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CROSSED SQUARES AND 2-CROSSED MODULES OF COMMUTATIVE ALGEBRAS

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ABSTRACT. In this paper, we construct a neat description of the passage from crossed squares of commutative algebras to 2-crossed modules analogous to that given by Conduché in the group case. We also give an analogue, for commutative algebra, of T.Porter's [13] simplicial groups to n-cubes of groups which implies an inverse functor to Conduché's one.

Introduction

Simplicial commutative algebras play an important role in homological algebras, homotopy theory and algebraic K-theory. In each theory, the internal structure has been studied relatively little. The present article intends to study the 3-types of simplicial algebras.

Crossed modules were initially defined by Whitehead (cf. [14]) as a model for 2-types. Conduché has defined a 2-crossed module as a model for 3-types. His unpublished work determines that there exists an equivalence (up to homotopy) between the category of crossed squares of groups and that of 2-crossed modules of groups. This construction was used by T.Porter in the work of n-Type of simplicial groups and crossed n-cubes, [13]. This result is true for the commutative algebra case.

In [3], we showed how to go from a simplicial algebra to a 2-crossed module of algebras and back to a truncated form of the simplicial algebra, and the link between simplicial algebras and crossed squares is explicitly given. It is natural for us to ask if the passage from a crossed square to a simplicial algebra and hence a 2-crossed module can be performed.

The main points of this paper are thus:

i) to construct a functor from simplicial commutative algebras to crossed n-cubes analogous to that given by T.Porter in the group case.

ii) to give a full description of the passage from a crossed square to a 2-crossed module of commutative algebras by using the Artin-Mazur "codiagonal" functor.

iii) to prove directly 2-crossed modules in a neat way.

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1. Preliminaries

1.1. Simplicial algebras. Let $\mathbf{k}$ be a commutative ring with identity. We will use the term *commutative algebra* to mean a commutative algebra over $\mathbf{k}$. The category of commutative algebras will be denoted by **CA**.

A *simplicial (commutative) algebra* $\mathbf{E}$ consists of a family of algebras $\{E_n\}$ together with face and degeneracy maps $d_i = d_i^n : E_n \to E_{n-1}$, $\quad 0 \le i \le n$, $\quad (n \neq 0)$ and $s_i = s_i^n : E_n \to E_{n+1}$, $\quad 0 \le i \le n$, satisfying the usual simplicial identities given in Illusie [10] for example. It can be completely described as a functor $\mathbf{E}$: $\Delta^{op} \to \mathbf{CommAlg}_k$ where Δ is the category of finite ordinals $[n] = \{0 < 1 < \cdots < n\}$ and increasing maps. We will denote the category of simplicial commutative algebras by **SimpAlg**. We have for each $k \ge 0$ a subcategory $\Delta_{\le k}$ determined by the objects $[j]$ of Δ with $j \le k$. A k-truncated simplicial commutative algebra is a functor from $(\Delta_{\le k}^{op})$ to **CommAlg**.

Consider the product $\Delta \times \Delta$ whose objects are pairs $([p],[q])$ and whose maps are pairs of weakly increasing maps. A functor $\mathbf{E}_{\cdot,\cdot}$: $(\Delta \times \Delta)^{op} \to \mathbf{CommAlg}_k$ is called a *bisimplicial algebra* with value in **CommAlg.** To give $\mathbf{E}_{\cdot,\cdot}$ is equivalent with giving for each (p,q) algebra $E_{p,q}$ and homomorphisms

$$\begin{array}{ll} d_i^h : E_{p,q} \to E_{p-1,q} & \\ s_i^h : E_{p,q} \to E_{p+1,q} & i : 0,\ldots,p \\ d_j^v : E_{p,q} \to E_{p,q-1} & \\ s_j^v : E_{p,q} \to E_{p,q+1} & j : 0,\ldots,q \end{array}$$

such that the maps d_i^h, s_i^h commute with d_j^v, s_j^v and d_i^h, s_i^h (resp. d_j^v, s_j^v) satisfy the usual simplicial identities. Here d_i^h, s_i^h denote the horizontal operators and d_j^v, s_j^v denote the vertical operators.

By an *ideal chain complex* of algebras, (X,d) we mean one in which each $\mathrm{Im} d_{i+1}$ is an ideal of X_i. Given any ideal chain complex (X,d) of algebras and an integer n the truncation, $t_{n]}X$, of X at level n will be defined by

$$(t_{n]}X)_i = \begin{cases} X_i & \text{if } i < n \\ X_i/\mathrm{Im} d_{n+1} & \text{if } i = n \\ 0 & \text{if } i > n. \end{cases}$$

The differential d of $t_{n]}X$ is that of X for $i < n$, $\; d_n$ is induced by the nth differential of X and all other are zero.

1.2. The Moore complex of simplicial algebra. We recall that given a simplicial algebra $\mathbf{E}$, *the Moore complex* $(\mathbf{NE}, \partial)$ *of* $\mathbf{E}$ is the chain complex defined by

$$(\mathbf{NE})_n = \bigcap_{i=0}^{n-1} \mathrm{Ker} d_i^n$$

with $\partial_n : NE_n \to NE_{n-1}$ induced by d_n^n by restriction.

The n^{th} *homotopy module* $\pi_n(\mathbf{E})$ of $\mathbf{E}$ is the nth homology of the Moore complex of $\mathbf{E}$, i.e.,

$$\pi_n(\mathbf{E}) \;\cong\; H_n(\mathbf{NE}, \partial) = \bigcap_{i=0}^{n} \mathrm{Ker} d_i^n / d_{n+1}^{n+1}(\bigcap_{i=0}^{n} \mathrm{Ker} d_i^{n+1}).$$

We say that the Moore complex $\mathbf{NE}$ of a simplicial algebra is of *length* k if $NE_n = 0$ for all $n \geq k+1$ so that a Moore complex is of length k also of length r for $r \geq k$.

A simplicial map $f : \mathbf{E} \to \mathbf{E}'$ is called a n- *equivalence* if it induces isomorphisms

$$\pi_n(\mathbf{E}) \cong \pi_n(\mathbf{E}') \qquad \text{for } n \geq 0.$$

Two simplicial algebras $\mathbf{E}$ and $\mathbf{E}'$ are said to be have the same *n-type* if there is a chain of n-equivalences linking them. A simplicial commutative algebra $\mathbf{E}$ is an n-type if $\pi_i(\mathbf{E}) = 0$ for $i > n$.

2. 2-crossed modules from simplicial algebras

Crossed modules techniques give a very efficient way of handling informations about on a homotopy type. They correspond to a 2-type (see [3]). As mentioned the above, Conduché, [5], in 1984 described the notion of 2-crossed module as a model for 3-types.

Throughout this paper we denote an action of $r \in R$ on $c \in C$ by $r \cdot c$.

A *crossed module* is an algebra morphism $\partial : C \to R$ with an action of R on C satisfying $\partial(r \cdot c) = r\partial c$ and $\partial(c) \cdot c' = cc'$ for all $c, c' \in M, r \in R$. The last condition is called the *Peiffer identity*. We will denote such a crossed module by (C, R, ∂). A *morphism* of crossed modules from (C, R, ∂) to (C', R', ∂') is a pair of $\mathbf{k}$-algebra morphisms, $\phi : C \to C'$, $\psi : R \to R'$ such that $\phi(r \cdot c) = \psi(r)\phi(c)$. We thus get a category $\mathbf{XMod}$ of crossed modules.

