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LAX OPERAD ACTIONS AND COHERENCE FOR MONOIDAL N-CATEGORIES, A_∞ RINGS AND MODULES

GERALD DUNN

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ABSTRACT. We establish a general coherence theorem for lax operad actions on an n-category which implies that an n-category with such an action is lax equivalent to one with a strict action. This includes familiar coherence results (e.g. for symmetric monoidal categories) and many new ones. In particular, any braided monoidal n-category is lax equivalent to a strict braided monoidal n-category. We also obtain coherence theorems for A_∞ and E_∞ rings and for lax modules over such rings. Using these results we give an extension of Morita equivalence to A_∞ rings and some applications to infinite loop spaces and algebraic K-theory.

Introduction

The purpose of this paper is to introduce the idea of a lax operad action on an n-category and to show how it can be used to simplify the coherence conditions for an n-category with an algebraic structure. This approach leads to coherence theorems for lax monoidal n-categories (Theorems 1.7, 3.13 and 3.14). Coherence for braidings and other forms of higher commutativity for lax monoidal structures are also covered by these theorems.

It is well known that the categories **Mon**, **SymMon** and **BrMon** of monoidal, symmetric monoidal and braided monoidal categories each has the structure of a 2-category. We show in Theorem 1.6 that the 2-category structure can be described more simply in terms of lax braided cat-operad actions. For example, there is a braided cat-operad $\widetilde{\mathcal{B}}$ (see 1.2) such that giving a lax action by $\widetilde{\mathcal{B}}$ on a category A is equivalent to giving a braided monoidal structure on A. Similarly, a strict $\widetilde{\mathcal{B}}$ action on A corresponds to a strict braided monoidal structure on A. Thus a coherence theorem for braided monoidal categories can be approached via $\widetilde{\mathcal{B}}$ actions. The monoidal and symmetric monoidal cases arise from different choices of operad. In Theorem 1.7 we establish a coherence result for lax $\mathcal{C}$-categories for any braided cat-operad $\mathcal{C}$. In particular this proves coherence for monoidal, symmetric monoidal and braided monoidal categories.

In section 2 we extend these results to categories with a coherent ring structure which we call lax $\mathcal{G}$-ring categories. The main coherence result here, Theorem 2.5, implies that a braided (resp. symmetric) bimonoidal category is lax equivalent to a strict braided bimonoidal (resp. bipermutative) category. We also give some applications of Theorem

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2.5. For example we show that the K-theory of a lax $\mathcal{G}$-ring category is a ring spectrum (Theorem 2.8).

In [8] Kapranov and Voevodsky define the notions of lax monoidal 2-category and of braiding of such an object. They show that this is a natural setting for defining Yang-Baxter and tetrahedra equations and that lax structures provide solutions for these equations. The idea of lax operad action allows these structures to be defined in a much simpler and more conceptual way (see Remarks 3.10). In section 3 we define lax monoidal n-category in this way and establish some coherence results, Theorems 3.13 and 3.14. Not only does this point of view simplify some of the basic notions of [8], but it also seems to be the proper setting for higher dimensional Yang-Baxter equations.

The proofs of the coherence theorems in sections 1 through 3 follow the same pattern since in each case the coherence conditions are encoded by a lax operad action. For example, the standard coherence result for monoidal categories due to Isbell (see [12;4.2]) is the special case of Theorem 1.7 for the trivial A_∞ operad $\mathcal{M}$, [11]. Essentially, the proof of Theorem 1.7 (in the monoidal case) is the proof of [12;4.2] "paramaterized" by the operad $\mathcal{M}$. Theorem 1.7 follows from the observation that this works more generally for any braided cat-operad and some additional considerations for the 2-category structure. Similarly, Theorem 2.5 is a parameterized version of [13;VI,3.5]. This idea also extends to n-categories with a coherent algebraic structure (Theorems 3.13 and 3.14). All n-categories considered in this paper are assumed to be topological (see 3.1) because of the following considerations. If $\mathcal{C}$ is a braided cat-operad and the n-category A has a lax $\mathcal{C}$ action, then our coherence theorems give an n-category $\mathcal{P}A$ having a strict $\mathcal{C}$ action which is lax $\mathcal{C}$-equivalent to A. The n-category $\mathcal{P}A$ will not in general be topologically discrete even if A is. Moreover, the applications in section 2 and elsewhere require the topological structure. However, the reader not interested in the topological case need only assume that the categories $\mathcal{C}_j$ comprising $\mathcal{C}$ are topologically discrete, for then $\mathcal{P}A$ will be discrete whenever A is. In particular there are nontopological versions of the coherence theorems for monoidal, braided monoidal and symmetric monoidal n-categories.

In section 4 we define (lax) modules over A_∞ and E_∞ rings R and develop some of their basic properties. The coherence conditions for a lax R-module are not encoded by an operad action, but can be expressed very simply using k-symmetric monoidal functors and natural transformations. These are functors and natural transformations of $k$variables that are symmetric monoidal in each variable and satisfy some additional compatibility conditions. We rely heavily on k-symmetric monoidal structures throughout this section.

The main goal is to set up a useful 2-category of modules having tensor product and hom 2-functors with reasonable properties. The first step is to prove a coherence result (Theorem 4.7). This and the use of k-symmetric monoidal structures makes the construction of tensor product and hom 2-functors more manageable. Next, one would like to establish some basic properties, such as adjointness, but virtually all of the expected results are unavailable using just the 2-category structure. For example, the tensor and hom 2-functors are lax (but not strictly) 2-adjoint, so we must also consider the 3-category structure. The relevant terminology is given in Definition 4.9. It should be pointed out that all of the coherence results of this paper, except Theorem 3.13, also refer to a 3-category structure, though this has been suppressed (through Theorem 4.7) for the sake of concreteness. See the discussion after Definition 4.9 for the proper statements.

In section 5 we use the constructions of section 4 to prove a version of Morita equivalence for A_∞ rings, Theorems 5.8 and 5.9. These results also depend on a 3-category structure, but unlike the coherence theorems they are essentially impossible to state without the terminology of Definition 4.9.

The material on modules was developed with a view towards applications in algebraic K-theory and stable homotopy, some of which will appear in [5] and are indicated in brief remarks at the ends of sections 4 and 5.

Throughout this paper **Cat** will denote the 2-category of (small) based topological categories; functors and natural transformations are required to be based and continuous. The symbols $\dot{\cong}$ and $\cong$ will denote equivalence and isomorphism of 2-categories respectively. The 2-categories with 0-cells the strict monoidal, permutative and strict braided monoidal categories are denoted st**Mon**, **Perm** and **Braid** respectively. The 1-cells are the strict morphisms. The 1-cells F of **Mon**, **SymMon** and **BrMon** satisfy $F(a \oplus b) \cong Fa \oplus Fb$ and (since F is based) $F0 = 0$. We assume the reader is familiar with the basic ideas concerning 2-categories, [10], and operads, [11], although very little of either is actually needed here. In the latter case all of the relevant definitions are given in section 1.

1. Lax braided cat-operad actions

Let $p : B_j \longrightarrow \Sigma_j$ be the usual homomorphism from the jth braid group to the jth symmetric group.

1.1. DEFINITION. *A* braided cat-operad $\mathcal{C}$ *consists of unbased categories* $\mathcal{C}_j$, $j \geq 0$ *with* $\mathcal{C}_0$ *the trivial category, functors* $\gamma : \mathcal{C}_k \times \mathcal{C}_{j_1} \times \ldots \times \mathcal{C}_{j_k} \longrightarrow \mathcal{C}_{\Sigma j_i}$ *and functorial actions* $\mathcal{C}_j \times B_j \longrightarrow \mathcal{C}_j$ *satisfying*

(i) $\gamma(u; \gamma(u_1; u_{11}, \ldots, u_{1\,j_1}), \ldots, \gamma(u_k; u_{k1}, \ldots, u_{k\,j_k})) = \gamma(\gamma(u; u_1, \ldots, u_k); u_{11}, \ldots, u_{k\,j_k})$

(ii) there is an object 1 *in* $\mathcal{C}_1$ *such that* $\gamma(1_1; u) = u$ *and* $\gamma(u; 1_1, \ldots 1_1) = u$

(iii) $\gamma(u\,\sigma; u_1, \ldots, u_k) = \gamma(u; u_{p\sigma^{-1}(1)}, \ldots, u_{p\sigma^{-1}(k)})\,\sigma(j_1, \ldots, j_k)$
$\gamma(u; u_1\,\tau_1, \ldots, u_k\,\tau_k) = \gamma(u; u_1, \ldots, u_k)\,(\tau_1 \oplus \cdots \oplus \tau_k)$

for morphisms u *in* $\mathcal{C}_k$, u_i *in* $\mathcal{C}_{j_i}$ *and* u_{pq} *in* $\mathcal{C}_{r_{pq}}$, *and* $\sigma \in B_k$, $\tau_i \in B_{j_i}$. *Here* $\sigma(j_1, \ldots, j_k) \in B_{\Sigma\, j_i}$ *is obtained by replacing the* i*th strand of* σ *with* j_i *parallel strands and* $\tau_1 \oplus \cdots \oplus \tau_k \in B_{\Sigma\, j_i}$ *is the usual sum of braids.*

$\mathcal{C}$ *is a* braided operad, *[6], if all morphisms are identities.* $\mathcal{C}$ *is a* cat-operad *if the braid group actions factor through* $p : B_j \longrightarrow \Sigma_j$ *and is an* operad *if in addition all morphisms are identities.*

A morphism $F : \mathcal{C} \longrightarrow \mathcal{C}'$ *of braided cat-operads consists of* B_j*-equivariant functors* $F_j : \mathcal{C}_j \longrightarrow \mathcal{C}'_j$ *such that* $F_1(1) = 1'$ *and* $F(\gamma(u; u_1, \ldots, u_k)) = \gamma'(F(u); F(u_1), \ldots, F(u_k))$. *Morphisms for the other types of operads are defined similarly.*

1.2. Examples. (i) Let $\mathcal{S}_j = \widetilde{\Sigma}_j$, the translation category of Σ_j ; the objects are the elements of Σ_j and there is a unique morphism $\sigma \to \tau$ for $\sigma, \tau \in \Sigma_j$. Thus it suffices to define the cat-operad structure just on objects. The Σ_j-action is given by right multiplication. If $e_j \in \Sigma_j$ is the identity element, let $\gamma(e_k; e_{j_1}, \ldots, e_{j_k}) = e_{\Sigma j_i}$. This determines γ on all objects by the equivariance conditions 1.1 (iii).

(ii) There is a braided cat-operad $\widetilde{\mathcal{B}}$ with $\widetilde{\mathcal{B}}_j = \widetilde{B}_j$, the translation category of B_j. The composition γ for $\widetilde{\mathcal{B}}$ is as for the cat-operad $\mathcal{S}$.

(iii) If $\mathcal{C}$ is any (braided) cat-operad, then taking objects in each degree determines a (braided) operad. For the cat-operad $\mathcal{S}$ this gives the A_∞ operad $\mathcal{M}$ with $\mathcal{M}(j) = \Sigma_j$.

Further examples and relations among the four categories of operads can be found in [3].

For any category A let $t : \mathcal{C}_k \times \mathcal{C}_{j_1} \times \cdots \times \mathcal{C}_{j_k} \times A^{\Sigma j_i} \longrightarrow \mathcal{C}_k \times \mathcal{C}_{j_1} \times A^{j_1} \times \cdots \times \mathcal{C}_{j_k} \times A^{j_k}$ be the obvious isomorphism.

1.3. Definition. *Let* $\mathcal{C}$ *be a braided cat-operad. A* lax $\mathcal{C}$-object *in* **Cat** *is a category* A *with functors* $\theta_j : \mathcal{C}_j \times A^j \longrightarrow A$ *and natural isomorphisms*

$$\sigma : \theta_1(1; -) \longrightarrow 1_A \qquad \alpha : \theta_k \circ (1 \times \theta_{j_1} \times \cdots \times \theta_{j_k}) \circ t \longrightarrow \theta_{\Sigma j_i} \circ (\gamma \times 1)$$

such that

(i) $\theta_k(u\tau; f) = \theta_k(u; \tau \cdot f)$, *for* $\tau \in B_k$, *and morphisms* u *in* $\mathcal{C}_k$ *and* f *in* A^k

(ii) $\theta_0(*; 0) = 0$

(iii) $\alpha((c; 1, \ldots, 1); a_1, \ldots, a_j) = \theta_j(1_c; \sigma(a_1), \ldots, \sigma(a_j))$
$\alpha((1; c); a_1, \ldots, a_j) = \sigma(\theta_j(c; a_1, \ldots, a_j))$

(iv) $\alpha((\gamma(c; c_1, \ldots, c_k); c_{11}, \ldots, c_{k\,j_k}); a_1^{11}, \ldots, a^{k\,j_k}_{r_{k\,j_k}}) \circ$
$\alpha((c; c_1, \ldots, c_k); x_{11}, \ldots, x_{1\,j_1}, \ldots, x_{k1}, \ldots, x_{k\,j_k}) =$
$\alpha((c; \gamma(c_1; c_{11}, \ldots, c_{1\,j_1}), \ldots, \gamma(c_k; c_{k1}, \ldots, c_{k\,j_k})); a_1^{11}, \ldots, a^{k\,j_k}_{r_{k\,j_k}}) \circ$
$\theta_k(1_c; \alpha_1, \ldots, \alpha_k)$

where $x_{pq} = \theta_{r_{pq}}(c_{pq}; a_1^{pq}, \ldots, a^{pq}_{r_{pq}})$ *and* $\alpha_i = \alpha((c_i; c_{i1}, \ldots, c_{i\,j_i}); a_1^{i1}, \ldots, a^{i\,j_i}_{r_{i\,j_i}})$.

A *is a (*strict*)* $\mathcal{C}$-object *if the components of* σ *and* α *are identities.*

1.4. Definition. *Let* A *and* A' *be lax* $\mathcal{C}$*-categories. A functor* $F : A \longrightarrow A'$ *is a* lax $\mathcal{C}$-functor *if there are natural isomorphisms*

$$h : F \circ \theta_j \longrightarrow \theta'_j \circ (1 \times F^j)$$

such that

(i) $h(*;0) = 1_{0'}$

(ii) $\sigma'(Fa) \circ h(1;a) = F\sigma(a)$

(iii) $h(\gamma(c;c_1,\ldots,c_k);a_{11},\ldots,a_{1j_1},\ldots,a_{k1},\ldots,a_{kj_k}) \circ$
$F\alpha((c;c_1,\ldots,c_k);a_{11},\ldots,a_{1j_1},\ldots,a_{k1},\ldots,a_{kj_k}) =$
$\alpha'((c;c_1,\ldots,c_k);Fa_{11},\ldots,Fa_{1j_1},\ldots,Fa_{k1},\ldots,Fa_{kj_k}) \circ$
$\theta'_k(1_c;h(c_1;a_{11},\ldots,a_{1j_1}),\ldots,h(c_k;a_{k1},\ldots,a_{kj_k})) \circ$
$h(c;\theta_{j_1}(c_1;a_{11},\ldots,a_{1j_1}),\ldots,\theta_{j_k}(c_k;a_{k1},\ldots,a_{kj_k}))$

*F is a (*strict*) $\mathcal{C}$-*functor* if the components of h are identities.*

1.5. Definition. *Let $F, F' : A \longrightarrow A'$ be lax $\mathcal{C}$-functors. A* lax $\mathcal{C}$-natural transformation *is a natural transformation $\tau : F \longrightarrow F'$ such that*

$$h'(c;a_1,\ldots,a_j) \circ \tau(\theta_j(c;a_1,\ldots,a_j)) = \theta'_j(1_c;\tau(a_1),\ldots,\tau(a_j)) \circ h(c;a_1,\ldots,a_j)$$

*τ is a (*strict*) $\mathcal{C}$-*natural transformation* if F and F' are $\mathcal{C}$-functors.*

The lax $\mathcal{C}$-categories, lax $\mathcal{C}$-functors and lax $\mathcal{C}$-natural transformations are the 0, 1 and 2-cells of a 2-category denoted $\mathcal{C}(\mathbf{Cat})$. The strict $\mathcal{C}$-categories, $\mathcal{C}$-functors and $\mathcal{C}$-natural transformations form a sub 2-category denoted $\mathcal{C}[\mathbf{Cat}]$.

1.6. Theorem. *There are equivalences and isomorphisms of 2-categories*

(i) $\mathcal{M}(\mathbf{Cat}) \dot{\cong} \mathbf{Mon}$ *and* $\mathcal{M}[\mathbf{Cat}] \cong st\mathbf{Mon}$

(ii) $\mathcal{S}(\mathbf{Cat}) \dot{\cong} \mathbf{SymMon}$ *and* $\mathcal{S}[\mathbf{Cat}] \cong \mathbf{Perm}$

(iii) $\widetilde{\mathcal{B}}(\mathbf{Cat}) \dot{\cong} \mathbf{BrMon}$ *and* $\widetilde{\mathcal{B}}[\mathbf{Cat}] \cong \mathbf{Braid}$

each the identity on underlying 0, 1 and 2-cells.

Proof. We show $\widetilde{\mathcal{B}}(\mathbf{Cat}) \dot{\cong} \mathbf{BrMon}$. If (A,θ) is a lax $\widetilde{\mathcal{B}}$-category define a monoidal structure $\Box$ on A by $a \Box b = \theta_2(e_2;a,b)$ on objects and $f \Box g = \theta_2(1_{e_2};f,g)$ on morphisms. The associativity isomorphism $a\Box(b\Box c) \cong (a\Box b)\Box c$ is given by the isomorphisms

$$\theta_2(e_2;a,\theta_2(e_2;b,c)) \xrightarrow{\alpha_1} \theta_3(e_3;a,b,c) \xleftarrow{\alpha_2} \theta_2(e_2;\theta_2(e_2;a,b),c)$$

where $\alpha_1 = \alpha((e_2;1,e_2);a,b,c)$ and $\alpha_2 = \alpha((e_2;e_2,1);a,b,c)$.

The unit isomorphism $0\Box a \cong a$ is $\sigma(a) \circ \alpha((e_2;*,1);0,a) \circ \theta_2(1_{e_2};1_0,\sigma(a))^{-1}$ and similarly for $a\Box 0 \cong a$. The braiding is the isomorphism

$$\theta_2(e_2 \to \tau;1_a,1_b) : \theta_2(e_2;a,b) \longrightarrow \theta_2(\tau;a,b) = \theta_2(e_2;b,a)$$

where τ is the generator of B_2. This defines a braided monoidal structure on A.

Now suppose $(A,\Box)$ is braided monoidal and define a lax $\widetilde{\mathcal{B}}$-action θ on A as follows. Let $\theta_1(1;a) = a$, $\theta_2(e_2;a_1,a_2) = a\Box b$ and $\theta_j(e_j;a_1,\ldots,a_j) = \theta_2(e_2;a_1,\theta_{j-1}(e_{j-1};a_2,\ldots,a_j))$ for $j \geq 3$.

In view of the equivariance conditions 1.1(iii), in order to define the natural isomorphisms α we need only specify the components

$$\alpha((e_k; e_{j_1}, \ldots, e_{j_k}); a_{11}, \ldots, a_{1\,j_1}, \ldots, a_{k1}, \ldots, a_{k\,j_k}).$$

This is taken to be the unique isomorphism

$$\theta_k(e_k; \theta_{j_1}(e_{j_1}; a_{11}, \ldots, a_{1\,j_1}), \ldots, \theta_{j_k}(e_{j_k}; a_{k1}, \ldots, a_{k\,j_k})) \longrightarrow \theta_j(e_j; a_{11}, \ldots, a_{k\,j_k})$$

provided by coherence, where $j = \Sigma j_i$. The natural isomorphisms σ are taken to be the identity.

