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MULTILINEARITY OF SKETCHES

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ABSTRACT. We give a precise characterization for when the models of the tensor product of sketches are structurally isomorphic to the models of either sketch in the models of the other. For each base category $\mathcal{K}$ call the just mentioned property (sketch) $\mathcal{K}$-multilinearity. Say that two sketches are $\mathcal{K}$-compatible with respect to base category $\mathcal{K}$ just in case in each $\mathcal{K}$-model, the limits for each limit specification in each sketch commute with the colimits for each colimit specification in the other sketch and all limits and colimits are pointwise. Two sketches are $\mathcal{K}$-multilinear if and only if the two sketches are $\mathcal{K}$-compatible. This property then extends to strong Colimits of sketches.

We shall use the technically useful property of limited completeness and completeness of every category of models of sketches. That is, categories of sketch models have all limits commuting with the sketched colimits and and all colimits commuting with the sketched limits. Often used implicitly, the precise statement of this property and its proof appears here.

1. Introduction

As Ageron mentions in [2], the equivalence of the models of one sketch in the models of another to the models of the second sketch in the models of the first sketch depends upon the commutivity of the limits and colimits specified. More is required to obtain this equivalence in a structural manner. What is required is the pointwise construction of the limits and colimits in the model categories, these being full subcategories of functor categories. We develop these conditions, both necessary and sufficient, through the use of tensor sketches, [2]. The tensor sketch formed from two sketches inherently contains only the acceptable limits and colimits, that is, those formed pointwise. We define a tensor sketch of two sketches to be $\mathcal{K}$-multilinear if and only if it is structurally isomorphic to the models of the first sketch in the models of the second with respect to base category $\mathcal{K}$.

We shall use the fact that, provided the base category $\mathcal{K}$ for models has enough limits and colimits, the category of models $\textsc{Mod}(s, \mathcal{K})$ for sketch s has all limits which commute with the colimits specified by the sketch colimit specifications in $\mathcal{C}_s$ and has all the colimits which commute with the limits specified by the limit specifications $\mathcal{L}_s$. This fact is implicit in studies such as [5, 1] and is stated here as a lemma.

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Two sketches are said to be $\mathcal{K}$-compatible when, indeed, the specified limits in models of the first sketch commute with the specified colimits in models of the second sketch and the specified limits in models of the second sketch commute with the specified colimits in models of the first sketch and also all the specified limits and colimits are acceptable. In section 3 we show that a tensor sketch is $\mathcal{K}$-multilinear if and only if the two sketches involved are $\mathcal{K}$-compatible. In section 4 we outline the fact that the tensor product of sketches commutes with the formation of any strong Colimit of a small diagram of sketches.

2. Preliminaries

Fix a base category $\mathcal{K}$. Throughout, $\mathcal{K}$ is assumed to have enough limits and colimits. In particular, in every subcategory of functors $\mathcal{F} \subseteq \mathcal{K}^{\mathcal{A}}$ that we consider, determining which limits and colimits are pointwise will be of central importance.

In each diagram in any category $\mathcal{B}$, $D \colon \mathbf{D} \longrightarrow \mathcal{B}$, the shape category $\mathbf{D}$ is small. For simplicity of notation, limits and colimits are written as class representatives rather than the class of isomorphic objects. So we write in the style of $\rho = \mathit{colim}\ D$ for the apex of the representative colimiting cone of the diagram $D \colon \mathbf{D} \longrightarrow \mathcal{B}$, and say that ρ is the colimit of D. We shall use any of the following notations for the limit of a diagram $D \colon \mathbf{D} \longrightarrow \mathcal{B}$: $\mathit{lim}\ D = \mathit{lim}_{\mathbf{D}}\ D = \mathit{lim}_{d \in \mathbf{D}}\ D(d)$. In some places the limits and colimits are taken with a parameter, [4], not indicated in the subscripting but clear from the diagram form.

