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DOCTRINES WHOSE STRUCTURE FORMS A FULLY FAITHFUL ADJOINT STRING

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Abstract. We pursue the definition of a KZ-doctrine in terms of a fully faithful adjoint string $Dd \dashv m \dashv dD$. We give the definition in any Gray-category. The concept of algebra is given as an adjunction with invertible counit. We show that these doctrines are instances of more general pseudomonads. The algebras for a pseudomonad are defined in more familiar terms and shown to be the same as the ones defined as adjunctions when we start with a KZ-doctrine.

1. Introduction

Free co-completions of categories under suitable classes of colimits were the motivating examples for the definition of KZ-doctrines. We introduce in this paper a not-strict version of such doctrines defined via a fully faithful adjoint string. Thus, a non-strict KZ-doctrine on a 2-category $\mathcal{K}$ consists of a normal endo homomorphism $D : \mathcal{K} \longrightarrow \mathcal{K}$, and strong transformations $d : 1_{\mathcal{K}} \longrightarrow D$, and $m : DD \longrightarrow D$ in such a way that $Dd \dashv m \dashv dD$ forms a fully faithful adjoint string, satisfying one equation involving the unit of $Dd \dashv m$ and the counit of $m \dashv dD$. Given an object C in $\mathcal{K}$, we think of $D\mathsf{C}$ as the co-completion of C, consisting of suitable diagrams over C, $d\mathsf{C} : \mathsf{C} \longrightarrow D\mathsf{C}$ as the functor that assigns to every object of C the diagram on that object with identities for every arrow in the diagram, and $m\mathsf{C} : DD\mathsf{C} \longrightarrow D\mathsf{C}$ as a colimit functor. The idea of pursuing the adjoint string as definition follows in the steps of [3] and was suggested by R. J. Wood.

Now, $Dd \dashv m \dashv dD$ being a fully faithful adjoint string means that the counit $\beta : m \circ dD \longrightarrow Id$ of $m \dashv dD$ is invertible (equivalently, the unit $\eta : Id \longrightarrow m \circ Dd$ is invertible [7]).

Recall that A. Kock's algebraic presentation of KZ-doctrines [9] asks for equalities $m \circ dD = Id$ and $Id = m \circ Dd$, and for a 2-cell $\delta : Dd \longrightarrow dD$ satisfying four equations.

We can produce from the adjoint string a 2-cell $\delta : Dd \longrightarrow dD$, namely, the pasting of β^{-1} and the unit for the adjunction $Dd \dashv m$. This δ satisfies similar ('non-strict' versions of) the conditions required for a KZ-doctrine in [9]. Thus, the KZ-doctrines of [9] are particular instances of our KZ-doctrines.

Since the algebras for a KZ-doctrine are given in terms of adjunctions it seems reasonable to define the doctrine in terms of adjunctions. Instead of having equality as in [9] we have the invertible 2-cells β and η. This laxification is justified if only because associativity

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in [9] is deduced up to isomorphism, but that paper also mentions some shortcomings of insisting on *normalized* algebras. We believe also that the approach via the adjoint string gives us a better insight into the nature of $\delta : Dd \longrightarrow dD$.

We work in the framework of enriched category theory [2], where the category $\mathbf{V}$ is equal to the category Gray with strict tensor product [5] (see [4] as well). By working in the context of Gray-categories we are developing the 'formal theroy of KZ-doctrines' in the way that, by working in a 2-category, [13] develops the 'formal theory of monads'. Notice that this is a very general setting since every tricategory is equivalent to a Gray-category [5]. The idea of defining KZ-doctrines in an enriched setting is also suggested in [9].

We adopt the definition of a pseudomonoid given in [1]. We show that every KZ-doctrine is a pseudomonad (pseudomonoid in the Gray monoid determined by an object of the Gray-category), and that the 2-categories of algebras defined as adjunctions coincide with the classical algebras for a pseudomonad (Theorem 10.7). We follow [13] in defining the algebras for a pseudomonad and the algebras for a KZ-doctrine with arbitrary objects of the Gray-category as domains.

R. Street [13] gives a conceptual global account of KZ-doctrines in terms of the simplicial category Δ. Recall that in that context a doctrine on a bicategory $\mathcal{K}$ is a homomorphism of bicategories $\Delta \longrightarrow \mathrm{Hom}(\mathcal{K}, \mathcal{K})$ that preserves the monoid structure (ordinal addition on the domain and composition on the codomain), with Δ considered as a locally discrete 2-category. A KZ-doctrine is a doctrine that agrees in the common domain with a homomorphism of bicategories $\Delta^+ \longrightarrow \mathrm{Hom}(\mathcal{K}, \mathcal{K})$ where Δ^+ is the 2-category of non-empty finite ordinals, order and last element preserving functions and inequalities. As pointed out in [9] this definition explicitly excludes the left most adjoint $Dd \dashv m$, without any indication as to whether it can be put back on. We show that, for a pseudomonad to be a KZ-doctrine either one of the adjunctions $Dd \dashv m$ or $m \dashv dD$ is enough.

For examples of free cocompletions of categories under different kinds of colimits we refer the reader to the bibliography of [9].

I would like express my thanks to R. J. Wood who not only provided ideas for this paper but also agreed to discuss them with me. I would like to thank Dalhousie University for its hospitality. I would like thank the referee as well, whose revisions of the different versions of this paper were very helpful on the one hand, and very fast on the other.

2. Background

We work in the context of Gray-categories, where Gray is the symmetric monoidal closed category whose underlying category is 2-Cat with the tensor product as in [5]. A Gray-category is then a category enriched in the category Gray as in [2]. If $\mathbf{A}$ is a Gray-category and $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are objects of $\mathbf{A}$, then the multiplication

$$\mathbf{A}(\mathcal{A}, \mathcal{B}) \otimes \mathbf{A}(\mathcal{B}, \mathcal{C}) \longrightarrow \mathbf{A}(\mathcal{A}, \mathcal{C})$$

corresponds to a cubical functor of two variables

$$M : \mathbf{A}(\mathcal{A}, \mathcal{B}) \times \mathbf{A}(\mathcal{B}, \mathcal{C}) \longrightarrow \mathbf{A}(\mathcal{A}, \mathcal{C}).$$

We denote the action of M by juxtaposition $M(F,G) = GF$. Given $f : F \longrightarrow F'$ in $\mathbf{A}(\mathcal{A},\mathcal{B})$ and $g : G \longrightarrow G'$ in $\mathbf{A}(\mathcal{B},\mathcal{C})$ we denote the invertible 2-cell $M_{f,g}$ by

$$\begin{tikzcd} GF \arrow[r, "gF"] \arrow[d, "Gf"'] & G'F \arrow[d, "G'f"] \arrow[dl, phantom, "\swarrow g_f"] \\ GF' \arrow[r, "gF'"'] & G'F'. \end{tikzcd}$$

M being a cubical functor implies that $(\_)F : \mathbf{A}(\mathcal{B},\mathcal{C}) \longrightarrow \mathbf{A}(\mathcal{A},\mathcal{C})$ and

$$G(\_) : \mathbf{A}(\mathcal{A},\mathcal{B}) \longrightarrow \mathbf{A}(\mathcal{A},\mathcal{C})$$

are 2-functors. It also implies that $(\_)f : (\_)F \longrightarrow (\_)F'$ and $g(\_) : G(\_) \longrightarrow G'(\_)$ are strong transformations. Furthermore, if $\varphi : f \longrightarrow f'$ and $\gamma : g \longrightarrow g'$ then $(\_)\varphi : (\_)f \longrightarrow (\_)f'$ and $\gamma(\_) : g(\_) \longrightarrow g'(\_)$ are modifications. Given $f'' : F' \longrightarrow F$ and $g'' : G' \longrightarrow G''$ we also have that $g_{(f''\circ f)} = (G'f'' \circ g_f)\cdot(g_{f''} \circ Gf)$ and $(g' \circ g)_f = (g''_f \circ gF)\cdot(g''F' \circ g_f)$. If $h : H \longrightarrow H'$ is a 1-cell in $\mathbf{A}(\mathcal{C},\mathcal{D})$, then properties like $h_{gF} = h_g F$ follow from the pentagon, and properties like $1_{\mathcal{A}}F = F$ follow from the triangle that define a Gray-category. We will use these properties in the sequel without explicit mention.

3. KZ-Doctrines

Let $\mathbf{A}$ be a Gray-category and $\mathcal{K}$ be an object in $\mathbf{A}$.

3.1. Definition. *A* KZ-doctrine D *on* $\mathcal{K}$ *consists of an object* D, *1-cells* $d : 1_{\mathcal{K}} \longrightarrow D$, *and* $m : DD \longrightarrow D$ *in* $\mathbf{A}(\mathcal{K},\mathcal{K})$ *and a fully-faithful adjoint string* $\eta,\epsilon : Dd \dashv m$; *and* $\alpha,\beta : m \dashv dD : D \longrightarrow DD$ *such that*

$$\begin{tikzcd} & & DD \arrow[dr, "m"] \arrow[d, phantom, "\beta\Downarrow"] & \\ 1_{\mathcal{K}} \arrow[r, "d"] & D \arrow[ur, "dD"] \arrow[rr, "Id_D"] \arrow[dr, "Dd"'] & \arrow[d, phantom, "\eta\Downarrow"] & D \\ & & DD \arrow[ur, "m"'] & \end{tikzcd} \quad = \quad \begin{tikzcd} & D \arrow[dr, "dD"] & & \\ 1_{\mathcal{K}} \arrow[ur, "d"] \arrow[dr, "d"'] \arrow[r, phantom, "(d_d)^{-1}\Downarrow"] & & DD \arrow[r, "m"] & D. \\ & D \arrow[ur, "Dd"'] & & \end{tikzcd} \qquad (1)$$

The adjoint string being fully-faithful means that the counit β is invertible. It follows from a folklore result, whose statement and proof can be found in [7], that this is the case if and only if the unit η is also invertible.

