NE_0,$$

where $NE_n = \bigcap_{i=0}^{n-1} \mathrm{Ker} d_i$ and $NE_0 = E_0$. Hence any element x in E_n can be written in the following form

$$x = e_n + s_{n-1}(x_{n-1}) + s_{n-2}(x'_{n-1}) + s_{n-1}s_{n-2}(x_{n-2}) + \ldots + s_{n-1}s_{n-2}\ldots s_0(x_0),$$

with $e_n \in NE_n$, $x_{n-1}, x'_{n-1} \in NE_{n-1}$, $x_{n-2} \in NE_{n-2}$, $x_0 \in NE_0$, etc.

We start by comparing I_n with NE_n. We show $NE_n = I_n$. It is enough to prove that, equivalently, any element in E_n/I_n can be written

$$s_{n-1}(x_{n-1}) + s_{n-2}(x'_{n-1}) + s_{n-1}s_{n-2}(x_{n-2}) + \ldots + s_{n-1}s_{n-2}\ldots s_0(x_0) + I_n$$

which implies, for any $b \in E_n$,

$$b + I_n = s_{n-1}(x_{n-1}) + s_{n-2}(x'_{n-1}) + \ldots + s_{n-1}s_{n-2}\ldots s_0(x_0) + I_n.$$

for some $x_{n-1} \in NE_{n-1}$ etc.

If $b \in E_n$, it is a sum of products of degeneracies so first of all assume it to be a product of degeneracies and that will suffice for the general case.

If b is itself a degenerate element, it is obvious that it is in some semidirect factor $s_\alpha(E_{n-\#\alpha})$. Assume therefore that provided an element b can be written as a product of $k-1$ degeneracies it has the desired form mod I_n, now for an element b which needs k degenerate elements

$$b = s_\beta(y_\beta)b' \qquad \text{with } y_\beta \in NE_{n-\#\beta}$$

where b' needs fewer than k and so

$$\begin{aligned} b + I_n &= s_\beta(y_\beta)(b' + I_n) \\ &= s_\beta(y_\beta)(s_{n-1}(x_{n-1}) + s_{n-2}(x'_{n-1}) + \ldots + s_{n-1}s_{n-2}\ldots s_0(x_0) + I_n) \\ &= \sum_{\alpha \in S(n)} s_\beta(y_\beta)s_\alpha(x_\alpha) + I_n. \end{aligned}$$

Next we ignore this summation and just look at the product

$$s_\alpha(x_\alpha)s_\beta(y_\beta) \qquad (*).$$

We check this product case by case as follows:

If $\alpha \cap \beta = \emptyset$, then by lemmas 2.7 and 2.8, there exists an element $s_\alpha(x_\alpha)s_\beta(y_\beta) - \sum_{k=1}^{n-1} s_k(z_k)$ in I_n with $z_k \in E_{n-1}$ and $k \in \alpha$ so that

$$s_\alpha(x_\alpha)s_\beta(y_\beta) \equiv \sum_{k=1}^{n-1} s_k(z_k) \bmod I_n.$$

If $\alpha \cap \beta \neq \emptyset$, then one gets, from lemma 2.9, the following

$$s_\alpha(x_\alpha)s_\beta(y_\beta) = s_{\alpha\cap\beta}(z_{\alpha\cap\beta})$$

where $z_{\alpha\cap\beta} = (s_{\alpha'}x_\alpha)(s_{\beta'}y_\beta) \in E_{n-t}$, with $t \in I\!N$. Since $\alpha' \cap \beta' = \emptyset$, we can use lemma 2.8 to form an equality

$$s_{\alpha'}(x_\alpha)s_{\beta'}(y_\beta) \equiv \sum_{k'=0}^{n-1} s_{k'}(z_{k'}) \mod I_n$$

where $z_{k'} \in E_{n-1}$. It then follows that

$$\begin{aligned} s_{\alpha\cap\beta}(z_{\alpha\cap\beta}) &= s_{\alpha\cap\beta}((s_{\alpha'}x_\alpha)(s_{\beta'}y_\beta)) \\ &\equiv \sum_{k'=0}^{n-1} s_{\alpha\cap\beta}s_{k'}(z_{k'}) \mod I_n. \end{aligned}$$

Thus we have shown that every product which can be formed in the required form is in I_n. Therefore $\partial_n(I_n) = \partial_n(NE_n)$. ■

3. Products of Kernels Elements and Boundaries in the Moore Complex

By way of illustration of potential applications of the above proposition we look at the case of $n = 2$.

Case $n = 2$

We know that any element e_2 of E_2 can be expressed in the form

$$e_2 = b + s_1y + s_0x + s_0u$$

with $b \in NE_2, x, y \in NE_1$ and $u \in s_0E_0$. We suppose $D_2 = E_2$. For $n = 1$, we take $\alpha = (1)$, $\beta = (0)$ and $x, y \in NE_1 =$Kerd_0. The ideal I_2 is generated by elements of the form

$$C_{(1)(0)}(x \otimes y) = s_1x(s_0y - s_1y).$$

The image of I_2 by ∂_2 is known to be Kerd_0Kerd_1 by direct calculation. Indeed,

$$\begin{aligned} d_2[C_{(1)(0)}(x \otimes y)] &= d_2[s_1x(s_0y - s_1y)] \\ &= x(s_0d_1y - y) \end{aligned}$$

where $x \in$ Kerd_0 and $(s_0d_1y - y)x \in$ Kerd_1 and all elements of Kerd_1 have this form due to lemma 2.1.