2.1. Closures of Limits and Colimits. As usual for small category $\mathcal{A}$, $\varepsilon \colon \mathcal{A} \times \mathcal{K}^{\mathcal{A}} \longrightarrow \mathcal{K}$ is the evaluation functor. For each subcategory of functors $\mathcal{F} \subseteq \mathcal{K}^{\mathcal{A}}$, $\varepsilon \colon \mathcal{A} \times \mathcal{F} \longrightarrow \mathcal{K}$ is the domain restriction of the evaluation functor.

2.2. Definition. *Let $\mathcal{F}$ be a subcategory of functors $\mathcal{F} \subseteq \mathcal{K}^{\mathcal{A}}$. The diagram $C \colon \mathbf{C} \longrightarrow \mathcal{F}$ is said to be an acceptable colimit diagram if the colimit of C, colim C, exists in $\mathcal{F}$ and moreover*

$$\mathit{colim}\ C = \mathit{colim}_{\mathbf{C}}\ \varepsilon \circ (\mathcal{A} \times C),$$

that is, the colimit of C is a pointwise colimit. The definition of acceptable limit diagram is entirely similar.

2.3. Definition. *Let $\mathcal{A}$ be a small category and let $\mathcal{F} \subseteq \mathcal{K}^{\mathcal{A}}$ be a category of functors. A diagram $L \colon \mathbf{L} \longrightarrow \mathcal{A}$ is said to be a limit constraint for $\mathcal{F}$ if every diagram $F \circ L$, for $F \in \mathcal{F}$, has a limit. Similarly for colimit constraints. A limit constraint establishes a functor $\mathit{Lim}(L) \colon \mathcal{F} \longrightarrow \mathcal{K}$ with value for each $F \in \mathcal{F}$ being lim $F \circ L$. Similarly for colimit constraints.*

It is clear that $\mathit{Lim}(L) = \mathit{lim}_{\mathbf{L}}\ \varepsilon \circ (L \times \mathcal{F})$ although complete precision requires another universe since $\mathcal{F}$ is not necessarily small. Our purpose is simply explicating the various notations for the limits and colimits of immediate interest.

2.4. Definition. *With the notation just established, let $\overline{\mathcal{L}}$ denote the class of all acceptable colimit diagrams in $\mathcal{F}$ with colimits which in fact commute, up to isomorphism in $\mathcal{K}$, with every limit constraint in the class of limit constraints $\mathcal{L}$. Precisely,*

$$\overline{\mathcal{L}} = \{C\colon \mathbf{C} \longrightarrow \mathcal{F} \mid C \text{ acceptable and } \mathit{colim}_{\mathbf{C}}\ \mathit{Lim}(L) \circ C \cong \mathit{lim}_{\mathbf{L}}\ (\mathit{colim}\ C) \circ L \quad \text{for all } \ L\colon \mathbf{L} \longrightarrow \mathcal{A} \in \mathcal{L}\ \}$$

Similarly, $\overline{\mathcal{C}}$ is the class of all acceptable limit diagrams with limits which commute, up to isomorphism in $\mathcal{K}$, with every colimit constraint in the class of colimit constraints $\mathcal{C}$.

The isomorphism defining $\overline{\mathcal{L}}$ can also be written in the style

$$\mathit{colim}_{c \in \mathbf{C}}\ \mathit{lim}_{l \in \mathbf{L}}\ C(c)(L(l)) \cong \mathit{lim}_{l \in \mathbf{L}}\ \mathit{colim}_{c \in \mathbf{C}}\ C(c)(L(l))$$

since the functor colimits are pointwise.