This defines 2-functors $\widetilde{\mathcal{B}}(\mathbf{Cat}) \to \mathbf{BrMon} \to \widetilde{\mathcal{B}}(\mathbf{Cat})$ on objects. The correspondence on functors and natural transformations is similar and is left to the reader. ∎

1.7. THEOREM. *Let $\mathcal{U} : \mathcal{C}[\mathbf{Cat}] \longrightarrow \mathcal{C}(\mathbf{Cat})$ be the inclusion. There is a 2-functor $\mathcal{P} : \mathcal{C}(\mathbf{Cat}) \longrightarrow \mathcal{C}[\mathbf{Cat}]$ such that $(\mathcal{P}, \mathcal{U})$ is a 2-adjoint pair. Moreover the unit and counit $\eta(A) : A \longrightarrow \mathcal{UP}(A)$ and $\varepsilon(X) : \mathcal{PU}(X) \longrightarrow X$ are equivalences in $\mathcal{C}(\mathbf{Cat})$ (more precisely $\mathcal{U}(\varepsilon)$ is an equivalence).*

PROOF. For $A \in \mathcal{C}(\mathbf{Cat})$ the category $\mathcal{P}A$ has objects the objects of $\coprod_{j \geq 0} \mathcal{C}_j \times_{B_j} A^j$ and morphisms the triples $(\,(c; a_1, \ldots, a_j), f, (d; b_1, \ldots, b_l)\,)$ with $f : \theta_j(c; a_1, \ldots, a_j) \longrightarrow \theta_l(d; b_1, \ldots, b_l)$ in A. Composition and identities in $\mathcal{P}A$ are induced by that in A.

Define a $\mathcal{C}$-action $\overline{\theta}$ on $\mathcal{P}A$ by

$$\overline{\theta}_k(c; (c_1; a_{11}, \ldots, a_{1\,j_1}), \ldots, (c_k; a_{k1}, \ldots, a_{k\,j_k})) = (\gamma(c; c_1, \ldots, c_k); a_{11}, \ldots, a_{k\,j_k})$$

and for $\overline{f}_i = (\,(c_i; a_{i1}, \ldots, a_{i\,j_i}), f_i, (d_i; b_{i1}, \ldots, b_{i\,l_i})\,)$, $i = 1, \ldots, k$, and $u : c \to d$ in $\mathcal{C}_k$, let

$$\overline{\theta}_k(u; \overline{f}_1, \ldots, \overline{f}_k) = ((\gamma(c; c_1, \ldots, c_k); a_{11}, \ldots, a_{k\,j_k}), \overline{f}, (\gamma(d; d_1, \ldots, d_k); b_{11}, \ldots, b_{k\,l_k})),$$

where

$$\overline{f} = \alpha((d; d_1, \ldots, d_k); b_{11}, \ldots, b_{k\,l_k}) \circ \theta_k(u; f_1, \ldots, f_k) \circ \alpha((c; c_1, \ldots, c_k); a_{11}, \ldots, a_{k\,j_k})^{-1}.$$

If $F : A \longrightarrow A'$ is a lax $\mathcal{C}$-functor, define a strict $\mathcal{C}$-functor $\mathcal{P}F : \mathcal{P}A \longrightarrow \mathcal{P}A'$ by $\mathcal{P}F((c; a_1, \ldots, a_j), f, (d; b_1, \ldots, b_l)) = ((c; Fa_1, \ldots, Fa_j), f', (d; Fb_1, \ldots, Fb_l))$ where $f' = h(d; b_1, \ldots, b_l) \circ Ff \circ h(c; a_1, \ldots, a_j)^{-1}$.

If $\tau : F \longrightarrow F'$ is a lax $\mathcal{C}$-natural transformation, define a strict $\mathcal{C}$-natural transformation $\mathcal{P}\tau : \mathcal{P}F \longrightarrow \mathcal{P}F'$ by

$$\mathcal{P}\tau(c; a_1, \ldots, a_j) = ((c; Fa_1, \ldots, Fa_j), \theta'_j(1_c; \tau(a_1), \ldots, \tau(a_j)), (c; F'a_1, \ldots, F'a_j)).$$

Define 2-natural transformations $\eta : \mathrm{Id} \longrightarrow \mathcal{UP}$ and $\varepsilon : \mathcal{PU} \longrightarrow \mathrm{Id}$ as follows. For a lax $\mathcal{C}$-category A, let $\eta_A(f : a \to b) = ((1; a), \theta_1(1_1; f), (1; b))$, and for a $\mathcal{C}$-category X, let $\varepsilon_X((c; a_1, \ldots, a_j), g, (d; b_1, \ldots, b_l)) = g$.

It is straightforward to verify that $(\mathcal{P}, \mathcal{U})$ is a 2-adjoint pair with unit η and counit ε having the stated properties. ∎

A braided cat-operad determines a monad in **Cat** as follows. This will be needed in section 4.

1.8. Definition. *If $\mathcal{C}$ is a braided cat-operad, the* associated monad $C : \mathbf{Cat} \longrightarrow \mathbf{Cat}$ *is defined on objects by*

$$CA = \coprod_{j \geq 0} \mathcal{C}_j \times_{B_j} A^j / \sim$$

where $(c; a_1, \ldots, a_j) \sim (\gamma(c; 1^{i-1} \times * \times 1^{j-i}); a_1, \ldots, \hat{a}_i, \ldots a_j)$ *if* $a_i = 0$.

The definitions of the unit $\eta : \mathrm{Id} \longrightarrow C$ and multiplication $\mu : CC \longrightarrow C$ are similar to the operad case, [11]. We also note that CA is the free $\mathcal{C}$-object on the category A.

2. Lax $\mathcal{G}$-ring categories

2.1. Definition. *Let $\mathcal{G}$ be a braided cat-operad. A* lax $\mathcal{G}$-ring category $(A, \oplus, \theta)$ *consists of a symmetric monoidal category* $(A, \oplus, 0, \alpha', \gamma')$ *and a lax $\mathcal{G}$-category* $(A, \theta, 1, \sigma, \alpha)$ *satisfying*

(i) $\theta_j(g; f_1, \ldots, f_j) = 1_0$ *if* $f_i = 1_0$ *for some* i.
$\alpha((y; x_1, \ldots, x_k); a_{11}, \ldots, a_{kj_k}) = 1_0$ *if* $a_{rs} = 0$ *for some* r, s.

(ii) There are natural distributivity isomorphisms $\delta^i_j(x; a_1, \ldots, a_i, a'_i, \ldots, a_j)$,$1 \leq i \leq j$,

$$\theta_j(x; a_1, \ldots, a_i \oplus a'_i, \ldots, a_j) \xrightarrow{\delta^i_j} \theta_j(x; a_1, \ldots, a_i, \ldots, a_j) \oplus \theta_j(x; a_1, \ldots, a'_i, \ldots, a_j)$$

such that

(a)
$$\begin{aligned}
&[\alpha((y; x_1, \ldots, x_k); a_{11}, \ldots, a_{kj_k}) \oplus \alpha((y; x_1, \ldots, x_k); a_{11}, \ldots, a'_{il}, \ldots, a_{kj_k})] \circ \\
&\quad \delta^i_k(y; a_1, \ldots, a_i, \theta_{j_i}(x_i; a_{i1}, \ldots, a'_{il}, \ldots, a_{ij_i}), \ldots, a_k) \circ \\
&\quad \theta_k(1_y; 1_{a_1}, \ldots, \delta^l_{j_i}(x_i; a_{i1}, \ldots, a_{il}, a'_{il}, \ldots, a_{ij_i}), \ldots, 1_{a_k}) = \\
&\quad \delta^t_j(\gamma_{\mathcal{G}}(y; x_1, \ldots, x_k); a_{11}, \ldots, a_{il}, a'_{il}, \ldots, a_{kj_k}) \circ \\
&\quad \alpha((y; x_1, \ldots, x_k); a_{11}, \ldots, a_{il} \oplus a'_{il}, \ldots, a_{kj_k})
\end{aligned}$$

where $a_r = \theta_{j_r}(x_r; a_{r1}, \ldots, a_{rj_r})$, $t = \Sigma_{r=1}^{i-1} j_r + l$ *and* $\gamma_{\mathcal{G}}$ *is the operad composition in* $\mathcal{G}$.

(b) $(\sigma a \oplus \sigma b) \circ \delta^1_1(1; a, b) = \sigma(a \oplus b)$

(c) $\delta^i_j(x\tau; a_1, \ldots, a_i, a'_i, \ldots, a_j) = \delta^{p\tau(i)}_j(x; a_{p\tau^{-1}(1)}, \ldots, a_{p\tau^{-1}(k)}, a'_{p\tau^{-1}(k)}, \ldots, a_{p\tau^{-1}(j)})$

where $\tau \in B_j$, $p\tau$ *is the associated permutation and* $p\tau(i) = k$.

(d) $\rho'(\theta_j(x; a_1, \ldots, a_j)) \circ \delta^i_j(x; a_1, \ldots, a_i, 0, \ldots, a_j) = \theta_j(1_x; 1_{a_1}, \ldots, \rho'(a_i), \ldots, 1_{a_j})$
$\lambda'(\theta_j(x; a_1, \ldots, a_j)) \circ \delta^i_j(x; a_1, \ldots, 0, a_i, \ldots, a_j) = \theta_j(1_x; 1_{a_1}, \ldots, \lambda'(a_i), \ldots, 1_{a_j})$
$\delta^i_j(x; a_1, \ldots, a_i, a'_i, \ldots, a_j) = \lambda'(0)^{-1} = \rho'(0)^{-1}$ *if* $a_r = 0$, *for some* $r \neq i$

where $\lambda'(a) : 0 \oplus a \to a$ *and* $\rho'(a) : a \oplus 0 \to a$ *are the unit isomorphisms.*

(e) $\delta_j^i(x;a_1,\dots,a_i',a_i,\dots,a_j)\circ\theta_j(1_x;1_{a_1},\dots,\gamma'(a_i,a_i'),\dots,1_{a_j}) =$
$\gamma'(a,a')\circ\delta_j^i(x;a_1,\dots,a_i,a_i',\dots,a_j)$

where $a=\theta_j(x;a_1,\dots,a_j)$ *and* $a'=\theta_j(x;a_1,\dots,a_i',\dots,a_j)$.

(f) $(\delta_j^i(x;a_1,\dots,a_i,a_i',\dots,a_j)\oplus 1_{a''})\circ\delta_j^i(x;a_1,\dots,a_i\oplus a_i',a_i'',\dots,a_j)\circ$
$\theta_j(1_x;1_{a_1},\dots,\alpha'(a_i,a_i',a_i''),\dots,1_{a_j})=\alpha'(a,a',a'')\circ$
$(1_a\oplus\delta_j^i(x;a_1,\dots,a_i',a_i'',\dots,a_j))\circ\delta_j^i(x;a_1,\dots,a_i,a_i'\oplus a_i'',\dots,a_j)$

with a,a' *as in (e) and* $a''=\theta_j(x;a_1,\dots,a_i'',\dots,a_j)$.

(g) $\alpha'(a\oplus c,b,d)\circ(\delta_j^l(x;a_1,\dots,a_l,a_l',\dots,a_j)\oplus\delta_j^l(x;a_1,\dots,a_k',\dots,a_l,a_l',\dots,a_j))\circ$
$\delta_j^k(x;a_1,\dots,a_k,a_k',\dots,a_l\oplus a_l',\dots,a_j)=$
$(\alpha'(a,c,b)\oplus 1_d)\circ(1_a\oplus\gamma'(b,c)\oplus 1_d)\circ(\alpha'(a,b,c)^{-1}\oplus 1_d)\circ\alpha'(a\oplus b,c,d)\circ$
$(\delta_j^k(x;a_1,\dots,a_k,a_k',\dots,a_j)\oplus\delta_j^k(x;a_1,\dots,a_k,a_k',\dots,a_l',\dots,a_j))\circ$
$\delta_j^l(x;a_1,\dots,a_k\oplus a_k',\dots,a_l,a_l',\dots,a_j)$

for $k<l$*, where* $a=\theta_j(x;a_1,\dots,a_j)$*,* $b=\theta_j(x;a_1,\dots,a_k',\dots,a_j)$*,*
$c=\theta_j(x;a_1,\dots,a_l',\dots,a_j)$ *and* $d=\theta_j(x;a_1,\dots,a_k',\dots,a_l',\dots,a_j)$.

A is a (strict) $\mathcal{G}$-ring category *if $(A,\oplus)$ is permutative and (A,θ) is a $\mathcal{G}$-category.*

2.2. Definition. *A* lax $\mathcal{G}$-ring functor $(F,w,h):A\longrightarrow A'$ *is a symmetric monoidal functor (F,w) and a lax $\mathcal{G}$-functor (F,h) satisfying:*

(i) $h(x;a_1,\dots,a_j)=1_0$*, if* $a_i=0$ *for some* i

(ii) $[h(x;a_1,\dots,a_j)\oplus h(x;a_1,\dots,a_i',\dots,a_j)]\circ$
$w(\theta_j(x;a_1,\dots,a_j),\theta_j(x;a_1,\dots,a_i',\dots,a_j))\circ F\delta_j^i(x;a_1,\dots,a_i,a_i',\dots,a_j)=$
$(\delta')_j^i(x;Fa_1,\dots,Fa_i,Fa_i',\dots,Fa_j)\circ\theta_j'(1_x;1_{Fa_1},\dots,w(a_i,a_i'),\dots,1_{Fa_j})\circ$
$h(x;a_1,\dots,a_i\oplus a_i',\dots,a_j)$

F is a (strict) $\mathcal{G}$-ring functor *of $\mathcal{G}$-ring categories if F is permutative and a $\mathcal{G}$-functor.*

2.3. Definition. *A* lax $\mathcal{G}$-ring natural transformation *of lax $\mathcal{G}$-ring functors is a natural transformation $\tau:F\longrightarrow F'$ such that $\tau:(F,w)\longrightarrow(F',w')$ is a symmetric monoidal natural transformation and $\tau:(F,h)\longrightarrow(F',h')$ is a lax $\mathcal{G}$-natural transformation.*

τ is strict *if F and F' are strict $\mathcal{G}$-ring functors.*

The 2-category of lax $\mathcal{G}$-ring categories is denoted $\mathcal{G}-\mathbf{Rng}(\mathbf{Cat})$ and the 2-category of (strict) $\mathcal{G}$-ring categories is denoted $\mathcal{G}-\mathbf{RngCat}$.

2.4. Examples. (i) Let **SymBiMon** be the 2-category with 0-cells the symmetric bimonoidal categories (see LaPlaza [9], except that here we assume distributivity isomorphisms). We also have the 2-category **BrBiMon** having 0-cells the braided bimonoidal categories. It contains **SymBiMon** as a full sub 2-category. Similarly the 2-category **BiPerm** with 0-cells the bipermutative categories is a full sub 2-category of **BrPerm** with 0-cells the braided permutative categories (such a category is additively permutative and multiplicatively strict braided monoidal). For the braided cat-operads $\mathcal{S}$ and $\widetilde{\mathcal{B}}$ we have equivalences and isomorphisms of 2-categories

$$\mathcal{S}-\mathbf{Rng(Cat)} \mathrel{\dot{\cong}} \mathbf{SymBiMon} \qquad \mathcal{S}-\mathbf{RngCat} \cong \mathbf{BiPerm}$$

$$\widetilde{\mathcal{B}}-\mathbf{Rng(Cat)} \mathrel{\dot{\cong}} \mathbf{BrBiMon} \qquad \tilde{\mathcal{B}}-\mathbf{RngCat} \cong \mathbf{BrPerm}$$

each the identity on underlying 0, 1 and 2-cells. Coherence for these 2-categories is a special case of Theorem 2.5 below.

(ii) If R is a bialgebra over a commutative ring k, then the category R-**Mod** of R-modules is a monoidal category under the tensor product $\otimes = \otimes_k$. Drinfel'd, [2], has shown that R-**Mod** is braided monoidal if there is an invertible element β in $R \otimes R$ satisfying certain conditions. In fact there is a one-to-one correspondence between braidings on R-**Mod** and such elements β, [7]. We say (R, β) is a *braided bialgebra.* In this case the category R-**Mod** is braided bimonoidal with additive operation the direct sum of R-modules and hence is a (large) lax $\widetilde{\mathcal{B}}$-ring category. If we drop the braiding on R, then R-**Mod** is a lax $\mathcal{M}$-ring category.

2.5. Theorem. *Let $\mathcal{U} : \mathcal{G}-\mathbf{RngCat} \longrightarrow \mathcal{G}-\mathbf{Rng(Cat)}$ be the inclusion. There is a 2-functor $\mathcal{P} : \mathcal{G}-\mathbf{Rng(Cat)} \longrightarrow \mathcal{G}-\mathbf{RngCat}$ such that $(\mathcal{P},\mathcal{U})$ is a 2-adjoint pair and the unit η and counit ε are equivalences in $\mathcal{G}-\mathbf{Rng(Cat)}$ (i.e. $\mathcal{U}(\varepsilon)$ is an equivalence).*

Proof. Denote the functors of Theorem 1.7 by $\mathcal{P}'$ and $\mathcal{U}'$. For a lax $\mathcal{G}$-ring category A define the category $\mathcal{P}A$ as follows. Let A_0 be the space *obj* $(\mathcal{P}'A)/\sim$ where $\sim$ is the relation $(c; a_1, \ldots, a_j) \sim 0 \equiv (*; 0)$ if $a_i = 0$ for some i. Let *obj* $(\mathcal{P}A)$ be the free based monoid on A_0 and define $\pi :$ *obj*$(\mathcal{P}A) \longrightarrow$ *obj*A by

$$\pi((c_1; a_{11}, \ldots, a_{1j_1})\Box \cdots \Box(c_k; a_{k1}, \ldots, a_{kj_k})) = a_1 \oplus (a_2 \oplus \cdots (a_{k-1} \oplus a_k) \cdots),$$

where $a_r = \theta_{j_r}(c_r; a_{r1}, \ldots, a_{rj_r})$ for $r = 1, \ldots, k$ and the sum is taken over the $a_r \neq 0$.

For objects a, b in $\mathcal{P}A$ let $\mathcal{P}A(a, b) = \{a\} \times A(\pi(a), \pi(b)) \times \{b\}$. Composition is induced by the composition in A. Note that coherence determines isomorphisms $\pi(a\Box b) \to \pi(a) \oplus \pi(b)$.

For morphisms (a, f, a'),(b, g, b') in $\mathcal{P}A$ let $(a, f, a')\Box(b, g, b') = (a\Box b, f\Box g, a'\Box b')$, where $f\Box g$ is the composite

$$\pi(a\Box b) \longrightarrow \pi(a) \oplus \pi(b) \xrightarrow{f\oplus g} \pi(a') \oplus \pi(b') \longrightarrow \pi(a'\Box b')$$

$\mathcal{P}A$ is a permutative category with commutativity isomorphisms $\overline{\gamma}(a, b)$ the composites

$$\pi(a\Box b) \longrightarrow \pi(a) \oplus \pi(b) \xrightarrow{\gamma'} \pi(b) \oplus \pi(a) \longrightarrow \pi(b\Box a)$$

where $\gamma' = \gamma'(\pi(a), \pi(b))$ is the symmetry of A.

Define a strict $\mathcal{G}$ action $\overline{\theta}$ on $\mathcal{P}A$ as follows. The $\mathcal{G}$ action on $\mathcal{P}'A$ determined by θ as in the proof of Theorem 1.7 passes to an action θ' on A_0. Let $a_1 \ldots, a_k$ be objects in $\mathcal{P}A$, say $a_i = \Box_{j=1}^{r_i} b_{ij}$ with b_{ij} an object of A_0. Now for $c \in \mathcal{G}_k$ let

$$\overline{\theta}_k(c; a_1, \ldots, a_k) = \Box_{(i_1,\ldots,i_k)}\, \theta'_k(c; b_{1i_1}, \ldots, b_{ki_k})$$

where $1 \le i_j \le r_j$ and the $\Box$ sum has the lexicographic order. Then coherence provides isomorphisms $\pi(\overline{\theta}_k(c; a_1, \ldots, a_k)) \to \theta_k(c; \pi(a_1), \ldots, \pi(a_k))$.

For morphisms $\overline{f}_i = (a_i, f_i, a'_i)$ in $\mathcal{P}A$, $i = 1, \ldots, k$ and $u : c \to d$ in $\mathcal{G}_k$ let

$$\overline{\theta}_k(u; \overline{f}_1, \ldots, \overline{f}_k) = (a, f, a')$$

where $a = \overline{\theta}_k(c; a_1, \ldots, a_k)$, $a' = \overline{\theta}_k(d; a'_1, \ldots, a'_k)$ and f is the composite

$$\pi(a) \longrightarrow \theta_k(c; \pi(a_1), \ldots, \pi(a_k)) \xrightarrow{f'} \theta_k(d; \pi(a'_1), \ldots, \pi(a'_k)) \longrightarrow \pi(a')$$

in which $f' = \theta_k(u; f_1, \ldots, f_k)$.