Compare this condition with condition T0 of [9], in which strict equality $m \circ Dd = m \circ dD = Id$ is asked for. As a matter of fact that paper points out some limitations that arise by requiring commutativity on the nose. Furthermore, associativity of m is deduced there only up to isomorphism.

The other piece of information given in [9] is a 2-cell from Dd to dD. In our case, this 2-cell comes from the adjoint string.

Define $\delta : Dd \longrightarrow dD$ to be the pasting

$$
\begin{tikzcd}
D \arrow[rr, "Id_D"] \arrow[dr, "dD"'] \arrow[rr, phantom, shift right=3ex, "\beta^{-1}\Downarrow"] & & D \arrow[dr, "Dd"] & \\
& DD \arrow[ur, "m"'] \arrow[rr, "Id_{DD}"'] \arrow[rr, phantom, shift left=3ex, "\epsilon\Downarrow"] & & DD.
\end{tikzcd}
\tag{2}
$$

We know from [12] that δ is equal to the pasting $(dD \circ \eta^{-1}) \cdot (\alpha \circ Dd)$ and that it is unique with the property $m \circ \delta = \beta^{-1} \cdot \eta^{-1}$.

Condition T1 from [9] now takes the form:

3.2. PROPOSITION.

$$
\begin{tikzcd}
1_{\mathcal{K}} \arrow[r, "d"] & D \arrow[r, bend left, "Dd"] \arrow[r, bend right, "dD"'] \arrow[r, phantom, "\delta\Downarrow"] & DD
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& D \arrow[dr, "Dd"] & \\
1_{\mathcal{K}} \arrow[ur, "d"] \arrow[dr, "d"'] \arrow[rr, phantom, "d_d\Downarrow"] & & DD. \\
& D \arrow[ur, "dD"'] &
\end{tikzcd}
$$

Proof. Observe that as a consequence of (1), d_d is equal to the pasting

$$
\begin{tikzcd}
& D \arrow[dr, "dD"] \arrow[drr, bend left, "Id_D"] \arrow[drr, phantom, "\beta^{-1}\Downarrow"] & & & \\
1_{\mathcal{K}} \arrow[ur, "d"] \arrow[dr, "d"] \arrow[ddr, "d"'] \arrow[r, phantom, "(d_d)^{-1}\Downarrow"] & & DD \arrow[r, "m"] & D \arrow[r, "Dd"] & DD. \\
& D \arrow[ur, "Dd"'] \arrow[urr, bend right, "Id_D"'] \arrow[urr, phantom, "\eta^{-1}\Downarrow"] \arrow[d, phantom, "d_d\Downarrow"] & & & \\
& D \arrow[uurrr, bend right, "dD"'] & & &
\end{tikzcd}
\tag{3}
$$

Notice that $\epsilon \circ Dd = Dd \circ \eta^{-1}$ (consequence of one of the triangular identities). Cancel d_d with its inverse. Finally observe that $\delta \circ d = (\epsilon \circ Dd \circ d) \cdot (Dd \circ \beta^{-1} \circ d)$. ∎

The condition T2 of [9] takes the form of the uniqueness property for δ mentioned above. We write it as a lemma.

3.3. LEMMA. $m \circ \delta = \beta^{-1} \cdot \eta^{-1}$ ∎

We define the algebras for a KZ-doctrine with an arbitrary object of the Gray-category **A** as domain. This is in agreement with [8], where the algebras for a monad on a 2-category are defined over arbitrary objects of the 2-category.

3.4. DEFINITION. *Let $\mathcal{X}$ be an object of* **A**. *A* D-algebra *with domain $\mathcal{X}$ is an adjunction*

$$\varphi, \psi : x \dashv dX : X \longrightarrow DX$$

in $\mathbf{A}(\mathcal{X}, \mathcal{K})$, with the counit ψ invertible.

A D-algebra as above, produces a co-fully-faithful adjoint string $Dx \dashv DdX \dashv mX$. As in the definition of δ, we obtain

$$\begin{tikzcd}
& DX \arrow[rr, "Id_{DX}"] \arrow[dr, "DdX"'] \arrow[d, phantom, "\epsilon X \Downarrow"] & & DX \\
DDX \arrow[ur, "mX"] \arrow[rr, "Id_{DDX}"'] & & DDX \arrow[ur, "Dx"'] \arrow[u, phantom, "D\psi^{-1} \Downarrow"] &
\end{tikzcd}
=
\begin{tikzcd}
DDX \arrow[rr, "Id_{DDX}"] \arrow[dr, "Dx"'] & & DDX \arrow[dr, "mX"] \arrow[d, phantom, "\eta X^{-1} \Downarrow"] & \\
& DX \arrow[ur, "DdX"'] \arrow[rr, "Id_{DX}"'] \arrow[u, phantom, "D\varphi \Downarrow"] & & DX.
\end{tikzcd} \tag{4}$$

The following proposition tells us that for a D-algebra, the unit is uniquely determined by the counit

3.5. PROPOSITION. *If $\varphi, \psi : x \dashv dX : X \longrightarrow DX$ is a D-algebra, then φ is equal to the pasting*

$$\begin{tikzcd}
DX \arrow[r, bend left=20, "DdX"] \arrow[r, bend right=20, "dDX"'] \arrow[r, phantom, "\delta X \Downarrow"] \arrow[rr, bend left=50, "Id_{DX}"] \arrow[dr, "x"'] & DDX \arrow[r, "Dx"] \arrow[d, phantom, "d_x \Downarrow"] \arrow[r, phantom, bend left=30, "D\psi^{-1}\Downarrow"] & DX. \\
& X \arrow[ur, "dX"'] &
\end{tikzcd} \tag{5}$$

Proof. Start with the above pasting. Replace δX by $(\epsilon X \circ dDX) \cdot (DdX \circ \beta X^{-1})$. Use (4). Since the pasting of d_x and d_{dX} is equal to $d_{(dX \circ x)}$, we have that

$$\begin{tikzcd}
DDX \arrow[r, "Dx"'] \arrow[rr, bend left=30, ""] \arrow[rr, phantom, bend left=15, "D\varphi\Downarrow"] & DX \arrow[r, "DdX"'] & DDX \\
DX \arrow[u, "dDX"] \arrow[r, "x"'] \arrow[ur, phantom, "\overset{d_x}{\Longrightarrow}"] & X \arrow[u, "dX"'] \arrow[r, "dX"'] \arrow[ur, phantom, "\overset{d_{dX}}{\Longrightarrow}"] & DX \arrow[u, "dDX"']
\end{tikzcd}
=
\begin{tikzcd}
DX \arrow[rr] \arrow[dr, "x"'] & {} \arrow[d, phantom, "\varphi\Downarrow"] & DX \arrow[r, "dDX"] & DDX. \\
& X \arrow[ur, "dX"'] & &
\end{tikzcd}$$

Therefore we have that $(D\varphi \circ dDX) \cdot (DdX \circ d_x) = (dDX \circ \varphi) \cdot (d^{-1}_{dX} \circ x)$. Make this last substitution. As a consequence of (1) we have that the pasting of d^{-1}_{dX}, βX^{-1} and ηX^{-1} is the identity. ∎

Observe that, for any invertible 2-cell $\psi : x \circ dX \longrightarrow Id_X$ the pasting (5) is always defined. Denote this pasting by $\widehat{\psi}$. Now we show that one of the triangular identities is always satisfied.

3.6. LEMMA. *If $\psi : x \circ dX \longrightarrow Id_X$ is an invertible 2-cell in $\mathbf{A}(\mathcal{X}, \mathcal{A})$, then the pasting*

$$\begin{tikzcd}
& DX \arrow[rr, "Id_{DX}"] \arrow[dr, "x"] \arrow[d, phantom, "\psi \Downarrow"] & {} \arrow[d, phantom, "\widehat{\psi} \Downarrow"] & DX \\
X \arrow[ur, "dX"] \arrow[rr, "Id_X"'] & {} & X \arrow[ur, "dX"'] &
\end{tikzcd}$$

is the identity on dX.

Proof. We know from Proposition 3.2 that $\delta X \circ dX = d_{dX}$. The pasting of d_{dX} with d_x is $d_{(x \circ dX)}$. The pasting of this last 2-cell with $D\psi^{-1}$ is equal to $dX \circ \psi^{-1}$. ∎

So, in order to see if an invertible $\psi : x \circ dX \longrightarrow Id_X$ determines a (necessarily unique, in view of Proposition 3.5) D-algebra, all we have to do is to check the other triangular identity.

3.7. Proposition. *An invertible 2-cell $\psi : x \circ dX \longrightarrow Id_X$ in $\mathbf{A}(\mathcal{X}, \mathcal{A})$ is the counit of an adjunction $x \dashv dX$ if and only if the pasting*

$$\begin{tikzcd}
DX \arrow[rr, "Id_{DX}"] \arrow[dr, "x"'] & {} \arrow[d, phantom, "\widehat{\psi}\,\Downarrow"] & DX \arrow[dr, "x"] & {} \arrow[d, phantom, "\psi\,\Downarrow"] \\
& X \arrow[ur, "dX"] \arrow[rr, "Id_X"'] & & X
\end{tikzcd}$$

is the identity on x. ∎

Since we have $m \dashv dD$ with invertible counit β, we have as a corollary the condition corresponding to condition T3 in [9]

3.8. Corollary. *The pasting*

$$\begin{tikzcd}
DD \arrow[r, bend left, "DdD"] \arrow[r, bend right, "dDD"'] \arrow[r, phantom, "\delta D\Downarrow"] \arrow[rr, bend left=40, "Id_{DD}"] \arrow[dr, "m"'] & DDD \arrow[r, "Dm"'] \arrow[r, phantom, shift left=3, "D\beta^{-1}\Downarrow"] \arrow[d, phantom, "d_m\Downarrow"] & DD \arrow[dr, "m"] & {} \arrow[d, phantom, "\beta\Downarrow"] \\
& D \arrow[ur, "dD"] \arrow[rr, "Id_D"'] & & D
\end{tikzcd}$$

is the identity on m. ∎

Observe that a KZ-doctrine in $\mathbf{A}$, gives with the same data a KZ-doctrine in $\mathbf{A}^{trop}$ but with the roles of α, ϵ and β, η interchanged. Here $\mathbf{A}^{op}$ is the dual in the enriched sense, whereas $\mathbf{A}^{tr}$ is such that for every $\mathcal{A}$ and $\mathcal{B}$ in $\mathbf{A}$, we have $\mathbf{A}^{tr}(\mathcal{A}, \mathcal{B}) = \mathbf{A}(\mathcal{A}, \mathcal{B})^{co}$. We thus obtain the condition corresponding to T3* of [9]