An example for the base category Sets: Let $\mathcal{F}$ be a Diers category, [5], a subcategory of $\mathrm{Sets}^{\mathcal{A}}$. Such a category is given by limit constraints and coproduct constraints. It is known that each such Diers category has all connected limits which are formed pointwise in $\mathrm{Sets}^{\mathcal{A}}$, since connected limits commute with coproducts in Sets. Symbolically,

$$(\text{connected limits}) \subseteq \overline{(\text{coproducts})}.$$

2.5. Sketch Definitions.

2.6. Definition. *A* sketch $s = (\mathcal{A}_s, \mathcal{L}_s, \mathcal{C}_s, \sigma_s)$ *consists of a small category $\mathcal{A}_s$, a class of diagrams $\mathcal{L}_s$ called* limit specifications, *a class of diagrams $\mathcal{C}_s$ called* colimit specifications, *and a function $\sigma = \sigma_s$ assigning a cone to every limit specification in $\mathcal{L}_s$ and a cocone to every colimit specification in $\mathcal{C}_s$.*

2.7. Definition. *A* sketch map *from sketch s to sketch t is a functor $F\colon \mathcal{A}_s \longrightarrow \mathcal{A}_t$ such that F carries $\mathcal{L}_s$ into $\mathcal{L}_t$ and carries $\mathcal{C}_s$ to $\mathcal{C}_t$ homomorphically with respect to σ.*

2.8. Definition. *A* model *of sketch s in category $\mathcal{K}$ is a functor $M\colon \mathcal{A}_s \longrightarrow \mathcal{K}$ such that*

$$\begin{aligned} \mathit{lim}\ M \circ L &= M(\sigma(L)) && \text{for all } L \text{ in } \mathcal{L}_s, \\ \mathit{colim}\ M \circ C &= M(\sigma(C)) && \text{for all } C \text{ in } \mathcal{C}_s. \end{aligned}$$

The category of all models of sketch s in category $\mathcal{K}$ is denoted by $\mathrm{Mod}(s, \mathcal{K})$ *and typically by* Mod *s when $\mathcal{K} =$* Sets. *The category of models* $\mathrm{Mod}(s, \mathcal{K})$ *is a full subcategory of $\mathcal{K}^{\mathcal{A}_s}$.*

In the subcategory of functors $\mathrm{Mod}(s, \mathcal{K}) \subseteq \mathcal{K}^{\mathcal{A}_s}$, each limit specification of s is a limit constraint and each colimit specification of s is a colimit constraint.

2.9. Lemma. *Let s be a sketch. For each diagram $L\colon \mathbf{L} \longrightarrow \textsc{Mod}(s,\mathcal{K})$ such that $lim_{\mathbf{L}}\ Colim(C) \circ L \cong colim_{\mathbf{C}}\ (lim\ L) \circ C$ for all $C \in \mathcal{C}_s$, that is, such that the limit of L commutes with all the colimits specified in $\mathcal{C}_s$,*

$$L \in \overline{\mathcal{C}_s}.$$

Similarly, for each diagram C such that the colimit of C commutes with all the limits specified in $\mathcal{L}_s$, $C \in \overline{\mathcal{L}_s}$.

Proof. Let $L\colon \mathbf{L} \longrightarrow \textsc{Mod}(s,\mathcal{K})$ be a diagram with limit commuting with all the colimits specified in $\mathcal{C}_s$. The pointwise limit of the diagram, $lim_{l\in\mathbf{L}}\ L(l)$, is given in the functor category $\mathcal{K}^{\mathcal{A}_s}$ by $\alpha(x) = lim_{l\in\mathbf{L}}\ L(l)(x)$, x in s. It remains to show that $\alpha \in \textsc{Mod}(s,\mathcal{K})$. It suffices to demonstrate, for each colimit specification $C\colon \mathbf{C} \longrightarrow \mathcal{A}_s$ in $\mathcal{C}_s$, that $colim\ \alpha \circ C = \alpha(\sigma(C))$. We have $colim\ L(l) \circ C = L(l)(\sigma(C)), l \in \mathbf{L}$, since each $L(l)$ is a model in $\textsc{Mod}(s,\mathcal{K})$. Therefore,

$$\begin{aligned} \alpha(\sigma(C)) &= lim_{l\in\mathbf{L}}\ L(l)(\sigma(C)) = lim_{l\in\mathbf{L}}\ colim_{c\in\mathbf{C}}\ L(l)(C(c)) \\ &\cong colim_{c\in\mathbf{C}}\ lim_{l\in\mathbf{L}}\ L(l)(C(c)) = colim_{c\in\mathbf{C}}\ \alpha(C(c)). \end{aligned}$$