The bottom, $\partial : NE_1 \to NE_0$, of the Moore complex of $\mathbf{E}$ is always a precrossed module, that is it satisfies CM1 where $r \in NE_0$ operates on $c \in NE_1$ via s_0, $r \cdot c = s_0(r)c$. The elements $\partial y \cdot x - yx$ are called the Peiffer elements.

As ∂ is the restriction of ∂_1 to $NE1$, these are precisely the $d_2(C_{(1)(0)}(x \otimes y))$. In other words the ideal ∂I_2 is the 'Peiffer ideal' of the precrossed module $d_1 : NE_1 \to NE_0$, whose vanishing is equivalent to this being a crossed module. The description of ∂I_2

as $\mathrm{Ker}d_0\mathrm{Ker}d_1$ gives that its vanishing in this situation is module-like behaviour since a module, M, is an algebra with $MM = 0$. Thus if $(\mathbf{NE}, \partial)$ yields a crossed module this fact will be reflected in the internal structure of $\mathbf{E}$ by the vanishing of $\mathrm{Ker}d_0\mathrm{Ker}d_1$. Because the image of this $C_{(1)(0)}(x \otimes y)$ is the Peiffer element determined by x and y, we will call the $C_{\alpha,\beta}(x \otimes y)$ in general *higher dimensional Peiffer elements* and will seek similar internal conditions for their vanishing.

We have seen that in all dimensions

$$\sum_{I,J} K_I K_J \subseteq \partial_n(NE_n) = \partial I_n$$

and we will show shortly that this inclusion is an equality, not only in dimension 2 (as above), but in dimensions 3 and 4. The arguments are calculatory and do not generalise in an obvious way to higher dimensions although similar arguments can be used to get partial results there.

4. Case $n = 3$

The analogue of the '$\mathrm{Ker}d_0\mathrm{Ker}d_1$' result here is:

4.1. Proposition.

$$\partial_3(NE_3) = \sum_{I,J} K_I K_J + K_{\{0,1\}}K_{\{0,2\}} + K_{\{0,2\}}K_{\{1,2\}} + K_{\{0,1\}}K_{\{1,2\}}$$

where $I \cup J = [2],\ I \cap J = \emptyset$ *and*

$$\begin{aligned} K_{\{0,1\}}K_{\{0,2\}} &= (\mathrm{Ker}d_0 \cap \mathrm{Ker}d_1)(\mathrm{Ker}d_0 \cap \mathrm{Ker}d_2) \\ K_{\{0,2\}}K_{\{1,2\}} &= (\mathrm{Ker}d_0 \cap \mathrm{Ker}d_2)(\mathrm{Ker}d_1 \cap \mathrm{Ker}d_2) \\ K_{\{0,1\}}K_{\{1,2\}} &= (\mathrm{Ker}d_0 \cap \mathrm{Ker}d_1)(\mathrm{Ker}d_1 \cap \mathrm{Ker}d_2) \end{aligned}$$

Proof. By proposition 2.8, we know the generator elements of the ideal I_3 and $\partial_3(I_3) = \partial_3(NE_3)$. The image of all the listed generator elements of the ideal I_3 is summarised in the following table.

	α	β	I, J
1	(1,0)	(2)	{2} {0,1}
2	(2,0)	(1)	{1} {0,2}
3	(2,1)	(0)	{0} {1,2}
4	(2)	(1)	{0,1} {0,2}
5	(2)	(0)	{0,1} {1,2}+{0,1} {0,2}
6	(1)	(0)	{0,2} {1,2}+{0,1} {1,2}+{0,1} {0,2}

The explanation of this table is the following:

$\partial_3 C_{\alpha,\beta}(x \otimes y)$ is in $K_I K_J$ in the simple cases corresponding to the first 4 rows. In row 5, $\partial_3 C_{(2)(0)}(x \otimes y) \in K_{\{0,1\}}K_{\{1,2\}} + K_{\{0,1\}}K_{\{0,2\}}$ and similarly in row 6, the higher Peiffer element is in the sum of the indicated $K_I K_J$.

To illustrate the sort of argument used we look at the case of $\alpha = (1,0)$ and $\beta = (2)$, i.e. row 1. For $x \in NE_1$ and $y \in NE_2$,

$$\begin{aligned} d_3[C_{(1,0)(2)}(x \otimes y)] &= d_3[(s_1 s_0 x - s_2 s_0 x) s_2 y] \\ &= (s_1 s_0 d_1 x - s_0 x) y \end{aligned}$$

and so

$$d_3[C_{(1,0)(2)}(x \otimes y)] = (s_1 s_0 d_1 x - s_0 x) y \in \mathrm{Ker} d_2(\mathrm{Ker} d_0 \cap \mathrm{Ker} d_1).$$

We have denoted $\mathrm{Ker} d_2(\mathrm{Ker} d_0 \cap \mathrm{Ker} d_1)$ by $K_{\{2\}}K_{\{0,1\}}$ where $I = \{2\}$ and $J = \{0,1\}$.

Rows 2, 3 and 4 are similar.