Examples of crossed modules: (i) any ideal I in R gives an inclusion map, inc: $I \to R$ which is a crossed module. Conversely given an arbitrary crossed module $\partial : C \to R$, one can easily sees that the Peiffer identity implies that ∂C is an ideal in R. (ii) Given any R-module, M, one can give M an algebra structure by giving it the zero multiplication. With this structure the zero morphism $0 : M \to R$ is a crossed module.

We recall from Grandjeán and Vale [8] the definition of 2-crossed module:

A 2-crossed module of k-algebras consists of a complex of C_0-algebras

$$C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

with ∂_2, ∂_1 morphisms of C_0-algebras, where the algebra C_0 acts on itself by multiplication, such that

$$C_2 \xrightarrow{\partial_2} C_1$$

is a crossed module in which C_1 acts on C_2, (we require thus that for all $x \in C_2,\ y \in C_1$ and $z \in C_0$ that $(xy)z = x(yz)$), further, there is a C_0-bilinear function giving

$$\{\quad \otimes \quad\} : C_1 \otimes_{C_0} C_1 \longrightarrow C_2,$$

called a Peiffer lifting, which satisfies the following axioms:

$$\begin{array}{lrcl}
PL1: & \partial_2\{y_0\otimes y_1\} & = & y_0y_1 - y_0\cdot\partial_1(y_1),\\
PL2: & \{\partial_2(x_1)\otimes\partial_2(x_2)\} & = & x_1x_2,\\
PL3 & \{y_0\otimes y_1y_2\} & = & \{y_0y_1\otimes y_2\}+\partial_1y_2\cdot\{y_0\otimes y_1\},\\
PL4: & \{y\otimes\partial_2(x)\}+\{\partial_2(x)\otimes y\} & = & \partial_1(y)\cdot x,\\
PL5: & \{y_0\otimes y_1\}\cdot z & = & \{y_0\cdot z\otimes y_1\}=\{y_0\otimes y_1\cdot z\},
\end{array}$$

for all $x, x_1, x_2 \in C_2,\ y,\ y_0,\ y_1,\ y_2 \in C_1$ and $z \in C_0$.

A morphism of 2-crossed modules of algebras can be defined in a obvious way. We thus define the category of 2-crossed module denoting it by $\mathbf{X_2Mod}$.

We denote the category of simplicial algebras with Moore complex of length n by $\mathbf{SimpAlg}_{\leq n}$ in the following.

In [3], we studied the truncated simplicial algebras and saw what properties that has.

Later, we turned to a simplicial algebra $\mathbf{E}$ which is 2-truncated, i.e., its Moore complex look like:

$$\cdots \longrightarrow 0 \longrightarrow NE_2 \xrightarrow{\partial_2} NE_1 \xrightarrow{\partial_1} NE_0$$

and we showed the following result:

2.1. Theorem. *The category* $\mathbf{X_2Mod}$ *of 2-crossed modules is equivalent to the category* $\mathbf{SimpAlg}_{\leq 2}$ *of simplicial algebras with Moore complex of length* 2.

That is, this result can be pictured by the following diagram:

$$\begin{tikzcd}
\mathbf{SimpAlg}_{\leq 2} \arrow[r, shift left, "\mathbf{C^{(2)}}"] & \mathbf{X_2Mod} \arrow[l, shift left, "\mathbf{K^{(2)}}"] \\
& \mathbf{Tr_2SimpAlg}. \arrow[ul, "t_{2]}"] \arrow[u]
\end{tikzcd}$$

where $\mathbf{Tr_2SimpAlg}$ denotes the category of 2-truncated simplicial algebras.

3. Cat^2-algebras and crossed squares

We will recall the definition (due to Ellis [7]) of a crossed square of commutative algebras.

A crossed square of commutative algebras is a commutative diagram

$$\begin{tikzcd}
L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\
N \arrow[r, "\nu"'] & R
\end{tikzcd}$$

together with actions of R on L, M and N. There are thus commutative actions of M on L and N via μ, and N acts on L and M via ν and a function $h : M \times N \to L$, the h-map. These data have to satisfy the following axioms: for all $m, m' \in M,\ n, n' \in N,\ r \in R,\ l \in L,\ k \in \mathbf{k}$;

1. each of the maps $\lambda,\ \lambda',\ \mu,\ \nu'$ and the composite $\mu\lambda = \nu\lambda'$ are crossed modules,
2. the maps λ, λ' preserve the action of R,
3. $kh(m, n) = h(km, n) = h(m, kn)$,
4. $h(m + m', n) = h(m, n) + h(m', n)$,
5. $h(m, n + n') = h(m, n) + h(m, n')$,
6. $r \cdot h(m, n) = h(r \cdot m, n) = h(m, r \cdot n)$,
7. $\lambda h(m, n) = m \cdot n$,
8. $\lambda' h(m, n) = n \cdot m$,
9. $h(m, \lambda' l) = m \cdot l$,
10. $h(\lambda l, n) = n \cdot l$.

In the simplest example of crossed squares (cf.[3]) μ and ν are ideal inclusions and $L = M \cap N$, with h being the multiplication map. We also mentioned that if

$$\begin{array}{ccc} \mathbf{M} \cap \mathbf{N} & \longrightarrow & \mathbf{M} \\ \downarrow & & \downarrow \\ \mathbf{N} & \longrightarrow & \mathbf{E} \end{array}$$

was a simplicial crossed square constructed from a simplicial algebra $\mathbf{E}$ and two simplicial ideals $\mathbf{M}$ and $\mathbf{N}$ then applying π_0 to the square gives a crossed square and that up to isomorphism all crossed squares arise his way, see [3].

Although when first introduced by Loday and Walery [9], the notion of crossed squares of groups was not linked to that of cat^2-groups, it was in this form that Loday gave their generalisation to an n-fold structure, cat^n-groups (see [11]).

We recall for low dimension Ellis equivalence between the category of crossed n-cubes and that of cat^n-algebras is given by Ellis (cf. [7]) it was not given an explicit description in low dimensions. We now examine this for low dimensions.

Recall from [12] that a cat^1-alg is a triple, (E, s, t), where E is an k-algebra and s, t are endomorphisms of E satisfying the following conditions:

(i) $st = t$ and $ts = s$

(ii) Ker$s$Ker$t = 0$.

It was shown in [12] that setting $C=$Ker$s$, $B=$Ims and $\partial = t \mid C$, then the action of B on C within E makes

$$\partial : C \longrightarrow B$$

into a crossed module. Conversely if $\partial : C \to B$ is a crossed module, then setting $E = C \rtimes B$ and letting s, t be defined by

$$s(c,b) = (0,b)$$

and

$$t(c,b) = (0, \partial(c) + b)$$

for $c \in C$, $b \in B$, then (E,s,t) is a cat^1-algebra.

For a cat^2-algebra, we again have an algebra, E, but this time with two independent cat^1-algebra structures on it. Explicitly:

A cat^2-algebra is a 5-tuple (E, s_1, t_1, s_2, t_2), where (E, s_i, t_i), $i = 1,2$, are cat^1-algebras and

$$s_i s_j = s_j s_i, \quad t_i t_j = t_j t_i, \quad s_i t_j = t_j s_i$$

for $i,j = 1,2$, $i \neq j$.