An entirely similar argument applies to diagrams which commute with all the limits specified in $\mathcal{L}_s$. ■

From this lemma, we may say that the category of models $\textsc{Mod}(s,\mathcal{K})$ is $\overline{\mathcal{C}_s}$-*replete* and also $\overline{\mathcal{L}_s}$-*coreplete.*

For properties of sketches and models in Sets see [1, 5, 3].

2.10. Definition. *Sketch s said to be $\mathcal{K}$-*compatible *with sketch t just in case both (i) and (ii) hold for every model $M \in \textsc{Mod}(s,\textsc{Mod}(t,\mathcal{K}))$: (i) For every limit specification $L\colon \mathbf{L} \longrightarrow \mathcal{A}_s$ in $\mathcal{L}_s$, $M \circ L \in \overline{\mathcal{C}_t}$. (ii) Symmetrically with regard to colimit specifications in s and $\overline{\mathcal{L}_t}$.*

When sketch s is $\mathcal{K}$-compatible with sketch t, for each model M the M-limits determined by the limit specifications of sketch s commute with the colimits determined by the colimit specifications of sketch t. To give details, consider model $M \in \textsc{Mod}(s,\textsc{Mod}(t,\mathcal{K}))$ and limit specification $L \in \mathcal{L}_s$. We have, for each $C \in \mathcal{C}_t$,

$$lim_{\mathbf{L}}\ Colim(C) \circ (M \circ L) \cong colim\ (lim\ M \circ L) \circ C$$

which also may be written as

$$lim_{l\in\mathbf{L}}\ colim_{c\in\mathbf{C}}\ M(L(l))(C(c)) \cong colim_{c\in\mathbf{C}}\ lim_{l\in\mathbf{L}}\ M(L(l))(C(c)).$$

Similarly with regard to colimit specifications in s and limit specifications in t.

In the next section we show that $\mathcal{K}$-compatibility is symmetric with respect to the two sketches.

Here are some symbolic examples: A limit sketch, [1], is a sketch s in which the colimit specification set is empty, $\mathcal{C}_s = \emptyset$. We shall say that such a sketch s is a (limit, $\emptyset$)-sketch. Clearly every pair of (limit, $\emptyset$)-sketches is $\mathcal{K}$-compatible for every base category $\mathcal{K}$ with sufficient limits.

A (connected limit, $\emptyset$)-sketch is a (limit, $\emptyset$)-sketch s in which the shape of every limit specification in $\mathcal{L}_s$ is nonempty and connected. A (limit, coproduct)-sketch is a sketch t in which the shape of every colimit specification in $\mathcal{C}_t$ is discrete. For each such sketch t, $\mathrm{MOD}(t, \mathrm{SETS})$ is a Diers category. Each (connected limit, $\emptyset$)-sketch s is SETS-compatible with each (limit, coproduct)-sketch t.

The notion of a (connected limit, coproduct)-sketch is clear by example. Every pair of (connected limit, coproduct)-sketches is SETS-compatible.

2.11. Sketch Tensor Products.

2.12. Definition. *Let s and t be sketches. The* tensor product *of s and t is the sketch*

$$s \otimes t = (\mathcal{A}_s \times \mathcal{A}_t, \mathcal{L}_{s\otimes t}, \mathcal{C}_{s\otimes t}, \sigma_{s\otimes t})$$

where the limit specification and colimit specification data in the tensor product sketch $s \otimes t$ is given as follows: For $L \in \mathcal{L}_s$, $L\colon \mathbf{D} \longrightarrow \mathcal{A}_s$, and each object y of $\mathcal{A}_t$, written y in t, let L_y denote the diagram