We next define distributivity isomorphisms $\delta_k^i : U \to V$ where $U = \overline{\theta}_k(x; a_1, \ldots, a_i \Box a'_i, \ldots, a_k)$ and $V = \overline{\theta}_k(x; a_1, \ldots, a_i, \ldots, a_k) \Box \overline{\theta}_k(x; a_1, \ldots, a'_i, \ldots, a_k)$. First suppose that each a_r and a'_i are in A_0. Then

$$\begin{aligned}\overline{\theta}_k(x; a_1, \ldots, a_i \Box a'_i, \ldots, a_k) &= \theta'_k(x; a_1, \ldots, a_k) \Box \theta'_k(x; a_1, \ldots, a'_i, \ldots, a_k)\\ &= \overline{\theta}_k(x; a_1, \ldots, a_k) \Box \overline{\theta}_k(x; a_1, \ldots, a'_i, \ldots, a_k)\end{aligned}$$

and we let $\delta_k^i(x; a_1, \ldots, a_i, a'_i, \ldots, a_k) = \mathrm{id}$ for objects in A_0.

In general expanding U and V using the definition of $\overline{\theta}$ we see that each is a $\Box$ sum of objects of $\mathcal{P}'A$ with exactly the same terms, but in different orders. It follows that coherence determines an isomorphism $d : \pi(U) \longrightarrow \pi(V)$ in A and we let $\delta_k^i = (U, d, V)$.

Given a lax $\mathcal{G}$-ring functor $(F, w, h) : A \longrightarrow A'$ define a $\mathcal{G}$-ring functor $\mathcal{P}F : \mathcal{P}A \longrightarrow \mathcal{P}A'$ as follows. On objects let $\mathcal{P}F(\Box_{i=1}^k (c_i; a_{i1}, \ldots, a_{i\,j_i})) = \Box_{i=1}^k (c_i; Fa_{i1}, \ldots, Fa_{i\,j_i})$.

Note that the natural isomorphisms w and h determine an isomorphism $\pi'(\mathcal{P}F(a)) \to F(\pi(a))$. Now for a morphism (a, f, b) in $\mathcal{P}A$ let $\mathcal{P}F(a, f, b) = (\mathcal{P}F(a), f', \mathcal{P}F(b))$ where f' is the composite

$$\pi'(\mathcal{P}F(a)) \longrightarrow F(\pi(a)) \xrightarrow{Ff} F(\pi(b)) \longrightarrow \pi'(\mathcal{P}F(b))$$

If $\tau : F \longrightarrow F'$ is a lax $\mathcal{G}$-ring natural transformation, then for $a = \Box_{i=1}^k (c_i; a_{i1}, \ldots, a_{i\,j_i})$ let $\mathcal{P}\tau(a) = (\mathcal{P}F(a), g, \mathcal{P}F'(a))$ where g is the composite

$$\pi'(\mathcal{P}F(a)) \longrightarrow F(\pi(a)) \xrightarrow{\tau(\pi(a))} F'(\pi(a)) \longrightarrow \pi'(\mathcal{P}F'(a))$$

This defines a strict natural transformation $\mathcal{P}\tau : \mathcal{P}F \longrightarrow \mathcal{P}F'$.

It is straightforward to check that $\mathcal{P} : \mathcal{G}\text{-}\mathbf{Rng}(\mathbf{Cat}) \longrightarrow \mathcal{G}\text{-}\mathbf{RngCat}$ is a 2-functor. The unit $\eta(A) : A \longrightarrow \mathcal{UP}(A)$ and counit $\varepsilon(X) : \mathcal{PU}(X) \longrightarrow X$ are given by $\eta(f : a \to b) = ((1; a), \theta_1(1_1; f), (1; b))$ and $\varepsilon(x, g, y) = g$. ∎

The previous theorem can be used to unify and extend the results of [3] and [4]. It is possible to prove significantly stronger results than those stated here. We only give the weaker versions below in order to avoid excessive technicalities. The reader should refer to [3] and [4] for additional details.

If $\mathcal{G}$ is a braided cat-operad the category of $\mathcal{G}$-spectra is denoted $\mathcal{G}\mathrm{Sp}$ and the zeroth space of a $\mathcal{G}$-spectrum E is denoted E_0. Let $\mathcal{L}$ be the linear isometries operad, [13].

2.6. Theorem. *There is a functor* $W : \mathcal{G}\text{-}\mathbf{Rng}\ (\mathbf{Cat}) \longrightarrow (\mathcal{G} \times \mathcal{L})Sp$ *and an additive group completion* $BA \longrightarrow (WA)_0$.

Proof. A functor $W' : \mathcal{G}\text{-}\mathbf{RngCat} \longrightarrow (\mathcal{G} \times \mathcal{L})\mathrm{Sp}$ and an additive group completion $\eta : BA \longrightarrow (W'A)_0$ were constructed in [3]. Let $W = W' \circ \mathcal{P}$. The group completion is obtained by composing η with the unit of the $(\mathcal{P},\mathcal{U})$ adjunction. ∎

If A is a category with cofibrations and weak equivalences, write $co\,A$ and $w\,A$ for the subcategories of cofibrations and weak equivalences respectively. The K-theory of A is denoted $K(w; A)$.

2.7. Definition. *Let A be a category with cofibrations and weak equivalences, and let $\vee$ denote the symmetric monoidal structure induced by the cofibrations. A* lax $\mathcal{G}$-ring category with cofibrations and weak equivalences *is an object* $(A, \oplus, \theta) \in \mathcal{G}\text{-}\mathbf{Rng}(\mathbf{Cat})$ *such that* $\mathrm{Id} : (A, \oplus) \cong (A, \vee)$ *is a symmetric monoidal natural isomorphism and $co\,A$, $w\,A$ are sub lax $\mathcal{G}$-ring categories of A. The category having these objects is denoted* $\mathcal{G}\text{-}\mathbf{Rng}(\mathbf{Cat})_{cw}$*; the morphisms are the exact functors in* $\mathcal{G}\text{-}\mathbf{Rng}(\mathbf{Cat})$*. The subcategory of objects and morphisms with strict $\mathcal{G}$-structure is denoted* $\mathcal{G}\text{-}\mathbf{RngCat}_{cw}$.

2.8. Theorem. *There is a functor* $K_{\mathcal{G}} : \mathcal{G}\text{-}\mathbf{Rng}(\mathbf{Cat})_{cw} \longrightarrow (\mathcal{G} \times \mathcal{L})Sp$ *such that the additive spectrum of $K_{\mathcal{G}}(w; A)$ is equivalent to the spectrum of $K(w; A)$ (as spectra).*

Proof. This was shown for $\mathcal{G}\text{-}\mathbf{RngCat}_{cw}$ in [4]. The present case is immediate from Theorem 2.5. ∎

3. Lax $\mathcal{C}$ n-categories

The first three definitions are from [14] with slight changes for the topological case. This approach to n-categories is well-suited for our development of lax n-functors and lax n-natural transformations in this section.

3.1. Definition. *A* category *is a space A with maps* $s, t : A \longrightarrow A$ *and* $* : \{(a, b) \in A \times A : t(a) = s(b)\} \longrightarrow A$ *satisfying:*

(i) $ss = ts = s$, $tt = st = t$ *and the inclusion* $A_0 \subseteq A$ *is a cofibration, where* $A_0 = s(A) = t(A)$

(ii) $s(b * a) = s(a)$ *and* $t(b * a) = t(b)$

(iii) $s(a) = t(v) = v$ *implies* $a * v = a$
$u = s(u) = t(a)$ *implies* $u * a = a$

(iv) $s(a) = t(b),\ s(b) = t(c)$ *imply* $a * (b * c) = (a * b) * c$

where $b * a$ *denotes the value of* $*$ *at* (a, b).

We will write (A, μ) for the category $(A, s, t, *)$ where $\mu = (s, t, *)$ is thought of as a partially defined operation on A.

3.2. DEFINITION. *A* 2*-category* (A, μ_0, μ_1) *with* $\mu_r = (s_r, t_r, *_r)$ *consists of categories* (A, μ_0) *and* (A, μ_1) *satisfying:*

(i) $s_1 s_0 = s_0 = s_0 s_1 = s_0 t_1,\ t_0 = t_0 s_1 = t_0 t_1$ *and the inclusions* $A_0 \subseteq A_1 \subseteq A$ *are cofibrations, where* $A_r = s_r(A) = t_r(A)$.

(ii) $s_0(a) = t_0(a')$ *implies* $s_1(a *_0 a') = s_1(a) *_0 s_1(a')$ *and* $t_1(a *_0 a') = t_1(a) *_0 t_1(a')$

(iii) $s_1(a) = t_1(b),\ s_1(a') = t_1(b'),\ s_0(a) = t_0(a')$ *imply*
$$(a *_1 b) *_0 (a' *_1 b') = (a *_0 a') *_1 (b *_0 b')$$

3.3. DEFINITION. *An* n-category $(A, \{\mu_r\}_{r=0}^{n-1})$ *consists of categories* (A, μ_r) *such that* (A, μ_r, μ_s) *is a* 2*-category for* $0 \le r < s \le n-1$*.* $A_r = s_r(A) = t_r(A)$ *is called the space of* r-cells*.*

An n-category $(A, \{\mu_r\}_{r=0}^{n-1})$ determines an $(n-1)$-category $\sigma A = (A, \{\mu_r\}_{r=1}^{n-1})$ with r-cells the $(r+1)$-cells of A. From this we get an $(n-1)$-category $(\sigma A \times_0 \sigma A, \{\mu_r\}_{r=1}^{n-1})$ the pullback of $\sigma A \xrightarrow{t_0} A_0 \xleftarrow{s_0} \sigma A$ with ith composition coordinatewise (and still denoted μ_i). Note that $\mu_0 : \sigma A \times_0 \sigma A \longrightarrow \sigma A$ is an $(n-1)$-functor. We also regard the space of 0-cells as an $(n-1)$-category $(A_0, \{\mu_r\}_{r=1}^{n-1})$ with all cells identities. The inclusion $I_0 : A_0 \longrightarrow \sigma A$ with $I_0(a) = 1_a$ is an $(n-1)$-functor. The full sub $(n-1)$-category of A_0 with one object (the basepoint) is denoted A^0. A *map* $F : A \longrightarrow A'$ of n-categories is a based map compatible with sources and targets. F a based map means $F(1_a) = 1_{Fa}$ for a in A^0. F determines maps $\sigma F : \sigma A \longrightarrow \sigma A'$, $\sigma F \times_0 \sigma F : \sigma A \times_0 \sigma A \longrightarrow \sigma A' \times_0 \sigma A'$ and $F_0 : A_0 \longrightarrow A'_0$ with $F_0(1_a) = 1_{F_0(a)}$.

If $\alpha : A_0 \longrightarrow \sigma A'$ is a map of $(n-1)$-categories we can form composite maps

$$\sigma_s(\alpha, F) : \sigma A \xrightarrow{(s_0, 1_{\sigma A})} A_0 \times_0 \sigma A \xrightarrow{\alpha \times_0 \sigma F} \sigma A' \times_0 \sigma A' \xrightarrow{\mu'_0} \sigma A'$$

$$\sigma_t(F, \alpha) : \sigma A \xrightarrow{(1_{\sigma A}, t_0)} \sigma A \times_0 A_0 \xrightarrow{\sigma F \times_0 \alpha} \sigma A' \times_0 \sigma A' \xrightarrow{\mu'_0} \sigma A'$$

In what follows F will be a lax n-functor and α will be the zeroth component of a lax n-natural transformation. The composite maps above will then be lax $(n-1)$-functors.

To help motivate the definitions of lax n-functor and lax n-natural transformation we first reformulate them in the case $n = 2$. Vertical and horizontal composition in the 2-category **Cat** are denoted by $\bullet$ and $\circ$ respectively.

3.4. Definition. *A* lax 2-functor $(F,\varphi,\eta) : (A,\mu_0,\mu_1) \longrightarrow (A',\mu'_0,\mu'_1)$ *consists of a map* $F : A \longrightarrow A'$ *of 2-categories such that* $\sigma F : \sigma A \longrightarrow \sigma A'$ *is a functor, and natural isomorphisms* $\varphi : \sigma F \circ \mu_0 \longrightarrow \mu'_0 \circ (\sigma F \times_0 \sigma F)$ *and* $\eta : \sigma F \circ I_0 \longrightarrow I'_0 \circ F_0$ *satisfying*

(i) $(\mu'_0 \circ (\varphi \times_0 \sigma F)) \bullet (\varphi \circ (\mu_0 \times_0 1_{\sigma A})) = (\mu'_0 \circ (\sigma F \times_0 \varphi)) \bullet (\varphi \circ (1_{\sigma A} \times_0 \mu_0))$

(ii) $\sigma_t(F,\eta) \bullet (\varphi \circ (1_{\sigma A}, I_0 t_0)) = 1_{\sigma F} = \sigma_s(\eta, F) \bullet (\varphi \circ (I_0 s_0, 1_{\sigma A}))$

F *is a* strict 2-functor *if* $\varphi = \mathrm{id}$ *and* $\eta = \mathrm{id}$.

3.5. Definition. *Let* (F,φ,η), $(F',\varphi',\eta') : (A,\mu) \longrightarrow (A',\mu')$ *be lax 2-functors. A* lax 2-natural transformation $\alpha : (F,\varphi,\eta) \longrightarrow (F',\varphi',\eta')$ *consists of a functor* $\alpha^0 : A_0 \longrightarrow \sigma A'$ *with* $\alpha^0(a) : Fa \longrightarrow F'a$ *and a natural isomorphism* $\sigma\alpha = \alpha^1 : \sigma_s(\alpha^0, F') \longrightarrow \sigma_t(F,\alpha^0)$ *satisfying*

(i) $(\mu'_0 \circ (\sigma F \times_0 \sigma\alpha)) \bullet (\mu'_0 \circ (\sigma\alpha \times_0 \sigma F')) \bullet (\mu'_0 \circ (\alpha^0 s_0, \varphi')) = (\mu'_0 \circ (\varphi, \alpha^0 t_0)) \bullet (\sigma\alpha \circ \mu_0)$

(ii) $(\mu'_0 \circ (\eta, \alpha^0)) \bullet (\sigma\alpha \circ I_0) = \mu'_0 \circ (\alpha^0, \eta')$

α *is a (*strict*)* 2-natural transformation *if the components of* α^1 *are identities in* (A',μ'_1), *i.e.* $\sigma\alpha : \sigma_s(\alpha^0,F') \longrightarrow \sigma_t(F,\alpha^0)$ *is the identity transformation.* α *is a (*lax *or* strict*)* 2-natural isomorphism *if the components of* α^0 *are isomorphisms in* (A',μ'_0).

3.6. Definition. *Let* $\alpha, \beta : (F,\varphi,\eta) \longrightarrow (F',\varphi',\eta')$ *be lax 2-natural transformations. A* modification $\rho : \alpha \longrightarrow \beta$ *consists of a map* $\rho : A_0 \longrightarrow \sigma A'$ *with* $\rho(a) : \alpha^0(a) \longrightarrow \beta^0(a)$ *such that* $\sigma\beta \bullet \sigma_s(\rho, F') = \sigma_t(F,\rho) \bullet \sigma\alpha$.

The definition of modification is included here for convenience and will not be needed until section 4.

We next define lax n-functor, lax n-natural transformation and certain composites of the latter simultaneously by induction on n.

3.7. Definition. *A* lax n-functor $(F,\varphi,\eta) : A \longrightarrow A'$ *consists of a map* $F : A \longrightarrow A'$ *of* n*-categories such that* $\sigma F = (F, (\varphi_i)_{i=1}^{n-2}, (\eta_i)_{i=1}^{n-2}) : \sigma A \longrightarrow \sigma A'$ *is a lax* $(n-1)$*-functor, and lax* $(n-1)$*-natural isomorphisms* $\varphi_0 : \sigma F \circ \mu_0 \longrightarrow \mu'_0 \circ (\sigma F \times_0 \sigma F)$ *and* $\eta_0 : \sigma F \circ I_0 \longrightarrow I'_0 \circ F_0$ *satisfying*

(i) $(\mu'_0 \circ (\varphi_0 \times_0 \sigma F)) \bullet (\varphi_0 \circ (\mu_0 \times_0 1_{\sigma A})) = (\mu'_0 \circ (\sigma F \times_0 \varphi_0)) \bullet (\varphi_0 \circ (1_{\sigma A} \times_0 \mu_0))$

(ii) $\sigma_t(F,\eta_0) \bullet (\varphi_0 \circ (1_{\sigma A}, I_0 t_0)) = 1_{\sigma F} = \sigma_s(\eta_0, F) \bullet (\varphi_0 \circ (I_0 s_0, 1_{\sigma A}))$

F *is a* strict n-functor *if* $\varphi = \mathrm{id}$ *and* $\eta = \mathrm{id}$.

3.8. Definition. *Let* (F,φ,η), $(F',\varphi',\eta') : (A,\mu) \longrightarrow (A',\mu')$ *be lax* n*-functors. A* lax n-natural transformation $\alpha = (\alpha^i)_{i=0}^{n-1} : (F,\varphi,\eta) \longrightarrow (F',\varphi',\eta')$ *consists of an* $(n-1)$*-functor* $\alpha^0 : A_0 \longrightarrow \sigma A'$ *with* $\alpha^0(a) : Fa \longrightarrow F'a$ *and a lax* $(n-1)$*-natural isomorphism* $\sigma\alpha = (\alpha^i)_{i=1}^{n-1} : \sigma_s(\alpha^0, F') \longrightarrow \sigma_t(F,\alpha^0)$ *satisfying*

(i) $(\mu'_0 \circ (\sigma F \times_0 \sigma\alpha)) \bullet (\mu'_0 \circ (\sigma\alpha \times_0 \sigma F')) \bullet (\mu'_0 \circ (\alpha^0 s_0, \varphi'_0)) = (\mu'_0 \circ (\varphi_0, \alpha^0 t_0)) \bullet (\sigma\alpha \circ \mu_0)$

(ii) $(\mu_0' \circ (\eta_0, \alpha^0)) \bullet (\sigma\alpha \circ I_0) = \mu_0' \circ (\alpha^0, \eta_0')$

α is a (strict*)* n-natural transformation *if the components of α^i are identities in (A', μ_i') for $i \geq 1$, i.e. $\sigma\alpha : \sigma_s(\alpha^0, F') \longrightarrow \sigma_t(F, \alpha^0)$ is the identity transformation. α is a* lax *(or* strict*)* n-natural isomorphism *if the components of α^0 are isomorphisms in (A', μ_0').*

The compositions $\bullet$ and $\circ$ in (i) and (ii) are defined inductively as follows. Suppose lax n-functor, lax n-natural transformation and the compositions $\bullet$ and $\circ$ for lax $(n-1)$-natural transformations are defined. If $F, F', F'' : A \longrightarrow A'$ are lax n-functors and $\alpha : F \longrightarrow F'$, $\beta : F' \longrightarrow F''$ are lax n-natural transformations, then the vertical composite $\beta \bullet \alpha : F \longrightarrow F''$ is the lax n-natural transformation defined by $(\beta \bullet \alpha)^0 = \mu_0' \circ (\alpha^0, \beta^0)$ and $\sigma(\beta \bullet \alpha) = \sigma_t(\alpha, \beta^0) \bullet \sigma_s(\alpha^0, \beta)$, where the latter is vertical composition of lax $(n-1)$-natural transformations each of which is the horizontal composite of lax $(n-1)$-natural transformations. For example $\sigma_t(\alpha, \beta^0) = \mu_0' \circ (\sigma\alpha \times_0 \beta^0) \circ (1_{\sigma A}, t_0)$.