3.1. Proposition. *There is an equivalence of categories between the category of cat^2-algebras and that of crossed squares.*

Proof. The cat^1-algebra (E, s_1, t_1) will give us a crossed module

$$\partial : C \longrightarrow B$$

with $C=$Ker$s$, $B=$Ims and $\partial = t \mid C$, but as the two cat^1-algebra structures are independent, (E, s_2, t_2) restricts to give cat^1-algebra structures on C and B and makes ∂ a morphism of cat^1-algebras. We thus get a morphism of crossed modules

$$\begin{array}{ccc} \mathrm{Ker}s_1 \cap \mathrm{Ker}s_2 & \longrightarrow & \mathrm{Im}s_1 \cap \mathrm{Ker}s_2 \\ \downarrow & & \downarrow \\ \mathrm{Ker}s_2 \cap \mathrm{Im}s_1 & \longrightarrow & \mathrm{Im}s_1 \cap \mathrm{Im}s_2 \end{array}$$

where each morphism is a crossed module for the natural action, i.e. multiplication in E. It remains to produce an h-map, but it is given by the multiplication within E since if $x \in$Ker$s_2 \cap$Ims_1 and $y \in$Im$s_2 \cap$Kers_1 then $xy \in$Ker$s_1 \cap$Kers_2. It is easy to check the crossed squares axioms.

Conversely, if

$$\begin{array}{ccc} L & \longrightarrow & M \\ \downarrow & & \downarrow \\ N & \longrightarrow & R \end{array}$$

is a crossed square, then we can think of it as a morphism of crossed modules

$$\begin{array}{ccc} L & & M \\ \Big\downarrow & \longrightarrow & \Big\downarrow \\ N & & R. \end{array}$$

Using the equivalence between crossed modules and cat^1-algebras this gives a morphism

$$\partial : (L \rtimes N, s, t) \longrightarrow (M \rtimes R, s', t')$$

of cat^1-algebras. There is an action of $(m, r) \in M \rtimes R$ on $(l, n) \in L \rtimes N$ given by

$$(m, r) \cdot (l, n) = (r \cdot l + \partial m \cdot l + h(m, n), r \cdot n + mn).$$

Using this action, we thus form its associated cat^1-algebra with big algebra $(L \rtimes N) \rtimes (M \rtimes R)$ and induced endomorphisms s_1, t_1, s_2, t_2. ∎

4. Crossed n-cubes and simplicial algebras

Crossed n-cubes in algebraic settings such as commutative algebras, Jordan algebras, Lie algebras have been defined by Ellis [7].

A crossed n-cube of commutative algebras is a family of commutative algebras, M_A for $A \subseteq < n >= \{1, ..., n\}$ together with homomorphisms $\mu_i : M_A \to M_{A-\{i\}}$ for $i \in < n >$ and for $A, B \subseteq < n >$, functions

$$h : M_A \times M_B \longrightarrow M_{A \cup B}$$

such that for all $k \in \mathbf{k},\ a, a' \in M_A,\ b, b' \in M_B,\ c \in M_C,\ i, j \in < n >$ and $A \subseteq B$

$$\begin{array}{lrcl}
1) & \mu_i a & = & a \quad \text{if } i \notin A \\
2) & \mu_i \mu_j a & = & \mu_j \mu_i a \\
3) & \mu_i h(a, b) & = & h(\mu_i a, \mu_i b) \\
4) & h(a, b) & = & h(\mu_i a, b) = h(a, \mu_i b) \quad \text{if } i \in A \cap B \\
5) & h(a, a') & = & aa' \\
6) & h(a, b) & = & h(b, a) \\
7) & h(a + a', b) & = & h(a, b) + h(a', b) \\
8) & h(a, b + b') & = & h(a, b) + h(a, b') \\
9) & k \cdot h(a, b) & = & h(k \cdot a, b) = h(a, k \cdot b) \\
10) & h(h(a, b), c) & = & h(a, h(b, c)) = h(b, h(b, c)).
\end{array}$$

A morphism of crossed n-cubes is defined in the obvious way: It is a family of commutative algebra homomorphisms, for $A \subseteq < n >,\ f_A : M_A \longrightarrow M'_A$ commuting with the μ_i's and h's. We thus obtain a category of crossed n-cubes denoted by $\mathbf{Crs^n}$.

Examples. (a) For $n = 1$, a crossed 1-cube is the same as a crossed module.

(b) For $n = 2$, one has a crossed square:

$$\begin{tikzcd} M_{<2>} \arrow[r, "\mu_2"] \arrow[d, "\mu_1"'] & M_{\{1\}} \arrow[d, "\mu_1"] \\ M_{\{2\}} \arrow[r, "\mu_2"'] & M_{\emptyset}. \end{tikzcd}$$

Each μ_i is a crossed module as is $\mu_1\mu_2$. The h-functions give actions and a pairing

$$h : M_{\{1\}} \times M_{\{2\}} \longrightarrow M_{<2>}.$$

The maps μ_2 (or μ_1) also define a map of crossed modules. In fact a crossed square can be thought of as a crossed module in the category of crossed modules.

(c) Let $\mathbf{E}$ be a simplicial algebra. Then the following diagram

$$\begin{tikzcd} NE_2/\partial_3 NE_3 \arrow[r, "\delta"] \arrow[d, "\delta'"'] & NE_1 \arrow[d, "\partial'"] \\ \overline{NE}_1 \arrow[r, "\partial"'] & E_1 \end{tikzcd}$$

is a crossed square. Here $NE_1 = \mathrm{Ker} d_0^1$ and $\overline{NE}_1 = \mathrm{Ker} d_1^1$.

Since E_1 acts on $NE_2/\partial_3 NE_3$, $\overline{NE}_1$ and NE_1, there are actions of $\overline{NE}_1$ on $NE_2/\partial_3 NE_3$ and NE_1 via ∂, and NE_1 acts on $NE_2/\partial_3 NE_3$ and $\overline{NE}_1$ via ∂'. As ∂ and ∂' are inclusions, all actions can be given by multiplication. The h-map is

$$\begin{array}{ccc} NE_1 \times \overline{NE}_1 & \longrightarrow & NE_2/\partial_3 NE_3 \\ (x, \overline{y}) & \longmapsto & h(x, \overline{y}) = s_1 x(s_1 y - s_0 y) + \partial_3 NE_3, \end{array}$$

which is bilinear. Here x and y are in NE_1 as there is a natural bijection between NE_1 and $\overline{NE}_1$ (by lemma 2.3.1 in [2]). The element $\bar{y}$ is the image of y under this. The reader is referred to the author's thesis, [2], for verifying of the crossed square axioms.

This last example effectively introduces the functor

$$\mathbf{M(-,2)} \ : \ \mathbf{SimpAlg} \longrightarrow \mathbf{Crs^2}.$$

This is the 2-dimensional case of a general construction to which we turn next.

By *an ideal* $(n+1)$*-ad* will be meant an algebra with n selected ideals (possibly with repeats), $(R, I_1, \ldots, I_n)$.

Let R be an algebra with ideals $I_1, \ldots, I_n$ of R. Let

$$M_A = \bigcap\{I_i : i \in A\} \quad \text{and} \quad M_\emptyset = R$$

with $A \subseteq < n >$. If $i \in < n >$, then M_A is an ideal of $M_{A-\{i\}}$. Define $\mu_i : M_A \longrightarrow M_{A-\{i\}}$ to be the inclusion. If $A, B \subseteq < n >$, then $M_{A\cup B} = M_A \cap M_B$, let

$$\begin{array}{rlcl} h: & M_A \times M_B & \longrightarrow & M_{A\cup B} \\ & (a,b) & \longmapsto & ab \end{array}$$

as $M_A M_B \subseteq M_A \cap M_B$, where $a \in M_A,\ b \in M_B$. Then $\{M_A :\ A \subseteq < n >,\ \mu_i,\ h\}$ is a crossed n-cube, called the *inclusion crossed n-cube* given by the ideal $(n+1)$-ad of commutative algebras $(R;\ I_1, \ldots, I_n)$.

