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SIMPLICIAL AND CATEGORICAL DIAGRAMS, AND THEIR EQUIVARIANT APPLICATIONS

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Abstract. We show that the homotopy category of simplicial diagrams $\mathbb{I}$-$\mathbb{SS}$ indexed by a small category $\mathbb{I}$ is equivalent to a homotopy category of $\mathbb{SS}\downarrow N\mathbb{I}$ simplicial sets over the nerve $N\mathbb{I}$. Then their equivalences, by means of the nerve functor $N : \mathbb{C}at \to \mathbb{SS}$ from the category $\mathbb{C}at$ of small categories, with respective homotopy categories associated to $\mathbb{C}at$ are established. Consequently, an equivariant simplicial version of the Whitehead Theorem is derived.

In his remarkable paper [14], Thomason shows the equivalence of the homotopy categories of $\mathbb{C}at$, the category of small categories, and $\mathbb{SS}$, the category of simplicial sets, by means of the nerve functor $N : \mathbb{C}at \to \mathbb{SS}$ and one of its homotopy inverses (see [7, 9] for details). By [11], the homotopy structure on $\mathbb{C}at$ induces, for every small category $\mathbb{I}$, homotopy structures on the category $\mathbb{C}at \downarrow \mathbb{I}$ of small categories over $\mathbb{I}$ and the category $\mathbb{I}$-$\mathbb{C}at$ of contravariant functors from $\mathbb{I}$ to $\mathbb{C}at$. From [10], it follows that there is a pair of adjoint functors $D : \mathbb{C}at \downarrow \mathbb{I} \to \mathbb{I}$-$\mathbb{C}at$ and $\mathbb{I}\int$- $: \mathbb{I}$-$\mathbb{C}at \to \mathbb{C}at \downarrow \mathbb{I}$ which establishes an equivalence of respective homotopy categories. Similarly, by [5] the homotopy category of simplicial sets on which a fixed simplicial group G acts is equivalent to the homotopy category of simplicial sets over the classifying complex $\overline{W}G$. From this it follows the well-known fact that the homotopy category of topological spaces on which a fixed discrete group G acts is equivalent to the homotopy category of spaces over the classifying space $K(G, 1)$.

We were influenced by these papers to search for a link between the comma category $\mathbb{SS} \downarrow N\mathbb{I}$ and the category $\mathbb{I}$-$\mathbb{SS}$ of contravariant functors from $\mathbb{I}$ to $\mathbb{SS}$. In Section 1 we define, by means of [3, p.327], a pair of adjoint functors $A : \mathbb{SS} \downarrow N\mathbb{I} \to \mathbb{I}$-$\mathbb{SS}$ and $B : \mathbb{I}$-$\mathbb{SS} \to \mathbb{SS} \downarrow N\mathbb{I}$, and examine in Proposition 1.4 their homotopy properties. Let $\overline{N} : \mathbb{C}at \downarrow \mathbb{I} \to \mathbb{SS} \downarrow N\mathbb{I}$ and $\widetilde{N} : \mathbb{I}$-$\mathbb{C}at \to \mathbb{I}$-$\mathbb{SS}$ be the associated functors to the nerve one $N : \mathbb{C}at \to \mathbb{SS}$. Then by Theorem 1.5, from the diagram of functors

$$\begin{tikzcd}
\mathbb{C}at \downarrow \mathbb{I} \arrow[r, shift left, "D"] \arrow[d, "\overline{N}"'] & \mathbb{I}\text{-}\mathbb{C}at \arrow[l, shift left, "\mathbb{I}\int -"] \arrow[d, "\widetilde{N}"] \\
\mathbb{SS} \downarrow N\mathbb{I} \arrow[r, shift left, "A"] & \mathbb{I}\text{-}\mathbb{SS} \arrow[l, shift left, "B"]
\end{tikzcd}$$

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we derive equivalences of respective homotopy categories to generalize the result presented in [8].