If $G : A''' \longrightarrow A$ and $H : A' \longrightarrow A''$ are lax n-functors, then we can define horizontal composites $\alpha \circ G$ and $H \circ \alpha$ as follows. First $\alpha \circ G : F \circ G \longrightarrow F' \circ G$ is the lax n-natural transformation defined by $(\alpha \circ G)^0 = \alpha^0 \circ G_0$ and

$$\sigma(\alpha \circ G) : \sigma_s(\alpha^0 \circ G_0, F' \circ G) = \sigma_s(\alpha^0, F') \circ \sigma G \stackrel{\sigma\alpha\circ\sigma G}{\longrightarrow} \sigma_t(F, \alpha^0) \circ \sigma G = \sigma_t(F \circ G, \alpha^0 \circ G_0)$$

where the identities for σ_s and σ_t are equalities of lax $(n-1)$-functors.

$H \circ \alpha : H \circ F \longrightarrow H \circ F'$ is the lax n-natural transformation defined by $(H \circ \alpha)^0 = \sigma H \circ \alpha^0$ and

$$\sigma(H \circ \alpha) : \sigma_s(\sigma H \circ \alpha^0, H \circ F') \stackrel{g}{\longrightarrow} \sigma H \circ \sigma_s(\alpha^0, F') \stackrel{\sigma H \circ \sigma\alpha}{\longrightarrow} \sigma H \circ \sigma_t(F, \alpha^0) \stackrel{h}{\longrightarrow} \sigma_t(H \circ F, \sigma H \circ \alpha^0)$$

where $g = (\varphi_0)^{-1} \circ (\alpha^0 \times_0 \sigma H) \circ (s_0, 1_{\sigma A'})$, $h = \varphi_0 \circ (\sigma F \times_0 \alpha^0) \circ (1_{\sigma A}, t_0)$ and the composite is vertical composition of lax $(n-1)$-natural transformations.

In defining these compositions for degree n we have implicitly used the fact that each is strictly associative in degree $n-1$. This is obvious for $n = 2$ and can be verified by induction in general.

Let $n-\mathbf{Cat}$ denote the category with objects the n-categories and morphisms the lax n-functors. It is not a 2-category with the lax n-natural transformations as 2-cells. We do get a 2-category $n-\mathbf{Cat}_{st}$ by restricting to strict n-functors and strict n-natural transformations.

If $\mathcal{C}$ is a braided cat-operad let $\gamma(j_1, \ldots, j_k) : \mathcal{C}_k \times \mathcal{C}_{j_1} \times \cdots \times \mathcal{C}_{j_k} \longrightarrow \mathcal{C}_{\Sigma j_i}$ be the composition of $\mathcal{C}$ and $t(j_1, \ldots, j_k) : \mathcal{C}_k \times \mathcal{C}_{j_1} \times \cdots \times \mathcal{C}_{j_k} \times A^{\Sigma j_i} \longrightarrow \mathcal{C}_k \times \mathcal{C}_{j_1} \times A^{j_1} \times \cdots \times \mathcal{C}_{j_k} \times A^{j_k}$ the obvious isomorphism. Also let $I_A : A \longrightarrow \mathcal{C}_1 \times A$ be the n-functor given by $I_A(f) = (1_1, f)$.

3.9. Definition. *Let $\mathcal{C}$ be a braided cat-operad. A* lax $\mathcal{C}$-object in n–$\mathbf{Cat}$ *is an n-category (A, μ) with lax n-functors $(\theta_j, \varphi_j, \eta_j) : \mathcal{C}_j \times A^j \longrightarrow A$ and lax n-natural isomorphisms*

$$\sigma : \theta_1 \circ I_A \longrightarrow 1_A$$

$$\alpha(j_1, \ldots, j_k) : \theta_k \circ (1_{\mathcal{C}_k} \times \theta_{j_1} \times \cdots \times \theta_{j_k}) \circ t(j_1, \ldots, j_k) \longrightarrow \theta_{\Sigma j_i} \circ (\gamma(j_1, \ldots, j_k) \times 1_{A^{\Sigma j_i}})$$

such that

(i) $\theta_k(u\sigma; a) = \theta_k(u; \sigma a)$, *for* $\sigma \in \Sigma_k$, *and morphisms* u *in* $\mathcal{C}_k$ *and* a *in* A^k

(ii) $\theta_0(0; *) = *$ *and* θ_0 *is a strict* n*-functor*

(iii) $\sigma \circ \theta_j = \alpha(j) \circ I$ *where* $I : \mathcal{C}_j \times A^j \longrightarrow \mathcal{C}_1 \times \mathcal{C}_j \times A^j$ *with* $I(u, f) = (1_1, u, f)$

(iv) $\theta_j \circ (\sigma \times \cdots \times \sigma) = \alpha(1, \ldots, 1) \circ J$ *where* $J : \mathcal{C}_j \times A^j \longrightarrow \mathcal{C}_j \times (\mathcal{C}_1)^j \times A^j$ *with* $J(u, f) = (u, (1_1)^j, f)$

(v)
$$\begin{aligned}
&[\alpha(r_1, \ldots, r_k) \circ (1_{\mathcal{C}_k} \times \gamma(r_{11}, \ldots, r_{1j_1}) \times 1_{A^{r_1}} \times \cdots \times \gamma(r_{k1}, \ldots, r_{kj_k}) \times 1_{A^{r_k}})] \bullet \\
&[\theta_k \circ (1_{\mathcal{C}_k} \times \alpha(r_{11}, \ldots, r_{1j_1}) \times \cdots \times \alpha(r_{k1}, \ldots, r_{kj_k}))] = \\
&[\alpha(r_{11}, \ldots, r_{kj_k}) \circ (\gamma(j_1, \ldots, j_k) \times 1_{\mathcal{C}_{r_{11}} \times A^{r_{11}} \times \cdots \times \mathcal{C}_{r_{kj_k}} \times A^{r_{kj_k}}}) \circ T^{-1} \circ \\
&(1_{\mathcal{C}_k} \times t(r_{11}, \ldots, r_{1j_1}) \times \cdots \times t(r_{k1}, \ldots, r_{kj_k}))] \bullet \\
&[\alpha(r_1, \ldots, r_k) \circ (1_{\mathcal{C}_k} \times \theta_{r_{11}} \times \cdots \times \theta_{r_{kj_k}}) \circ (1_{\mathcal{C}_k} \times t(r_{11}, \ldots, r_{1j_1}) \times \cdots \times t(r_{k1}, \ldots, r_{kj_k}))]
\end{aligned}$$

where $T : \mathcal{C}_k \times \mathcal{C}_{j_1} \times \cdots \times \mathcal{C}_{j_k} \times \mathcal{C}_{r_{11}} \times A^{r_{11}} \times \cdots \times \mathcal{C}_{r_{kj_k}} \times A^{r_{kj_k}} \longrightarrow \mathcal{C}_k \times \mathcal{C}_{j_1} \times \mathcal{C}_{r_{11}} \times A^{r_{11}} \times \cdots \times \mathcal{C}_{r_{1j_1}} \times A^{r_{1j_1}} \times \cdots \times \mathcal{C}_{j_k} \times \mathcal{C}_{r_{k1}} \times A^{r_{k1}} \times \cdots \times \mathcal{C}_{r_{kj_k}} \times A^{r_{kj_k}}$ *is the obvious isomorphism and* $r_i = \Sigma_{l=1}^{j_i} r_{il}$.

We say A *is a* strict $\mathcal{C}$ n-category *if each* θ_j *is a strict* n*-functor,* $\sigma = \mathrm{id}$ *and* $\alpha = \mathrm{id}$.

3.10. Remarks. (i) A lax $\mathcal{C}$-object in $1-$**Cat** is the same thing as a lax $\mathcal{C}$-object in **Cat**, so there is no conflict with the terminology of section 1.

(ii) The notion of lax (braided) monoidal 2-category as defined in [8] is a special case of Definition 3.9. A lax monoidal 2-category is just a lax $\mathcal{M}$-object in $2-$**Cat** and a lax braided monoidal 2-category is just a lax $\widetilde{\mathcal{B}}$-object in $2-$**Cat**.

3.11. Definition. *Let* A *and* A' *be lax* $\mathcal{C}$ n*-categories. A lax* n*-functor* $F : A \longrightarrow A'$ *is a* lax $\mathcal{C}$ lax n-functor *if there are lax* n*-natural isomorphisms*

$$h(j) : F \circ \theta_j \longrightarrow \theta'_j \circ (1_{\mathcal{C}_j} \times F^j)$$

such that

(i) $h(0) = \mathrm{id}$

(ii) $(\sigma' \circ F) \bullet (h(1) \circ I) = F \circ \sigma$ *where* $I : A \longrightarrow \mathcal{C}_1 \times A$ *with* $I(f) = (1_1, f)$

(iii)
$$\begin{aligned}
&[h(j) \circ (\gamma(j_1, \ldots, j_k) \times 1_{A^j})] \bullet [F \circ \alpha(j_1, \ldots, j_k)] = \\
&[\alpha'(j_1, \ldots, j_k) \circ (1_{\mathcal{C}_j} \times F^j)] \bullet [\theta'_k \circ (1_{\mathcal{C}_k} \times h(j_1) \times \cdots \times h(j_k)) \circ t(j_1, \ldots, j_k)] \bullet \\
&[h(k) \circ (1_{\mathcal{C}_k} \times \theta_{j_1} \times \cdots \times \theta_{j_k}) \circ t(j_1, \ldots, j_k)]
\end{aligned}$$

where $j = \Sigma j_i$.

F *is a (*strict*)* $\mathcal{C}$ *lax* n-functor *if* $h = \mathrm{id}$.

3.12. Definition. *Let $F, F' : A \longrightarrow A'$ be lax $\mathcal{C}$ lax n-functors. A* lax $\mathcal{C}$ lax n-natural transformation *is a lax n-natural transformation $\tau : F \longrightarrow F'$ such that $h'(j) \bullet (\tau \circ \theta_j) = (\theta'_j \circ (1_{\mathcal{C}_j} \times \tau^j)) \bullet h(j)$.*

*If F and F' are strict $\mathcal{C}$ lax n-functors (so that h and h' are identities), then we call τ a (*strict*) $\mathcal{C}$* lax n-natural transformation.

Notation. A lax n-natural transformation $\alpha = (\alpha^i)_{i=0}^{n-1} : F \longrightarrow F'$ determines shifts $\sigma^k\alpha = (\alpha^i)_{i=k}^{n-1} : s_k\alpha \longrightarrow t_k\alpha$, a lax $(n-k)$-natural transformation, for $k = 0, \ldots, n-1$. Thus $s_0\alpha = F$, $s_1\alpha = \sigma_s(\alpha^0, F')$, etc.

The coherence theorem 3.13 below is based on the following construction of an n-category A'_F from a lax n-functor $F : A \longrightarrow A'$. First let $(A'_F)_{n-1} = A_{n-1}$ as $(n-1)$-categories. The n-cells of A'_F are the triples (f, x, g) where f and g are $(n-1)$-cells of A such that $s_r(f) = s_r(g)$ and $t_r(f) = t_r(g)$ for $r = 0, \ldots, n-2$ and $x : Ff \longrightarrow Fg$ is an n-cell of A'.

If (f, x, g) and (f', x', g') are n-cells such that $s_r(f', x', g') = t_r(f, x, g)$ the rth composition is defined as follows. If $r = n-1$, then $f' = g$ and we let $(f', x', g')\Box_r(f, x, g) = (f, x' *_r x, g')$. For $r < n-1$ let $(f', x', g')\Box_r(f, x, g) = (f' *_r f, y, g' *_r g)$ where y is the n-cell described below.

Consider the lax $(n-r-1)$-natural transformation $\varphi_r = (\varphi_r^i)_{i=0}^{n-r-2} : \sigma^{r+1}F \circ \mu_r \longrightarrow \mu'_r \circ (\sigma^{r+1}F \times_r \sigma^{r+1}F)$ and its shifts $\sigma^k\varphi_r = (\varphi_r^i)_{i=k}^{n-r-2} : s_k\varphi_r \longrightarrow t_k\varphi_r$ for $k = 0, \ldots, n-r-2$. We can apply the map $\mu'_r \circ (\sigma^{r+1}F \times_r \sigma^{r+1}F)$ to the pair $(p, p') = ((f, x, g), (f', x', g'))$ and obtain $\mu'_r(x, x') : \mu'_r(Ff, Ff') \longrightarrow \mu'_r(Fg, Fg')$ an n-cell of A'. In a similar way we can apply the shifts of this map to the pair (p, p'). These shifts are $t_0\varphi_r$, $s_1\varphi_r$, $t_2\varphi_r$, etc. Also, if $y' : F\mu_r(f, f') \longrightarrow F\mu_r(g, g')$ is an n-cell we can apply the shifts of $\sigma^{r+1}F \circ \mu_r$ to y'.

Now take $k = n-r-2$ above and consider the natural isomorphism $\sigma^{n-r-2}\varphi_r = \varphi_r^{n-r-2} : s_{n-r-2}\varphi_r \longrightarrow t_{n-r-2}\varphi_r$. If $n-r$ is even, then $s_{n-r-2}\varphi_r$ and $t_{n-r-2}\varphi_r$ are shifts of $\sigma^{r+1}F \circ \mu_r$ and $\mu'_r \circ (\sigma^{r+1}F \times_r \sigma^{r+1}F)$ respectively. In this case there is a unique n-cell $y : F\mu_r(f, f') \longrightarrow F\mu_r(g, g')$ satisfying

$$s_{n-r-2}\varphi_r(y) = \varphi_r^{n-r-2}(g, g')^{-1} *'_{n-1} t_{n-r-2}\varphi_r(p, p') *'_{n-1} \varphi_r^{n-r-2}(f, f').$$

The case when $n-r$ is odd is similar. In principle one could obtain an explicit description of y as a pasting composite by unravelling the iterated shift, but this is extremely tedious. The above descirption has the advantage that checking associativity and the interchange of compositions is relatively painless.

The n-cells that are identities for $\Box_r$ are easy to describe. If f is an r-cell the identity n-cell is $(1_f^{n-r-1}, F(1_f^{n-r-1}), 1_f^{n-r-1})$ where 1_f^{n-r-1} is the identity $(n-1)$-cell on f. To see that this is an identity for $\Box_r$ requires unravelling the shifts for the lax $(n-r-1)$-natural transformation η_r. This completes the construction of A'_F.

Let $\mathcal{C}(n-\mathbf{Cat})$ be the category with 0-cells the lax $\mathcal{C}$ n-categories $(A, \theta, \alpha, \sigma)$ and 1-cells the lax $\mathcal{C}$ lax n-functors (F, h). The subcategory $\mathcal{C}{<}n{-}\mathbf{Cat}{>}$ has 0-cells the strict $\mathcal{C}$ n-categories and 1-cells the strict $\mathcal{C}$ lax n-functors. Let $\mathcal{U} : \mathcal{C}{<}n{-}\mathbf{Cat}{>} \longrightarrow \mathcal{C}(n{-}\mathbf{Cat})$ be the inclusion.

3.13. Theorem. *There is a functor $\mathcal{P} : \mathcal{C}(n\text{–}\mathbf{Cat}) \longrightarrow \mathcal{C}{<}n\text{–}\mathbf{Cat}{>}$ such that $(\mathcal{P},\mathcal{U})$ is an adjoint pair. Moreover the unit and counit $\eta(A) : A \longrightarrow \mathcal{UP}(A)$ and $\varepsilon(X) : \mathcal{PU}(X) \longrightarrow X$ are equivalences in $\mathcal{C}(n-\mathbf{Cat})$ (i.e. $\mathcal{U}(\varepsilon)$ is an equivalence).*

Proof. If A is a 0-cell in $\mathcal{C}(n-\mathbf{Cat})$ let $\mathcal{P}A = A_{\widehat{\theta}}$, the n-category associated to the lax n-functor $\widehat{\theta} : \coprod_{j\geq 0} \mathcal{C}_j \times_{B_j} A^j \longrightarrow A$ with $\widehat{\theta}(u; f_1, \ldots, f_j) = \theta_j(u; f_1, \ldots, f_j)$.

Define a $\mathcal{C}$-action $\overline{\theta}$ on $(\mathcal{P}A)_{n-1}$ as follows. On r-cells, $r < n$, let

$$\overline{\theta}_k(u; (u_1; f_{11}, \ldots, f_{1\,j_1}), \ldots, (u_k; f_{k1}, \ldots, f_{k\,j_k})) = (\gamma(u; u_1, \ldots, u_k); f_{11}, \ldots, f_{k\,j_k})$$

For n-cells $\widehat{x}_i = ((u_i; f_{i1}, \ldots, f_{i\,j_i}), x_i, (v_i; g_{i1}, \ldots, g_{i\,j_i}))$, $i = 1, \ldots, k$ with $s_{n-2}(u_i) = c_i$, $t_{n-2}(u_i) = d_i$, $s_{n-2}(f_{il}) = s_{n-2}(g_{il}) = a_{il}$ and $t_{n-2}(f_{il}) = t_{n-2}(g_{il}) = b_{il}$, and $u : c \longrightarrow d$ in $\mathcal{C}_k$, let $\overline{\theta}_k(u; \widehat{x}_1, \ldots, \widehat{x}_k) = (f, x, g)$ where $f = (\gamma(u; u_1, \ldots, u_k); f_{11}, \ldots, f_{k\,j_k})$, $g = (\gamma(u; v_1, \ldots, v_k); g_{11}, \ldots, g_{k\,j_k})$ and x is the unique n-cell of A such that

$$\begin{aligned}
&[\alpha^{n-2}((d; d_1, \ldots, d_k); b_{11}, \ldots, b_{k\,j_k}) *_{n-2} \theta_k(u; x_1, \ldots, x_k)] *_{n-1}\\
&\alpha^{n-1}((u; u_1, \ldots, u_k); f_{11}, \ldots, f_{k\,j_k}) =\\
&\alpha^{n-1}((u; v_1, \ldots, v_k); g_{11}, \ldots, g_{k\,j_k}) *_{n-1} [x *_{n-2} \alpha^{n-2}((c; c_1, \ldots, c_k); a_{11}, \ldots, a_{k\,j_k})]
\end{aligned}$$

If $(F, h) : A \longrightarrow A'$ is a lax $\mathcal{C}$ lax n-functor, define a strict $\mathcal{C}$ lax n-functor $\mathcal{P}F : \mathcal{P}A \longrightarrow \mathcal{P}A'$ as follows. On r-cells, $r < n$, let $\mathcal{P}F((u; f_1, \ldots, f_j)) = (u; Ff_1, \ldots, Ff_j)$. If $(f, x, g) = ((u; f_1, \ldots, f_j), x, (v; g_1, \ldots, g_j))$ is an n-cell, then we let $\mathcal{P}F(f, x, g) = ((u; Ff_1, \ldots, Ff_j), y,$
$(v; Fg_1, \ldots, Fg_j))$ where y is defined as follows. Consider the shift $\sigma^{n-1}h(j) = h(j)^{n-1} : s_{n-1}h(j) \longrightarrow t_{n-1}h(j)$. If n is even, then $s_{n-1}h(j)$ is a shift of $F \circ \theta_j$ and as in the construction prior to the theorem the n-cell $s_{n-1}h(j)(x)$ is defined. Then y is the unique n-cell such that $y *_{n-1} h(j)^{n-1}(u; f_1, \ldots, f_j) = h(j)^{n-1}(v; g_1, \ldots, g_j) *_{n-1} s_{n-1}h(j)(x)$. y is defined in a similar way if n is odd. The remaining verifications are straightforward. ∎

Theorem 3.13 has the following variant obtained by replacing the underlying category $n-\mathbf{Cat}$ with the 2-category $n-\mathbf{Cat}_{st}$. Thus let $\mathcal{C}(n-\mathbf{Cat}_{st})$ be the 2-category with 0,1 and 2-cells the lax $\mathcal{C}$-objects in $n-\mathbf{Cat}$, lax $\mathcal{C}$ strict n-functors and lax $\mathcal{C}$ strict n-natural transformations, and let $\mathcal{C}[n-\mathbf{Cat}_{st}]$ be the sub 2-category in which the $\mathcal{C}$-structures on 0,1 and 2-cells are strict as well. Let $\mathcal{V} : \mathcal{C}[n-\mathbf{Cat}_{st}] \longrightarrow \mathcal{C}(n-\mathbf{Cat}_{st})$ be the inclusion.