For Row 5, $\alpha = (2),\ \beta = (0)$ with $x,\ y \in NE_2 = \mathrm{Ker} d_0 \cap \mathrm{Ker} d_1$,

$$\begin{aligned} d_3[C_{(2)(0)}(x \otimes y)] &= d_3[s_2 x s_0 y] \\ &= x s_0 d_2 y. \end{aligned}$$

We can assume, for $x,\ y \in NE_2$,

$$x \in \mathrm{Ker} d_0 \cap \mathrm{Ker} d_1 \quad \text{and} \quad y + s_0 d_2 y - s_1 d_2 y \in \mathrm{Ker} d_1 \cap \mathrm{Ker} d_2$$

and, multiplying them together,

$$\begin{aligned} x(y + s_0 d_2 y - s_1 d_2 y) &= xy + x s_0 d_2 y - x s_1 d_2 y \\ &= x(y - s_1 d_2 y) + x s_0 d_2 y \\ &= d_3[C_{(2)(1)}(x \otimes y)] + d_3[C_{(2)(0)}(x \otimes y)] \end{aligned}$$

and so

$$\begin{aligned} d_3[C_{(2)(0)}(x \otimes y)] &\in K_{\{0,1\}}K_{\{1,2\}} + d_3[C_{(2)(1)}(x \otimes y)] \\ &\subseteq K_{\{0,1\}}K_{\{1,2\}} + K_{\{0,1\}}K_{\{0,2\}}. \end{aligned}$$

For Row 6, for $\alpha = (1),\ \beta = (0)$ and $x,\ y \in NE_2 = \mathrm{Ker} d_0 \cap \mathrm{Ker} d_1$,

$$\begin{aligned} d_3[C_{(1)(0)}(x \otimes y)] &= d_3[s_1 x s_0 y - s_1 x s_1 y + s_2 x s_2 y] \\ &= s_1 d_2 x s_0 d_2 y - s_1 d_2 x s_1 d_2 y + xy \end{aligned}$$

We can take the following elements

$$(s_0 d_2 y - s_1 d_2 y + y) \in \mathrm{Ker} d_1 \cap \mathrm{Ker} d_2 \quad \text{and} \quad (s_1 d_2 x - x) \in \mathrm{Ker} d_0 \cap \mathrm{Ker} d_2.$$

When we multiply them together, we get

$$\begin{aligned} (s_0 d_2 y - s_1 d_2 y + y)(s_1 d_2 x - x) &= [s_0 d_2 y s_1 d_2 x - s_1 d_2 y s_1 d_2 x + yx] \\ &\quad -[x s_0 d_2 y] + [x(s_1 d_2 y - y)] \\ &\quad +[y(s_1 d_2 x - x)] \\ &= d_3[C_{(1)(0)}(x \otimes y)] - d_3[C_{(2)(0)}(x \otimes y)] + \\ &\quad d_3[C_{(2)(1)}(x \otimes y) + C_{(2)(1)}(y \otimes x)] \end{aligned}$$

and hence

$$d_3[C_{(1)(0)}(x \otimes y)] \quad \in \quad K_{\{0,2\}}K_{\{1,2\}} + K_{\{0,1\}}K_{\{1,2\}} + K_{\{0,1\}}K_{\{0,2\}}.$$

So we have shown

$$\partial_3 I_3 \subseteq \sum_{I,J} K_I K_J + K_{\{0,1\}}K_{\{0,2\}} + K_{\{0,2\}}K_{\{1,2\}} + K_{\{0,1\}}K_{\{1,2\}}.$$

The opposite inclusion can be verified by using proposition 2.3. Therefore

$$\begin{aligned}\partial_3(NE_3) = \quad & \mathrm{Ker}d_2(\mathrm{Ker}d_0 \cap \mathrm{Ker}d_1) + \mathrm{Ker}d_1(\mathrm{Ker}d_0 \cap \mathrm{Ker}d_2) + \\ & \mathrm{Ker}d_0(\mathrm{Ker}d_1 \cap \mathrm{Ker}d_2) + (\mathrm{Ker}d_0 \cap \mathrm{Ker}d_1)(\mathrm{Ker}d_0 \cap \mathrm{Ker}d_2) + \\ & (\mathrm{Ker}d_1 \cap \mathrm{Ker}d_2)(\mathrm{Ker}d_0 \cap \mathrm{Ker}d_2) + (\mathrm{Ker}d_1 \cap \mathrm{Ker}d_2)(\mathrm{Ker}d_0 \cap \mathrm{Ker}d_1)\end{aligned}$$

This completes the proof of the proposition. ∎

5. Illustrative Application: 2-Crossed Modules of Algebras

5.1. Definition. (cf. [10]) *A 2-crossed module of* $\mathbf{k}$*-algebras consists of a complex of* C_0 *-algebras*

$$C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

and ∂_2, ∂_1 *morphisms of* C_0*-algebras, where the algebra* C_0 *acts on itself by multiplication such that*

$$C_2 \xrightarrow{\partial_2} C_1$$

is a crossed module in which C_1 *acts on* C_2 *via* C_0*, (we require thus that for all* $x \in C_2,\ y \in C_1$ *and* $z \in C_0$ *that* $(xy)z = x(yz)$*), further, there is a* C_0*-bilinear function*

$$\{\quad \otimes \quad\} : C_1 \otimes_{C_0} C_1 \longrightarrow C_2,$$

called a Peiffer lifting, which satisfies the following axioms:

$$\begin{array}{llrcl} PL1: & & \partial_2\{y_0 \otimes y_1\} & = & y_0y_1 - y_0 \cdot \partial_1(y_1), \\ PL2: & & \{\partial_2(x_1) \otimes \partial_2(x_2)\} & = & x_1x_2, \\ PL3 & & \{y_0 \otimes y_1y_2\} & = & \{y_0y_1 \otimes y_2\} + \partial_1 y_2 \cdot \{y_0 \otimes y_1\}, \\ PL4: & a) & \{\partial_2(x) \otimes y\} & = & y \cdot x - \partial_1(y) \cdot x, \\ & b) & \{y \otimes \partial_2(x)\} & = & y \cdot x, \\ PL5: & & \{y_0 \otimes y_1\} \cdot z & = & \{y_0 \cdot z \otimes y_1\} = \{y_0 \otimes y_1 \cdot z\}, \end{array}$$

for all $x, x_1, x_2 \in C_2,\ y,\ y_0,\ y_1,\ y_2 \in C_1$ *and* $z \in C_0$*.*

We denote such a 2-crossed module of algebras by $\{C_2, C_1, C_0, \partial_2, \partial_1\}$.