4.1. PROPOSITION. *Let* $(\mathbf{E};\ I_1, \ldots, I_n)$ *be a simplicial ideal* $(n+1)$*-ad of algebras and define for* $A \subseteq < n >$

$$M_A = \pi_0\Big(\bigcap_{i\in A} I_i\Big)$$

with homomorphisms $\mu_i : M_A \to M_{A-\{i\}}$ *and h-maps induced by the corresponding maps in the simplicial inclusion crossed n-cube, constructed by applying the previous example to each level. Then* $\{M_A :\ A \subseteq < n >,\ \mu_i,\ h\}$ *is a crossed n-cube.*

The proof is straightforward and so has been omitted.

Up to isomorphism, all crossed n-cubes arise in this way. In fact any crossed n-cube can be realised (up to isomorphism) as a π_0 of a simplicial inclusion crossed n-cube coming from a simplicial ideal $(n+1)$-ad in which π_0 is a non-trivial homotopy module.

In 1991, T. Porter described a functor, [13], from the category of simplicial groups to that of crossed n-cubes of groups, based on ideas of Loday. Here, we adapt that description to give an obvious analogue of this functor for the commutative algebra case. The functor here constructed is defined using the *décalage* functor studied by Illusie [10] and Duskin [6] and is a π_0-image of a functor taking values in a category of simplicial ideal $(n+1)$-ads.

The *décalage* functor forgets the last face operators at each level of a simplicial algebra $\mathbf{E}$ and moves everything down one level. It is denoted by Dec. Thus

$$(\mathrm{Dec}E)_n = E_{n+1}.$$

The last degeneracy of $\mathbf{E}$ yields a contraction of $\mathrm{Dec}^1\mathbf{E}$ as an augmented simplicial algebra,

$$\mathrm{Dec}^1\mathbf{E} \simeq \mathbf{K}(E_0, 0),$$

by an explicit natural homotopy equivalence (cf. [6]). The last face map will be denoted

$$\delta_0 : \mathrm{Dec}^1\mathbf{E} \longrightarrow \mathbf{E}.$$

Iterating the Dec construction gives an augmented bisimplicial algebra

$$[\dots\ \mathrm{Dec}^3\mathbf{E} \equiv\!\!\!\Rightarrow \mathrm{Dec}^2\mathbf{E} \underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}} \mathrm{Dec}^1\mathbf{E}]$$

which in expanded form is the total décalage of **E:**

$$[\dots\ \mathrm{Dec}^3\mathbf{E} \equiv\!\!\!\Rightarrow \mathrm{Dec}^2\mathbf{E} \underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}} \mathrm{Dec}^1\mathbf{E}] \xrightarrow{\quad\delta_0\quad} \mathbf{E}.$$

(see [6] or [10] for details). The maps from $\mathrm{Dec}^i\mathbf{E}$ to $\mathrm{Dec}^{i-1}\mathbf{E}$ coming from the i last face maps will be labelled $\delta_0, \dots, \delta_{i-1}$ so that $\delta_0 = d_{last}$, $\delta_1 = d_{last\ but\ one}$ and so on.

For a simplicial algebra $\mathbf{E}$ and a given n, we write $\mathbf{M}(\mathbf{E},\, n)$ for the crossed n-cube, arising from the functor

$$\mathbf{M(-,n) : SimpAlg \longrightarrow Crs^n},$$

which is given by $\pi_0(\mathrm{Dec}\mathbf{E}; \mathrm{Ker}\delta_0, \dots, \mathrm{Ker}\delta_{n-1})$. The following propositions and their proofs aim to explore this compressed definition, giving an 'elementary' specification of the construction.

4.2. Proposition. *If* $\mathbf{E}$ *be a simplicial algebra, then the crossed* n*-cube* $\mathbf{M}(\mathbf{E},n)$ *is determined by:*

(i) for $A \subseteq < n >$,

$$\mathbf{M}(\mathbf{E}, n)_A = \frac{\bigcap_{j\in A} \mathrm{Ker} d^n_{j-1}}{d^{n+1}_{n+1}(\mathrm{Ker} d^{n+1}_0 \cap \{\bigcap_{j\in A} \mathrm{Ker} d^{n+1}_j\})};$$

(ii) the inclusion

$$\bigcap_{j\in A} \mathrm{Ker} d^n_{j-1} \longrightarrow \bigcap_{j\in A-\{i\}} \mathrm{Ker} d^n_{j-1}$$

induces the morphism

$$\mu_i : \mathbf{M}(\mathbf{E}, n)_A \longrightarrow \mathbf{M}(\mathbf{E}, n)_{A-\{i\}};$$

(iii) the functions, for $A, B \subseteq < n >$,

$$h : \mathbf{M}(\mathbf{E}, n)_A \times \mathbf{M}(\mathbf{E}, n)_B \longrightarrow \mathbf{M}(\mathbf{E}, n)_{A\cup B}$$

given by

$$h(\bar{x}, \bar{y}) = \overline{xy},$$

where an element of $\mathbf{M}(\mathbf{E}, n)_A$ *is denoted by* $\bar{x}$ *with* $x \in \bigcap_{j\in A}\mathrm{Ker} d^n_{j-1}$.

Proof. First some explanation of the definition of $\mathcal{M}(-,n)$ as

$$\pi_0(\mathrm{Dec}\mathbf{E};\ \mathrm{Ker}\delta_0,\ \ldots,\ \mathrm{Ker}\delta_{n-1}).$$

For each simplicial algebra, $\mathbf{E}$, we start by looking at the canonical augmentation map δ_0 : $\mathrm{Dec}^1\mathbf{E} \longrightarrow \mathbf{E}$, which has kernel the simplicial algebra, $\mathrm{Ker}d_{last}$ mentioned above. Then take the simplicial inclusion crossed module $\mathrm{Ker}\delta_0 \longrightarrow \mathrm{Dec}^1\mathbf{E}$ to be $\mathcal{M}(\mathbf{E},1)$ defining thus a functor

$$\mathcal{M}(\quad,1) : \mathbf{SimpAlg} \longrightarrow \mathbf{Simp}(\mathbf{IncCrs^1}).$$

Then it is easy to show that

$$\pi_0(\mathrm{Ker}\delta_0) \longrightarrow \pi_0(\mathrm{Dec}^1\mathbf{E})$$

is precisely $\mathbf{M}(\mathbf{E},1)$. The higher order analogues $\mathbf{M}(\quad,n)$ are as follows:

For each simplicial algebra, $\mathbf{E}$, there is a functorial short exact sequence

$$\mathrm{Ker}\delta_0 \longrightarrow \mathrm{Dec}^1\mathbf{E} \xrightarrow{\delta_0} \mathbf{E}.$$

This corresponds to the 0-skeleton of the total décalage of $\mathbf{E}$,

$$[\ldots\ \mathrm{Dec}^3\mathbf{E} \equiv \mathrm{Dec}^2\mathbf{E} \underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}} \mathrm{Dec}^1\mathbf{E}] \xrightarrow{\delta_0} \mathbf{E}.$$