3.9. Corollary. *The pasting*

$$\begin{tikzcd}
DD \arrow[r, bend left, "DdD"] \arrow[r, bend right, "DDd"'] \arrow[r, phantom, "D\delta\Uparrow"] \arrow[rr, bend left=40, "Id_{DD}"] \arrow[dr, "m"'] & DDD \arrow[r, "mD"'] \arrow[r, phantom, shift left=3, "\eta D^{-1}\Uparrow"] \arrow[d, phantom, "m_d\Uparrow"] & DD \arrow[dr, "m"] & {} \arrow[d, phantom, "\eta\Uparrow"] \\
& D \arrow[ur, "Dd"] \arrow[rr, "Id_D"'] & & D
\end{tikzcd}$$

is the identity on m. ∎

4. Normalized KZ-doctrines vs. KZ-doctrines

In this section we make explicit the comparison between the definition of KZ-doctrines in [9] and the definition given in this paper. Notice first that our definition is given in a general Gray-category, whereas the definition in [9] is given in 2-Cat. Notice furthermore, that we have replaced invertible 2-cells where the definition in [9] asked for strict equalities.

The definition given in [9] makes sense in a general Gray-category provided that the 2-cell d_d is an identity. So what we do is to compare the definitions in this more general setting.

Let's assume first then, that we have a KZ-doctrine D in our sense, such that β, η and d_d are identities. Define $\delta = \epsilon \circ dD$ (pasting (2)). In this case the conditions corresponding to T1, T2 and T3 above are identical to the conditions T1, T2 and T3 of [9].

Conversely, assume we have (D, d, m, δ) a KZ-doctrine in the sense of [9] (where we are assuming that d_d is an identity). It follows from the work done in [9] that $Dd \dashv m$ with identity unit and $m \dashv dD$ with identity counit. We have therefore a KZ-doctrine in our sense. All we have to show now is that $\delta = \epsilon \circ dD$, where ϵ is the counit of $Dd \dashv m$. But this is clear since δ is unique with the property $m \circ \delta = \beta^{-1} \cdot \eta^{-1}$.

5. Associativity up to isomorphism for KZ-doctrines

We deduce associativity up to isomorphism for a KZ-doctrine D as a corollary to the following technical proposition. Recall that D-algebras have objects of $\mathbf{A}$ as domains.

5.1. Proposition. *Let $\psi : x \circ dX \longrightarrow Id_X$ and $\zeta : z \circ dZ \longrightarrow Id_Z$ be D-algebras with the same object $\mathcal{X}$ of $\mathbf{A}$ as domain. Let $h : X \longrightarrow Z$ be a 1-cell in $\mathbf{A}(\mathcal{X}, \mathcal{K})$. If h has a right adjoint then the pasting*

$$\begin{tikzcd}
DX \arrow[rr, "Id_{DX}"] \arrow[dr, "x"'] & & DX \arrow[rr, "Dh"] & & DZ \arrow[dr, "z"] & \\
& X \arrow[ur, "dX"'] \arrow[rr, "h"'] \arrow[u, phantom, "\hat{\psi}\Downarrow"] & & Z \arrow[ur, "dZ"] \arrow[rr, "Id_Z"'] \arrow[u, phantom, "d_h\Downarrow"] & \arrow[u, phantom, "\zeta\Downarrow"] & Z
\end{tikzcd}$$

is invertible.

Proof. Assume $\pi, \chi : h \dashv k$. The inverse of the above pasting is

$$\begin{tikzcd}
DX \arrow[rrrrr] \arrow[dr] & & & & \arrow[d, phantom, "D\pi\Downarrow"] & DX \arrow[ddr] & & \\
& DZ \arrow[rr] \arrow[ddr] & \arrow[d, phantom, "\hat{\zeta}\Downarrow"] & DZ \arrow[urr] & \arrow[r, phantom, "d_k\Downarrow"] & \arrow[d, phantom, "\psi\Downarrow"] & & \\
& & & & X \arrow[uur] \arrow[rr] & & X \arrow[dr] & \\
& & Z \arrow[uur] \arrow[urr] \arrow[rrrrr] & & & \arrow[u, phantom, "\chi\Downarrow"] & & Z.
\end{tikzcd}$$

■

As a corollary we have,

5.2. Proposition. *The pasting*

$$
\begin{tikzcd}
DDD \arrow[rr, "Id_{DDD}"] \arrow[dr, "mD"'] & & DDD \arrow[rr, "Dm"] & & DD \arrow[dr, "m"] & \\
& DD \arrow[ur, "dDD"'] \arrow[rr, "m"'] \arrow[u, phantom, "\alpha D \Downarrow"] & & D \arrow[ur, "dD"'] \arrow[rr, "Id_D"'] \arrow[u, phantom, "d_m \Downarrow"] & & D \arrow[ul, phantom, "\beta \Downarrow"]
\end{tikzcd}
\qquad (6)
$$

is invertible.

Proof. Apply 5.1 with $\psi = \beta D$, $\zeta = \beta$ and $h = m$. ■

As a corollary of the following lemma, we are able to write (6) in terms of $D\epsilon, m_d$ and η.

5.3. Lemma. *Denoting pasting (6) by μ, we have*

$$
\begin{tikzcd}
DD \arrow[d, "DDd"'] \arrow[dr, "Id_{DD}"] & \\
DDD \arrow[r, "Dm"'] \arrow[d, "mD"'] \arrow[ur, phantom, "\overset{D\eta}{\Leftarrow}"] & DD \arrow[d, "m"] \\
DD \arrow[r, "m"'] \arrow[ur, phantom, "\overset{\mu}{\Longleftarrow}"] & D
\end{tikzcd}
=
\begin{tikzcd}
DD \arrow[r, "m"] \arrow[d, "DDd"'] & D \arrow[d, "Dd"] \arrow[dr, "Id_D"] & \\
DDD \arrow[r, "mD"'] \arrow[ur, phantom, "\overset{m_d}{\Longleftarrow}"] & DD \arrow[r, "m"'] \arrow[ur, phantom, "\overset{\eta}{\Leftarrow}"] & D.
\end{tikzcd}
$$

Proof. Start on the left hand side. Substitute (6) for μ. Make the substitution

$$
\begin{tikzcd}
DD \arrow[r, "DDd"] & DDD \arrow[rr] \arrow[dr, "mD"'] & & DDD \\
& & DD \arrow[ur, "dDD"'] \arrow[u, phantom, "\alpha D \Downarrow"] &
\end{tikzcd}
=
\begin{tikzcd}
DD \arrow[r, "m"'] \arrow[d, "DDd"'] \arrow[rr, bend left, "\Downarrow \alpha"] & D \arrow[r, "dD"'] \arrow[d, "Dd"] & DD \arrow[d, "DDd"] \\
DDD \arrow[r, "mD"'] \arrow[ur, phantom, "\overset{m_d}{\Longleftarrow}"] & DD \arrow[r, "dDD"'] \arrow[ur, phantom, "\overset{dD_d}{\Longleftarrow}"] & DDD.
\end{tikzcd}
$$

Then the substitution

$$
\begin{tikzcd}
DD \arrow[r, "DDd"'] \arrow[rr, bend left, "D\eta \Downarrow"] & DDD \arrow[r, "Dm"'] & DD \\
D \arrow[u, "dD"] \arrow[r, "Dd"'] \arrow[ur, phantom, "\overset{dD_d}{\Longrightarrow}"] & DD \arrow[u, "dDD"] \arrow[r, "m"'] \arrow[ur, phantom, "\overset{d_m}{\Longrightarrow}"] & D \arrow[u, "dD"']
\end{tikzcd}
=
\begin{tikzcd}
D \arrow[rr] \arrow[dr, "Dd"'] & & D \arrow[r, "dD"] & DD \\
& DD \arrow[ur, "m"'] \arrow[u, phantom, "\eta \Downarrow"] & &
\end{tikzcd}
$$

recalling that $dD_d = d_{Dd}$. Finally, use the fact that α and β define an adjunction. ■

5.4. Corollary. *Pasting (6) equals*

$$
\begin{tikzcd}
 & DD \arrow[dr, "DDd"] \arrow[rr, "m"] \arrow[d, phantom, "D\epsilon \Downarrow"] & & D \arrow[rr, "Id_D"] \arrow[dr, "Dd"] \arrow[d, phantom, "m_d \Downarrow"] & \arrow[d, phantom, "\eta \Downarrow"] & D \\
DDD \arrow[ur, "Dm"] \arrow[rr, "Id_{DDD}"'] & \, & DDD \arrow[rr, "mD"'] & \, & DD. \arrow[ur, "m"'] &
\end{tikzcd}
$$

∎

Another corollary to Proposition 5.1 is

5.5. Proposition. *For any* D*-algebra* (X, x, ψ)*, the pasting*

$$
\begin{tikzcd}
DDX \arrow[rr, "Id_{DDX}"] \arrow[dr, "mX"'] & \arrow[d, phantom, "\alpha X \Downarrow"] & DDX \arrow[rr, "Dx"] & \arrow[d, phantom, "d_x \Downarrow"] & DX \arrow[dr, "x"] \arrow[d, phantom, "\psi \Downarrow"] & \\
 & DX \arrow[ur, "dDX"'] \arrow[rr, "x"'] & & X \arrow[ur, "dX"] \arrow[rr, "Id_X"'] & & X
\end{tikzcd}
\tag{7}
$$

is invertible.