3.14. Theorem. *There is a 2-functor $\mathcal{P} : \mathcal{C}(n-\mathbf{Cat}_{st}) \longrightarrow \mathcal{C}[n-\mathbf{Cat}_{st}]$ such that $(\mathcal{P}, \mathcal{V})$ is a 2-adjoint pair. Moreover the unit and counit $\eta(A) : A \longrightarrow \mathcal{VP}(A)$ and $\varepsilon(X) : \mathcal{PV}(X) \longrightarrow X$ are equivalences in $\mathcal{C}(n-\mathbf{Cat}_{st})$ (i.e. $\mathcal{V}(\varepsilon)$ is an equivalence).*

Proof. The functor $\mathcal{P}$ of Theorem 3.13 is defined on the 0 and 1-cells of $\mathcal{C}(n-\mathbf{Cat}_{st})$, and it extends to the 2-cells in an obvious way. ∎

4. Tensor products for A_∞ rings and modules

4.1. Definition. *Let A,A_1 and A_2 be symmetric monoidal categories. A functor $F : A_1 \times A_2 \longrightarrow A$ is* 2-symmetric monoidal *if there are natural isomorphisms*

$$w^1(a_1, a_2, b) : F(a_1 \oplus a_2, b) \longrightarrow F(a_1, b) \oplus F(a_2, b)$$

$$w^2(a, b_1, b_2) : F(a, b_1 \oplus b_2) \longrightarrow F(a, b_1) \oplus F(a, b_2)$$

such that

(i) $(F(-, b), w^1(-, b))$ and $(F(a, -), w^2(a, -))$ are symmetric monoidal functors

(ii) $w^1(a_1, a_2, -) : F(a_1 \oplus a_2, -) \longrightarrow F(a_1, -) \oplus F(a_2, -)$ is a symmetric monoidal natural transformation (equivalently $w^2(-, b_1, b_2) : F(-, b_1 \oplus b_2) \longrightarrow F(-, b_1) \oplus F(-, b_2)$ is a symmetric monoidal natural transformation)

for all objects $a, a_1, a_2 \in A_1$ and $b, b_1, b_2 \in A_2$.

4.2. Definition. *Let (F, w^1, w^2), $(G, v^1, v^2) : A_1 \times A_2 \longrightarrow A$ be 2-symmetric monoidal functors. A* 2-symmetric monoidal natural transformation *is a natural transformation $h : F \longrightarrow G$ such that $h(-, b) : (F(-, b), w^1(-, b)) \longrightarrow (G(-, b), v^1(-, b))$ and $h(a, -) : (F(a, -), w^2(a, -)) \longrightarrow (G(a, -), v^2(a, -))$ are symmetric monoidal natural transformations for all objects $a \in A_1$ and $b \in A_2$.*

h is a 2-symmetric monoidal natural isomorphism *if h is also a natural isomorphism.*

k-*symmetric monoidal functor* $(F, w^1, \dots, w^k) : A_1 \times \cdots \times A_k \longrightarrow A$ and k-*symmetric monoidal natural transformation* are defined similarly.

If $F_i : A_{i1}, \times \cdots \times A_{i r_i} \longrightarrow B_i$ is an r_i-symmetric monoidal functor for $i = 1, \dots, k$ and $H : B_1 \times \cdots \times B_k \longrightarrow A$ is a k-symmetric monoidal functor, then $H \circ (F_1 \times \cdots \times F_k)$ is a $\Sigma\, r_i$-symmetric monoidal functor. The verification is straightforward.

The use of k-symmetric monoidal structures simplifies the coherence conditions for a lax $\mathcal{G}$-ring category (Definition 2.1).

4.3. Definition. *Let $\mathcal{G}$ be a braided cat-operad. A* lax $\mathcal{G}$-ring category $(A, \oplus, \theta)$ *consists of a symmetric monoidal category $(A, \oplus, 0, \alpha', \gamma')$ and a lax $\mathcal{G}$-category $(A, \theta, 1, \sigma, \alpha)$ satisfying*

(i) $\theta_j(g; f_1, \dots, f_j) = 1_0$ if $f_i = 1_0$ for some i.
$\alpha((y; x_1, \dots, x_k); a_{11}, \dots, a_{k j_k}) = 1_0$ if $a_{rs} = 0$ for some r, s.

(ii) There are natural distributivity isomorphisms $\delta^i_j(x; a_1, \dots, a_i, a'_i, \dots, a_j)$,$1 \le i \le j$,

$$\theta_j(x; a_1, \dots, a_i \oplus a'_i, \dots, a_j) \xrightarrow{\delta^i_j} \theta_j(x; a_1, \dots, a_i, \dots, a_j) \oplus \theta_j(x; a_1, \dots, a'_i, \dots, a_j)$$

such that

(a) $(\theta_j(x; -), \delta^1_j(x; -), \dots, \delta^j_j(x; -))$ is a j-symmetric monoidal functor.

(b) $\alpha : \theta_k \circ (1 \times \theta_{j_1} \times \cdots \times \theta_{j_k}) \circ t \longrightarrow \theta_{\Sigma j_i} \circ (\gamma \times 1)$ *is a* Σj_i*-symmetric monoidal natural transformation.*

(c) $\sigma : (\theta_1(1; -), \delta_1^1(1; -)) \longrightarrow 1_A$ *is a symmetric monoidal natural transformation.*

(d) $(\theta_j(x\tau; -), \delta_j^1(x\tau; -), \ldots, \delta_j^j(x\tau; -)) =$
$(\theta_j(x; p\tau(-)), \delta_j^{p\tau(1)}(x; p\tau(-)), \ldots, \delta_j^{p\tau(j)}(x; p\tau(-)))$
as j*-symmetric monoidal functors, where* $\tau \in B_j$.

*A is a (*strict*)* $\mathcal{G}$-ring category *if* $(A, \oplus)$ *is permutative and* (A, θ) *is a* $\mathcal{G}$*-category, i.e.* $\sigma = \mathrm{id}$ *and* $\alpha = \mathrm{id}$.

From now on we consider only strict $\mathcal{G}$-ring categories which can always be arranged by Theorem 2.5. We also assume $\mathcal{G}$ is one of the braided cat-operads $\mathcal{M}$, $\widetilde{\mathcal{B}}$ or $\mathcal{S}$ of section one. In each case there is a morphism of braided cat-operads $\mathcal{M} \longrightarrow \mathcal{G}$, so any $\mathcal{G}$-ring category has an underlying $\mathcal{M}$-ring category. This assumption is not overly restrictive by a standard change of operad argument, [3;§4]. In fact the most important examples of A_∞ and E_∞ rings are already covered by the above list (*cf.* Examples 2.4). We say a $\mathcal{G}$-ring category satisfies *strict left distributivity* if $\delta_j^i(e_j; a_1, \ldots, a_i, a_i', \ldots, a_j) = \mathrm{id}$ when $a_k = 1$ for $k > i$ (notation as in 4.3). Imposing this condition results in no loss of generality since it holds for the $\mathcal{G}$-ring category $\mathcal{P}A$ of Theorem 2.5.

Let R be a $\mathcal{G}$-ring category. The multiplicative operation $\theta_2(e_2; -) : R \times R \to R$ will be denoted $\mu(r, s)$ or $r \otimes s$. Note that this is a 2-symmetric monoidal functor.

4.4. Definition. *A* lax *(*left*)* R- module *is a symmetric monoidal category* M *with a 2-symmetric monoidal functor* $(\alpha, w^1, w^2) : R \times M \longrightarrow M$*, a symmetric monoidal natural isomorphism* $\sigma : (\alpha(1, -), w^2(1, -)) \longrightarrow Id$ *and a3-symmetric monoidal natural isomorphism* $v : \alpha \circ (1_R \times \alpha) \longrightarrow \alpha \circ (\mu \times 1_M)$ *such that:*

(i) $v(1, s, a) = \sigma(\alpha(s, a))$ *and* $v(r, 1, a) = \alpha(1_r, \sigma(a))$

(ii) $v(r \otimes s, t, a) \circ v(r, s, \alpha(t, a)) = v(r, s \otimes t, a) \circ \alpha(1_r, v(s, t, a))$

M *is an* R-module *if* M *is a permutative category and* $\sigma = \mathrm{id}$*;* M *is a* strict R*-module if in addition* $v = \mathrm{id}$ *and* $w^2 = \mathrm{id}$.

4.5. Definition. *A* lax R-module morphism *is a symmetric monoidal functor* $F : M \longrightarrow M'$ *with a 2-symmetric monoidal natural isomorphism* $h : F \circ \alpha \longrightarrow \alpha' \circ (1_R \times F)$ *such that*

(i) $\sigma'(Fa) \circ h(1, a) = F\sigma(a)$

(ii) $h(r \otimes s, a) \circ Fv(r, s, a) = v'(r, s, Fa) \circ \alpha'(1_r, h(s, a)) \circ h(r, \alpha(s, a))$

F *is an* R-module morphism *if* M *and* M' *are* R*-modules.* F *is a* strict R*-module morphism if* M *and* M' *are strict* R*-modules,* F *is permutative and* $h = \mathrm{id}$.

4.6. Definition. *Let $F, G : M \longrightarrow M'$ be lax R-module morphisms. A* lax R-natural transformation *is a symmetric monoidal natural transformation $\tau : F \longrightarrow G$ such that $h_G(r,a) \circ \tau(\alpha(r,a)) = \alpha'(1_r, \tau(a)) \circ h_F(r,a)$. τ is a* lax R-natural isomorphism *if its components are isomorphisms in M'. τ is an* R-natural transformation *if F and G are R-module morphisms, and τ is* strict *if F and G are strict.*

The 2-categories of lax R-modules, R-modules and strict R-modules are denoted by $R-(\mathrm{Mod})$, $R-\mathrm{Mod}$ and $R-[\mathrm{Mod}]$ respectively. Similarly there are 2-categories of right modules $(\mathrm{Mod})-R$ and $\mathrm{Mod}-R$. Regarding R as a left R-module, the condition of strict left distributivity on R is equivalent to saying R is a strict left R-module, and similarly for right modules. Since a $\mathcal{G}$-ring category will rarely satisfy strict right distributivity at the same time, it is not useful to consider strict right modules.

Let $\mathcal{U} : R-[\mathrm{Mod}] \longrightarrow R-(\mathrm{Mod})$ and $\mathcal{V} : R-[\mathrm{Mod}] \longrightarrow R-\mathrm{Mod}$ be the inclusions.

4.7. Theorem. *There is a 2-functor $\mathcal{P} : R-(\mathrm{Mod}) \longrightarrow R-[\mathrm{Mod}]$ such that $(\mathcal{P},\mathcal{U})$ is a 2-adjoint pair and the unit η and counit ε are equivalences in $R-(\mathrm{Mod})$ (i.e. $\mathcal{U}(\varepsilon)$ is an equivalence). Moreover, $(\mathcal{P},\mathcal{U})$ restricts to a 2-adjoint pair $(\mathcal{P},\mathcal{V})$ between $R-\mathrm{Mod}$ and $R-[\mathrm{Mod}]$ with unit and counit equivalences in $R-\mathrm{Mod}$.*

Proof. For a lax R-module (M, α, σ, v) let M_0 be the smash product $(\mathrm{obj}\, R) \wedge (\mathrm{obj}\, M)$ formed using the basepoints 0_R and 0_M. The space of objects of $\mathcal{P}M$ is the free based monoid on M_0 and we define a map $\pi : \mathrm{obj}\,(\mathcal{P}M) \longrightarrow \mathrm{obj}\, M$ by $\pi((r_1,a_1)\Box\cdots\Box(r_k,a_k)) = \alpha(r_1,a_1) \oplus (\alpha(r_2,a_2) \oplus \cdots (\alpha(r_{k-1},a_{k-1}) \oplus \alpha(r_k,a_k))\cdots)$. $\mathrm{mor}\,(\mathcal{P}M)$ is the space of triples (a,f,b) with $a, b \in \mathrm{obj}\,(\mathcal{P}M)$ and $f : \pi(a) \longrightarrow \pi(b)$ a morphism of M. For objects a, b in $\mathcal{P}M$ coherence determines an isomorphism $\pi(a\Box b) \longrightarrow \pi(a) \oplus \pi(b)$. $\mathcal{P}M$ is a permutative category via $(a,f,b) \oplus (a',f',b') = (a\Box a', g, b\Box b')$ where g is the composite

$$\pi(a\Box a') \longrightarrow \pi(a) \oplus \pi(a') \xrightarrow{f\oplus f'} \pi(b) \oplus \pi(b') \longrightarrow \pi(b\Box b')$$

The commutativity isomorphism $\overline{\gamma}$ has components $(a\Box b, \overline{\gamma}(a,b), b\Box a)$ where $\overline{\gamma}(a,b)$ is the composite

$$\pi(a\Box b) \longrightarrow \pi(a) \oplus \pi(b) \xrightarrow{g} \pi(b) \oplus \pi(a) \longrightarrow \pi(b\Box a)$$

with $g = \gamma(\pi(a), \pi(b))$, the commutativity isomorphism of M.

R acts on the objects of $\mathcal{P}M$ by $\overline{\alpha}(r, (r_1,a_1)\Box\cdots\Box(r_k,a_k)) = (r\otimes r_1, a_1)\Box\cdots\Box(r\otimes r_k, a_k)$, and coherence gives isomorphisms $\pi(\overline{\alpha}(r,a)) \longrightarrow \alpha(r, \pi(a))$. On morphisms we let $\overline{\alpha}(u, (a,f,b)) = (\overline{\alpha}(r,a), \overline{f}, \overline{\alpha}(s,b))$ where $u : r \to s$ in R and $\overline{f}$ is the composite

$$\pi(\overline{\alpha}(r,a)) \longrightarrow \alpha(r,\pi(a)) \xrightarrow{\alpha(u,f)} \alpha(s,\pi(b)) \longrightarrow \pi(\overline{\alpha}(s,b))$$

We have a 2-symmetric monoidal functor $(\overline{\alpha}, \overline{w}^1, \overline{w}^2)$ with $\overline{w}^2 = id$ and $\overline{w}^1$ having components $\overline{w}^1(r,s,a) = (\overline{\alpha}(r\oplus s, a), g, \overline{\alpha}(r,a)\Box\overline{\alpha}(s,a))$ where g is the composite

$$\pi(\overline{\alpha}(r\oplus s,a)) \longrightarrow \alpha(r\oplus s, \pi(a)) \xrightarrow{w^1(r,s,\pi(a))} \alpha(r,\pi(a)) \oplus \alpha(s,\pi(a))$$
$$\longrightarrow \pi(\overline{\alpha}(r,a)) \oplus \pi(\overline{\alpha}(s,a)) \longrightarrow \pi(\overline{\alpha}(r,a)\Box(\overline{\alpha}(s,a))$$

This defines a strict R-module structure on $\mathcal{P}M$.

If $F : M \longrightarrow M'$ is a lax R-module morphism define $\mathcal{P}F : \mathcal{P}M \longrightarrow \mathcal{P}M'$ on objects by $\mathcal{P}F((r_1, a_1)\Box \ldots \Box(r_k, a_k)) = (r_1, Fa_1)\Box \ldots \Box(r_k, Fa_k)$ and note that coherence provides isomorphisms $\pi'(\mathcal{P}F(a)) \longrightarrow F(\pi(a))$. For (a, f, b) in $\mathcal{P}M$ let $\mathcal{P}F(a, f, b) = (\mathcal{P}F(a), \overline{f}, \mathcal{P}F(b))$ where $\overline{f}$ is the composite

$$\pi'(\mathcal{P}F(a)) \longrightarrow F(\pi(a)) \xrightarrow{F(f)} F(\pi(b)) \longrightarrow \pi'(\mathcal{P}F(b))$$

$\mathcal{P}F$ is a strict R-module morphism.

If $\tau : F \longrightarrow G$ is a lax R-natural transformation let $\mathcal{P}\tau : \mathcal{P}F \longrightarrow \mathcal{P}G$ be the strict R-natural transformation with components $\mathcal{P}\tau(a) = (\mathcal{P}F(a), g, \mathcal{P}G(a))$ where g is the composite

$$\pi'(\mathcal{P}F(a)) \longrightarrow F(\pi(a)) \xrightarrow{\tau(\pi(a))} G(\pi(a)) \longrightarrow \pi'(\mathcal{P}G(a))$$

The unit and counit are defined as in the proof Theorem 2.5 or 1.7. The remaining verifications are routine. ■

4.8. Example. The commutative monoid $S^0 = \{0, 1\}$ is an $\mathcal{S}$-category with zero and hence $S(S^0)$ is an $\mathcal{S}$-ring category (see [3;§3]) where S is the monad associated to $\mathcal{S}$. It satisfies strict left distributivity. $S(S^0)$ is the analogue for (lax) modules of the ring of integers. If R is a $\mathcal{G}$-ring category (satisfying strict left distributivity), then the inclusion $S^0 \hookrightarrow R$ extends to a permutative functor $S(S^0) \longrightarrow R$ which is in fact a morphism of $\mathcal{G}$-ring categories. Thus an R-module has an underlying $S(S^0)$-module. Also, there are equivalences of 2-categories

$$S(S^0)-(\mathrm{Mod}) \dot{\cong} \mathbf{SymMon} \qquad\qquad S(S^0)-\mathrm{Mod} \dot{\cong} \mathbf{Perm}$$

For the latter let $J : \mathrm{Mod}-S(S^0) \longrightarrow \mathbf{Perm}$ be the 2-functor that sends a right $S(S^0)$-module to its underlying permutative category and define $I : \mathbf{Perm} \longrightarrow \mathrm{Mod}-S(S^0)$ by $I(M) = (M, \alpha)$ where $\alpha(a, [e_n; 1, \ldots, 1]) = a \oplus \cdots \oplus a$ (n terms). $I(M)$ is an $S(S^0)$-module with $v = id$ (but is not strict). Then $J \circ I = id$ and $I \circ J \cong id$. Now combine this equivalence with the isomorphism $S(S^0)-\mathrm{Mod} \cong \mathrm{Mod}-S(S^0)$ (see the discussion prior to 4.16).

4.9. Definition. *(i) Let $F, F' : A \longrightarrow A'$ be lax 2-functors and $\sigma : F \longrightarrow F'$ a lax 2-natural transformation. σ is a* lax 2-natural equivalence *if there is a lax 2-natural transformation $\tau : F' \longrightarrow F$ and modifications $\rho : \tau \circ \sigma \longrightarrow 1_F$ and $\lambda : \sigma \circ \tau \longrightarrow 1_{F'}$ whose component 2-cells $\rho(a)$, $\lambda(a')$ are isomorphisms in (A', μ'_1).*

(ii) Let $F : A \longrightarrow A'$ and $G : A' \longrightarrow A$ be lax 2-functors. We say (F, G) is a lax 2-adjoint pair *if there are lax 2-natural transformations $\eta : Id \longrightarrow G \circ F$ and $\varepsilon : F \circ G \longrightarrow Id$, and modifications $\rho : \varepsilon F \circ F\eta \longrightarrow 1_F$ and $\lambda : G\varepsilon \circ \eta G \longrightarrow 1_G$ whose components $\rho(a)$, $\lambda(a')$ are isomorphisms in (A', μ'_1) and (A, μ_1) respectively. Note that for F, G strict 2-functors, (F, G) is a 2-adjoint pair when η and ε are strict and the components $\rho(a)$, $\lambda(a')$ are identities.*

If also η and ε are lax 2-natural equivalences we say (F, G) is a lax 2-adjoint, lax 2-equivalence. *If we only have $F\eta$ and $G\varepsilon$ lax 2-natural equivalences we say (F, G) is a* lax 2-adjoint, weak lax 2-equivalence. *Finally, if we drop the adjointness condition, but still assume η and ε are lax 2-natural equivalences, we say F (or G) is a* lax 2-equivalence *of the 2-categories A and A'. If the functors or transformations are strict we drop the adjective "lax".*

The coherence theorems in sections 1 through 4 (except for Theorem 3.13) say that $(\mathcal{P}, \mathcal{U})$ is a 2-adjoint, weak 2-equivalence. In Theorem 4.7, for example, the unit η is a 2-equivalence while $\mathcal{U}\varepsilon$ is a 2-equivalence but ε is not. Note also that equivalence of 2-categories is the special case of 2-equivalence when all four modifications are identities.