For $n=2$, the 1-skeleton of that total décalage gives a commutative diagram

$$\begin{tikzcd}
\mathrm{Dec}^2\mathbf{E} \arrow[r, "\delta_0"] \arrow[d, "\delta_1"'] & \mathrm{Dec}^1\mathbf{E} \arrow[d, "\delta_0"] \\
\mathrm{Dec}^1\mathbf{E} \arrow[r, "\delta_0"] & \mathbf{E}.
\end{tikzcd}$$

Here δ_1 is d^n_{n-1} in dimension n whilst δ_0 is d^n_n. Forming the square of kernels gives

$$\begin{tikzcd}
\mathrm{Ker}\delta_0 \cap \mathrm{Ker}\delta_1 \arrow[r] \arrow[d] & \mathrm{Ker}\delta_1 \arrow[d] \\
\mathrm{Ker}\delta_0 \arrow[r] & \mathrm{Dec}^2\mathbf{E}.
\end{tikzcd}$$

Again, π_0 of this gives $\mathbf{M}(\mathbf{E},2)$. In general, we use the $(n-1)$-skeleton of the total décalage to form an n-cube and thus a simplicial inclusion crossed n-cube corresponding to the simplicial ideal $(n+1)$-ad

$$(\mathrm{Dec}^n E;\ \mathrm{Ker}\delta_{n-1},\ \ldots,\ \mathrm{Ker}\delta_0).$$

This simplicial inclusion n-cube will be denoted by $\mathcal{M}(\mathbf{E}, n)$, and its associated crossed n-cube by

$$\pi_0(\mathcal{M}(\mathbf{E}, n)) = \mathbf{M}(\mathbf{E}, n).$$

The result now follows by direct calculation on examining the construction of π_0 as the zeroth homology of the Moore complex of each term in the inclusion crossed n-cube, $\mathcal{M}(\mathbf{E}, n)$. ■

4.3. Lemma. *If* $\mathbf{E}$ *is a simplicial algebra with* $A \subseteq < n >$, $A \neq < n >$, *then*

$$d_n\Big(\bigcap_{i\in A} \mathrm{Ker} d_i^n\Big) = \bigcap_{i\in A} \mathrm{Ker} d_{i-1}^{n-1}.$$

Proof. If $i \in A$, then

$$d_n\Big(\bigcap_{i\in A} \mathrm{Ker} d_i^n\Big) \subseteq \bigcap_{i\in A} \mathrm{Ker} d_{i-1}^{n-1},$$

since $d_{i-1}d_n = d_{n-1}d_{i-1}$.

Conversely, we suppose that x is an element in $\bigcap_{i\in A} \mathrm{Ker} d_{i-1}^{n-1}$ and consider the element

$$y = s_n x - s_{n-1}x + \ldots + (-1)^{n-k} s_k x = \sum_{i=0}^{n-k} (-1)^{i+1} s_{i+k} x,$$

where k is the first integer in $< n > \backslash A$. Then $d_n y = x$ and $d_i y = 0$for all $i \in A$ and hence $y \in \bigcap_{i\in A} \mathrm{Ker} d_i^n$ implies $x \in d_n(\bigcap_{i\in A} \mathrm{Ker} d_i^n)$ as required. ■

This lemma gives the following proposition:

4.4. Proposition. *If* $\mathbf{E}$ *is a simplicial algebra, then*

i) for $A \subseteq < n >$, $A \neq < n >$,

$$\mathbf{M}(\mathbf{E}, n)_A \cong \bigcap_{i\in A} \mathrm{Ker} d_{i-1}^{n-1}$$

so that in particular, $\mathbf{M}(\mathbf{E}, n)_\emptyset \cong E_{n-1}$; *in every case the isomorphism is induced by* d_n,

ii) if $A \neq < n >$ *and* $i \in < n >$,

$$\mu_i : \mathbf{M}(\mathbf{E}, n)_A \longrightarrow \mathbf{M}(\mathbf{E}, n)_{A\backslash\{i\}}$$

is the inclusion of an ideal,

iii) for $j \in < n >$,

$$\mu_j : \mathbf{M}(\mathbf{E}, n)_{<n>} \longrightarrow \bigcap_{i\neq j} \mathrm{Ker} d_i^{n+1}$$

is induced by d_n.

Proof. By proposition 4.2 and the previous lemma, one can obtain, for $A \neq < n >$,

$$\mathbf{M}(\mathbf{E},n)_A = \frac{\bigcap_{i\in A}\mathrm{Ker} d^n_{i-1}}{d_{n+1}(\mathrm{Ker} d^{n+1}_0 \cap \{\bigcap_{i\in A}\mathrm{Ker} d^{n+1}_i\})} = \frac{\bigcap_{i\in A}\mathrm{Ker} d^n_{i-1}}{\mathrm{Ker} d^n_0 \cap (\bigcap_{i\in A}\mathrm{Ker} d^n_{i-1}).}$$

The epimorphism $d_n : E_n \to E_{n-1}$, which is split, $d_n s_{n-1} = \mathrm{id}$, can be restricted to an epimorphism

$$\bigcap_{i\in A}\mathrm{Ker} d^n_{i-1} \longrightarrow \bigcap_{i\in A}\mathrm{Ker} d^{n-1}_{i-1},$$

by lemma 4.3 It follows then that

$$\mathrm{Ker}\Big(\bigcap_{i\in A}\mathrm{Ker} d^n_{i-1} \xrightarrow{d_n} \bigcap_{i\in A}\mathrm{Ker} d^{n-1}_{i-1}\Big) = \mathrm{Ker} d^n_0 \cap \Big(\bigcap_{i\in A}\mathrm{Ker} d^n_{i-1}\Big).$$

which completes the proof of (i).
(ii) and (iii) are now consequences. ∎

Expanding this out for low values of n gives:
1) For $n = 0$, $\mathbf{M}(\mathbf{E},0) = E_0/d_1(\mathrm{Ker} d_0) \cong \pi_0(\mathbf{E}) = H_0(\mathbf{E})$.
2) For $n = 1$, $\mathbf{M}(\mathbf{E},n)$ is the crossed module $\mu_1 : \mathrm{Ker} d^1_0/d^2_2(NE_2) \to E_1/d^2_2(\mathrm{Ker} d^2_0)$. Since $d^2_2(NE_2) = \mathrm{Ker} d^1_0 \mathrm{Ker} d^1_1$, this implies

$$\mu : NE_1/\mathrm{Ker} d^1_0\mathrm{Ker} d^1_1 \longrightarrow E_0.$$

3) For $n = 2$, $\mathbf{M}(\mathbf{E},n)$ is

$$\begin{tikzcd}
\mathrm{Ker} d^2_0 \cap \mathrm{Ker} d^2_1/d^3_3(\mathrm{Ker} d^3_0 \cap \mathrm{Ker} d^3_1 \cap \mathrm{Ker} d^3_2) \arrow[r, "\mu_2"] \arrow[d, "\mu_1"] & \mathrm{Ker} d^2_0/d^3_3(\mathrm{Ker} d^3_0 \cap \mathrm{Ker} d^3_1) \arrow[d, "\mu_1"] \\
\mathrm{Ker} d^2_1/d^3_3(\mathrm{Ker}\ d^3_0 \cap \mathrm{Ker} d^3_2) \arrow[r, "\mu_2"] & E_2/d^3_3(\mathrm{Ker} d^3_0).
\end{tikzcd}$$

By proposition 4.4, this is isomorphic to

$$\begin{tikzcd}
NE_2/d^3_3(NE_3) \arrow[r, "\mu_2"] \arrow[d, "\mu_1"'] & \mathrm{Ker} d^1_0 \arrow[d, "\mu_1"] \\
\mathrm{Ker} d^1_1 \arrow[r, "\mu_2"'] & E_1
\end{tikzcd}$$

Here the h-map is

$$h : \mathrm{Ker}d_0^1 \times \mathrm{Ker}d_1^1 \longrightarrow NE_2/d_3^3(NE_3)$$

given by

$$h(x,y) = s_1x(s_1y - s_0y) + \partial_3 NE_3.$$

where $x, y \in NE_1$.