Proof. Apply 5.1 with $\psi = \beta X$, $\zeta = \psi$ and $h = x$. ∎

Denote pasting (7) by χ_ψ.

5.6. Proposition. *For any* D*-algebra* (X, x, ψ)*, we have that*

$$
\begin{tikzcd}
DX \arrow[r, "DdX"] \arrow[dr, "Id_{DX}"'] \arrow[dr, phantom, "(\eta X)^{-1} \Leftarrow"] & DDX \arrow[r, "Dx"] \arrow[d, "mX"] \arrow[dr, phantom, "\chi_\psi \Longleftarrow"] & DX \arrow[d, "x"] \\
 & DX \arrow[r, "x"'] & X
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & DDX \arrow[dr, "Dx"] \arrow[d, phantom, "D\psi \Downarrow"] & \\
DX \arrow[ur, "DdX"] \arrow[r, "Id_{DX}"'] & DX \arrow[r, "x"] & X.
\end{tikzcd}
$$

Proof. Replace χ_ψ by (7). Notice then that the pasting of ηX with αX produces δX. Now paste with $D\psi$ and its inverse and use 3.7. ∎

6. 2-categories of algebras for a KZ-doctrine

Fix an object $\mathcal{X}$ in $\mathbf{A}$. Define the 2-category $\mathsf{D}\text{-}Alg_{\mathcal{X}}$ of D-algebras with domain $\mathcal{X}$ as follows: The objects of $\mathsf{D}\text{-}Alg_{\mathcal{X}}$ are D-algebras $\psi : x \circ dX \longrightarrow Id_X$ with domain $\mathcal{X}$. Given another D-algebra $\zeta : z \circ dZ \longrightarrow Id_Z$ with domain $\mathcal{X}$, define $\mathsf{D}\text{-}Alg_{\mathcal{X}}(\psi, \zeta)$ to be the full subcategory of $\mathbf{A}(\mathcal{X}, \mathcal{K})(X, Z)$ determined by those 1-cells $h : X \longrightarrow Z$ with the property that

$$
\begin{tikzcd}
DX \arrow[rr, "Id_{DX}"] \arrow[dr, "x"'] & \arrow[d, phantom, "\widehat{\psi}\Downarrow"] & DX \arrow[rr, "Dh"] & \arrow[d, phantom, "d_h \Downarrow"] & DZ \arrow[dr, "z"] & \\
& X \arrow[ur, "dX"'] \arrow[rr, "h"'] & & Z \arrow[ur, "dZ"'] \arrow[rr, "Id_Z"'] & \arrow[u, phantom, "\zeta \Downarrow"] & Z
\end{tikzcd}
\tag{8}
$$

is invertible. The horizontal composite of $h : \psi \longrightarrow \zeta$ and $k : \zeta \longrightarrow \tau$ is $k \circ h$.

There is a forgetful 2-functor $U_{\mathcal{X}} : \mathsf{D}\text{-}Alg_{\mathcal{X}} \longrightarrow \mathbf{A}(\mathcal{X}, \mathcal{K})$ with $U_{\mathcal{X}}(\psi) = X$. The left biadjoint $F_{\mathcal{X}} : \mathbf{A}(\mathcal{X}, \mathcal{K}) \longrightarrow \mathsf{D}\text{-}Alg_{\mathcal{X}}$ is defined as follows: For every X in $\mathbf{A}(\mathcal{X}, \mathcal{K})$ define $F_{\mathcal{X}}(X) = \beta X$. If $\gamma : h \longrightarrow h' : X \longrightarrow Z$, define $F_{\mathcal{X}}(h) = Dh$ and $F_{\mathcal{X}}(\gamma) = D\gamma$. It is straightforward to show that $F_{\mathcal{X}}$ is a 2-functor provided we know that $Dh : \beta X \longrightarrow \beta Z$ is a 1-cell in $\mathsf{D}\text{-}Alg_{\mathcal{X}}$. To see this we need a lemma.

6.1. Lemma. *For every 1-cell $h : X \longrightarrow Z$ in $\mathbf{A}(\mathcal{X}, \mathcal{K})$ we have that the pasting*

$$
\begin{tikzcd}
DDX \arrow[rr, "Id_{DDX}"] \arrow[dr, "mX"'] & \arrow[d, phantom, "\alpha X\Downarrow"] & DDX \arrow[rr, "DDh"] & \arrow[d, phantom, "d_{Dh}\Downarrow"] & DDZ \arrow[dr, "mZ"] & \\
& DX \arrow[ur, "dDX"'] \arrow[rr, "Dh"'] & & DZ \arrow[ur, "dDZ"'] \arrow[rr, "Id_{DZ}"'] & \arrow[u, phantom, "\beta Z \Downarrow"] & DZ
\end{tikzcd}
$$

is equal to m_h^{-1}.

Proof. Since

$$
\begin{tikzcd}
DX \arrow[r, "dDX"] \arrow[d, "Dh"'] \arrow[dr, phantom, "\overset{dD_h}{\Longleftarrow}"] & DDX \arrow[r, "mX"] \arrow[d, "DDh"] \arrow[dr, phantom, "\overset{m_h}{\Longleftarrow}"] & DX \arrow[d, "Dh"] \\
DZ \arrow[r, "dDZ"] \arrow[rr, bend right, ""{name=B}] & DDZ \arrow[r, "mZ"] \arrow[d, phantom, "\beta Z\Downarrow"] & DZ \\
& &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& DDX \arrow[dr, "mX"] \arrow[d, phantom, "\beta X\Downarrow"] & & \\
DX \arrow[ur, "dDX"] \arrow[rr] & & DX \arrow[r, "Dh"] & DZ
\end{tikzcd}
$$

we have that $(\beta Z \circ Dh) \cdot (mZ \circ dD_h) = (Dh \circ \beta X) \cdot (m_h^{-1} \circ dDX)$. Make this last substitution on the pasting of the lemma, and use the fact that α and β define an adjunction. ∎

Notice that $F_{\mathcal{X}} \circ U_{\mathcal{X}} = D(\_) : \mathbf{A}(\mathcal{X}, \mathcal{K}) \longrightarrow \mathbf{A}(\mathcal{X}, \mathcal{K})$. The unit for the biadjunction $F_{\mathcal{X}} \dashv U_{\mathcal{X}}$ is $d(\_) : 1_{\mathbf{A}(\mathcal{X}, \mathcal{K})} \longrightarrow D(\_)$. The counit $s : F_{\mathcal{X}} \circ U_{\mathcal{X}} \longrightarrow 1_{\mathsf{D}\text{-}Alg_{\mathcal{X}}}$ is given by the structure maps, that is to say, for $\psi : x \circ dX \longrightarrow Id_X$ we put $s_\psi = x : \beta X \longrightarrow \psi$. Notice that Proposition 5.5 says that x is a 1-cell in $\mathsf{D}\text{-}Alg_{\mathcal{X}}$. Given $h : \psi \longrightarrow \zeta$ in $\mathsf{D}\text{-}Alg_{\mathcal{X}}$, we define the transition 2-cell s_h as the inverse of (8).

The invertible modification $Id_{F_{\mathcal{X}}} \longrightarrow (sF_{\mathcal{X}}) \circ (F_{\mathcal{X}} d(\_))$ is defined to be ηX at every X in $\mathbf{A}(\mathcal{X}, \mathcal{K})$. The invertible modification $(U_{\mathcal{X}} s) \circ (d(\_) U_{\mathcal{X}}) \longrightarrow Id_{U_{\mathcal{X}}}$ is defined to be ψ at every ψ in $\mathsf{D}\text{-}Alg_{\mathcal{X}}$. To see that this defines a modification we have to show:

6.2. Lemma. *$h \circ \psi$ is equal to the pasting*

$$
\begin{tikzcd}
X \arrow[r, "dX"] \arrow[d, "h"'] & DX \arrow[r, "x"] \arrow[d, "Dh"] & X \arrow[d, "h"] \\
Z \arrow[r, "dZ"] \arrow[ru, phantom, "\overset{d_h}{\Leftarrow}"] \arrow[rr, bend right, ""{name=B}] & DZ \arrow[r, "mZ"] \arrow[ru, phantom, "\overset{s_h}{\Leftarrow}"] \arrow[d, phantom, "\zeta\Downarrow"] & Z. \\
& {} &
\end{tikzcd}
$$

Proof. Consider the inverse of the above pasting composite and use the definition of s_h. Notice that $\widehat{\psi} \circ dX = dX \circ \psi^{-1}$. ∎

Change of base. Assume that we have two objects $\mathcal{X}$ and $\mathcal{Z}$ of $\mathbf{A}$, and H an object in $\mathbf{A}(\mathcal{X}, \mathcal{Z})$. Then the 2-functor $(\_)H : \mathbf{A}(\mathcal{Z}, \mathcal{K}) \longrightarrow \mathbf{A}(\mathcal{X}, \mathcal{K})$ induces a change of base 2-functor $\widehat{H} : \mathsf{D}\text{-}Alg_{\mathcal{X}} \longrightarrow \mathsf{D}\text{-}Alg_{\mathcal{Z}}$ such that

$$
\begin{tikzcd}
\mathsf{D}\text{-}Alg_{\mathcal{Z}} \arrow[r, "\widehat{H}"] \arrow[d, "U_{\mathcal{Z}}"'] & \mathsf{D}\text{-}Alg_{\mathcal{X}} \arrow[d, "U_{\mathcal{X}}"] \\
\mathbf{A}(\mathcal{Z}, \mathcal{K}) \arrow[r, "(\_)H"'] & \mathbf{A}(\mathcal{X}, \mathcal{K})
\end{tikzcd}
$$

commutes.

7. The Gray-category of D-algebras

We can, by allowing the domain to change, define the Gray-category $\mathsf{D}\text{-}Alg$ made up of D-algebras for a KZ-doctrine D.