Let G be a discrete group, $\mathcal{O}(G)$ the associated category of canonical orbits (see e.g., [1, 2]) and G-$\mathbb{SS}$ the category of G-simplicial sets. Then, there is a canonical functor $\Phi : G\text{-}\mathbb{SS} \to \mathcal{O}(G)\text{-}\mathbb{SS}$ which establishes (cf. [4, 6]) an equivalence of respective homotopy categories. In Section 2, we apply our previous results to deduce an equivalence of the homotopy categories of G-$\mathbb{SS}$ and $\mathbb{SS} \downarrow N\mathcal{O}(G)$. Furthermore, a G-simplicial version of the Whitehead Theorem is derived (cf. [2]).

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1. Main results

Let $\mathbb{I}$ be a small category. As mentioned in the introduction, in [10] a pair of adjoint functors is given

$$\mathbb{C}at \downarrow \mathbb{I} \underset{\mathbb{I}\int\text{-}}{\overset{D}{\rightleftarrows}} \mathbb{I}\text{-}\mathbb{C}at$$

such that the adjunction maps $\eta : \mathrm{id} \to (\mathbb{I}\int\text{-})D$ and $\xi : D(\mathbb{I}\int\text{-}) \to \mathrm{id}$ are weak equivalences. The functor $\mathbb{I}\int\text{-}$ is defined by the Grothendieck construction (see e.g., [13]). If $F : \mathbb{I} \to \mathbb{C}at$ is a contravariant functor then objects in $\mathbb{I}\int F$ are pairs (i, X) with i an object in $\mathbb{I}$ and X an object in $F(i)$, and with morphisms $(\alpha, x) : (i_1, X_1) \to (i_2, X_2)$ consisting of a morphism $\alpha : i_1 \to i_2$ in $\mathbb{I}$ and a morphism $x : X_1 \to F(\alpha)(X_2)$. Composition is defined by $(\alpha, x)(\alpha', x') = (\alpha\alpha', F(\alpha')(x)x')$. For $\alpha : \mathbb{C} \to \mathbb{I}$ (an object in $\mathbb{C}at \downarrow \mathbb{I}$) and an object i in $\mathbb{I}$, $D\alpha(i) = i \downarrow \alpha$ is the comma category and the functor D is defined in an obvious way on morphisms in $\mathbb{I}$.

By [11], the simplicial closed model category structure in the sense of Quillen on the category $\mathbb{C}at$ considered in [14] determines such a structure on the categories $\mathbb{C}at \downarrow \mathbb{I}$ and $\mathbb{I}$-$\mathbb{C}at$, respectively. More precisely, from [11] one deduces immediately

1.1. Proposition. *The categories $\mathbb{C}at \downarrow \mathbb{I}$ and $\mathbb{I}$-$\mathbb{C}at$ with fibrations, cofibrations and weak equivalences of small categories as defined below, are simplicial closed model categories in the sense of Quillen* (*see* [11]) :

1) *the model category structure on $\mathbb{C}at \downarrow \mathbb{I}$ is the one induced by the usual one on the category of small categories $\mathbb{C}at$* (*see* [14]). *In particular, the weak equivalences are the weak equivalences over $\mathbb{I}$;*

2) *a map in $\mathbb{I}$-$\mathbb{C}at$ is a fibration if it is an object-wise fibration* (*resp. weak equivalence*) *in $\mathbb{C}at$* (*see* [14]) *and is a cofibration if it has the left lifting property with respect to maps which are simultaneously fibrations and weak equivalences.*

Let now [1] denote the small category associated with the ordered set $\{0 < 1\}$. Then, any natural transformation $\phi : F_0 \to F_1$ of functors $F_0, F_1 : \mathbb{B} \to \mathbb{C}$ determines a functor $\Theta : \mathbb{B} \times [1] \to \mathbb{C}$. Therefore, for an object $\alpha : \mathbb{B} \to \mathbb{I}$ in $\mathbb{C}at \downarrow \mathbb{I}$ define the functor