5. 2-crossed modules from crossed squares

Conduché constructed (in a letter to R.Brown in 1984) the 2-crossed module from a crossed square

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & R \end{tikzcd}$$

as

$$L \xrightarrow{(-\lambda, \lambda')} M \rtimes N \xrightarrow{\mu + \nu} R.$$

This construction can be briefly described as follows: apply the nerve in the both directions so as to get a bisimplicial algebra, then apply either the diagonal or the Artin-Mazur 'codiagonal' functor to get to a simplicial algebra and take the Moore complex.

We saw above the category of crossed modules is equivalent to that of cat^1-algebras. Ellis (cf. [7]) generalised this equivalence to the dimension n, i.e., the equivalence between crossed n-cubes and cat^n-algebras.

We form the associated cat^2-algebra. This is

$$\begin{tikzcd} (L \rtimes N) \rtimes (M \rtimes R) \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[d, shift right, "s'"'] \arrow[d, shift left, "t'"] & N \rtimes R \arrow[d, shift right, "s_N"'] \arrow[d, shift left, "t_N"] \\ M \rtimes R \arrow[r, shift left, "s_M"] \arrow[r, shift right, "t_M"'] & R. \end{tikzcd}$$

The source and target maps are defined as follows:

$$\begin{array}{ll} s((l,n),(m,r)) = (n,r) & t((l,n),(m,r)) = (\lambda' l + \mu(m)n, \mu(m) + r) \\ s'((l,n),(m,r)) = (m,r) & t'((l,n),(m,r)) = (\lambda l + m, \nu(n) + r) \\ s_M(m,r) = r & t_M(m,r) = \mu(m) + r \\ s_N(n,r) = r & t_N(n,r) = \nu(n) + r \end{array}$$

for $l \in L, m \in M, n \in N$ and $r \in R$.

We now take the binerve, that is the nerves in the both directions of the cat^2-algebra constructed. This is a bisimplicial algebra. The first few entries in the bisimplicial array are given below (where $A^B := A \rtimes B$)

$$\begin{array}{ccccc}
\cdots & \Rrightarrow & (L^N) \rtimes ((L^N) \rtimes (M^R)) & \rightrightarrows & N \rtimes (N \rtimes R)\\
\downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow\\
((L^L) \rtimes N) \rtimes ((M^M) \rtimes R) & \Rrightarrow & (L^N) \rtimes (M^R) & \rightrightarrows & N \rtimes R\\
\downdownarrows & & \downdownarrows & & \downdownarrows\\
M \rtimes (M \rtimes R) & \Rrightarrow & M \rtimes R & \rightrightarrows & R
\end{array}$$

Some reduction has already been done. For example, the double semi-direct product represents the algebra of pairs of elements $((m_1, r_1), (m_2, r_2)) \in M \rtimes R$ where $\mu(m_1) + r_1 = r_2$. This is the algebra $M \rtimes (M \rtimes R)$ where the action is $M \rtimes R$ on M given by $(m, r) \cdot m' = (\mu m + r)m'$.

We will recall the 'Artin-Mazur' codiagonal functor ∇ (cf. [1]) from bisimplicial algebras to simplicial algebras.

Let $\mathbf{E}.,.$ be a bisimplicial algebra. Put

$$E_{(n)} = \bigoplus_{p+q=n} E_{p,q}$$

and define $\nabla_n \subset E_{(n)}$ as follows: An element $(x_0, \ldots, x_n)$ of $E_{(n)}$ with $x_p \in E_{p,n-p}$ is in ∇_n if and only if

$$d_0^v x_p = d_{p+1}^h x_{p+1} \quad \text{for } p = 0, \ldots, n-1.$$

Next, define the faces and degeneracies

$$\begin{array}{ll} D_j : \nabla_n \to \nabla_{n-1} & \\ S_j : \nabla_n \to \nabla_{n+1} & \qquad j = 0, \ldots, n \end{array}$$

by

$$\begin{array}{rcl} D_j(x) & = & (d_j^v x_0, d_{j-1}^v x_1, \ldots, d_1^v x_{j-1}, d_j^h x_{j+1}, d_j^h x_{j+2}, \ldots, d_j^h x_n) \\ S_j(x) & = & (s_j^v x_0, s_{j-1}^v x_1, \ldots, s_0^v x_{j-1}, s_j^h x_j, s_j^h x_{j+1}, \ldots, s_j^h x_n). \end{array}$$

Thus $\nabla(\mathbf{E}.,.) = \{\nabla_n : D_j, S_j\}$ is a simplicial algebra.

We now examine this construction in low dimensions:

Example: For $n = 0, E_{(0)} = E_{0,0}$. For $n = 1$, we have

$$\nabla_1 \subset E_{(1)} = E_{1,0} \oplus E_{0,1}$$

where

$$\nabla_1 = \left\{(e_{1,0}, e_{0,1}) : d_0^h e_{1,0} = d_1^v e_{0,1}\right\}$$

together with the homomorphisms

$$\begin{array}{l} D_0^1(e_{1,0}, e_{0,1}) = (d_0^v e_{1,0}, d_0^h e_{0,1}) \\ D_1^1(e_{1,0}, e_{0,1}) = (d_1^v e_{1,0}, d_1^h e_{0,1}) \\ S_0^0(e_{0,0}) = (s_0^v e_{0,0}, s_0^h e_{0,0}). \end{array}$$

For $n = 2$, we have

$$\nabla_2 \subset E_{(2)} = \bigoplus_{p+q=2} E_{p,q} = E_{2,0} \oplus E_{1,1} \oplus E_{0,2}$$

where

$$\nabla_2 = \left\{(e_{2,0}, e_{1,1}, e_{0,2}) : d_0^h e_{2,0} = d_1^v e_{1,1}, \quad d_0^h e_{1,1} = d_2^v e_{0,2}\right\}.$$

We use the 'Artin-Mazur' codiagonal functor to obtain a simplicial algebra $\mathbf{A}$ (of some complexity);

The base algebra is still $A_0 \cong R$. However the algebra of 1-simplices is subset of

$$E_{1,0} \oplus E_{0,1} = (M \rtimes R) \oplus (N \rtimes R),$$

consisting of elements $((m, r), (n, r'))$ where $\mu(m) + r = r'$, i.e.,

$$A_1 = \{((m, r), (n, r')) : \mu(m) + r = r'\}.$$

Let us see what the composite of two such elements

$$((m_1, \mu(m_1) + r_1), (n_1, r_1))\text{and } ((m_2, \mu(m_2) + r_2)), (m_2, r_2))$$

becomes:

$$\begin{array}{l} ((m_1, \mu(m_1) + r_1), (n_1, r_1)) \circ ((m_2, \mu(m_2) + r_2)), (m_2, r_2)) \\ = (m_1, \mu(m_1) + r_1) \cdot (m_2, \mu(m_2) + r_2) \circ ((n_1, r_1) \cdot (m_2, r_2))\text{by the inter-change law} \\ = ((m_1 m_2 + m_1(\mu(m_2) + r_2) + m_2(\mu(m_1) + r_1), (\mu(m_1) + r_1)(\mu(m_2) + r_2)), \\ n_1 n_2 + n_1 r_2 + n_1 r_1, r_1 r_2). \end{array}$$