The objects of $\mathsf{D}\text{-}Alg$ are D-algebras with any object of $\mathbf{A}$ as domain. Given D-algebras $\psi : x \circ dX \longrightarrow Id_X$ with domain $\mathcal{X}$ and $\zeta : z \circ dZ \longrightarrow Id_Z$ with domain $\mathcal{Z}$, the 2-category $\mathsf{D}\text{-}Alg(\psi, \zeta)$ is defined as follows:

The objects of $\mathsf{D}\text{-}Alg(\psi, \zeta)$ are pairs (N, h), where N is an object in $\mathbf{A}(\mathcal{X}, \mathcal{Z})$ and $h : X \longrightarrow ZN$ is a 1-cell in $\mathbf{A}(\mathcal{X}, \mathcal{K})$, such that the pasting

$$
\begin{tikzcd}
DX \arrow[rr, "Id_{DX}"] \arrow[dr, "x"'] & & DX \arrow[rr, "Dh"] & & DZN \arrow[dr, "zN"] \arrow[d, phantom, "\zeta N\Downarrow"] & \\
& X \arrow[ur, "dX"'] \arrow[rr, "h"'] \arrow[u, phantom, "\widehat{\psi}\Downarrow"] & & ZN \arrow[ur, "dZN"] \arrow[rr, "Id_{ZN}"'] \arrow[u, phantom, "d_h\Downarrow"] & {} & ZN
\end{tikzcd}
$$

is invertible.

A 1-cell $(n, \bar{n}) : (N, h) \longrightarrow (N', h')$ in $\mathsf{D}\text{-}Alg(\psi, \zeta)$ consists of a 1-cell $n : N \longrightarrow N'$ in $\mathbf{A}(\mathcal{X}, \mathcal{Z})$ and a 2-cell $\bar{n} : Zn \circ h \longrightarrow h'$ in $\mathbf{A}(\mathcal{X}, \mathcal{K})$.

A 2-cell $\nu : (n, \bar{n}) \longrightarrow (n', \bar{n}')$ is a 2-cell $\nu : n \longrightarrow n'$ in $\mathbf{A}(\mathcal{X}, \mathcal{Z})$ such that $\bar{n} = \bar{n}' \cdot (Y\nu \circ h)$. Vertical composition is the obvious one.

Define $Id_{(N,h)} = (Id_N, id_h)$.

Given $(n,\bar{n}) : (N,h) \longrightarrow (N',h')$, and $(\ell,\bar{\ell}) : (N',h') \longrightarrow (N'',h'')$ define $(\ell,\bar{\ell}) \circ (n,\bar{n}) = (\ell \circ n, \bar{\ell} \cdot (Z\ell \circ \bar{n}))$. If $\lambda : (\ell,\bar{\ell}) \longrightarrow (\ell',\bar{\ell}')$ and $\nu : (n,\bar{n}) \longrightarrow (n',\bar{n}')$ define $\lambda \circ (n,\bar{n}) = \lambda \circ n$ and $(\ell,\bar{\ell}) \circ \nu = \ell \circ \nu$. This completes the definition of the 2-category $\mathsf{D}\text{-}Alg(\psi,\zeta)$.

Define $1_\psi = (1_{\mathcal{X}}, Id_X)$.

For another D-algebra $\tau : y \circ dY \longrightarrow Id_Y$ with domain $\mathcal{Y}$, we define the cubical functor

$$M : \mathsf{D}\text{-}Alg(\psi,\zeta) \times \mathsf{D}\text{-}Alg(\zeta,\tau) \longrightarrow \mathsf{D}\text{-}Alg(\psi,\tau)$$

denoted by juxtaposition as for $\mathbf{A}$, as follows:

Given (N,h) in $\mathsf{D}\text{-}Alg(\psi,\zeta)$ and $\omega : (o,\bar{o}) \to (o',\bar{o}') : (O,g) \longrightarrow (O',g')$ in $\mathsf{D}\text{-}Alg(\zeta,\tau)$, define $(O,g)(N,h) = (ON, gN \circ h)$, and $(o,\bar{o})(N,h) = (oN, \bar{o}N \circ h)$, and $\omega(N,h) = \omega N$.

On the other hand, given $\nu : (n,\bar{n}) \longrightarrow (n',\bar{n}') : (N,h) \longrightarrow (N',h')$ in $\mathsf{D}\text{-}Alg(\psi,\zeta)$ and (O,g) in $\mathsf{D}\text{-}Alg(\zeta,\tau)$ we define $(O,g)(N,h) = (ON, gN \circ h)$, and $(O,g)(n,\bar{n}) = (On, (gN' \circ \bar{n}) \cdot (g_n \circ h))$, and $(O,g)\nu = O\nu$. The proof that we obtain 2-functors with these definitions is fairly straightforward.

For $(n,\bar{n}) : (N,h) \longrightarrow (N',h')$ and $(o,\bar{o}) : (O,g) \longrightarrow (O',g')$ we define the invertible 2-cell $(o,\bar{o})_{(n,\bar{n})} = o_n : (O',g')(n,\bar{n}) \circ (o,\bar{o})(N,h) \longrightarrow (o,\bar{o})(N',h') \circ (O,g)(n,\bar{n})$.

These definitions give us a cubical functor since we have a cubical functor $\mathbf{A}(\mathcal{X},\mathcal{Z}) \times \mathbf{A}(\mathcal{Z},\mathcal{Y}) \longrightarrow \mathbf{A}(\mathcal{X},\mathcal{Y})$.

We have to show now that the diagrams required for a Gray-category are satisfied. We only do the pentagon. Given another D-algebra $\theta : w \circ dW \longrightarrow Id_W$ with domain $\mathcal{W}$, we have that the pentagon commutes if and only if the diagram of cubical functors

$$\begin{tikzcd}
\mathsf{D}\text{-}Alg(\psi,\zeta) \times \mathsf{D}\text{-}Alg(\zeta,\tau) \times \mathsf{D}\text{-}Alg(\tau,\theta) \arrow[r, "M \times \mathsf{D}\text{-}Alg(\tau,\theta)"] \arrow[d, "\mathsf{D}\text{-}Alg(\psi,\zeta)\times M"'] & \mathsf{D}\text{-}Alg(\psi,\tau) \times \mathsf{D}\text{-}Alg(\tau,\theta) \arrow[d, "M"] \\
\mathsf{D}\text{-}Alg(\psi,\zeta) \times \mathsf{D}\text{-}Alg(\zeta,\theta) \arrow[r, "M"'] & \mathsf{D}\text{-}Alg(\psi,\theta)
\end{tikzcd}$$

commutes. This is equivalent to the following six conditions for $(n,\bar{n}) : (N,h) \longrightarrow (N',h')$ in $\mathsf{D}\text{-}Alg(\psi,\zeta)$, $(o,\bar{o}) : (O,g) \longrightarrow (O',g')$ in $\mathsf{D}\text{-}Alg(\zeta,\tau)$ and $(p,\bar{p}) : (P,k) \longrightarrow (P',k')$ in $\mathsf{D}\text{-}Alg(\tau,\theta)$:

1. $((\_)(N,h)) \circ ((\_)(O,g)) = (\_)((O,g)(N,h)) : \mathsf{D}\text{-}Alg(\tau,\theta) \longrightarrow \mathsf{D}\text{-}Alg(\psi,\theta)$.
2. $((P,k)(\_)) \circ ((\_)(N,h)) = ((\_)(N,h)) \circ ((P,k)(\_)) : \mathsf{D}\text{-}Alg(\zeta,\tau) \longrightarrow \mathsf{D}\text{-}Alg(\psi,\theta)$.
3. $(P,k)(\_)) \circ (O,g)(\_)) = ((P,k)(O,g))(\_) : \mathsf{D}\text{-}Alg(\psi,\zeta) \longrightarrow \mathsf{D}\text{-}Alg(\psi,\theta)$.
4. $(p,\bar{p})_{(o,\bar{o})(N,h)} = ((p,\bar{p})_{(o,\bar{o})})(N,h)$.
5. $(p,\bar{p})_{(O,g)(n,\bar{n})} = ((p,\bar{p})(O,g))_{(n,\bar{n})}$.
6. $(P,k)((o,\bar{o})_{(n,\bar{n})}) = ((P,k)(o,\bar{o}))_{(n,\bar{n})}$.

All the above conditions follow from the definitions and the corresponding facts for the Gray-category $\mathbf{A}$.

8. Pseudomonads

We adopt the definition of pseudomonoid given in [1]. That is, given a Gray-category $\mathbf{A}$, and an object $\mathcal{K}$ in $\mathbf{A}$, we define a *pseudomonad* $\mathbb{D}$ on $\mathcal{K}$ to be a pseudomonoid in the Gray monoid $\mathbf{A}(\mathcal{K},\mathcal{K})$. Explicitly, $\mathbb{D}$ consists of an object D in $\mathbf{A}(\mathcal{K},\mathcal{K})$ together with 1-cells $d : 1_{\mathcal{K}} \longrightarrow D$ and $m : DD \longrightarrow D$ and invertible 2-cells

$$\begin{tikzcd}
D \arrow[r, "dD"] \arrow[dr, "Id_D"'] & DD \arrow[d, "m"] \arrow[dl, phantom, "\overset{\beta}{\Leftarrow}"] \arrow[dr, phantom, "\overset{\eta}{\Leftarrow}"] & D \arrow[l, "Dd"'] \arrow[dl, "Id_D"] \\
& D &
\end{tikzcd}
\qquad
\begin{tikzcd}
DDD \arrow[r, "Dm"] \arrow[d, "mD"'] \arrow[dr, phantom, "\overset{\mu}{\Longleftarrow}"] & DD \arrow[d, "m"] \\
DD \arrow[r, "m"'] & D
\end{tikzcd}$$