5.2. PROPOSITION. *Let* $\mathbf{E}$ *be a simplicial algebra with the Moore complex* $\mathbf{NE}$. *Then the complex of algebras*

$$NE_2/\partial_3(NE_3\cap D_3) \xrightarrow{\overline{\partial}_2} NE_1 \xrightarrow{\partial_1} NE_0$$

is a 2-crossed module of algebras, where the Peiffer map is defined as follows:

$$\begin{aligned} \{\ \otimes\ \}:\ NE_1\otimes NE_1 &\longrightarrow NE_2/\partial_3(NE_3\cap D_3)\\ (y_0\otimes y_1) &\longmapsto \overline{s_1y_0(s_1y_1-s_0y_1)}. \end{aligned}$$

Here the right hand side denotes a coset in $NE_2/\partial_3(NE_3\cap D_3)$ *represented by the corresponding element in* NE_2.

PROOF. We will show that all axioms of a 2-crossed module are verified. It is readily checked that the morphism $\overline{\partial}_2 : NE_2/\partial_3(NE_3\cap D_3)\to NE_1$ is a crossed module. (In the following calculations we display the elements omitting the overlines.)

PL1:

$$\begin{aligned} \overline{\partial}_2\{y_0\otimes y_1\} &= \partial_2(s_1y_0(s_1y_1-s_0y_1))\\ &= y_0y_1-y_0\cdot\partial_1y_1. \end{aligned}$$

PL2: From $\partial_3(C_{(1)(0)}(x_1\otimes x_2)) = s_1d_2(x_1)s_0d_2(x_2)-s_1d_2(x_1)s_1d_2(x_2)+x_1x_2$, one obtains

$$\begin{aligned} \{\overline{\partial}_2(x_1)\otimes\overline{\partial}_2(x_2)\} &= s_1d_2x_1(s_1d_2x_2-s_0d_2x_2)\\ &\equiv x_1x_2 \mod \partial_3(NE_3\cap D_3). \end{aligned}$$

PL3:

$$\begin{aligned} \{y_0\otimes y_1y_2\} &= s_1y_0[s_1(y_1y_2)-s_0(y_1y_2)]\\ &= s_1y_0[s_1y_1(s_1y_2-s_0y_2)]+[s_1y_0(s_1y_1-s_0y_1)]s_0y_2\\ &= s_1(y_0y_1)(s_1y_2-s_0y_2)+\{y_0\otimes y_1\}s_0y_2 \end{aligned}$$

but $\partial_3(C_{(1,0)(2)}(y\otimes x)) = (s_1s_0d_1y-s_0y)x$, so this implies

$$\begin{aligned} \{y_0\otimes y_1y_2\} &\equiv s_1(y_0y_1)(s_1y_2-s_0y_2)+s_1s_0d_1(y_2)\{y_0\otimes y_1\} \mod \partial_3(NE_3\cap D_3)\\ &= \{y_0y_1\otimes y_2\}+\partial_1y_2\cdot\{y_0\otimes y_1\} \text{ by the definition of the action.} \end{aligned}$$

PL4: a)

$$\{\overline{\partial}_2(x)\otimes y\} = s_1\partial_2x(s_1y-s_0y),$$

but

$$\partial_3(C_{(2,0)(1)}(y\otimes x)) = (s_0y-s_1y)s_1d_2x-(s_0y-s_1y)x\in\partial_3(NE_3\cap D_3)$$

and

$$\partial_3(C_{(1,0)(2)}(y\otimes x)) = (s_1s_0d_1y-s_0y)x\in\partial_3(NE_3\cap D_3),$$

so then

$$\begin{array}{rcll}\{\overline{\partial}_2(x)\otimes y\} & \equiv & s_1(y)x - s_0(y)x & \quad \text{mod } \partial_3(NE_3\cap D_3)\\ & = & y\cdot x - \partial_1(y)\cdot x & \quad \text{by the definition of the action,}\end{array}$$

b) since $\partial_3(C_{(2,1)(0)}(y\otimes x)) = s_1y(s_0d_2x - s_1d_2x) + s_1(y)x$,

$$\begin{array}{rcl}\{y\otimes\overline{\partial}_2(x)\} & = & s_1y(s_1\partial_2x - s_0\partial_2x)\\ & \equiv & s_1(y)x \quad \text{mod } \partial_3(NE_3\cap D_3)\\ & = & y\cdot x \qquad \text{by the definition of the action.}\end{array}$$

PL5:

$$\begin{array}{rcl}\{y_0\otimes y_1\}\cdot z & = & (s_1y_0(s_1y_1 - s_0y_1))\cdot z\\ & = & s_1(s_0(z)y_0)(s_1y_1 - s_0y_1)\\ & = & s_1(y_0\cdot z)(s_1y_1 - s_0y_1) \quad \text{by the definition of the action}\\ & = & \{y_0\cdot z\otimes y_1\}.\end{array}$$

Clearly the same sort of argument works for

$$\{y_0\cdot z\otimes y_1\} = \{y_0\otimes y_1\cdot z\}$$

with $x, x_1, x_2 \in NE_2/\partial_3(NE_3\cap D_3)$, $y, y_0, y_1, y_2 \in NE_1$ and $z\in NE_0$. This completes the proof of the proposition. ∎

This only used the higher dimensional Peiffer elements. A result in terms of K_IK_J vanishing can also be given:

5.3. PROPOSITION. *If in a simplicial algebra* $\mathbf{E}$, $K_IK_J = 0$ *in the following cases:* $I\cup J = [2]$, $I\cap J = \emptyset$; $I = \{0,1\}$ $J = \{0,2\}$ *or* $I = \{1,2\}$*; and* $I = \{0,2\}$, $J = \{1,2\}$*, then*

$$NE_2 \longrightarrow NE_1 \longrightarrow NE_0$$

can be given the structure of a 2-crossed module. ∎

6. The case $n = 4$

With dimension 4, the situation is more complicated, but is still manageable.