$\alpha \times [1] : \mathbb{B} \times [1] \to \mathbb{I}$ such that $\alpha \times [1] = \alpha\pi$, where $\pi : \mathbb{B} \times [1] \to \mathbb{B}$ is the projection functor. Furthermore, let $j_0, j_1 : \alpha \to \alpha \times [1]$ denote the canonical inclusions and let $\alpha : \mathbb{B} \to \mathbb{I}$ and $\beta : \mathbb{C} \to \mathbb{I}$ be objects in $\mathbb{C}at \downarrow \mathbb{I}$. We say that maps $F_0, F_1 : \alpha \to \beta$ are *homotopic* if there is a map $\Theta : \alpha \times [1] \to \beta$ such that $\Theta j_0 = F_0$ and $\Theta j_1 = F_1$. Similarly, for an object $G : \mathbb{I} \to \mathbb{C}at$ in the category $\mathbb{I}$-$\mathbb{C}at$ let $G \times [1] : \mathbb{I} \to \mathbb{C}at$ denote the contravariant functor which is defined by $(G \times [1])(i) = G(i) \times [1]$, for any object i in $\mathbb{I}$ and in the obvious way on morphisms in $\mathbb{I}$. For functors $G, H : \mathbb{I} \to \mathbb{C}at$, two maps ψ_0, $\psi_1 : G \to H$ are *homotopic* if there is a map $\Psi : G \times [1] \to H$ in the category $\mathbb{I}$-$\mathbb{C}at$ such that $\Psi j_0 = \psi_0$ and $\Psi j_1 = \psi_1$, where $j_0, j_1 : G \to G \times [1]$ are the canonical inclusions. Then, we may state the following result.

1.2. Proposition. 1) *The functors* $D : \mathbb{C}at \downarrow \mathbb{I} \longrightarrow \mathbb{I}$-$\mathbb{C}at$ *and* $\mathbb{I} \int$ - : $\mathbb{I}$-$\mathbb{C}at \longrightarrow \mathbb{C}at \downarrow \mathbb{I}$ *preserve weak equivalences.*

2) *If* $\alpha : \mathbb{B} \to \mathbb{I}$ *and* $\beta : \mathbb{C} \to \mathbb{I}$ *are objects in* $\mathbb{C}at \downarrow \mathbb{I}$ *and maps* F_0, $F_1 : \alpha \to \beta$ *are homotopic then the induced functors* DF_0, $DF_1 : \mathbb{I} \to \mathbb{C}at$ *are also homotopic. Conversely, for* $G, H : \mathbb{I} \to \mathbb{C}at$ *and homotopic maps* ψ_0, $\psi_1 : G \to H$ *the induced maps* $\mathbb{I} \int \psi_0 : \mathbb{I} \int G \to \mathbb{I}$ *and* $\mathbb{I} \int \psi_1 : \mathbb{I} \int H \to \mathbb{I}$ *are also homotopic.*

Proof. 1) Let $F_0, F_1 : \mathbb{I} \to \mathbb{C}at$ be contravariant functors with a natural transformation $F_0 \to F_1$ and such that the functors $F_0(i) \to F_1(i)$ are weak equivalences, for any object i in the category $\mathbb{I}$ and consider the induced functor $\mathbb{I} \int F_0 \to \mathbb{I} \int F_1$. Put N for the nerve functor from the category $\mathbb{C}at$ to the category of simplicial sets $\mathbb{SS}$ and hocolim for the homotopy colimit functor (see [1]) on the category of diagrams of simplicial sets. Then, there is a commutative diagram

$$\begin{tikzcd} \operatorname{hocolim} NF_0 \arrow[r] \arrow[d] & N(\mathbb{I} \int F_0) \arrow[d] \\ \operatorname{hocolim} NF_1 \arrow[r] & N(\mathbb{I} \int F_1) \end{tikzcd}$$

with the horizontal maps as homotopy equivalences in the light of [13]. But by [3, p.335], the map $\operatorname{hocolim} NF_0 \to \operatorname{hocolim} NF_1$ is a weak equivalence. Thus, the map $N(\mathbb{I} \int F_0) \to N(\mathbb{I} \int F_1)$ is a weak equivalence as well.