satisfying the following two conditions

$$\begin{tikzcd}
DDDD \arrow[rr, "DDm"] \arrow[dd, "mDD"'] \arrow[dr, "DmD"] & \arrow[d, phantom, "\overset{D\mu}{\Longleftarrow}"] & DDD \arrow[dr, "Dm"] & \\
& DDD \arrow[rr, "Dm"'] \arrow[dd, "mD"] & & DD \arrow[dd, "m"] \\
DDD \arrow[r, phantom, "\overset{\mu D}{\Longleftarrow}"] \arrow[dr, "mD"'] & & \overset{\mu}{\Longleftarrow} & \\
& DD \arrow[rr, "m"'] & & D
\end{tikzcd}
=
\begin{tikzcd}
DDDD \arrow[rr, "DDm"] \arrow[dd, "mDD"'] & & DDD \arrow[dd, "mD"] \arrow[dr, "Dm"] & \\
& \overset{m_m^{-1}}{\Longleftarrow} & & DD \arrow[dd, "m"] \\
DDD \arrow[rr, "Dm"'] \arrow[dr, "mD"'] & & DD \arrow[r, phantom, "\overset{\mu}{\Longleftarrow}"] \arrow[dr, "m"] & \\
& DD \arrow[u, phantom, "\overset{\mu}{\Longleftarrow}"] \arrow[rr, "m"'] & & D
\end{tikzcd}
\quad (9)$$

$$\begin{tikzcd}
& & DD \arrow[dr, "m"] \arrow[dd, phantom, "\mu\Downarrow"] & \\
DD \arrow[r, "DdD"] & DDD \arrow[ur, "Dm"] \arrow[dr, "mD"'] & & D \\
& & DD \arrow[ur, "m"'] &
\end{tikzcd}
=
\begin{tikzcd}
& DDD \arrow[dr, "Dm"] \arrow[d, phantom, "D\beta\Downarrow"] & & \\
DD \arrow[ur, "DdD"] \arrow[rr] \arrow[dr, "DdD"'] & {} \arrow[d, phantom, "\eta D\Downarrow"] & DD \arrow[r, "m"] & D. \\
& DDD \arrow[ur, "mD"'] & &
\end{tikzcd}$$

(10)

Warning: The direction of the arrows η and μ is the opposite to that given in [1]. Since they are invertible this represents no problem.

As pointed out in [1], a pseudomonoid in the cartesian closed 2-category Cat of categories, functors and natural transformations is precisely a monoidal category, where condition (9) corresponds to the pentagon and condition (10) corresponds to the triangle that has the distinguished object I in the middle. It is well known that in this case the commutativity of these diagrams implies the commutativity of the two triangles that have I on one extreme or the other, and that the 'right' and 'left' arrows $I \otimes I \longrightarrow I$ coincide [6]. (This in turn implies the commutativity of all the diagrams [11]). Results like those of [6] can be shown in the present context.

8.1. Proposition. *If $\mathbb{D} = (D, d, m, \beta, \eta, \mu)$ is a pseudomonad on an object $\mathcal{K}$, then we have the following equalities:*

1.
$$\begin{tikzcd}
& & DD \arrow[dr, "m"] \arrow[d, phantom, "\beta\Downarrow"] & \\
1_{\mathcal{K}} \arrow[r, "d"] & D \arrow[ur, "dD"] \arrow[rr, "Id_D"] \arrow[dr, "Dd"'] & {} & D \\
& & DD \arrow[ur, "m"'] \arrow[u, phantom, "\eta\Downarrow"] &
\end{tikzcd}
=
\begin{tikzcd}
& D \arrow[dr, "dD"] & & \\
1_{\mathcal{K}} \arrow[ur, "d"] \arrow[dr, "d"'] \arrow[rr, phantom, "(d_d)^{-1}\Downarrow"] & & DD \arrow[r, "m"] & D. \\
& D \arrow[ur, "Dd"'] & &
\end{tikzcd}$$

2.
$$\begin{tikzcd}
DD \arrow[r, "dDD"] \arrow[dr, "Id_{DD}"'] & DDD \arrow[r, "Dm"] \arrow[d, "mD"] \arrow[dl, phantom, "\overset{\beta D}{\Leftarrow}"] \arrow[dr, phantom, "\overset{\mu}{\Leftarrow}"] & DD \arrow[d, "m"] \\
{} & DD \arrow[r, "m"'] & D
\end{tikzcd}
=
\begin{tikzcd}
DD \arrow[r, "dDD"] \arrow[d, "m"'] \arrow[dr, phantom, "\overset{d_m}{\Leftarrow}"] & DDD \arrow[d, "Dm"] \\
D \arrow[r, "dD"] \arrow[dr, "Id_D"'] & DD \arrow[d, "m"] \arrow[dl, phantom, "\overset{\beta}{\Leftarrow}"] \\
{} & D.
\end{tikzcd}$$

3.
$$\begin{tikzcd}
DD \arrow[d, "DDd"'] \arrow[dr, "Id_{DD}"] & {} \\
DDD \arrow[r, "Dm"'] \arrow[d, "mD"'] \arrow[ur, phantom, "\overset{D\eta}{\Leftarrow}"] \arrow[dr, phantom, "\overset{\mu}{\Leftarrow}"] & DD \arrow[d, "m"] \\
DD \arrow[r, "m"'] & D
\end{tikzcd}
=
\begin{tikzcd}
DD \arrow[r, "m"] \arrow[d, "DDd"'] \arrow[dr, phantom, "\overset{m_d}{\Leftarrow}"] & D \arrow[d, "Dd"] \arrow[dr, "Id_D"] & {} \\
DDD \arrow[r, "mD"'] & DD \arrow[r, "m"'] \arrow[ur, phantom, "\overset{\eta}{\Leftarrow}"] & D.
\end{tikzcd}$$

Proof. To show 2 start with the following pasting

$$\begin{tikzcd}
& & & DD \arrow[ddr, "m"] \arrow[d, phantom, "\mu\Downarrow"] & & \\
& DDD \arrow[urr, "Dm"] \arrow[dr, "mD"] \arrow[d, phantom, "\beta D\Downarrow"] & & {} & & \\
DD \arrow[ur, "dDD"] \arrow[rr] \arrow[d, "dDD"'] & {} & DD \arrow[rr, "m"'] \arrow[dr, phantom, "\overset{d_m^{-1}}{\Leftarrow}"] & & D \arrow[d, "dD"] \arrow[ddr] & \\
DDD \arrow[rr] \arrow[dr, "DdDD"'] & {} & DDD \arrow[rr, "Dm"] \arrow[drr, "mD"'] & {} & DD \arrow[dr, "m"] \arrow[r, phantom, "\overset{\beta^{-1}}{\Leftarrow}"] \arrow[dl, phantom, "\mu\Downarrow"] & {} \\
& DDD \arrow[ur, "mDD"'] \arrow[u, phantom, "\eta DD\Downarrow"] & & {} & DD \arrow[r, "m"'] & D.
\end{tikzcd}$$

Make the substitution

$$
\begin{tikzcd}[column sep=small]
 & & & DD \arrow[dr, "m"] & \\
 & DDD \arrow[urr, "Dm"] \arrow[dr, "mD"] \arrow[d, phantom, "\beta D\Downarrow"] & & \arrow[d, phantom, "\mu\Downarrow"] & \\
DD \arrow[ur, "dDD"] \arrow[rr] \arrow[d, "dDD"'] & \, & DD \arrow[rr, "m"'] \arrow[d, phantom, "\overset{d_m^{-1}}{\Leftarrow}"] & & D \arrow[d, "dD"] \\
DDD \arrow[rr] & & DDD \arrow[rr, "Dm"] & & DD
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
 & & & DD \arrow[dd, "dDD"] \arrow[dddr, "m"] & \\
 & DDD \arrow[urr, "Dm"] \arrow[ddd, "dDDD"] \arrow[r, phantom, "\overset{d^{-1}_{Dm}}{\Leftarrow}"] & \, & & \\
 & & & DDD \arrow[dddr, "Dm"] & \\
DD \arrow[uur, "dDD"] \arrow[dd, "dDD"'] \arrow[r, phantom, "\overset{d^{-1}_{dDD}}{\Leftarrow}"] & \, & & \arrow[r, phantom, "\overset{d_m^{-1}}{\Leftarrow}"] & D \arrow[dd, "dD"] \\
 & DDDD \arrow[uurr, "DDm"] \arrow[dr, "DmD"] \arrow[d, phantom, "D\beta D\Downarrow"] & & \arrow[l, phantom, "\overset{D\mu}{\Leftarrow}"] & \\
DDD \arrow[ur, "DdDD"] \arrow[rr] & \, & DDD \arrow[rr, "Dm"] & & DD
\end{tikzcd}
$$

(using the fact that $d^{-1}_{m\circ Dm\circ dDD}$ is equal to the pasting of d^{-1}_{dDD}, d^{-1}_{Dm} and d^{-1}_m). Make the substitution (10) multiplied on the right by D. Now make the substitution (9). Make the substitution

$$
\begin{tikzcd}
DD \arrow[r, "dDD"] \arrow[d, "dDD"'] \arrow[dr, phantom, "\overset{d^{-1}_{dDD}}{\Leftarrow}"] & DDD \arrow[r, "Dm"] \arrow[d, "dDDD"] \arrow[dr, phantom, "\overset{d^{-1}_{Dm}}{\Leftarrow}"] & DD \arrow[d, "dDD"] \\
DDD \arrow[r, "DdDD"'] & DDDD \arrow[r, "DDm"'] & DDD
\end{tikzcd}
=
\begin{tikzcd}
 & & DDD \arrow[drr, "Dm"] \arrow[d, phantom, "d_m\Downarrow"] & & \\
DD \arrow[urr, "dDD"] \arrow[rr, "m"] \arrow[d, "dDD"'] \arrow[drr, phantom, "\overset{d_m^{-1}}{\Leftarrow}"] & & D \arrow[rr, "dD"] \arrow[d, "dD"] \arrow[drr, phantom, "\overset{d^{-1}_{dD}}{\Leftarrow}"] & & DD \arrow[d, "dDD"] \\
DDD \arrow[rr, "Dm"'] \arrow[drr, "DdDD"'] & & DD \arrow[rr, "DdD"] \arrow[d, phantom, "Dd_m^{-1}\Downarrow"] & & DDD \\
 & & DDD \arrow[urr, "DDm"'] & &
\end{tikzcd}
$$