We will shortly construct tensor products for R-modules. This is much simpler than for lax R-modules and is no real limitation by the previous theorem. On the other hand, some basic properties of the tensor product fail in R–[Mod], but hold in R–Mod (e.g. 4.17 (i)). The reverse is true in other situations, free modules for example, so it is necessary to work in both categories.

4.10. Definition. *Let $R, T \in \mathcal{G}-\mathbf{RngCat}$ with both multiplicative operations denoted by $\otimes$. An* R–T-bimodule *is a permutative category M that is a left R-module (M, α_R, v_R) and a right T-module (M, α_T, v_T) with a 3-symmetric monoidal natural isomorphism $v_{R,T}$: $\alpha_T \circ (\alpha_R \times 1_T) \longrightarrow \alpha_R \circ (1_R \times \alpha_T)$ such that:*

(i) $v_{R,T}(r, a, 1) = 1_{\alpha_R(r,a)}$ *and* $v_{R,T}(1, a, t) = 1_{\alpha_T(a,t)}$

(ii) $v_{R,T}(r_1 \otimes r_2, a, t_2 \otimes t_1) \circ v_T(\alpha_R(r_1 \otimes r_2, a), t_2, t_1) \circ \alpha(\alpha_T(v_R(r_1, r_2, a), 1_{t_2}, 1_{t_1}) =$
$v_R(r_1, r_2, \alpha_T(a, t_2 \otimes t_1)) \circ \alpha_R(1_{r_1}, \alpha_R(1_{r_2}, v_T(a, t_2, t_1))) \circ$
$\alpha_R(1_{r_1}, v_{R,T}(r_2, \alpha_T(a, t_2), t_1)) \circ \alpha_R(1_{r_1}, \alpha_T(v_{R,T}(r_2, a, t_2), 1_{t_1})) \circ$
$v_{R,T}(r_1, \alpha_T(\alpha_R(r_2, a), t_2), t_1) \circ \alpha_T(v_{R,T}(r_1, \alpha_R(r_2, a), t_2), 1_{t_1})$

$F : M \longrightarrow M'$ *is an* $R-T$-bimodule morphism *if it is a left R-module morphism (F, h_R) and a right T-module morphism (F, h_T) such that*

$$v'_{R,T}(r, Fa, t) \circ \alpha'_T(h_R(r, a), 1_t) \circ h_T(\alpha_R(r, a), t) = \alpha'_R(1_r, h_T(a, t)) \circ h_R(r, \alpha_T(a, t)) \circ Fv_{R,T}(r, a, t)$$

An $R-T$-natural transformation *is a symmetric monoidal natural transformation τ : $F \longrightarrow G$ of $R-T$-bimodule morphisms that is both R-natural and T-natural.*

The 2-category of $R-T$-bimodules is denoted $R-\mathrm{Mod}-T$.

4.11. Definition. *Let M be a right R-module, N a left R-module and P a permutative category. An* R-biadditive functor *is a 2-symmetric monoidal functor $(F, w^1, w^2) : M \times N \longrightarrow P$ with a 3-symmetric monoidal natural isomorphism*

$$h : F \circ (\alpha_M \times 1_N) \longrightarrow F \circ (1_M \times \alpha_N)$$

such that

(i) $h(-,1,-) = 1_F$

(ii) $h(a, r\otimes s, b)\circ F(v_M(a,r,s),1_b) =$
$F(1_a, v_N(r,s,b))\circ h(a,r,\alpha_N(s,b))\circ h(\alpha_M(a,r),s,b)$

*Suppose R is an $\mathcal{S}$-ring category, P is a left R-module and let $t : M\times R\times N \longrightarrow R\times M\times N$ be the obvious isomorphism. An R-*bilinear functor *is a 2-symmetric monoidal functor $(F, w^1, w^2) : M\times N \longrightarrow P$ with 3-symmetric monoidal natural isomorphisms*

$$h^1 : F\circ(\alpha_M\times 1_N) \longrightarrow \alpha_P\circ(1_R\times F)\circ t$$

$$h^2 : F\circ(1_M\times \alpha_N) \longrightarrow \alpha_P\circ(1_R\times F)\circ t$$

such that F is R-biadditive via $h = (h^2)^{-1}\circ h^1$ and

(i) $h^1(-,1,-) = 1_F = h^2(-,1,-)$

(ii) $v_P(s,r,F(a,b))\circ\alpha_P(1_s,h^1(a,r,b))\circ h^2(\alpha_M(a,r),s,b) =$
$\alpha_P(\gamma_R(r,s),1_{F(a,b)})\circ v_P(r,s,F(a,b))\circ\alpha_P(1_r,h^2(a,s,b))\circ h^1(a,r,\alpha_N(s,b))$

where γ_R is the multiplicative commutativity isomorphism of R.

4.12. Example. A $\mathcal{G}$-ring category R becomes an $R-R$-bimodule via the multiplicative operation $\mu : R\times R \longrightarrow R$. In this case $v_{R,R} = \mathrm{id}$. Moreover μ is R-biadditive and if R is an $\mathcal{S}$-ring category, then μ is R-bilinear.

If we take $R = S(S^0)$, then R-biadditive is equivalent to R-bilinear and they imply 2-symmetric monoidal, but not conversely.

We now construct a tensor product functor $\otimes_R : \mathrm{Mod}-R\times R-\mathrm{Mod} \longrightarrow \mathbf{Perm}$. Let M be a right R-module, N a left R-module and $A = S(M\times N)$, the free permutative category on $M\times N$, where S is the monad associated to the cat-operad $\mathcal{S}$. Now let T be a set of formal morphisms between objects of A of the following types:

$$w_i^1((a_1,b_1),\dots,(a_i,a_i',b_i),\dots,(a_j,b_j)) : [e_j;(a_1,b_1),\dots,(a_i\oplus a_i',b_i),\dots,(a_j,b_j)] \longrightarrow [e_{j+1};(a_1,b_1),\dots,(a_i,b_i),(a_i',b_i),\dots,(a_j,b_j)]$$

$$w_i^2((a_1,b_1),\dots,(a_i,b_i,b_i'),\dots,(a_j,b_j)) : [e_j;(a_1,b_1),\dots,(a_i,b_i\oplus b_i'),\dots,(a_j,b_j)] \longrightarrow [e_{j+1};(a_1,b_1),\dots,(a_i,b_i),(a_i,b_i'),\dots,(a_j,b_j)]$$

$$h_i((a_1,b_1),\dots,(a_i,r,b_i),\dots,(a_j,b_j)) : [e_j;(a_1,b_1),\dots,(\alpha_M(a_i,r),b_i),\dots,(a_j,b_j)] \longrightarrow [e_j;(a_1,b_1),\dots,(a_i,\alpha_N(r,b_i)),\dots,(a_j,b_j)]$$

Let A_T be the graph with the same objects as A and $\mathrm{mor}\, A_T = \mathrm{mor}\, A\coprod T\coprod T^{\mathrm{op}}$. Now form the "localization" $A[T^{-1}]$ of A with respect to T as the free category on the graph A_T modulo the identifications

(i) $(\, x_0 \xleftarrow{u_1} x_1 \xleftarrow{u_2} \cdots \xleftarrow{u_n} x_n\,) = (\, x_0 \xleftarrow{u_1} \cdots \xleftarrow{u_{i-1}} x_{i-1} \xleftarrow{u_i\circ u_{i+1}} x_{i+1} \xleftarrow{u_{i+2}} \cdots \xleftarrow{u_n} x_n\,)$

if u_i and u_{i+1} are morphisms in A

(ii) $(x \xleftarrow{t} y \xleftarrow{t^{\mathrm{op}}} x) = 1_x$ and $(y \xleftarrow{t^{\mathrm{op}}} x \xleftarrow{t} y) = 1_y$, for $t \in T$

(iii) $(x \xleftarrow{u} y \xleftarrow{1_y} y) = (x \xleftarrow{u} y)$ and $(x \xleftarrow{1_x} x \xleftarrow{u} y) = (x \xleftarrow{u} y)$, for $u \in \mathrm{mor}\, A_T$.

A is a subcategory of $A[T^{-1}]$ and we partially extend the permutative operation $\oplus$ to the morphisms in $T \coprod T^{\mathrm{op}}$. Let $a = [e_j; (a_1, b_1), \ldots, (a_j, b_j)]$ and define

$$1_a \oplus w_i^l((c_1, d_1), \ldots, (c_k, d_k)) = w_{i+j}^l((a_1, b_1), \ldots, (a_j, b_j), (c_1, d_1), \ldots, (c_k, d_k))$$

$$w_i^l((c_1, d_1), \ldots, (c_k, d_k)) \oplus 1_a = w_i^l((c_1, d_1), \ldots, (c_k, d_k), (a_1, b_1), \ldots, (a_j, b_j))$$

$$1_a \oplus h_i((c_1, d_1), \ldots, (c_k, d_k)) = h_{i+j}((a_1, b_1), \ldots, (a_j, b_j), (c_1, d_1), \ldots, (c_k, d_k))$$

$$h_i((c_1, d_1), \ldots, (c_k, d_k)) \oplus 1_a = h_i((c_1, d_1), \ldots, (c_k, d_k), (a_1, b_1), \ldots, (a_j, b_j))$$

and similarly for morphisms $t^{\mathrm{op}} = t^{-1}$ in T^{op}. For example if $1_a \oplus t = u$, then u is an isomorphism and we let $1_a \oplus t^{-1} = u^{-1}$.

Let $\eta_j : (M \times N)^j \xrightarrow{\eta^j} A^j \xrightarrow{\oplus} A \hookrightarrow A[T^{-1}]$ where $\eta : \mathrm{Id} \longrightarrow S$ is the unit of the monad S. Also let Δ denote a diagonal functor and $P : M^2 \times N^2 \longrightarrow (M \times N)^2$ the obvious isomorphism. Define the tensor product $M \otimes_R N$ to be the quotient of $A[T^{-1}]$ by the (implied) identifications

(i) $w_i^1 : \eta_j \circ [1_{(M\times N)^{i-1}} \times (\oplus_M \times 1_N) \times 1_{(M\times N)^{j-i}}] \longrightarrow$
$$\eta_{j+1} \circ [1_{(M\times N)^{i-1}} \times (P \circ (1_{M^2} \times \Delta_N)) \times 1_{(M\times N)^{j-i}}]$$

$w_i^2 : \eta_j \circ [1_{(M\times N)^{i-1}} \times (1_M \times \oplus_N) \times 1_{(M\times N)^{j-i}}] \longrightarrow$
$$\eta_{j+1} \circ [1_{(M\times N)^{i-1}} \times (P \circ (\Delta_M \times 1_{N^2})) \times 1_{(M\times N)^{j-i}}]$$

$h_i : \eta_j \circ [1_{(M\times N)^{i-1}} \times (\alpha_M \times 1_N) \times 1_{(M\times N)^{j-i}}] \longrightarrow$
$$\eta_j \circ [1_{(M\times N)^{i-1}} \times (1_M \times \alpha_N) \times 1_{(M\times N)^{j-i}}]$$

are natural transformations

(ii) (η, w_1^1, w_1^2) is a 2-symmetric monoidal functor and $h_1 : \eta \circ (\alpha_M \times 1_N) \longrightarrow \eta \circ (1_M \times \alpha_N)$ is a 3-symmetric monoidal natural transformation such that η is R-biadditive

(iii) $(u_1 \oplus 1_{y_2}) \circ (1_{x_1} \oplus u_2) = (1_{y_1} \oplus u_2) \circ (u_1 \oplus 1_{x_2})$

for morphisms $u_i : x_i \longrightarrow y_i$ in A_T.

Now for morphisms $u = (x_0 \xleftarrow{u_1} x_1 \xleftarrow{u_2} \cdots \xleftarrow{u_n} x_n)$ and $v = (y_0 \xleftarrow{v_1} y_1 \xleftarrow{v_2} \cdots \xleftarrow{v_m} y_m)$ in $M \otimes_R N$ define $u \oplus v$ as follows. The identifications above imply that each diagram

$$
\begin{tikzcd}
x_i \oplus y_j & x_{i+1} \oplus y_j \arrow[l, "u_{i+1} \oplus 1_{y_j}"'] \\
x_i \oplus y_{j+1} \arrow[u, "1_{x_i} \oplus v_{j+1}"] & x_{i+1} \oplus y_{j+1} \arrow[l, "u_{i+1} \oplus 1_{y_{j+1}}"'] \arrow[u, "1_{x_{i+1}} \oplus v_{j+1}"']
\end{tikzcd}
$$

commutes. Thus the $m \times n$ diagram formed from all these squares commutes and so there is a uniquely determined morphism $u \oplus v : x_n \oplus y_m \longrightarrow x_0 \oplus y_0$. This defines the operation $\oplus$ and ensures it is a functor. It is clear from the construction that $M \otimes_R N$ is a permutative category and that the composite

$$M \times N \xrightarrow{\eta} A \longrightarrow A[T^{-1}] \longrightarrow M \otimes_R N$$

also denoted by η, is R-biadditive.

4.13. THEOREM. *Let M be a right R-module, N a left R-module and P a permutative category. If $(F, w_F^1, w_F^2, h_F) : M \times N \longrightarrow P$ is R-biadditive, then there is a unique permutative functor $\overline{F} : M \otimes_R N \longrightarrow P$ such that $\overline{F} \circ \eta = F$. Moreover, if $G : M \otimes_R N \longrightarrow P$ is symmetric monoidal with $G \circ \eta = F$, then there is a symmetric monoidal natural isomorphism $G \longrightarrow \overline{F}$.*

PROOF. Let $A = S(M \times N)$ and T be as above. F induces a permutative functor $F' : A \longrightarrow P$ which extends to a morphism of graphs $F'' : A_T \longrightarrow P$ as follows. For morphisms of $T \coprod T^{\mathrm{op}}$ we let

$$F''(w_i^1((a_1, b_1), \ldots, (a_i, a_i', b_i), \ldots, (a_j, b_j))) = 1_{F(a_1,b_1) \oplus \cdots \oplus F(a_{i-1},b_{i-1})} \oplus w_F^1(a_i, a_i', b_i) \oplus 1_{F(a_{i+1},b_{i+1}) \oplus \cdots \oplus F(a_j,b_j)}$$

$$F''(w_i^2((a_1, b_1), \ldots, (a_i, b_i, b_i'), \ldots, (a_j, b_j))) = 1_{F(a_1,b_1) \oplus \cdots \oplus F(a_{i-1},b_{i-1})} \oplus w_F^2(a_i, b_i, b_i') \oplus 1_{F(a_{i+1},b_{i+1}) \oplus \cdots \oplus F(a_j,b_j)}$$

$$F''(h_i((a_1, b_1), \ldots, (a_i, r, b_i), \ldots, (a_j, b_j))) = 1_{F(a_1,b_1) \oplus \cdots \oplus F(a_{i-1},b_{i-1})} \oplus h_F(a_i, r, b_i) \oplus 1_{F(a_{i+1},b_{i+1}) \oplus \cdots \oplus F(a_j,b_j)}$$

$$F''(w_i^1((a_1, b_1), \ldots, (a_i, a_i', b_i), \ldots, (a_j, b_j))^{\mathrm{op}}) = 1_{F(a_1,b_1) \oplus \cdots \oplus F(a_{i-1},b_{i-1})} \oplus w_F^1(a_i, a_i', b_i)^{-1} \oplus 1_{F(a_{i+1},b_{i+1}) \oplus \cdots \oplus F(a_j,b_j)}$$

$$F''(w_i^2((a_1, b_1), \ldots, (a_i, b_i, b_i'), \ldots, (a_j, b_j))^{\mathrm{op}}) = 1_{F(a_1,b_1) \oplus \cdots \oplus F(a_{i-1},b_{i-1})} \oplus w_F^2(a_i, b_i, b_i')^{-1} \oplus 1_{F(a_{i+1},b_{i+1}) \oplus \cdots \oplus F(a_j,b_j)}$$

$$F''(h_i((a_1, b_1), \ldots, (a_i, r, b_i), \ldots, (a_j, b_j))^{\mathrm{op}}) = 1_{F(a_1,b_1) \oplus \cdots \oplus F(a_{i-1},b_{i-1})} \oplus h_F(a_i, r, b_i)^{-1} \oplus 1_{F(a_{i+1},b_{i+1}) \oplus \cdots \oplus F(a_j,b_j)}$$

Now F'' induces a functor $A[T^{-1}] \longrightarrow P$ which in turn induces a functor $\overline{F} : M \otimes_R N \to P$ such that $\overline{F} \circ \eta = F$. It is straightforward to check that $\overline{F}$ is permutative and has the stated uniqueness property.

Given (G, w_G) with $G \circ \eta = F$, w_G determines a symmetric monoidal natural isomorphism $G \longrightarrow \overline{F}$. ∎

4.14. COROLLARY. *$\otimes_R$ is a 2-functor* Mod$-R$ $\times$ $R-$Mod $\longrightarrow$ **Perm**.

PROOF. If $F : M \longrightarrow M'$ and $G : N \longrightarrow N'$ are morphisms of right and left R-modules respectively, then

$$M \times N \xrightarrow{F \times G} M' \times N' \xrightarrow{\eta'} M' \otimes_R N'$$

is R-biadditive, hence induces a permutative functor $F \otimes_R G : M \otimes_R N \longrightarrow M' \otimes_R N'$.

If $\tau : F \longrightarrow F'$ and $\sigma : G \longrightarrow G'$ are R-natural transformations, let $(\tau \otimes_R \sigma)([e_j; (a_1, b_1), \ldots, (a_j, b_j)]) = [e_j \to e_j; (\tau a_1, \sigma b_1), \ldots, (\tau a_j, \sigma b_j)]$. This defines a permutative natural transformation $F \otimes_R G \longrightarrow F' \otimes_R G'$. ∎

The proof of the expected result for bimodules is straightforward and is left to the reader.

4.15. COROLLARY. *The tensor product $\otimes_R$ induces a 2-functor*

$$T-\text{Mod}-R \times R-\text{Mod}-T' \longrightarrow T-\text{Mod}-T'$$

If R is a $\mathcal{G}$-ring category, then its (ring) *opposite* R^o has objects and morphisms a^o for a in R. R^o is a $\mathcal{G}$-ring category with $a^o \oplus b^o = (a \oplus b)^o$, $\theta^o_j(x; a^o_1, \ldots, a^o_j) = \theta_j(x\tau_j; a_1, \ldots, a_j)^o$ and $(\delta^o)^i_j(x; a^o_1, \ldots, a^o_i, (a'_i)^o, \ldots, a^o_j) = \delta^i_j(x\tau_j; a_1, \ldots, a_i, a'_i, \ldots, a_j)^o$, where $\tau_j(i) = j+1-i$.

If T is an $\mathcal{S}$-ring category, then T and T^o are isomorphic in $\mathcal{S}$–**Rng**(**Cat**). $F : T \longrightarrow T^o$ defined by $F(a) = a^o$ is permutative and a lax $\mathcal{S}$-morphism where $h : F \circ \theta_j \longrightarrow \theta^o_j \circ (1 \times F^j)$ has components $h(\sigma; a_1, \ldots, a_j) = \theta_j(\sigma \to \sigma\tau_j; 1_{a_1}, \ldots, 1_{a_j})^o$.

For a $\mathcal{G}$-ring category R and an $\mathcal{S}$-ring category T, there are isomorphisms of 2-categories $R-$Mod $\cong$ Mod$-R^o$, Mod$-T^o \cong$ Mod$-T$ (by Proposition 4.16) and $T-$Mod $\cong$ Mod$-T$. Also note that a (left or right) T-module is a $T-T$-bimodule in the obvious way, hence for T-modules M and N Corollary 4.15 implies $M \otimes_T N$ is a $T-T$-bimodule.