$$L_y\colon \mathbf{D} \longrightarrow \mathcal{A}_s \times \mathcal{A}_t : d \longmapsto (L(d), y)$$

and let

$$\sigma_{s\otimes t}(L_y) = (\sigma_s(L), y).$$

For $L \in \mathcal{L}_t$, $L\colon \mathbf{D} \longrightarrow \mathcal{A}_t$, and each object x of $\mathcal{A}_s$, written x in s, let L_x denote the diagram

$$L_x\colon \mathbf{D} \longrightarrow \mathcal{A}_s \times \mathcal{A}_t : d \longmapsto (x, L(d))$$

and let

$$\sigma_{s\otimes t}(L_x) = (x, \sigma_t(L)).$$

The class of limit specifications for $s \otimes t$ is

$$\mathcal{L}_{s\otimes t} = \{L_x | L \in \mathcal{L}_t, x \text{ in } s\} \cup \{L_y | L \in \mathcal{L}_s, y \text{ in } t\}.$$

The construction of the colimit specification data is entirely similar.

One easily checks that $s \otimes t$ is the sketch in the product category $\mathcal{A}_s \times \mathcal{A}_t$ with the coarest limit and colimit specification data such that the canonical functors

$$(x, -)\colon \mathcal{A}_t \longrightarrow \mathcal{A}_s \times \mathcal{A}_t, \qquad x \text{ in } s,$$

$$(-, y)\colon \mathcal{A}_s \longrightarrow \mathcal{A}_s \times \mathcal{A}_t, \qquad y \text{ in } t,$$

are sketch maps, written

$$(x,-)\colon t \longrightarrow s\otimes t, \qquad \text{viz,} \qquad (-,y)\colon s \longrightarrow s\otimes t.$$

This definition of tensor product thus agrees with that in [2].

Note that in every model M of $s\otimes t$ the M-limit of every limit specification L_x for $L\in\mathcal{L}_t$ and x in s commutes with the M-colimit of every colimit specification C_y for $C\in\mathcal{C}_s$ and y in t, and similarly regarding $\mathcal{C}_t$ and $\mathcal{L}_s$. The sketch object $(\sigma_s(C),\sigma_t(L))$ of $s\otimes t$ provides, simultaneously, the M-limit of $L_{\sigma_s(C)}$ and the M-colimit of $C_{\sigma_t(L)}$ in every model M of $s\otimes t$. Here is the calculation for $M\in \mathrm{M{\small OD}}(s\otimes t,\mathcal{K})$, limit specification $L\colon \mathbf{L}\longrightarrow \mathcal{A}_s \in \mathcal{L}_s$ and colimit specification $C\colon \mathbf{C}\longrightarrow \mathcal{A}_t \in \mathcal{C}_t$:

$$\begin{aligned}
\mathit{colim}_{c\in\mathbf{C}}\ \mathit{lim}\ M\circ L_{C(c)} &= \mathit{colim}_{c\in\mathbf{C}}\ M(\sigma_{s\otimes t}(L_{C(c)}))\\
&= \mathit{colim}_{c\in\mathbf{C}}\ M(\sigma_s(L),C(c))\\
&= M(\sigma_s(L),\sigma_t(C))\\
&= \mathit{lim}_{l\in\mathbf{L}}\ M(L(l),\sigma_t(C))\\
&= \mathit{lim}_{l\in\mathbf{L}}\ M(\sigma_{s\otimes t}(C_{L(l)}))\\
&= \mathit{lim}_{l\in\mathbf{L}}\ \mathit{colim}\ M\circ C_{L(l)}.
\end{aligned}$$

Therefore the model category $\mathrm{M{\small OD}}(s\otimes t,\mathcal{K})$ is empty if the limits and colimits specified by one of the sketches do not commute, in $\mathcal{K}$, with the colimits and limits specified by the other. An example is found in [2].