6.1. PROPOSITION.

$$\partial_4(NE_4) = \sum_{I,J} K_IK_J$$

where $I\cup J = [3]$, $I = [3]-\{\alpha\}$, $J = [3]-\{\beta\}$ *and* $(\alpha,\beta)\in P(4)$.

PROOF. There is a natural isomorphism

$$
\begin{aligned}
E_4 \quad \cong \quad & NE_4 \rtimes s_3NE_3 \rtimes s_2NE_3 \rtimes s_3s_2NE_2 \rtimes s_1NE_3 \rtimes \\
& s_3s_1NE_2 \rtimes s_2s_1NE_2 \rtimes s_3s_2s_1NE_1 \rtimes s_0NE_3 \rtimes \\
& s_3s_0NE_2 \rtimes s_2s_0NE_2 \rtimes s_3s_2s_0NE_1 \rtimes \\
& s_1s_0NE_2 \rtimes s_3s_1s_0NE_1 \rtimes s_3s_2s_1s_0NE_0.
\end{aligned}
$$

We firstly see what the generator elements of the ideal I_4 look like. One gets

$$
\begin{aligned}
S(4) \quad = \quad & \{\emptyset_4 < (3) < (2) < (3,2) < (1) < (3,1) < (2,1) < (3,2,1) < (0) < \\
& (3,0) < (2,0) < (3,2,0) < (1,0) < (3,1,0) < (3,2,1,0)\}.
\end{aligned}
$$

The bilinear morphisms are the following:

$$
\begin{array}{llll}
C_{(3,2,1)(0)} & C_{(3,2,0)(1)} & C_{(3,1,0)(2)} & C_{(2,1,0)(3)} \\
C_{(3,2)(1,0)} & C_{(3,1)(2,0)} & C_{(3,0)(2,1)} & C_{(3,2)(1)} \\
C_{(3,2)(0)} & C_{(3,1)(2)} & C_{(3,1)(0)} & C_{(3,0)(2)} \\
C_{(3,0)(1)} & C_{(2,1)(3)} & C_{(2,1)(0)} & C_{(2,0)(3)} \\
C_{(2,0)(1)} & C_{(1,0)(3)} & C_{(1,0)(2)} & C_{(3)(2)} \\
C_{(3)(1)} & C_{(3)(0)} & C_{(2)(1)} & C_{(2)(0)} \\
C_{(1)(0)}. & & &
\end{array}
$$

For $x_1, y_1 \in NE_1$, $x_2, y_2 \in NE_2$ and $x_3, y_3 \in NE_3$, the generator elements of the ideal I_4 are

$$
\begin{array}{rrcl}
1) & C_{(3,2,1)(0)}(x_1 \otimes y_3) & = & s_3s_2s_1x_1(s_0y_3 - s_1y_3 + s_2y_3 - s_3y_3) \\
2) & C_{(3,2,0)(1)}(x_1 \otimes y_3) & = & (s_3s_2s_0x_1 - s_1s_2s_1x_1)(s_1y_3 - s_2y_3 + s_3y_3) \\
3) & C_{(3,1,0)(2)}(x_1 \otimes y_3) & = & (s_3s_1s_0x_1 - s_2s_2s_0x_1)(s_2y_3 - s_3y_3) \\
4) & C_{(2,1,0)(3)}(x_1 \otimes y_3) & = & (s_2s_1s_0x_1 - s_3s_1s_0x_1)s_3y_3 \\
5) & C_{(3,2)(1,0)}(x_2 \otimes y_2) & = & (s_1s_0x_2 - s_2s_0x_2 + s_3s_0x_2)s_3s_2y_2 \\
6) & C_{(3,1)(2,0)}(x_2 \otimes y_2) & = & (s_3s_1x_2 - s_3s_0x_2 + s_2s_0x_2 - s_1s_1x_2) \\
 & & & (s_3s_1y_2 - s_3s_2y_2) \\
7) & C_{(3,0)(2,1)}(x_2 \otimes y_2) & = & (s_2s_1x_2 - s_3s_1x_2)(s_3s_0y_2 - s_1s_2y_2 + s_2s_2y_2) \\
8) & C_{(3,2)(1)}(x_2 \otimes y_3) & = & s_3s_2x_2(s_1y_3 - s_2y_3 + s_3y_3) \\
9) & C_{(3,2)(0)}(x_2 \otimes y_3) & = & s_3s_2x_2s_0y_3 \\
10) & C_{(3,1)(2)}(x_2 \otimes y_3) & = & (s_2y_3 - s_3y_3)(s_3s_1x_2 - s_2s_2x_2) \\
11) & C_{(3,1)(0)}(x_2 \otimes y_3) & = & s_3s_1x_2(s_0y_3 - s_1y_3) + s_3s_2x_2(s_2y_3 - s_3y_3) \\
12) & C_{(3,0)(2)}(x_2 \otimes y_3) & = & s_3s_0x_2(s_2y_3 - s_3y_3) \\
13) & C_{(3,0)(1)}(x_2 \otimes y_3) & = & s_1y_3(s_3s_0x_2 - s_1s_2x_2) + s_2s_2x_2(s_2y_3 - s_3y_3) \\
14) & C_{(2,1)(3)}(x_2 \otimes y_3) & = & (s_2s_1x_2 - s_3s_1x_2)s_3y_3 \\
15) & C_{(2,1)(0)}(x_2 \otimes y_3) & = & s_2s_1x_2(s_0y_3 - s_1y_3 + s_2y_3) + s_3s_1x_2s_3y_3 \\
16) & C_{(2,0)(3)}(x_2 \otimes y_3) & = & (s_2s_0x_2 - s_3s_0x_2)s_3y_3 \\
17) & C_{(2,0)(1)}(x_2 \otimes y_3) & = & (s_2s_0x_2 - s_1s_1x_2)(s_1y_3 - s_2y_3) + \\
 & & & (s_3s_1x_2 - s_3s_0x_2)s_3y_3 \\
18) & C_{(1,0)(3)}(x_2 \otimes y_3) & = & s_1s_0x_2s_3y_3 \\
19) & C_{(1,0)(2)}(x_2 \otimes y_3) & = & (s_1s_0x_2 - s_2s_0x_2)s_2y_3 + s_3s_0x_2s_3y_3 \\
20) & C_{(3)(2)}(x_3 \otimes y_3) & = & s_3x_3(s_2y_3 - s_3y_3) \\
21) & C_{(3)(1)}(x_3 \otimes y_3) & = & s_3x_3s_1y_3 \\
22) & C_{(3)(0)}(x_3 \otimes y_3) & = & s_3x_3s_0y_3 \\
23) & C_{(2)(1)}(x_3 \otimes y_3) & = & s_2x_3(s_1y_3 - s_2y_3) + s_3(x_3y_3) \\
24) & C_{(2)(0)}(x_3 \otimes y_3) & = & s_2x_3s_0y_3 \\
25) & C_{(1)(0)}(x_3 \otimes y_3) & = & s_1x_3(s_0y_3 - s_1y_3) + s_2(x_3y_3) - s_3(x_3y_3)
\end{array}
$$