4.16. PROPOSITION. *Let R and T be $\mathcal{G}$-ring categories. If R and T are equivalent (respectively isomorphic) in $\mathcal{G}$–**Rng**(**Cat**), then $R-$Mod and $T-$Mod are equivalent (respectively isomorphic) as 2-categories.*

PROOF. Let $(\varphi, w_\varphi, h_\varphi) : R \longrightarrow T$ and $(\psi, w_\psi, h_\psi) : T \longrightarrow R$ be morphisms in $\mathcal{G}-$**Rng**(**Cat**) with lax $\mathcal{G}$-ring natural isomorphisms $\tau : \varphi \circ \psi \cong \text{Id}$ and $\sigma : \psi \circ \varphi \cong \text{Id}$. Define 2-functors $I : R-\text{Mod} \longrightarrow T-\text{Mod}$ and $J : T-\text{Mod} \longrightarrow R-\text{Mod}$ as follows.

If (M, α_M, v_M) is an R-module let $I(M, \alpha_M, v_M) = (M, \widehat{\alpha}_M, \widehat{v}_M)$ where $\widehat{\alpha}_M = \alpha_M \circ (\psi \times 1_M)$ and $\widehat{v}_M(t_1, t_2, a) = \alpha_M(h_\psi(t_1, t_2)^{-1}, 1_a) \circ v_M(\psi(t_1), \psi(t_2), a)$. If $(F, w_F, h_F) : (M, \alpha_M, v_M) \longrightarrow (M', \alpha_{M'}, v_{M'})$ is an R-module morphism let $I(F, w_F, h_F) = (F, w_F, \widehat{h}_F)$ where $\widehat{h}_F(t, a) = h_F(\psi(t), a)$. If $\omega : F \longrightarrow G$ is an R-natural transformation let $(I\omega)(a) = \omega(a)$. J is defined similarly.

Define a 2-natural isomorphism $\overline{\sigma} : J \circ I \longrightarrow \mathrm{Id}$ with components $\overline{\sigma}(M) = (\iota, h_\iota) : (J \circ I)(M) \longrightarrow M$ where $\iota = 1_M$ and $h_\iota(r, a) = \alpha_M(\sigma(r), 1_a)$. A 2-natural isomorphism $\overline{\tau} : I \circ J \longrightarrow \mathrm{Id}$ is defined similarly. Thus $R-\mathrm{Mod}$ and $T-\mathrm{Mod}$ are equivalent as 2-categories.

If σ and τ are identities, then $I = J^{-1}$ so $R-\mathrm{Mod}$ and $T-\mathrm{Mod}$ are isomorphic. ∎

4.17. Proposition. *Let M be a right R-module and N a left R-module.*

(i) $M \otimes_R R \,\dot{\cong}\, M$ and $R \otimes_R N \,\dot{\cong}\, N$ in $R-\mathrm{Mod}$.

(ii) $M \otimes_R N \cong N \otimes_R M$ as $R-R$-bimodules, if R is an $\mathcal{S}$-ring category.

(iii) $(M \otimes_R N) \otimes_T P \cong M \otimes_R (N \otimes_T P)$ if N is an $R-T$-bimodule and P is a T-module.

Proof. $\alpha_M : M \times R \longrightarrow M$ is R-biadditive, so we get a permutative functor $\overline{\alpha}_M : M \otimes_R R \longrightarrow M$. Define $I : M \longrightarrow M \otimes_R R$ by $I(a) = \eta(a, 1) = [e_1; (a, 1)]$. Then $\overline{\alpha}_M$ and I are right R-module morphisms such that $\overline{\alpha}_M \circ I = \mathrm{Id}$ and $I \circ \overline{\alpha}_M \cong \mathrm{Id}$. We note that I is a 2-natural transformation, but $\overline{\alpha}$ is only lax 2-natural.

The R-bilinear morphism $M \times N \longrightarrow N \times M \longrightarrow N \otimes_R M$ induces a permutative functor $M \otimes_R N \longrightarrow N \otimes_R M$ which is easily seen to be an isomorphism of R–R-bimodules.

The proof of (iii) is straightforward (but lengthy) and is left to the reader. ∎

If M and N are R-modules let $\mathrm{Hom}_R(M, N)$ be the category with objects the R-module morphisms from M to N and with morphisms the R-natural transformations. The set $\mathcal{O}$ of objects is topologized as a subspace of $\mathrm{Map}(\mathrm{obj}\, M, \mathrm{obj}\, N) \times \mathrm{Map}(\mathrm{mor}\, M, \mathrm{mor}\, N)$ and the set of morphisms as a subspace of $\mathcal{O} \times \mathrm{Map}(\mathrm{obj}\, M, \mathrm{mor}\, N) \times \mathcal{O}$. This is a permutative category under pointwise operations. Moreover if M is an $R-T$-bimodule and N is an $R-T'$-bimodule, then $\mathrm{Hom}_R(M, N)$ is a $T-T'$-bimodule.

An elaboration of the usual argument gives the following adjointness property.

4.18. Proposition. *If M is an $R-T$-bimodule, then $(M \otimes_T (-), \mathrm{Hom}_R(M, -))$ is a lax 2-adjoint pair. In particular, if N is a T-module and P is an R-module, then there is an equivalence of permutative categories (hence of $S(S^0)$-modules)*

$$\mathrm{Hom}_R(M \otimes_T N, P) \,\dot{\cong}\, \mathrm{Hom}_T(N, \mathrm{Hom}_R(M, P))$$

If M is an R-module, then $\mathrm{Hom}_R(M, M)$ is an $\mathcal{M}$-ring category with multiplicative operation being composition of functors and natural transformations. Distributivity isomorphisms

$$F_1 \circ \cdots \circ (F_i \oplus F_i') \circ \cdots \circ F_j \xrightarrow{\delta_j^i} (F_1 \circ \cdots \circ F_i \circ \cdots \circ F_j) \oplus (F_1 \circ \cdots \circ F_i' \circ \cdots \circ F_j)$$

are defined inductively. For $j = 2$ let $\delta_2^1 = \mathrm{id} : (F_1 \oplus F_1') \circ F_2 \longrightarrow (F_1 \circ F_2) \oplus (F_1' \circ F_2)$ and $\delta_2^2 = w_{F_1}(F_2(-), F_2'(-)) : F_1 \circ (F_2 \oplus F_2') \longrightarrow (F_1 \circ F_2) \oplus (F_1 \circ F_2')$. Higher distributivity isomorphisms are composites of these. Note that $\mathrm{Hom}_R(M, M)$ satisfies strict right distributivity, i.e. $\delta_j^i(e_j; F_1, \ldots, F_i, F_i', \ldots, F_j) = \mathrm{id}$ when $F_k = 1_M$ for $k < i$.

The *endomorphism ring* $\mathrm{End}_R M$ is defined to be the $\mathcal{M}$-ring category $\mathrm{Hom}_R(M, M)^o$ which satisfies strict left distributivity.

If R is an $\mathcal{M}$-ring category, then R and $\mathrm{End}_R R$ are equivalent in $\mathcal{M}$–$\mathbf{Rng}(\mathbf{Cat})$. Define $\varphi : R \longrightarrow \mathrm{End}_R R$ by $\varphi(r) = \rho_r^o$ where ρ_r is right translation, and $\varphi(f : r \to s) = \tau_f^o$ where τ_f has components $\tau_f(a) = 1_a \otimes f$. Also define $\psi : \mathrm{End}_R R \longrightarrow R$ by $\psi(F^o) = F(1)$ and $\psi(\tau^o) = \tau(1)$. Then $\psi \circ \varphi = \mathrm{Id}$ and $\varphi \circ \psi \cong \mathrm{Id}$. The result is also true for a $\mathcal{G}$-ring category R when $\mathcal{G}$ is $\widetilde{\mathcal{B}}$ or $\mathcal{S}$.

We next construct a tensor product for algebras over an $\mathcal{S}$-ring category. Suppose A is a $\mathcal{G}$-ring category with $\psi : \mathrm{End}_A A \longrightarrow A$ as above and let $\mathrm{End}_A A_A$ be the sub $\mathcal{G}$-ring category consisting of the $A-A$-bimodule morphisms and $A-A$-natural transformations. It is in fact an $\mathcal{S}$-ring category.

4.19. Definition. *Let A be a $\mathcal{G}$-ring category. The* center *of A is the sub $\mathcal{G}$-ring category $Z(A) = \psi(\mathrm{End}_A A_A)$, a full subcategory of A.*

If $\mathcal{G}$ is $\widetilde{\mathcal{B}}$ or $\mathcal{S}$, then $Z(A)$ is the full subcategory with objects a such that $\gamma(a, x) = \gamma(x, a)^{-1}$ for all objects x in A, where γ is the commutativity isomorphism. Thus $Z(A) = A$ if A is an $\mathcal{S}$-ring category.

4.20. Definition. *Let R be an $\mathcal{S}$-ring category and A a $\mathcal{G}$-ring category. A is an R-*algebra in $\mathcal{G}-\mathbf{RngCat}$ *if there is a morphism $\varphi : R \longrightarrow A$ in $\mathcal{G}-\mathbf{Rng}(\mathbf{Cat})$ such that $\varphi(R)$ is contained in $Z(A)$ and $\varphi : R \longrightarrow Z(A)$ is a morphism in $\mathcal{S}-\mathbf{Rng}(\mathbf{Cat})$.*

A morphism *of R-algebras is a morphism $F : A \longrightarrow A'$ in $\mathcal{G}-\mathbf{RngCat}$ such that $F \circ \varphi = \varphi'$. An* R-algebra natural transformation *is a $\mathcal{G}$-ring natural transformation $\tau : F \longrightarrow F'$ of R-algebra morphisms such that $\tau(\varphi(r)) = 1_{\varphi'(r)}$. The 2-category of R-algebras is denoted $R-\mathbf{Alg}_{\mathcal{G}}$.*

4.21. Examples. (i) Any object A in $\mathcal{G}-\mathbf{RngCat}$ is a $Z(A)$-algebra in $\mathcal{G}-\mathbf{RngCat}$.

(ii) There is an isomorphism of 2-caregories $\mathcal{G}-\mathbf{RngCat} \cong S(S^0)-\mathbf{Alg}_{\mathcal{G}}$ analogous to the case of ordinary rings (ring = $\mathbf{Z}$-algebra).

(iii) An R-algebra A is an A–A-bimodule and hence an A–R, R–A and R–R-bimodule by restriction.

4.22. Proposition. *$\otimes_R$ is a 2-functor $R-\mathbf{Alg}_{\mathcal{G}} \times R-\mathbf{Alg}_{\mathcal{G}} \longrightarrow R-\mathbf{Alg}_{\mathcal{G}}$.*

Proof. Let (A, φ) and (A', φ') be R-algebras in $\mathcal{G}-\mathbf{RngCat}$. The construction follows that for the tensor product of modules with some additional requirements.

First $A \times A'$ is a $\mathcal{G}$-category with zero, where $\mathcal{G}$ acts diagonally, hence $S(A \times A')$ is a $\mathcal{G}$-ring category satisfying strict left distributivity, [3;§3]. Next, partially extend the multiplicative operation $\otimes$ (i.e. the $\mathcal{M}$-action underlying the $\mathcal{G}$-action) to the morphisms in $T \coprod T^{op}$. It suffices to define just the following products by strict left distributivity. Let $a = [e_j; (a_1, b_1), \ldots, (a_j, b_j)]$.

$$\begin{aligned} &1_a \otimes w_1^1((c, c', d)) = \\ &\quad [e_{2j} \to \sigma(2, j); (1_{a_1 \otimes c}, 1_{b_1 \otimes d}), (1_{a_1 \otimes c'}, 1_{b_1 \otimes d}), \ldots, (1_{a_j \otimes c}, 1_{b_j \otimes d}), (1_{a_j \otimes c'}, 1_{b_j \otimes d})] \circ \\ &\quad w_1^1((a_1 \otimes c, a_1 \otimes c', b_1 \otimes d)) \oplus \cdots \oplus w_1^1((a_j \otimes c, a_j \otimes c', b_j \otimes d)) \circ \end{aligned}$$

$$[e_j;(\delta_2^2(e_2;a_1,c,c'),1_{b_1\otimes d}),\ldots,(\delta_2^2(e_2;a_j,c,c'),1_{b_j\otimes d})]$$

where $\sigma(2,j)\in\Sigma_{2j}$ is the obvious shuffle permutation.

$$\begin{aligned}
&1_a\otimes w_1^2((c,d,d'))=\\
&\quad [e_{2j}\to\sigma(2,j);(1_{a_1\otimes c},1_{b_1\otimes d}),(1_{a_1\otimes c},1_{b_1\otimes d'}),\ldots,(1_{a_j\otimes c},1_{b_j\otimes d}),(1_{a_j\otimes c},1_{b_j\otimes d'})]\circ\\
&\quad w_1^2((a_1\otimes c,b_1\otimes d,b_1\otimes d'))\oplus\cdots\oplus w_1^2((a_j\otimes c,b_j\otimes d,b_j\otimes d'))\circ\\
&\quad [e_j;(1_{a_1\otimes c},\delta_2^2(e_2;b_1,d,d')),\ldots,(1_{a_j\otimes c},\delta_2^2(e_2;b_j,d,d'))]
\end{aligned}$$

$$\begin{aligned}
&w_i^1((c_1,d_1),\ldots,(c_k,d_k))\otimes 1_{[e_1;(a,b)]}=\\
&\quad w_i^1((c_1\otimes a,d_1\otimes b),\ldots,(c_i\otimes a,c_i'\otimes a,d_i\otimes b),\ldots,(c_k\otimes a,d_k\otimes b))\circ\\
&\quad [e_k\to e_k;1_{c_1\otimes a},1_{d_1\otimes b}),\ldots,(\delta_2^1(e_2;c_i,c_i',a),1_{d_i\otimes b}),\ldots,1_{c_k\otimes a},1_{d_k\otimes b})]
\end{aligned}$$

$$\begin{aligned}
&w_i^2((c_1,d_1),\ldots,(c_k,d_k))\otimes 1_{[e_1;(a,b)]}=\\
&\quad w_i^1((c_1\otimes a,d_1\otimes b),\ldots,(c_i\otimes a,d_i\otimes b,d_i'\otimes b),\ldots,(c_k\otimes a,d_k\otimes b))\circ\\
&\quad [e_k\to e_k;1_{c_1\otimes a},1_{d_1\otimes b}),\ldots,(1_{c_i\otimes a},\delta_2^1(e_2;d_i,d_i',b)),\ldots,1_{c_k\otimes a},1_{d_k\otimes b})]
\end{aligned}$$

$$\begin{aligned}
&1_a\otimes h_1((c,r,d))=\\
&\quad [e_j\to e_j;(1_{a_1\otimes c},\gamma_{A'}(\varphi'(r),b_1)\otimes 1_d),\ldots,(1_{a_j\otimes c},\gamma_{A'}(\varphi'(r),b_j)\otimes 1_d)]\circ\\
&\quad h_1((a_1\otimes c,r,b_1\otimes d))\oplus\cdots\oplus h_1((a_j\otimes c,r,b_j\otimes d))
\end{aligned}$$

$$\begin{aligned}
&h_i((c_1,d_1),\ldots,(c_i,r,d_i),\ldots,(c_k,d_k))\otimes 1_{[e_1;(a,b)]}=\\
&\quad h_i((c_1\otimes a,d_1\otimes b),\ldots,(c_i\otimes a,r,d_i\otimes b),\ldots,(c_k\otimes a,d_k\otimes b))\circ\\
&\quad [e_k\to e_k;(1_{c_1\otimes a},1_{d_1\otimes b}),\ldots,(1_{c_i}\otimes\gamma_A(\varphi(r),a),1_{d_i\otimes b}),\ldots,(1_{c_k\otimes a},1_{d_k\otimes b})]
\end{aligned}$$

and similarly for morphisms $t^{op}=t^{-1}$ in T^{op}.

We add to requirement (iii) above that $(u_1\otimes 1_{y_2})\circ(1_{x_1}\otimes u_2)=(1_{y_1}\otimes u_2)\circ(u_1\otimes 1_{x_2})$ for morphisms $u_i:x_i\longrightarrow y_i$ in $S(A\times A')_T$. The remainder of the construction is as for the module case. One can check that $A\otimes_R A'$ is a $\mathcal{G}$-ring category and that defining $\varphi'':R\longrightarrow A\otimes_R A'$ by $\varphi''(r)=[e_1;(\varphi(r),1)]$ makes $A\otimes_R A'$ into an R-algebra in $\mathcal{G}-\mathbf{RngCat}$. We note that $A\otimes_R A'$ satisfies strict left distributivity (even if A and A' do not), since this is true for $S(A\times A')_T$. For 1 and 2-cells, $\otimes_R$ is defined as for modules. ∎

Let R be a $\mathcal{G}$-ring category and let A be an $\mathcal{H}$-category with zero, [3;§3], where $\mathcal{H}$ is also one of the operads $\mathcal{M}$, $\widetilde{\mathcal{B}}$ or $\mathcal{S}$. Then $R\otimes S(A)$ is an $S(S^0)$-algebra in $\mathcal{M}-\mathbf{RngCat}$, where $\otimes=\otimes_{S(S^0)}$. In section five we show $R\otimes S(A)$ is the free strict R-module on the category A. If $\mathcal{H}=\mathcal{G}$, then $R\otimes S(A)$ is an $S(S^0)$-algebra in $\mathcal{G}-\mathbf{RngCat}$. If $\mathcal{G}=\mathcal{S}$, any $\mathcal{H}$, then $R\otimes S(A)$ is an R-algebra in $\mathcal{H}-\mathbf{RngCat}$.

For a category A let A_+ denote A with an object 0 and a morphism 1_0 adjoined. If A is an $\mathcal{H}$-category, then A_+ is an $\mathcal{H}$-category with zero and we define the A_∞ *monoid ring* of A to be $R[A]=R\otimes S(A_+)$. If R is an $\mathcal{S}$-ring category, then $R[A]$ is an object of $R-\mathbf{Alg}_{\mathcal{H}}$; $R[A]$ is called the $\mathcal{H}$-*monoid algebra*, or more loosely A_∞ *monoid algebra* of A.

In [5] we will show how the integral A_∞ monoid algebra $S(S^0)[A]$ can be used to construct algebraic K-theory of spaces and stable homotopy.

5. Morita equivalence for A_∞ rings

We begin with the construction of free modules in $R-[\mathrm{Mod}]$. Throughout this section $\otimes$ denotes the tensor product over $S(S^0)$. Let $\mathcal{U} : R-[\mathrm{Mod}] \longrightarrow \mathbf{Cat}$ be the underlying category functor and define $\mathcal{F} = R \otimes S(-) : \mathbf{Cat} \longrightarrow R-[\mathrm{Mod}]$. The following lemma implies that $(\mathcal{F},\mathcal{U})$ is a 2-adjoint pair.

5.1. Lemma. *$R \otimes S(A)$ is the free strict R-module on the category A. Moreover, any two R-module morphisms $R \otimes S(A) \longrightarrow N$ (N an R-module) that agree on A are R-naturally isomorphic.*

Proof. Let $\tau : A \longrightarrow R \otimes S(A)$ be the functor defined on objects by $\tau(a) = [e_1; (1, [e_1; a])]$. For M a strict R-module, a functor $F : A \longrightarrow M$ induces a permutative functor $\widetilde{F} : S(A) \longrightarrow M$ and regarding this as an $S(S^0)$-module morphism we have a strict R-module morphism $\overline{F} = \overline{\alpha}_M \circ (1_R \otimes \widetilde{F})$ such that $\overline{F} \circ \tau = F$. Here $\overline{\alpha}_M : R \otimes M \longrightarrow M$ is induced by the R-action α_M which is $S(S^0)$-biadditive. It is a strict R-module morphism since M is strict.

Let $\eta_A : R \times S(A) \longrightarrow R \otimes S(A)$ be the universal $S(S^0)$-biadditve functor. If $G : R \otimes S(A) \longrightarrow M$ is a strict R-module morphism such that $G \circ \tau = F$, then using the universal property of $S(-)$, we get $G \circ \eta_A = \overline{F} \circ \eta_A$ and therefore $G = \overline{F}$. Thus $R \otimes S(A)$ is the free strict R-module on A. The second statement is a consequence of Theorem 4.13. ∎

The adjoint pair $(\mathcal{F},\mathcal{U},\eta,\varepsilon)$ determines a comonad (i.e. cotriple) $(\mathbf{G},\delta,\varepsilon)$ in R–[Mod] where $\mathbf{G} = \mathcal{F} \circ \mathcal{U}$ and $\delta = \mathcal{F}\eta\mathcal{U}$. Moreover, it extends to a lax comonad in R–Mod which we also denote by $(\mathbf{G},\delta,\varepsilon)$. It fails to be a strict comonad only because ε is a lax 2-natural transformation.