2.13. LEMMA. *For any pair of sketches s and t, the functor* $\mathrm{C{\small URRY}}\colon \mathcal{K}^{\mathcal{A}_s\times\mathcal{A}_t}\longrightarrow {\mathcal{K}^{\mathcal{A}_t}}^{\mathcal{A}_s}$, *where* $\mathrm{C{\small URRY}}(M)(x)(y)=M(x,y)$, *domain and codomain restricts to*

$$\mathrm{C{\small URRY}}\colon \mathrm{M{\small OD}}(s\otimes t,\mathcal{K})\longrightarrow \mathrm{M{\small OD}}(s,\mathrm{M{\small OD}}(t,\mathcal{K}))\ .$$

This restricted functor CURRY *is injective on objects.*

Proof. For each $M\in\mathrm{M{\small OD}}(s\otimes t,\mathcal{K})$, $\mathrm{C{\small URRY}}(M)$ is a model of s in $\mathrm{M{\small OD}}(s,\mathcal{K}^{\mathcal{A}_t})$ since for each $L\colon\mathbf{L}\longrightarrow\mathcal{A}_s \in\mathcal{L}_s$ and each y in s,

$$\begin{aligned}
\mathit{lim}_{l\in\mathbf{L}}\ \mathrm{C{\small URRY}}(M)(L(l))(y) &= \mathit{lim}_{l\in\mathbf{L}}\ M(L(l),y)\\
&= \mathit{lim}\ M\circ L_y\\
&= M(\sigma_{s\otimes t}(L_y))\\
&= M(\sigma_s(L),y)\\
&= \mathrm{C{\small URRY}}(M)(\sigma_s(L))(y)
\end{aligned}$$

is pointwise so that $\mathit{lim}\,\mathrm{C{\small URRY}}(M)\circ L=\mathrm{C{\small URRY}}(M)(\sigma_s(L))$ and similarly for the colimit specifications in $\mathcal{C}_s$. Further, $\mathrm{C{\small URRY}}(M)$ is a model of s in $\mathrm{M{\small OD}}(s,\mathrm{M{\small OD}}(t,\mathcal{K}))$ as for each x in s, $M_x=\mathrm{C{\small URRY}}(M)(x)$ enjoys the properties that

$$\begin{aligned}
\mathit{lim}\ M_x\circ L &= M_x(\sigma_t(L)) &&\text{for all } L\in\mathcal{L}_t,\\
\mathit{colim}\ M_x\circ C &= M_x(\sigma_t(C)) &&\text{for all } C\in\mathcal{C}_t
\end{aligned}$$

since: for each $L \in \mathcal{L}_s$, $M_x(\sigma_t(L)) = M(\sigma_{s\otimes t}(L_x)) = \mathit{lim}\ M \circ L_x = \mathit{lim}\ M_x \circ L$ and similarly for colimit specifications. The injectivity follows immediately from the fact that the original CURRY is an isomorphism. ∎

2.14. DEFINITION. *For sketches s and t, the tensor sketch $s\otimes t$ is said to be $\mathcal{K}$-multilinear if*

$$\text{CURRY}\colon \text{MOD}(s \otimes t, \mathcal{K}) \cong \text{MOD}(s, \text{MOD}(t, \mathcal{K}))$$

is an isomorphism.

By the symmetry in the definition, $s \otimes t \cong t \otimes s$. To repeat, in the next section we show that multilinearity is also symmetric.