By proposition 2.8, we have $\partial_4(NE_4) = \partial_4(I_4)$. We take an image by ∂_4 of each $C_{\alpha,\beta}$, where $\alpha, \beta \in P(4)$. We summarise the images of all generator elements, which are listed early on, in the subsequent table.

	α	β	I,J
1	(3,2,1)	(0)	{0}{1,2,3}
2	(3,2,0)	(1)	{1}{0,2,3}
3	(3,1,0)	(2)	{2}{0,1,3}
4	(2,1,0)	(3)	{3}{0,1,2}
5	(3,2)	(1,0)	{0,1}{2,3}
6	(3,1)	(2,0)	{0,2}{1,3}
7	(3,0)	(2,1)	{1,2}{0,3}
8	(3,2)	(1)	{0,1}{0,2,3}
9	(3,2)	(0)	{0,1}{1,2,3}+{0,1}{0,2,3}
10	(3,1)	(2)	{0,2}{0,1,3}
11	(3,1)	(0)	{0,2}{1,2,3}+{0,2}{0,1,3}+{0,1}{1,2,3}+{0,1}{0,2,3}
12	(3,0)	(2)	{1,2}{0,1,3}+{0,2}{0,1,3}
13	(3,0)	(1)	{1,2}{0,2,3}+ {0,1}{0,2,3}+{1,2}{0,1,3}+{0,2}{0,1,3}
14	(2,1)	(3)	{0,3}{0,1,2}
15	(2,1)	(0)	{0,3}{1,2,3}+{0,3}{0,1,2}+{0,2}{1,2,3}+{0,2}{0,1,3}
16	(2,0)	(3)	{1,3}{0,1,2}+{0,3}{0,1,2}
17	(2,0)	(1)	{1,3}{0,2,3}+{0,3}{0,1,2}+{1,3}{0,1,2}+{1,2}{0,2,3}+
			{0,2}{0,1,3}+{1,2}{0,1,3}
18	(1,0)	(3)	{2,3}{0,1,2}+{1,3}{0,1,2}
19	(1,0)	(2)	{2,3}{0,1,3}+{1,2}{0,1,3}+{1,3}{0,1,2}+{2,3}{0,1,2}
20	(3)	(2)	{0,1,2}{0,1,3}
21	(3)	(1)	{0,1,2}{0,2,3}+{0,1,2}{0,1,3}
22	(3)	(0)	{0,1,2}{1,2,3}+{0,1,2}{0,2,3}+{0,1,2}{0,1,3}
23	(2)	(1)	{0,1,3}{0,2,3}+{0,1,2}{1,2,3}+{0,1,2}{0,2,3}+
			{0,1,2}{0,1,3}
24	(2)	(0)	{0,1,3}{1,2,3}+{0,1,3}{0,2,3}+{0,1,2}{1,2,3}+
			{0,1,2}{0,2,3}+{0,1,2}{0,1,3}
25	(1)	(0)	{0,2,3}{1,2,3}+{0,1,3}{1,2,3}+{0,1,3}{0,2,3}+
			{0,1,2}{1,2,3}+{0,1,2}{0,2,3}+{0,1,2}{0,1,3}

As the proofs are largely similar to those for $n = 3$ we leave most to the reader, limiting ourselves to one or two of the more complex cases by way of illustration.