Define an R-module M to be $\mathbf{G}$-*projective*, [1], if there is an R-module morphism $s : M \longrightarrow \mathbf{G}M$ such that $\varepsilon_M \circ s = 1_M$. If M is projective in the usual sense, then it is $\mathbf{G}$-projective, but not conversely.

5.2. Example. *The standard Morita context.* For an R-module M, let $R' = \mathrm{End}_R M$ and $M' = \mathrm{Hom}_R(M,R)$. Then M is an $R-R'$-bimodule where R' acts by $\alpha'_M(a, F^o) = F(a)$ and for $\tau^o : F^o \longrightarrow G^o$ and $f : a \longrightarrow b$, $\alpha'_M(f,\tau^o) = G(f) \circ \tau(a) = \tau(b) \circ F(f)$. It follows that M' is an $R'-R$-bimodule; the R'-action is denoted $\alpha_{M'}$. Note that M' is a strict R'-module, so $M' \otimes_R M$ is also a strict R'-module.

Define $\varphi : M \times M' \longrightarrow R$ by $\varphi(a,H) = H(a)$ and for $f : a \longrightarrow b$ in M and $\sigma : H_1 \longrightarrow H_2$ in M', $\varphi(f,\sigma) = H_2(f) \circ \sigma(a) = \sigma(b) \circ H_1(f)$. φ is clearly 2-symmetric monoidal and in fact is R'-biadditive with $h = \mathrm{id} : \varphi \circ (\alpha'_M \times 1_{M'}) \longrightarrow \varphi \circ (1_M \times \alpha_{M'})$. Thus φ induces an $R-R$-bimodule morphism $E : M \otimes_{R'} M' \longrightarrow R$.

Similarly there is an R-biadditive functor $\varphi' : M' \times M \longrightarrow R'$ defined on objects by $\varphi'(H,a) = \alpha_M(H(-),a)^o$, and this induces a morphism $E' : M' \otimes_R M \longrightarrow R'$ of $R'-R'$-bimodules.

A functor F is called *surjective* if it is surjective on both objects and morphisms.

5.3. Definition. *(i) An R-module M is a* generator *for $R-$Mod if for each R-module N there is a category A and a surjection $F : M \otimes S(A) \longrightarrow N$ in $R-$Mod with F permutative.*

(ii) M is finitely generated *if there is a subcategory A of M such that the composite*

$$R \otimes S(A) \stackrel{1_R \otimes S(I)}{\longrightarrow} R \otimes S(M) \stackrel{\varepsilon_M}{\longrightarrow} M$$

is surjective, where $I : A \hookrightarrow M$ is the inclusion. If all morphisms of A are identities we say M is generated by a finite subset.

(iii) M is finitely generated $\mathbf{G}$-projective *if it is $\mathbf{G}$-projective and is generated by a finite subset A for which $s : M \longrightarrow \mathbf{G}M$ factors as*

$$M \stackrel{\sigma}{\longrightarrow} R \otimes S(A) \stackrel{1_R \otimes S(I)}{\longrightarrow} R \otimes S(M)$$

for some σ in $R-$Mod.

M is a progenerator *if it is a generator and is finitely generated $\mathbf{G}$-projective.*

Recall the morphism $E : M \otimes_{R'} M' \longrightarrow R$ and let $I : M \otimes M' \longrightarrow M \otimes_{R'} M'$ be the obvious R-module morphism arising from the morphism $S(S^0) \longrightarrow R'$ of $\mathcal{G}$-ring categories (Examples 4.8).

5.4. Lemma. *The following are equivalent for an R-module M.*

(i) M is a generator.

(ii) $M \otimes M' \stackrel{I}{\longrightarrow} M \otimes_{R'} M' \stackrel{E}{\longrightarrow} R$ is surjective.

(iii) There is a category A and a surjection $F : M \otimes S(A) \longrightarrow R$ in $R-$Mod with F permutative.

Proof. Suppose M is a generator and let $F : M \otimes S(A) \longrightarrow R$ be a permutative surjection. By Proposition 4.18 we have a morphism of $S(S^0)$-modules $\widetilde{F} : S(A) \longrightarrow \mathrm{Hom}_R(M, R)$ and since F is permutative it equals the composite

$$M \otimes S(A) \stackrel{1_M \otimes \widetilde{F}}{\longrightarrow} M \otimes M' \stackrel{I}{\longrightarrow} M \otimes_{R'} M' \stackrel{E}{\longrightarrow} R$$

Since F is surjective it follows $E \circ I$ is also.

Now assume $E \circ I$ is surjective, let $A = M'$ and consider the composite

$$M \otimes S(A) \stackrel{1_M \otimes G}{\longrightarrow} M \otimes M' \stackrel{I}{\longrightarrow} M \otimes_{R'} M' \stackrel{E}{\longrightarrow} R$$

where G is the permutative extension of $id : A \longrightarrow M'$. This composite is surjective since G is surjective.

Suppose given $F : M \otimes S(A) \longrightarrow R$ in $R-$Mod with F permutative and surjective. If N is an R-module, then the composite

$$M \otimes S(S(A) \otimes N) \longrightarrow M \otimes (S(A) \otimes N) \cong (M \otimes S(A)) \otimes N \longrightarrow R \otimes N \longrightarrow N$$

is permutative and surjective, hence M is a generator. ∎

5.5. Examples. (i) If $F : M \longrightarrow N$ is a surjection in $R-\mathrm{Mod}$ and N is a generator, then so is M.

(ii) The (strict) R-module morphism $R\otimes S(S^0) \longrightarrow R$ is surjective, so R and $R\otimes S(S^0)$ are generators.

(iii) If A is a category and there is a surjection $A \longrightarrow S^0$ (i.e. A has an object not in the component of $*$), then we have a surjection $\mathcal{F}(A) = R \otimes S(A) \longrightarrow R \otimes S(S^0) \longrightarrow R$, so $\mathcal{F}(A)$ is a generator.

We will use the notation of Example 5.2 and Definition 5.3 throughout the proofs of 5.6 and 5.7. Let $\Phi = \varepsilon_M \circ (1_R \otimes S(I)) : \mathcal{F}(A) \longrightarrow M$ and $A = \{a_0, \ldots, a_n\}$ with $a_0 = *$. Notice that any two R-module morphisms $F, G : M \longrightarrow N$ that agree on A are R-naturally isomorphic. For then $F \circ \Phi$ and $G \circ \Phi$ agree on A and hence are R-naturally isomorphic by Lemma 5.1, so composing with σ gives the same for F and G. The functors $\pi_i^A : A \longrightarrow R$ with $\pi_i^A(a_i) = 1$ and $\pi_i^A(a_k) = 0$, $k \neq i$, induce R-module morphisms $\widehat{\pi}_i : \mathcal{F}(A) \longrightarrow R$ for $i = 1, \ldots, n$ and we let $\pi_i = \widehat{\pi}_i \circ \sigma$. Also let $I_j = \alpha_M(\pi_j(-), a_j) \in \mathrm{Hom}_R(M, M)$ and $I_A = I_1 \oplus \cdots \oplus I_n$. Then I_A and 1_M are R-naturally isomorphic by Lemma 5.1 since they agree on A.

5.6. Lemma. *If M is finitely generated* $\mathbf{G}$*-projective, then there is a lax 2-natural equivalence $\tau_M : \mathrm{Hom}_R(M, R)\otimes_R(-) \longrightarrow \mathrm{Hom}_R(M, -)$ of 2-functors $R-\mathrm{Mod} \longrightarrow R'-\mathrm{Mod}$.*

Proof. For an R-module N, $\tau_M^0(N)$ is induced by the R-biadditive morphism φ'_N : $\mathrm{Hom}_R(M, R) \times N \longrightarrow \mathrm{Hom}_R(M, N)$ with $\varphi'_N(H, b) = \alpha_N(H(-), b)$. For $G : N \longrightarrow P$ in $R-\mathrm{Mod}$, $\tau_M^1(G) : G_* \circ \tau_M^0(N) \longrightarrow \tau_M^0(P) \circ (1_{M'}\otimes_R G)$ is the R-natural transformation with components induced by h_G.

Define a (strict) 2-natural transformation $\gamma_M : \mathrm{Hom}_R(M, -) \longrightarrow \mathrm{Hom}_R(M, R)\otimes_R(-)$ by

$$\gamma_M(N)(F) = [e_n; (\pi_1, Fa_1), \ldots, (\pi_n, Fa_n)]$$

and

$$\gamma_M(N)(\delta) = [1_{e_n}; (1_{\pi_1}, \delta(a_1)), \ldots, (1_{\pi_n}, \delta(a_n))]$$

for $\delta : F \longrightarrow F'$ in $\mathrm{Hom}_R(M, N)$. To see $\gamma_M(N)$ is in $R'-\mathrm{Mod}$, it suffices by lax naturality to check for $N = M$. Letting $x = [e_n; (\pi_1, a_1), \ldots, (\pi_n, a_n)] \in M'\otimes_R M$, we have $\gamma_M(M)(F) = x \cdot F^o$, the right R'-action on $M'\otimes_R M$, so $\gamma_M(M)$ is a morphism of right R'-modules. It is also a morphism of left R'-modules as follows. In $M'\otimes_R M$ we have isomorphisms

$$\begin{aligned}
x \cdot F^o &\cong [e_n; (\pi_1, I_1F(a_1) \oplus \cdots \oplus I_nF(a_1)), \ldots, (\pi_n, I_1F(a_n) \oplus \cdots \oplus I_nF(a_n))] \\
&= [e_n; (\pi_1, \alpha_M(\pi_1F(a_1), a_1) \oplus \cdots \oplus \alpha_M(\pi_nF(a_1), a_n)), \ldots, \\
&\quad (\pi_n, \alpha_M(\pi_1F(a_n), a_1) \oplus \cdots \oplus \alpha_M(\pi_nF(a_n), a_n))] \\
&\cong \oplus_{i,j} [e_1; (\pi_i, \alpha_M(\pi_jF(a_i), a_j))] \\
&\cong \oplus_{i,j} [e_1; (\pi_i(-) \cdot \pi_jF(a_i), a_j)] \\
&\cong \oplus_{j=1}^n [e_1; (\pi_1(-) \cdot \pi_jF(a_1) \oplus \cdots \oplus \pi_n(-) \cdot \pi_jF(a_n), a_j)] \\
&\cong \oplus_{j=1}^n [e_1; (\pi_jF, a_j)]
\end{aligned}$$

$$\cong [e_n; (\pi_1 F, a_1), \ldots, (\pi_n F, a_n)] = F^o \cdot x$$

which can be used to give $\gamma_M(M)$ a left R'-module morphism structure; in fact $\gamma_M(M)$ is an $R'-R'$-bimodule morphism with this structure.

Define modifications $\rho : \tau_M \circ \gamma_M \longrightarrow \mathrm{id}$ and $\lambda : \gamma_M \circ \tau_M \longrightarrow \mathrm{id}$ with $\rho(N)$ and $\lambda(N)$ the R-natural transformations with components the following chains of isomorphisms, where $G \in \mathrm{Hom}_R(M, N)$ and $y = [e_m; (H_1, b_1), \ldots, (H_m, b_m)] \in M' \otimes_R M$.

$$\begin{aligned}
\rho(N)(G) : (\tau_M^0(N) \circ \gamma_M^0(N))(G) &= \alpha_N(\pi_1(-), G(a_1)) \oplus \cdots \oplus \alpha_N(\pi_n(-), G(a_n)) \\
&\cong G\alpha_M(\pi_1(-), a_1) \oplus \cdots \oplus G\alpha_M(\pi_n(-), a_n) \\
&\cong G(\alpha_M(\pi_1(-), a_1) \oplus \cdots \oplus \alpha_M(\pi_n(-), a_n)) \\
&= G\tau_M^0(M)(x) \cong G
\end{aligned}$$

$$\begin{aligned}
\lambda(N)(y) : (\gamma_M^0(N) \circ \tau_M^0(N))(y) &= [e_n; (\pi_1, \alpha_N(H_1(a_1), b_1) \oplus \cdots \oplus \alpha_N(H_m(a_1), b_m)), \ldots, \\
&\qquad (\pi_n, \alpha_N(H_1(a_n), b_1) \oplus \cdots \oplus \alpha_N(H_m(a_n), b_m))] \\
&\cong \oplus_{i,j} [e_1; (\pi_i, \alpha_N(H_j(a_i), b_j))] \\
&\cong \oplus_{i,j} [e_1; (\pi_i(-) \cdot H_j(a_i), b_j)] \\
&\cong \oplus_{j=1}^m [e_1; (\pi_1(-) \cdot H_j(a_1) \oplus \cdots \oplus \pi_n(-) \cdot H_j(a_n), b_j)] \\
&= y
\end{aligned}$$

This completes the proof. ■

5.7. PROPOSITION. *(i) If M is finitely generated $\mathbf{G}$-projective, then $E' : M' \otimes_R M \longrightarrow R'$ is an equivalence of $R'-R'$-bimodules.*

(ii) If M is a progenerator, then $E : M \otimes_{R'} M' \longrightarrow R$ is an equivalence of $R-R$-bimodules.

PROOF. (i). E' is $\tau_M^0(M)$ composed with the anti-isomorphism of A_∞-rings

$$\psi : \mathrm{Hom}_R(M, M) \cong R'$$

(an isomorphism of $R'-R'$-bimodules). Let $\Sigma' : R' \longrightarrow M' \otimes_R M$ be $\gamma_M(M) \circ \psi^{-1}$. From the proof of Lemma 5.6, Σ' is an $R'-R'$-bimodule inverse equivalence of E'.

(ii). E is surjective by Lemma 5.4, so there is $y = [e_m; (b_1, H_1), \ldots, (b_m, H_m)] \in M \otimes_R M'$ with $E(y) = 1$. Define $\Sigma : R \longrightarrow M \otimes_R M'$ by $\Sigma(r) = r \cdot y$, and note that $E \circ \Sigma = 1_R$. As in the proof of Lemma 5.6 one can show $r \cdot y \cong y \cdot r$ and obtain that Σ is an $R-R$-bimodule morphism. The diagram

$$\begin{tikzcd}
M' \otimes_R M \otimes_{R'} M' \arrow[r, "1_{M'} \otimes_R E"] \arrow[d, "E' \otimes_{R'} 1_{M'}"'] & M' \otimes_R R \arrow[d, "\overline{\beta}_{M'}"] \\
R' \otimes_{R'} M' \arrow[r, "\overline{\alpha}_{M'}"] & M'
\end{tikzcd}$$

commutes up to $R'-R$-natural isomorphism, where $\alpha_{M'}$ is the left R'-action and $\beta_{M'}$ is the right R-action. Since $(1_{M'} \otimes_R E) \circ (1_{M'} \otimes_R \Sigma) = 1_{M' \otimes_R R}$ we see that $1_{M'} \otimes_R \Sigma$

is an $R'-R$-bimodule inverse equivalence of $1_{M'}\otimes_R E$. By Lemma 5.6, we have that $\Sigma_* : \mathrm{Hom}_R(M,R) \longrightarrow \mathrm{Hom}_R(M, M\otimes_{R'}M')$ is an $R'-R$-bimodule inverse equivalence of $E_* : \mathrm{Hom}_R(M, M\otimes_{R'}M') \longrightarrow \mathrm{Hom}_R(M,R)$. Applying $\mathrm{Hom}_{R'}(M',-)$ to this pair of equivalences and using adjointness we now have

$$\Sigma_* : \mathrm{Hom}_R(M\otimes_{R'}M', R) \longrightarrow \mathrm{Hom}_R(M\otimes_{R'}M', M\otimes_{R'}M')$$

and

$$E_* : \mathrm{Hom}_R(M\otimes_{R'}M', M\otimes_{R'}M') \longrightarrow \mathrm{Hom}_R(M\otimes_{R'}M', R)$$

are $R-R$-bimodule inverse equivalences. This gives an $R-R$-natural isomorphism $\delta : \Sigma_* \circ E_* \longrightarrow \mathrm{Id}$ and hence an R–R-natural isomorphism $\delta(1_{M\otimes_{R'}M'}) : \Sigma \circ E \longrightarrow 1_{M\otimes_{R'}M'}$. ■

5.8. Theorem. *If M is a progenerator in R–*Mod*, then $M\otimes_{R'}(-)$ is a lax 2-equivalence of the 2-categories $R-$*Mod *and $R'-$*Mod.

Proof. We show $M'\otimes_R(-)$ is an inverse 2-equivalence of $M\otimes_{R'}(-)$. For an R-module N, let $\overline{\alpha}_N$ and I_N be the equivalences of Proposition 4.17. Define lax 2-natural transformations $\sigma : M\otimes_{R'}(-) \circ M'\otimes_R(-) \longrightarrow \mathrm{Id}$ and $\tau : M'\otimes_R(-) \circ M\otimes_{R'}(-) \longrightarrow \mathrm{Id}$ by:

$$\sigma^0(N) : M\otimes_{R'}(M'\otimes_R N) \overset{U_N}{\cong} (M\otimes_{R'}M')\otimes_R N \overset{E\otimes_R 1_N}{\longrightarrow} R\otimes_R N \overset{\overline{\alpha}_N}{\longrightarrow} N$$

$$\tau^0(P) : M'\otimes_R(M\otimes_{R'}P) \overset{V_P}{\cong} (M'\otimes_R M)\otimes_{R'} P \overset{E'\otimes_{R'}1_P}{\longrightarrow} R'\otimes_{R'} P \overset{\overline{\alpha}_P}{\longrightarrow} P$$

and for $F : N \longrightarrow L$ in $R-$Mod, $G : P \longrightarrow Q$ in $R'-$Mod, let $\sigma^1(F) = h_F *_0 1_{E\otimes_R 1_N}$, $\tau^1(G) = h_G *_0 1_{E'\otimes_{R'}1_P}$, horizontal composition of 2-cells.

σ and τ have "inverses" $\overline{\sigma}$ and $\overline{\tau}$, the (strict) 2-natural transformations with $\overline{\sigma}(N) = U_N^{-1} \circ (\Sigma\otimes_R 1_N) \circ I_N$ and $\overline{\tau}(P) = V_P^{-1} \circ (\Sigma'\otimes_{R'} 1_P) \circ I_P$. It is straightforward to write down the required modifications ($\overline{\sigma} \circ \sigma \longrightarrow \mathrm{id}$, etc.) and this is left to the reader. ■

5.9. Theorem. *If M is a progenerator in $R-$*Mod*, then $(M\otimes_{R'}(-), \mathrm{Hom}_R(M,-))$ is a lax 2-adjoint, lax 2-equivalence of the 2-categories $R-$*Mod *and $R'-$*Mod.

Proof. $(M\otimes_{R'}(-), \mathrm{Hom}_R(M,-))$ is a lax 2-adjoint pair by Proposition 4.18. The unit η and counit ε are the composites

$$\eta : \mathrm{Id} \overset{\overline{\tau}}{\longrightarrow} M'\otimes_R(-) \circ M\otimes_{R'}(-) \longrightarrow \mathrm{Hom}_R(M,-) \circ M\otimes_{R'}(-)$$

$$\varepsilon : M\otimes_{R'}(-) \circ \mathrm{Hom}_R(M,-) \longrightarrow M\otimes_{R'}(-) \circ M'\otimes_R(-) \overset{\sigma}{\longrightarrow} \mathrm{Id}$$

where σ and $\overline{\tau}$ are as in the proof of Theorem 5.8 and the unlabelled morphisms arise from the lax 2-natural equivalence of Lemma 5.6. η and ε are clearly lax 2-natural equivalences.

■

5.10. REMARK. A_∞ rings R and T are *Morita equivalent* if $R-$Mod and $T-$Mod are lax 2-equivalent. In [5] we will show that Morita equivalent A_∞ rings have the same K-theory.

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Department of Mathematical Sciences,
New Mexico State University,
Las Cruces, New Mexico 88003
U.S.A.

Email: `gdunn@nmsu.edu`

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