This means that the subalgebra A_1 is $M \rtimes (N \rtimes R)$ where the action of $N \rtimes R$ on M via R is the obvious one. Thus

$$A_1 \cong M \rtimes (N \rtimes R) = \{(m, n, r) : m \in M, n \in N, r \in R\}.$$

We next turned to the algebra of 2-simplices which is the subset A_2 of

$$E_{2,0} \oplus E_{1,1} \oplus E_{0,2} = (M \rtimes (M \rtimes R)) \oplus ((L \rtimes N) \rtimes (M \rtimes R)) \oplus (N \rtimes (N \rtimes R)),$$

whose elements $((m_2, m_1, r), (l, n, m, r'), (n_2, n_1, r''))$ are such that $(m_2, \mu m_1 + r) = (m, r')$ and $(\lambda' l + n, \mu m + r') = (n_1, r'')$.

By similar calculation given above, the composition of two such elements

$$((m_2, m_1, r), ((l, n), (m_2, \mu m_1 + r)), (n_2, (\lambda' l + n, \mu m + r')))$$

and

$$((m'_2, m'_1, r_1), ((l', n'), (m'_2, \mu m'_1 + r_1)), (n'_2, (\lambda' l' + n', \mu m' + r'_1)))$$

becomes such an element $(\mathbf{l}, \mathbf{m}, \mathbf{n}, \mathbf{r}) \in L \rtimes (M \rtimes (N \rtimes R))$ where $\mathbf{l} \in L$, $\mathbf{m} \in M$, $\mathbf{n} \in N$ and $\mathbf{r} \in R$. This is the algebra $L \rtimes (M \rtimes (N \rtimes R))$ where the action of $M \rtimes (N \rtimes R)$ on L is the obvious one. Thus

$$A_2 \cong L \rtimes (M \rtimes (N \rtimes R)).$$

Therefore we can get a 2-truncated simplicial algebra $\mathbf{A^{(2)}}$ that looks like

$$\mathbf{A^{(2)}}: \quad L \rtimes (M \rtimes (N \rtimes R)) \underset{s_0^1,\, s_1^1}{\overset{d_0^2,\, d_1^2,\, d_2^2}{\rightleftarrows}} M \rtimes (N \rtimes R) \underset{s_0^0}{\overset{d_0^1,\, d_1^1}{\rightleftarrows}} R.$$

with the faces and degeneracies

$$\begin{aligned} &d_0^1(m, n, r) = r \\ &d_1^1(m, n, r) = \mu(m) + r' \qquad \text{where } r' = \nu(n) + r \\ &s_0^0(r) = (0, n, r) \end{aligned}$$

and

$$\begin{aligned} &d_0^2(l, m, n, r) = (m, n, r) \\ &d_1^2(l, m, n, r) = (m, n, r) \\ &d_2^2(l, m, n, r) = (-\lambda(l) + m, \lambda'(l) + n, \mu(m) + r'); \quad \text{where } r' = \nu(n) + r \\ &s_0^1(m, n, r) = (0, m, n, r) \\ &s_1^1(m, n, r) = (0, m, n, r). \end{aligned}$$

Loday,[11], defined a mapping cone of a complex which is analogous to the construction of the Moore complex of a simplicial group. One can easily give an obvious description for commutative algebras.

We next give a mapping cone of a crossed square of algebras as follows:

5.1. Proposition. *The Moore complex of the simplicial algebra* $\mathbf{A^{(2)}}$ *is the mapping cone, in the sense of Loday, of the crossed square. Furthermore, this mapping cone complex has a structure of 2-crossed module of algebras.*

Proof. Given the 2-truncated simplicial algebra $\mathbf{A^{(2)}}$ described the above and look at its Moore complex:

Of course $NA_0 = A_0 \cong R$. The second term of the Moore complex is $NA_1 = \mathrm{Ker} d_0$. The d_0 map is d_0 on $N \rtimes R$, that is it picks out the point r, i.e., $d_0(n, r) = r$. Thus the kernel of d_0 is the subalgebra of $M \rtimes (N \rtimes R)$ when $r = 0$. This is the algebra

$M \rtimes N = NA_1 = \mathrm{Ker} d_0$. The map $d_1 |_{\mathrm{Ker} d_0}$ is $d_1(m, n, 0) = \mu(m) + \nu(n)$. This is d_1 on $M \rtimes R$, which is $\mu(m) + r'$ where $r' = \nu(n) + r$, and of course $r = 0$.

It is not hard to see that the kernel of d_0 and d_1 is the subalgebra of $L \rtimes (M \rtimes (N \rtimes R))$ where $m = 0$, $n = 0$, $r = 0$. This is the algebra L. In other words $NA_2 = L$. The map $d_2 |_{\mathrm{Ker} d_0 \cap \mathrm{Ker} d_1}$ is

$$d_2(l, 0, 0, 0) = (-\lambda l, \lambda' l, 0),$$

This is $\partial_2 = d_2$ on L, which is, $\partial_2 l = (-\lambda l, \lambda' l)$.

Next we will see that $\partial_1 \partial_2 = 0$,

$$\begin{aligned} \partial_1 \partial_2(l) &= \partial_1(-\lambda l, \lambda' l) \\ &= \mu(-\lambda l) + \nu \lambda' l \\ &= -\mu\lambda(l) + \mu\lambda(l) \quad \text{by } \mu\lambda = \nu\lambda' \\ &= 0. \end{aligned}$$

Thus, if given a crossed square

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & R \end{tikzcd}$$

then its mapping cone is

$$L \xrightarrow{\partial_2} M \rtimes N \xrightarrow{\partial_1} R$$

where $\partial_1(m, n) = \mu(m) + \nu(n)$ and $\partial_2 l = (-\lambda l, \lambda' l)$.

Now we will show the second part of the proof of result:

The semi-direct product $M \rtimes N$ can be formed by making N act on M via R,

$$n \cdot m = \nu(n) m$$

where the R-action is the given one. The fact that ∂_2 and ∂_1 are algebra homomorphisms:

$$\begin{aligned} (m, n)(m', n') &= (n \cdot m' + n' \cdot m + mm', nn') \\ &= (\nu(n)m' + \nu(n')m + mm', nn') \end{aligned}$$

Thus, (i)

$$\begin{aligned} \partial_1((m, n)(m', n')) &= \partial_1(\nu(n)m' + \nu(n')m + mm', nn') \\ &= \mu(\nu(n)m' + \nu(n')m + mm') + \nu(nn') \\ &= \nu(n)\mu(m') + \nu(n')\mu(m) + \mu(m)\mu(m') + \nu(n)\nu(n') \\ &= (\mu(m) + \nu(n))(\mu(m') + \nu(n')) \\ &= \partial_1(m, n)\partial_1(m', n') \end{aligned}$$

and similarly of course

$$\begin{aligned}\partial_1((m,n)+(m',n')) &= \partial_1(m+m',n+n')\\ &= \mu(m+m')+\nu(n+n')\\ &= (\mu(m)+\nu(n))+(\mu(m')+\nu(n'))\\ &= \partial_1(m,n)+\partial_1(m',n')\end{aligned}$$