Row: 17

$$\begin{aligned} d_4[C_{(2,0)(1)}(x_2 \otimes y_3)] &= (s_2 s_0 d_2 x_2 - s_1 s_1 d_2 x_2)(s_1 d_3 y_3 - s_2 d_3 y_3) \\ &\quad + y_3(s_1 x_2 - s_0 x_2). \end{aligned}$$

Take elements

$a = (s_2 s_0 d_2 x_2 - s_0 x_2 + s_1 x_2 - s_2 s_1 d_2 x_2) \in K_{\{1,3\}}$ and $b = (s_1 d_3 y_3 - s_2 d_3 y_3 + y_3) \in K_{\{0,2,3\}}$,

$$\begin{array}{rcl} d_4[C_{(2,0)(1)}(x_2\otimes y_3)] & = & ab - d_4[C_{(2,1)(3)}(x_2\otimes y_3)] + \\ & & d_4[C_{(2,0)(3)}(x_2\otimes y_3)] + d_4[C_{(3,0)(1)}(x_2\otimes y_3)] - \\ & & d_4[C_{(3,1)(2)}(x_2\otimes y_3)] + d_4[C_{(3,0)(2)}(x_2\otimes y_3)] \\ & \in & K_{\{1,3\}}K_{\{0,2,3\}} + K_{\{0,3\}}K_{\{0,1,2\}} + \\ & & K_{\{1,3\}}K_{\{0,1,2\}} + K_{\{1,2\}}K_{\{0,2,3\}} + \\ & & K_{\{0,2\}}K_{\{0,1,3\}} + K_{\{1,2\}}K_{\{0,1,3\}} \end{array}$$

by other results from earlier rows.

Row: 20

$$\begin{array}{rcl} d_4[C_{(3)(2)}(x_3\otimes y_3)] & = & x_3(s_2d_3y_3 - y_3) \\ & \in & (\mathrm{Ker}d_0\cap\mathrm{Ker}d_1\cap\mathrm{Ker}d_2)(\mathrm{Ker}d_0\cap\mathrm{Ker}d_1\cap\mathrm{Ker}d_3) \\ & = & K_{\{0,1,2\}}K_{\{0,1,3\}}. \end{array}$$

Row: 21

$$d_4[C_{(3)(1)}(x_3\otimes y_3)] = x_3s_1d_3(y_3).$$

Take elements $x_3\in NE_3 = K_{\{0,1,2\}}$ and $(s_1d_3y_3 - s_2d_3y_3 + y_3)\in K_{\{0,2,3\}}$. When multiplying them together, one gets

$$\begin{array}{rcl} d_4[C_{(3)(1)}(x_3\otimes y_3)] & \in & d_4[C_{(3)(2)}(x_3\otimes y_3)] + K_{\{0,1,2\}}K_{\{0,2,3\}} \\ & \subseteq & K_{\{0,1,2\}}K_{\{0,2,3\}} + K_{\{0,1,2\}}K_{\{0,1,3\}}. \end{array}$$

Row: 23

$$d_4[C_{(2)(1)}(x_3\otimes y_3)] = s_2d_3x_3(s_1d_3y_3 - s_2d_3y_3) + x_3y_3.$$

Take elements $a = (s_1d_3y_3 - s_2d_3y_3 + y_3)\in K_{\{0,2,3\}}$ and $b = (s_2d_3x_3 - x_3)\in K_{\{0,1,3\}}$. Putting them together, we obtain

$$\begin{array}{rcl} d_4[C_{(2)(1)}(x_3\otimes y_3)] & = & ab + d_4[C_{(3)(1)}(x_3\otimes y_3)] - d_4[C_{(3)(2)}(x_3\otimes y_3) + C_{(3)(2)}(y_3\otimes x_3)] \\ & \in & K_{\{0,1,3\}}K_{\{0,2,3\}} + K_{\{0,1,2\}}K_{\{0,2,3\}} + K_{\{0,1,2\}}K_{\{0,1,3\}}. \end{array}$$

Finally

Row: 25

$$d_4[C_{(1)(0)}(x_3\otimes y_3)] = s_1d_3(x_3)(s_0d_3y_3 - s_1d_3y_3) + s_2d_3(x_3y_3) - x_3y_3,$$

and

$$a = (s_1d_3x_3 - s_2d_3x_3 + x_3)\in K_{\{0,2,3\}} \text{ and } b = (s_2d_3y_3 - s_1d_3y_3 + s_0d_3y_3 - y_3)\in K_{\{1,2,3\}},$$

then one has

$$\begin{array}{rcl} d_4[C_{(1)(0)}(x_3\otimes y_3)] & = & d_4[C_{(3)(1)}(y_3\otimes x_3) + C_{(3)(1)}(x_3\otimes y_3)] - \\ & & d_4[C_{(3)(0)}(x_3\otimes y_3)] + d_4[C_{(2)(0)}(x_3\otimes y_3)] - \\ & & d_4[C_{(2)(1)}(x_3\otimes y_3) + C_{(2)(1)}(y_3\otimes x_3)] - \\ & & d_4[C_{(3)(2)}(x_3\otimes y_3) + C_{(3)(2)}(y_3\otimes x_3)] + ab \\ & \in & K_{\{0,1,2\}}K_{\{0,1,3\}} + \ldots + K_{\{0,2,3\}}K_{\{1,2,3\}}. \end{array}$$

So we have shown that for each $d_4C_{\alpha,\beta}(x\otimes y)\in\sum_{I,J}K_IK_J$ and hence $\partial_4(I_4)\subseteq\sum_{I,J}K_IK_J$. The opposite inclusion to this is again given by proposition 2.3. ∎

To summarise we have:

6.2. THEOREM. *Let* $n = 2, 3$, *or* 4 *and let* $\mathbf{E}$ *be a simplicial algebra with Moore complex* $\mathbf{NE}$ *in which* $E_n = D_n$, *Then*

$$\partial_n(NE_n) = \sum_{I,J} K_I K_J$$

for any $I, J \subseteq [n-1]$ *with* $I \cup J = [n-1]$, $I = [n-1] - \{\alpha\}$ *and* $J = [n-1] - \{\beta\}$, *where* $(\alpha, \beta) \in P(n)$. ∎

In more generality we can observe that only elements of $NE_n \cap D_n$ were used.