Then the substitution

$$
\begin{tikzcd}
D \arrow[r, "dD"] \arrow[d, "dD"'] \arrow[dr, phantom, "\overset{d^{-1}_{dD}}{\Leftarrow}"] & DD \arrow[r, "m"] \arrow[d, "dDD"] \arrow[dr, phantom, "\overset{d^{-1}_{m}}{\Leftarrow}"] & D \arrow[d, "dD"] \\
DD \arrow[r, "DdD"'] & DDD \arrow[r, "Dm"'] & DD
\end{tikzcd}
=
\begin{tikzcd}
 & DD \arrow[dr, "m"] \arrow[d, phantom, "\beta\Downarrow"] & \\
D \arrow[ur, "dD"] \arrow[rr] \arrow[d, "dD"'] & \, & D \arrow[d, "dD"] \\
DD \arrow[rr] \arrow[dr, "DdD"'] & \arrow[d, phantom, "D\beta^{-1}\Downarrow"] & DD. \\
 & DDD \arrow[ur, "Dm"'] &
\end{tikzcd}
$$

Notice that, as consequence of (10), the pasting of $D\beta^{-1}$ and μ is equal to $m \circ \eta D$. The pasting of ηD, Dd_m^{-1} and m_m^{-1} is equal to $Dm \circ \eta DD$. Observe that the bottom part of the resulting diagram is equal to the bottom part of the pasting we started from. Since all the 2-cells are invertible, we conclude 2.

3 can be proved similarly or by duality.

To show 1, we show first that the pasting

$$
\begin{tikzcd}
 & & DD \arrow[dr, "DdD"] \arrow[drr, bend left] & & & \\
1 \arrow[r, "d"] & D \arrow[ur, "dD"] \arrow[dr, "dD"'] \arrow[r, phantom, "d_{dD}\Downarrow"] & & DDD \arrow[u, phantom, "\eta D\Downarrow"] \arrow[d, phantom, "\beta D\Downarrow"] \arrow[r, "mD"'] & DD \arrow[r, "m"] & D \\
 & & DD \arrow[ur, "dDD"'] \arrow[urr, bend right] & & &
\end{tikzcd}
\tag{11}
$$

is the identity. To do this, replace $m \circ \eta D$ by a pasting of $D\beta^{-1}$ and μ, using (10). Use condition 2 of the proposition proved above. The pasting of $D\beta^{-1}$, d_{dD} and d_m is $dD \circ \beta^{-1}$. We thus obtain an identity.

Start again with (11). Paste d_d and its inverse on top of it. Now, $\eta D \circ Dd$ is equal to the pasting of Dd_d, m_d and η. The pasting of Dd_d, d_d and d_{dD} is equal to the pasting of d_d, dD_d and d_d. The pasting of dD_d, m_d and βD is $Dd \circ \beta$. Since (11) is an identity, the resulting pasting is an identity. We thus obtain another identity if we remove d_d and its inverse. Now paste with η and η^{-1}. ∎

9. 2-categories of algebras for a Pseudomonad

As in the case of algebras for a KZ-doctrine we define the algebras for a pseudomonad with an object of $\mathbf{A}$ for domain.

Let $\mathbb{D}$ be a pseudomonad on an object $\mathcal{K}$ of the Gray-category $\mathbf{A}$. Let $\mathcal{X}$ be an object of $\mathbf{A}$. We define the 2-category $\mathbb{D}\text{-}Alg_{\mathcal{X}}$ of $\mathbb{D}$-algebras with domain $\mathcal{X}$ as follows.

An object of $\mathbb{D}\text{-}Alg_{\mathcal{X}}$ consists of an object X in $\mathbf{A}(\mathcal{X},\mathcal{K})$, together with a 1-cell $x : DX \longrightarrow X$, and invertible 2-cells

$$
\begin{tikzcd}
X \arrow[r, "dX"] \arrow[dr, "Id_X"'] & DX \arrow[d, "x"] \arrow[dl, phantom, "\overset{\psi}{\Leftarrow}"] \\
 & X
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
DDX \arrow[r, "Dx"] \arrow[d, "mX"'] \arrow[dr, phantom, "\overset{\chi}{\Longleftarrow}"] & DX \arrow[d, "x"] \\
DX \arrow[r, "x"'] & X.
\end{tikzcd}
$$

This data must satisfy the following two conditions

$$
\begin{tikzcd}
DDDX \arrow[rr, "DDx"] \arrow[dd, "mDX"'] \arrow[dr, "DmX"] & & DDX \arrow[dr, "Dx"] \arrow[d, phantom, "\overset{D\chi}{\Longleftarrow}"] & \\
 & DDX \arrow[rr, "Dx"] \arrow[dd, "mX"] \arrow[dl, phantom, "\overset{\mu X}{\Longleftarrow}"] \arrow[ddrr, phantom, "\overset{\chi}{\Longleftarrow}"] & & DX \arrow[dd, "x"] \\
DDX \arrow[dr, "mX"'] & & & \\
 & DX \arrow[rr, "x"'] & & X
\end{tikzcd}
=
\begin{tikzcd}
DDDX \arrow[rr, "DDx"] \arrow[dd, "mDX"'] \arrow[ddrr, phantom, "\overset{m_x^{-1}}{\Longleftarrow}"] & & DDX \arrow[dd, "mX"] \arrow[dr, "Dx"] & \\
 & & & DX \arrow[dd, "x"] \arrow[dl, phantom, "\overset{\chi}{\Longleftarrow}"] \\
DDX \arrow[rr, "Dx"'] \arrow[dr, "mX"'] & & DX \arrow[dr, "x"] \arrow[d, phantom, "\overset{\chi}{\Longleftarrow}"] & \\
 & DX \arrow[rr, "x"'] & & X
\end{tikzcd}
\tag{12}
$$

$$
\begin{tikzcd}[column sep=small]
 & & DX \arrow[dr, "x"] & \\
DX \arrow[r, "DdX"] & DDX \arrow[ur, "Dx"] \arrow[dr, "mX"'] \arrow[rr, phantom, "\chi \Downarrow"] & & X \\
 & & DX \arrow[ur, "x"'] &
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
 & DDX \arrow[dr, "Dx"] \arrow[d, phantom, "D\psi \Downarrow"] & & \\
DX \arrow[ur, "DdX"] \arrow[dr, "DdX"'] \arrow[rr] & {} \arrow[d, phantom, "\eta X \Downarrow"] & DX \arrow[r, "x"] & X. \\
 & DDX \arrow[ur, "mX"'] & &
\end{tikzcd}
\tag{13}
$$

We denote an object in $\mathbb{D}$-$Alg_{\mathcal{X}}$ by the pair (ψ, χ).

Given another $\mathbb{D}$-algebra (ζ, θ) with $\zeta : z \circ dZ \longrightarrow Id_Z$, a 1-cell in $\mathbb{D}$-$Alg_{\mathcal{X}}$ is a pair $(h, \rho) : (\psi, \chi) \longrightarrow (\zeta, \theta)$, where $h : X \longrightarrow Z$ is a 1-cell in $\mathbf{A}(\mathcal{X}, \mathcal{K})$ and

$$
\begin{tikzcd}
DX \arrow[r, "Dh"] \arrow[d, "x"'] \arrow[dr, phantom, "\overset{\rho}{\Longleftarrow}"] & DZ \arrow[d, "z"] \\
X \arrow[r, "h"'] & Z
\end{tikzcd}
$$

is an invertible 2-cell in $\mathbf{A}(\mathcal{X}, \mathcal{K})$, such that the following two conditions are satisfied.

$$
\begin{tikzcd}
X \arrow[r, "dX"] \arrow[dr, "Id_X"'] & DX \arrow[r, "Dh"] \arrow[d, "x"] \arrow[dl, phantom, "\overset{\psi}{\Longleftarrow}" near start] \arrow[dr, phantom, "\overset{\rho}{\Longleftarrow}"] & DZ \arrow[d, "z"] \\
 & X \arrow[r, "h"'] & Z
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
 & DX \arrow[dr, "Dh"] & & \\
X \arrow[ur, "dX"] \arrow[dr, "h"'] & d_h \Downarrow & DZ \arrow[dr, "z"] & \\
 & Z \arrow[ur, "dZ"] \arrow[rr, "Id_Z"'] & \zeta \Downarrow & Z
\end{tikzcd}
\tag{14}
$$

$$
\begin{tikzcd}[column sep=small]
 & DDX \arrow[rr, "DDh"] \arrow[dl, "mX"'] \arrow[dr, "Dx"] & \overset{D\rho}{\Longleftarrow} & DDZ \arrow[dr, "Dz"] & \\
DX \arrow[dr, "x"'] & \overset{\chi}{\Longleftarrow} & DX \arrow[rr, "Dh"] \arrow[dl, "x"] & \overset{\rho}{\Longleftarrow} & DZ \arrow[dl, "z"] \\
 & X \arrow[rr, "h"'] & & Z &
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
 & DDX \arrow[rr, "DDh"] \arrow[dl, "mX"'] & \overset{m_h^{-1}}{\Longleftarrow} & DDZ \arrow[dl, "mZ"] \arrow[dr, "Dz"] & \\
DX \arrow[rr, "Dh"'] \arrow[dr, "x"'] & \overset{\rho}{\Longleftarrow} & DZ \arrow[dr, "z"'] & \overset{\theta}{\Longleftarrow} & DZ \arrow[dl, "z"] \\
 & X \arrow[rr, "h"'] & & Z. &
\end{tikzcd}
\tag{15}
$$

Given $(h, \rho), (h', \rho') : (\psi, \chi) \longrightarrow (\zeta, \theta)$, a 2-cell $\xi : (h, \rho) \longrightarrow (h', \rho')$ is a 2-cell $\xi : h \longrightarrow h'$ in $\mathbf{A}(\mathcal{X}, \mathcal{K})$ such that $(\xi \circ x) \cdot \rho = \rho' \cdot (z \circ D\xi)$. Vertical composition is the obvious one.