(ii) if $l_1, l_2 \in L$, then

$$\begin{aligned}\partial_2(l_1 l_2) &= (-\lambda(l_1 l_2), \lambda'(l_1 l_2))\\ &= (-(\lambda l_1 \lambda l_2), \lambda'(l_1)\lambda'(l_2))\end{aligned}$$

whilst

$$\begin{aligned}\partial_2(l_1)\partial_2(l_2) &= (-\lambda l_1, \lambda' l_1)(-\lambda l_2, \lambda' l_2)\\ &= ((\lambda' l_1)\cdot(-\lambda l_2)+(\lambda' l_2)\cdot(-\lambda l_1)+(-\lambda l_1)(-\lambda l_2), \lambda'(l_1)\lambda'(l_2))\end{aligned}$$

but $\nu\lambda' = \mu\lambda$, so the first coordinates are equal. Similarly

$$\begin{aligned}\partial_2(l_1+l_2) &= (-\lambda(l_1+l_2), \lambda'(l_1+l_2))\\ &= (-\lambda l_1, \lambda' l_1)+(-\lambda l_2, \lambda' l_2).\\ &= \partial_2(l_1)+\partial_2(l_2).\end{aligned}$$

These elementary calculations are useful as they pave the way for calculation of the Peiffer multiplication of $x=(m,n)$ and $y=(c,a)$ in the above complex:

$$\begin{aligned}\langle x,y\rangle &= \partial_1 x\cdot y - xy\\ &= (\mu(m)+\nu(n))\cdot(c,a)-(m,n)(c,a)\\ &= ((\mu(m)+\nu(n))\cdot c, (\mu(m)+\nu(n))\cdot a)-(n\cdot c+a\cdot m+mc, na)\\ &= (a\cdot m, (\mu(m)+\nu(n))\cdot a-na) \quad \text{since any dependence on } c \text{vanishes}\\ &= (\nu(a)m, \mu(m)\cdot a+\nu(n)\cdot a-na) \quad \text{by definition of action}\\ &= (\nu(a)m, \mu(m)a+na-na) \quad \text{since } \nu \text{ is a crossed module}\\ &= (\nu(a)m, \mu(m)a).\end{aligned}$$

Thus the Peiffer lifting

$$\{\ \otimes\ \} : (M \rtimes N)\otimes(M \rtimes N) \longrightarrow L$$

for this structure is given by

$$\{x\otimes y\} = h(m, na).$$

It is immediate that this works:

$$\begin{aligned}\partial_2\{x\otimes y\} &= (-\lambda h(m,na), \lambda' h(m,na))\\ &= ((-m)\cdot na, (na)\cdot m)\\ &= (-\nu(na)m, \mu(m)(na))\end{aligned}$$

by the axioms of a crossed square.

We will not check all the axioms for a 2-crossed module for this structure, but will note the proof for two or three of them as it illustrates the connection between the h-map and the Peiffer lifting.

PL2: $\{\partial_2(l_0)\otimes\partial_1(l_1)\}=l_0l_1$.

As $\partial_2 l=(-\lambda l,\lambda' l)$, this needs the calculation of

$$h(-\lambda l_0,\lambda'(l_0l_1))$$

but the crossed square axiom $h(\lambda l,n)=n\cdot l$, and $h(m,\lambda' l)=m\cdot l$, together with the fact that the map $\lambda:L\to M$ is a crossed module give:

$$\begin{array}{rcl}\{\partial_2(l_0)\otimes\partial_1(l_1)\} &=& \{(-\lambda l_0,\lambda' l_0)\otimes(-\lambda l_1,\lambda' l_1)\}\\ &=& h(-\lambda l_0,\lambda'(l_0l_1))\\ &=& -\lambda l_0\cdot(l_0l_1)\\ &=& l_0l_1\end{array}$$

PL3: $\{y_0\otimes y_1y_2\}=\{y_0\otimes y_1y_2\}+\partial_1 y_2\cdot\{y_0\otimes y_1y_2\}$, where $y_0=(m_0,n_0)$, $y_1=(m_1,n_1)$ and $y_2=(m_2,n_2)\in M\rtimes N$

$$\begin{array}{rcl}\{y_0\otimes y_1y_2\} &=& \{(m_0,n_0)\otimes(m_1,n_1)(m_2,n_2)\}\\ &=& \{(m_0,n_0)\otimes(n_1\cdot m_2+n_2\cdot m_1+m_1m_1,n_1n_2)\}\\ &=& h(m_0,n_0n_1n_2)\end{array}$$

whilst

$$\begin{array}{l}\{y_0\otimes y_1y_2\}+\partial_1 y_2\cdot\{y_0\otimes y_1y_2\}=\{(m_0,n_0)(m_1,n_1)\otimes(m_2,n_2)\}\\ +\partial_1 y_2\cdot\{(m_0,n_0)\otimes(m_1,n_1)(m_2,n_2)\}\\ =h(n_0\cdot m_1+n_1\cdot m_0,n_0n_1(n_2))+h(m_0,\partial_1 y_2\cdot(n_0n_1))\end{array}$$

and this simplifies as expected to give the correct result.

PL4:

$$\begin{array}{rcl}\{x\otimes\partial_2 l\}+\{\partial_2 l\otimes x\} &=& \{(m,n)\otimes(-\lambda l,\lambda' l)\}+\{(-\lambda l,\lambda' l)\otimes(m,n)\}\\ &=& h(m,n\lambda' l)+h(-\lambda l,\lambda'(l)n)\\ &=& h(m,\lambda'(n\cdot l))+h(-\lambda l,\lambda'(n\cdot l)) \text{ by } \lambda' \text{ cros. mod.}\\ &=& (\mu(m)+\nu(n))\cdot l\\ &=& \partial_1(m,n)\cdot l\\ &=& \partial_1 x\cdot l\end{array}$$

■

We thus have two ways of going from simplicial algebras to 2-crossed modules

(i) directly to get

$$\frac{NE_2}{\partial_2 NE_3}\longrightarrow NE_1\longrightarrow NE_0,$$

(ii) indirectly via the square axiom $\mathbf{M}(\mathbf{E},\mathbf{2})$ and then by the above construction to get

$$\frac{NE_2}{\partial_2 NE_3}\longrightarrow \mathrm{Ker}d_0\rtimes\mathrm{Ker}d_1\longrightarrow E_1,$$

and they clearly give the same homotopy type. More precisely E_1 decomposes as $\mathrm{Ker}d_1 \rtimes s_0E_0$ and the $\mathrm{Ker}d_0$ factor in the middle term of (ii) maps down to that in this decomposition by the identity map. Thus d_0 induces a quotient map from (ii) to (i) with kernel isomorphic to

$$0 \longrightarrow \mathrm{Ker}d_0 \stackrel{=}{\longrightarrow} \mathrm{Ker}d_0$$

which is thus contractible.

Remark. The situation in this paper can be summarised in the following diagram:

$$\begin{tikzcd}
 & \mathbf{SimpAlg}_{\leq 2} \arrow[dl, "\mathbf{M}(-,\mathbf{2})"'] \arrow[d, shift left, "\mathbf{C}_{(2)}"] \\
\mathbf{Crs^2} \arrow[r, "(\quad)_2"'] & \mathbf{X_2Mod}, \arrow[u, shift left, "\mathbf{K}_{(2)}"]
\end{tikzcd}$$

where $(\quad)_2$ is given by Proposition 5.1. The diagram is commutative, linking the constructions of section 4 and 5 with those given here.

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