3. Multilinearity is equivalent to compatibility

3.1. THEOREM. *For all sketches s and t, $s \otimes t$ is $\mathcal{K}$-multilinear if and only if s is $\mathcal{K}$-compatible with t.*

Proof. Assume that sketch s is $\mathcal{K}$-compatible with sketch t. From the previous lemma, (2.5), $\text{MOD}(s \otimes t, \mathcal{K}) \hookrightarrow \text{MOD}(s, \text{MOD}(t, \mathcal{K}))$. It remains to show that every model in $\text{MOD}(s, \text{MOD}(t, \mathcal{K}))$ is isomorphic to a model in $\text{MOD}(s \otimes t, \mathcal{K})$ via UNCURRY, where $\text{UNCURRY}(M)(x, y) = M(x)(y)$. To this end consider $M \in \text{MOD}(s, \text{MOD}(t, \mathcal{K}))$. First, M is a functor $M\colon \mathcal{A}_s \longrightarrow \mathcal{K}^{\mathcal{A}_t}$ with the property that for each $C\colon \mathbf{C} \longrightarrow \mathcal{A}_t$ in $\mathcal{C}_t$ and for each x in s, $\mathit{colim}_{\mathbf{C}}\ M(x) \circ C = M(x)(\sigma_t(C))$. Via UNCURRY, we have $\mathit{colim}_{\mathbf{C}}\ \text{UNCURRY}(M) \circ C_x = \text{UNCURRY}(M)(\sigma_{s\otimes t}(C_x))$. For each y in t the M-limit of specification $L\colon \mathbf{L} \longrightarrow \mathcal{A}_s$ in $\mathcal{L}_s$ is $\mathit{lim}_{l\in\mathbf{L}}\ M(L(l))(y) = M(\sigma_s(L))(y)$ by $\mathcal{K}$-compatibility. This establishes that $\mathit{lim}_{\mathbf{L}}\ \text{UNCURRY}(M) \circ L_y = \text{UNCURRY}(M)(\sigma_{s\otimes t}(L_y))$. The case of limit specifications in t and colimit specifications in s is entirely similar. Therefore $\text{UNCURRY}(M) \in \text{MOD}(s \otimes t, \mathcal{K})$. Clearly the functor UNCURRY is injective on objects. It is immediately that CURRY and UNCURRY are bijective on natural transformations and so CURRY is an isomorphism.

Now assume that $\text{MOD}(s, \text{MOD}(t, \mathcal{K})) \cong \text{MOD}(s \otimes t, \mathcal{K})$ via the functor UNCURRY. Consider any limit specification $L\colon \mathbf{L} \longrightarrow \mathcal{A}_s$ in $\mathcal{L}_s$ and $M \in \text{MOD}(s, \text{MOD}(t, \mathcal{K}))$. We see that $M \circ L$ is an acceptable limit diagram by the following in which for notational convenience we let $M' = \text{UNCURRY}(M)$. For each y in t

$$(\mathit{lim}\ M \circ L)(y) = M(\sigma_s(L))(y) = M'(\sigma_{s\otimes t}(L_y)) = \mathit{lim}\ M' \circ L_y = \mathit{lim}_{l\in\mathbf{L}}\ M(L(l))(y)$$

so that $\mathit{lim}\ M \circ L$ is indeed pointwise. Continuing the same notation, consider in addition any colimit specification $C\colon \mathbf{C} \longrightarrow \mathcal{A}_t$ in $\mathcal{C}_t$. We have, by taking the colimit of the above,

$$\mathit{colim}_{c\in\mathbf{C}}\ \mathit{lim}_{l\in\mathbf{L}}\ M'(L(l), C(c)) = M'(\sigma_s(L), \sigma_t(C))$$

while for each $l \in \mathbf{L}$,

$$\mathit{colim}_{c\in\mathbf{C}}\ M'(L(l), C(c)) = M'(L(l), \sigma_t(C))$$

and so

$$lim_{l\in\mathbf{L}}\ colim_{c\in\mathbf{C}}\ M'(L(l),C(c)) = M'(\sigma_s(L),\sigma_t(C)).$$

As the argument for colimit specifications in s is entirely similar, we have completed the proof that s is $\mathcal{K}$-compatible with t. ∎

3.2. Remark. Since the case of $\mathrm{Mod}(t, \mathrm{Mod}(s,\mathcal{K}))$ is symmetric to the situation just considered we have that t in $\mathcal{K}$-compatible with s if and only if s is $\mathcal{K}$-compatible with t.