Horizontal composition: for $(h, \rho) : (\psi, \chi) \longrightarrow (\zeta, \theta)$ and $(k, \pi) : (\zeta, \theta) \longrightarrow (\tau, \sigma)$ we define $(k, \pi) \circ (h, \rho) = (k \circ h, (k \circ \rho) \cdot (\pi \circ Dh))$.

This completes the definition of $\mathbb{D}$-$Alg_{\mathcal{X}}$.

A proof very similar to that of condition 2 of Proposition 8.1 produces:

9.1. Lemma. *For every $\mathbb{D}$-algebra (ψ, χ) we have*

$$
\begin{tikzcd}
DX \arrow[r, "dDX"] \arrow[dr, "Id_{DX}"'] & DDX \arrow[r, "Dx"] \arrow[d, "mX"] \arrow[dl, phantom, "\overset{\beta X}{\Leftarrow}"] & DX \arrow[d, "x"] \arrow[dl, phantom, "\overset{\chi}{\Longleftarrow}"] \\
& DX \arrow[r, "x"'] & X
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
DX \arrow[r, "dDX"] \arrow[d, "x"'] \arrow[dr, phantom, "\overset{d_x}{\Longleftarrow}"] & DDX \arrow[d, "Dx"] \\
X \arrow[r, "dX"'] \arrow[dr, "Id_X"'] & DX \arrow[d, "x"] \arrow[dl, phantom, "\overset{\psi}{\Leftarrow}"] \\
& X.
\end{tikzcd}
\tag{16}
$$

■

As a matter of fact, condition 2 of Proposition 8.1 is the above lemma applied to the $\mathbb{D}$-algebra (β, μ).

The Gray-category $\mathbb{D}$-Alg of algebras for a pseudomonad $\mathbb{D}$ can be defined along the same lines as the Gray-category D-Alg of algebras for a KZ-doctrine.

10. Every KZ-doctrine is a pseudomonad

Assume we have a KZ-doctrine $\mathsf{D} = (D, d, m, \alpha, \beta, \eta, \epsilon)$ as in Section 3. Define μ as pasting (6). We already know that μ is invertible.

10.1. Proposition. $\mathbb{D} = (D, d, m, \beta, \eta, \mu)$ *is a pseudomonad.*

Proof. Condition (10) is Proposition 5.6 applied to the D-algebra β. As for the other condition, start on the left hand side of (9). Substitute (6) and (6) multiplied by D on the right for μ and μD respectively. The pasting of βD and αD is the identity. The pasting of d_{mD}, d_m and $D\mu$ equals the pasting of d_{Dm}, d_m and μ. Paste with $(dDD \circ \beta) \cdot (\alpha D \circ dDD)$ in the middle. Use Lemma 6.1. ■

To be able to say anything meaningful on this connection between KZ-doctrines and pseudomonads, we must show first that the categories of algebras D-$Alg_{\mathcal{X}}$ and $\mathbb{D}$-$Alg_{\mathcal{X}}$ for any $\mathcal{X}$ are essentially the same. We devote the rest of this section to show that they are 2-isomorphic. So we fix an object $\mathcal{X}$ of $\mathbf{A}$, and a KZ-doctrine D on $\mathcal{K}$. We take $\mathbb{D}$ as the pseudomonad induced by D as in the above proposition.

We start by stating the recognition lemma [13] in the form we will use it

10.2. Lemma. *Given $\psi : x \circ dX \longrightarrow Id_X$ and $\zeta : z \circ dZ \longrightarrow Id_Z$ in D-$Alg_{\mathcal{X}}$, $h : X \longrightarrow Z$ a 1-cell in $\mathbf{A}(\mathcal{X}, \mathcal{K})$ and $\rho : z \circ Dh \longrightarrow h \circ x$ a 2-cell, we have that*

$$
\rho \quad = \quad
\begin{tikzcd}
DX \arrow[rr, "Id_{DX}"] \arrow[dr, "x"'] & \arrow[d, phantom, "\widehat{\psi}\Downarrow"] & DX \arrow[rr, "Dh"] & \arrow[d, phantom, "d_h\Downarrow"] & DZ \arrow[dr, "z"] & \\
& X \arrow[ur, "dX"'] \arrow[rr, "h"'] & & Z \arrow[ur, "dZ"'] \arrow[rr, "Id_Z"'] & \arrow[u, phantom, "\zeta\Downarrow"] & Z
\end{tikzcd}
$$

if and only if

$$\begin{tikzcd} X \arrow[r, "dX"] \arrow[dr, "Id_X"'] & DX \arrow[r, "Dh"] \arrow[d, "x"] & DZ \arrow[d, "z"] \arrow[dl, phantom, "\overset{\rho}{\Longleftarrow}"] \\ \arrow[ur, phantom, "\overset{\psi}{\Leftarrow}"] & X \arrow[r, "h"'] & Z \end{tikzcd} \quad = \quad \begin{tikzcd} & DX \arrow[dr, "Dh"] \arrow[dd, phantom, "d_h \Downarrow"] & & \\ X \arrow[ur, "dX"] \arrow[dr, "h"'] & & DZ \arrow[dr, "z"] \arrow[d, phantom, "\zeta \Downarrow"] & \\ & Z \arrow[ur, "dZ"] \arrow[rr, "Id_Z"'] & & Z. \end{tikzcd}$$

■

Let $\psi : x \circ dX \longrightarrow Id_X$ be an object in D-$Alg_{\mathcal{X}}$. Let χ_ψ be equal to pasting (7).

10.3. Lemma. *(ψ, χ_ψ) is a $\mathbb{D}$-algebra.*

Proof. Condition (12) is shown as condition (9) in Proposition 10.1. Condition (13) is Proposition 5.6. ■

Conversely

10.4. Lemma. *If (ψ, χ) is a $\mathbb{D}$-algebra with $\psi : x \circ dX \longrightarrow Id_X$, then ψ is a* D*-algebra and $\chi = \chi_\psi$ (pasting 7).*

Proof. To show that ψ is a D-algebra it suffices to show that the pasting in Proposition 3.7 is the identity on x. Substitute pasting (5) for $\widehat{\psi}$. Paste with χ and its inverse. Use (13) on the pasting of $D\psi^{-1}$ and χ. By Lemma 3.3 the pasting of ηX and δX is βX^{-1}. Now use (16). The condition for χ follows from Lemma 10.2 and (16). ■

10.5. Lemma. *Let $\psi : x \circ dX \longrightarrow Id_X$ and $\zeta : z \circ dZ \longrightarrow Id_Z$ be objects and $h : \psi \longrightarrow \zeta$ be a 1-cell in* D*-$Alg_{\mathcal{X}}$. Define ρ_h as pasting (8). Then we have that $(h, \rho_h) : (\psi, \chi_\psi) \longrightarrow (\zeta, \chi_\zeta)$ is a 1-cell in $\mathbb{D}$-$Alg_{\mathcal{X}}$.*

Proof. Condition (14) follows immediately from the definition of ρ_h. The proof of (15) is very similar to the proof of condition (9) in Proposition 10.1. ■

Conversely

10.6. Lemma. *If $(h, \rho) : (\psi, \chi) \longrightarrow (\zeta, \theta)$ is a 1-cell in $\mathbb{D}$-$Alg_{\mathcal{X}}$, then $h : \psi \longrightarrow \zeta$ is a 1-cell in* D*-$Alg_{\mathcal{X}}$ and $\rho = \rho_h$ (pasting (8)).* ■

The situation for 2-cells is similar. We thus have

10.7. Theorem. *If we define $\Phi :$ D-$Alg_{\mathcal{X}} \longrightarrow \mathbb{D}$-$Alg_{\mathcal{X}}$ such that for every $\xi : h \longrightarrow h' : \psi \longrightarrow \zeta$ in* D*-$Alg_{\mathcal{X}}$ we have $\Phi(\psi) = (\psi, \chi_\psi)$, $\Phi(h) = (h, \rho_h)$ and $\Phi(\xi) = \xi$, we obtain a 2-isomorphism.* ■

It can also be shown that the Gray-categories D-Alg and $\mathbb{D}$-Alg are isomorphic.

11. Pseudomonads vs. KZ-doctrines

In [13], the leftmost adjoint in the definition of KZ-doctrine is explicitly excluded. A question raised in [9] asks whether it can be put back on. The answer given here is in the affirmative.

11.1. Theorem. *If $\mathbb{D} = (D, d, m, \beta, \eta, \mu)$ is a pseudomonad on an object $\mathcal{K}$ of a* Gray-*category* $\mathbf{A}$*, then, the following statements are equivalent*

1. *$m \dashv dD$ with counit β.*
2. *$Dd \dashv m$ with unit η.*

Proof. Assume $\alpha, \beta : m \dashv dD$. Notice that we can still define δ as

$$
\begin{tikzcd}
& {} \arrow[rr] \arrow[dr] \arrow[d, phantom, "\eta^{-1}\Downarrow"] & & {} . \\
{} \arrow[ur] \arrow[rr] & {} & {} \arrow[ur] \arrow[u, phantom, "\alpha\Downarrow"] &
\end{tikzcd}
$$

Define ϵ as the pasting

$$
\begin{tikzcd}
& D \arrow[dr, "Dd"] \arrow[d, phantom, "m_d \Downarrow"] & \\
DD \arrow[ur, "m"] \arrow[r, bend left, "DDd"] \arrow[r, bend right, "DdD"'] \arrow[r, phantom, "D\delta\Downarrow"] \arrow[rr, bend right=40, "mD"'] & DDD \arrow[r] \arrow[d, phantom, "\eta D^{-1}\Downarrow"] & DD. \\
& {} &
\end{tikzcd}
$$

Then ϵ is the counit for an adjunction $\eta, \epsilon : Dd \dashv m$. The converse follows similarly or by duality. ∎

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