6.3. THEOREM. *If for any simplicial algebra* $\mathbf{E}$ *with Moore complex* $\mathbf{NE}$,

$$\partial_n(NE_n \cap D_n) = \sum_{I,J} K_I K_J \quad \textit{with } n = 2, 3, 4.$$

6.4. REMARK. In general for $n > 4$, we have only managed to prove

$$\sum_{I,J} K_I K_J \subseteq \partial_n(NE_n).$$

To prove the opposite inclusion, we have a general argument for $I \cap J = \emptyset$ and $I \cup J = [n-1]$, but for $I \cap J \neq \emptyset$, we as yet do not see the pattern. One should be able to extend this result by means of computer algebra software such as AXIOM or MAPLE, and this may help reveal what structure is lying behind the observed behaviour in low dimensions, but the overall pattern is still mysterious.

References

[1] M.ANDRÉ. Homologie des algèbres commutatives. *Die Grundlehren der Mathematischen Wissenchaften*, **206** *Springer-Verlag* (1970).

[2] Z.ARVASI. Applications in commutative algebra of the Moore complex of a simplicial algebra. *Ph.D. Thesis,* University of Wales, (1994).

[3] Z.ARVASI and T.PORTER. Simplicial and crossed resolutions of commutative algebras. *J. Algebra*, **181**, (1996) 426-448.

[4] H.J.BAUES. Combinatorial homotopy and 4-dimensional complexes. *Walter de Gruyter,* (1991).

[5] R.BROWN and J-L.LODAY. Van Kampen theorems for diagram of spaces. *Topology,* **26** (1987) 311-335.

[6] P.CARRASCO. Complejos hipercruzados, cohomologia y extensiones. *Ph.D. Thesis*, Univ. de Granada, (1987).

[7] P.CARRASCO and A.M.CEGARRA. Group-theoretic algebraic models for homotopy types. *Journal Pure Appl. Algebra,* **75** (1991) 195-235.

[8] D.Conduché. Modules croisés généralisés de longueur 2. *Journal Pure Appl. Algebra,* **34** (1984) 155-178.

[9] G.J.Ellis. Higher dimensional crossed modules of algebras. *Journal Pure Appl. Algebra,* **52** (1988) 277-282.

[10] A.R.Grandjeán and M.J.Vale. 2-Modulos cruzados en la cohomologia de André-Quillen. *Memorias de la Real Academia de Ciencias,* **22** (1986) 1-28.

[11] D.Guin-Walery and J.L.Loday. Obstructions à l'excision en K-théorie algèbrique. Springer Lecture Notes in Math. **854** (1981) 179-216.

[12] L.Illusie. Complex cotangent et deformations I, II. *Springer Lecture Notes in Math.,* **239** (1971) II **283** (1972).

[13] S. Lichtenbaum and M. Schlessinger, The cotangent complex of a morphism, *Trans. Amer. Math. Soc,* **128** (1967), 41-70.

[14] T. Porter, Homology of commutative algebras and an invariant of Simis and Vasconcelos, *J. Algebra*, **99** (1986) 458-465.

[15] D. Quillen, On the homology of commutative rings, *Proc. Sympos. Pure Math,* **17** (1970) 65-87.

[16] J.H.C. Whitehead. Combinatorial homotopy II. *Bull. Amer. Math. Soc.,* **55** (1949) 213-245.

Department of Mathematics,
Faculty of Science,
Osmangazi University,
Eskişehir, Turkey
and
School of Mathematics
University of Wales, Bangor
Gwynedd LL57 1UT, UK.

Email: `zarvasi@vm.baum.anadolu.edu.tr` and `t.porter@bangor.ac.uk`

This article may be accessed via WWW at `http://www.tac.mta.ca/tac/` or by anonymous ftp at `ftp://ftp.tac.mta.ca/pub/tac/html/volumes/1997/n1/n1.{dvi,ps}`

THEORY AND APPLICATIONS OF CATEGORIES (ISSN 1201-561X) will disseminate articles that significantly advance the study of categorical algebra or methods, or that make significant new contributions to mathematical science using categorical methods. The scope of the journal includes: all areas of pure category theory, including higher dimensional categories; applications of category theory to algebra, geometry and topology and other areas of mathematics; applications of category theory to computer science, physics and other mathematical sciences; contributions to scientific knowledge that make use of categorical methods.

Articles appearing in the journal have been carefully and critically refereed under the responsibility of members of the Editorial Board. Only papers judged to be both significant and excellent are accepted for publication.

The method of distribution of the journal is via the Internet tools `WWW/ftp`. The journal is archived electronically and in printed paper format.

Subscription information. Individual subscribers receive (by e-mail) abstracts of articles as they are published. Full text of published articles is available in .dvi and Postscript format. Details will be e-mailed to new subscribers and are available by `WWW/ftp`. To subscribe, send e-mail to `tac@mta.ca` including a full name and postal address. For institutional subscription, send enquiries to the Managing Editor, Robert Rosebrugh, `rrosebrugh@mta.ca`.

Information for authors. The typesetting language of the journal is TeX, and LaTeX is the preferred flavour. TeX source of articles for publication should be submitted by e-mail directly to an appropriate Editor. They are listed below. Please obtain detailed information on submission format and style files from the journal's WWW server at `URL http://www.tac.mta.ca/tac/` or by anonymous ftp from `ftp.tac.mta.ca` in the directory `pub/tac/info`. You may also write to `tac@mta.ca` to receive details by e-mail.