As a computer science application of this theorem, note that the natural join – in the sense of relational databases – is a pullback. Pullbacks may be specified in a (connected limit, $\emptyset$)-sketch. Lists and streams with `cons`, `hd`, and `tl` may be specified in a (limit, coproduct)-sketch. From the theorem we have that joins of lists are structurally isomorphic to lists of joins.

As another example, note that the tensor product of two (connected limit, coproduct)-sketches is again a (connected limit, coproduct)-sketch.

4. Colimits of Sketches commute with Tensor Products

The definition of the strong Colimit of sketches is given in Chapter 5 of [5] with respect to a small weight $W\colon \mathcal{I}^{op} \longrightarrow \mathrm{Cats}$. These W-Colimits do not change the shapes of the limit specifications or the colimit specifications in the sketches ΓI for any diagram of sketches $\Gamma\colon \mathcal{I} \longrightarrow \mathrm{Sketches}$.

4.1. Definition. *For each sketch s and each diagram of sketches $\Gamma\colon \mathcal{I} \longrightarrow \mathrm{Sketches}$, the diagram of sketches $s\otimes\Gamma$ is defined by*

$$(s\otimes\Gamma)I = s\otimes(\Gamma I), \qquad\qquad I\in\mathcal{I}.$$

We may then elide the redundant parenthesis to write $s\otimes\Gamma I$. The following result is entirely syntactic, that is, solely a property of sketches.

4.2. Theorem. *For each sketch s, each diagram of sketches $\Gamma\colon \mathcal{I} \longrightarrow \mathrm{Sketches}$ and each small weight $W\colon \mathcal{I}^{op} \longrightarrow \mathrm{Cats}$,*

$$s\otimes StrongColim_W\Gamma \cong StrongColim_W(s\otimes\Gamma).$$

Proof. (Outline.) The strong W-colimit construction has objects of the form

$$\gamma_I(X)(A) = \langle I, X, A\rangle$$

for I in $\mathcal{I}$, X in WI and A in ΓI, [5], p. 102. The Colimiting cocone is denoted by $\gamma_I(X)\colon \Gamma I \longrightarrow StrongColim_W\Gamma$.

In $s\otimes StrongColim_W\Gamma$ the limit specifications arising from $L\colon \mathbf{L} \longrightarrow \mathcal{A}_s$ in $\mathcal{L}_s$ have the form

$$L_{\langle I,X,A\rangle}\colon l \longmapsto (L(l), \langle I,X,A\rangle)$$

for each I in $\mathcal{I}$, each X in WI and each A in ΓI. The cone assigned is

$$\sigma_{s\otimes StrongColim_W\Gamma}(L_{\langle I,X,A\rangle}) = (\sigma_s(L), \langle I, X, A\rangle).$$

In $StrongColim_W(s \otimes \Gamma)$ the limit specifications arising from $L\colon \mathbf{L} \longrightarrow \mathcal{A}_s$ in $\mathcal{L}_s$ have the form

$$\gamma_I(X) \circ L_A \colon l \longmapsto \langle I, X, (L(l), A)\rangle$$

for each I in $\mathcal{I}$, each X in WI and each A in ΓI. The cone assigned is

$$\gamma_I(X)(\sigma_{s\otimes\Gamma I}(L_A)) = \langle I, X, (\sigma_s(L), A)\rangle.$$

These give equivalent constraints on the models. An entirely similar argument applies to the colimit specifications. ∎

4.3. Corollary. *If sketch s is $\mathcal{K}$-compatible with each sketch ΓI,*

$$\textsc{Mod}(StrongColim_W\Gamma, \textsc{Mod}(s,\mathcal{K})) \cong \textsc{Mod}(s \otimes StrongColim_W\Gamma, \mathcal{K}).$$

This corollary generalizes the usual notion of multilinearity in mathematical module theory and is the main motivation for calling these properties of compatible sketches $\mathcal{K}$-multilinearity.

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