Let now

$$\begin{tikzcd} \mathbb{C} \arrow[rr, "F"] \arrow[dr, "\alpha"'] & & \mathbb{D} \arrow[dl, "\beta"] \\ & \mathbb{I} & \end{tikzcd}$$

be a map in the category $\mathbb{C}at \downarrow \mathbb{I}$ with F as a weak equivalence. Then by [12], the comma category $d \downarrow F$ is contractible, for any object d in $\mathbb{D}$. For any object i in $\mathbb{I}$, consider the induced functor

$$i \downarrow \alpha \xrightarrow{i \downarrow F} i \downarrow \beta.$$

Then, for any object $\bar{d} = (d, i \to \beta(d))$ in $i \downarrow \beta$ we have $\bar{d} \downarrow (i \downarrow F) = d \downarrow F$. Hence, the comma category $\bar{d} \downarrow (i \downarrow F)$ is contractible, for any object $\bar{d}$ in the category $i \downarrow \beta$ and again by [12] we get that the functor $i \downarrow F$ is a weak homotopy equivalence.

2) The proof is straightforward. ∎

Denote by $\mathbb{SS} \downarrow N\mathbb{I}$ the category $\mathbb{SS}$ over the nerve $N\mathbb{I}$ of the small category $\mathbb{I}$ and by $\mathbb{I}$-$\mathbb{SS}$ the category of contravariant functors from a small category $\mathbb{I}$ to the category $\mathbb{SS}$ of simplicial sets. Then, the proposition below immediately follows from [11].

1.3. Proposition. *The categories $\mathbb{I}$-$\mathbb{SS}$ and $\mathbb{SS} \downarrow N\mathbb{I}$ with fibrations, cofibrations and weak equivalences as defined below, are simplicial closed model categories in the sense of Quillen (see [11]).*

1) *The model category structure on the category $\mathbb{SS} \downarrow N\mathbb{I}$ is the one induced by the usual one on the category of simplicial sets (see [11]). In particular, the weak equivalences are the weak homotopy equivalences over $N\mathbb{I}$;*

2) *a map in $\mathbb{I}$-$\mathbb{SS}$ is a fibration (resp. weak equivalence) if it is an object-wise fibration (resp. weak equivalence) in $\mathbb{SS}$ (see [11]) and is a cofibration if it has the left lifting property with respect to maps which are simultaneously fibrations and weak equivalences.*

Similarly as in the categories $\mathbb{C}at \downarrow \mathbb{I}$ and $\mathbb{I}$-$\mathbb{C}at$, we define a homotopy notion in the categories $\mathbb{SS} \downarrow N\mathbb{I}$ and $\mathbb{I}$-$\mathbb{SS}$. Then, we have the following

1.4. Proposition. *There is a pair of adjoint functors $A : \mathbb{SS} \downarrow N\mathbb{I} \to \mathbb{I}$-$\mathbb{SS}$ (the left adjoint) and $B : \mathbb{I}$-$\mathbb{SS} \to \mathbb{SS} \downarrow N\mathbb{I}$ (the right adjoint) such that in the terminology of the previous proposition:*

1) *both functors send weak equivalences into weak equivalences and homotopic maps into homotopic ones;*

2) *for every object Θ in $\mathbb{SS} \downarrow N\mathbb{I}$ and every object F in $\mathbb{I}$-$\mathbb{SS}$, the adjunction maps $\Theta \to BA(\Theta)$ and $AB(F) \to F$ are weak equivalences.*

Proof. Given an object $\Theta : \mathbb{X} \to N\mathbb{I}$ in the category $\mathbb{SS} \downarrow N\mathbb{I}$, one defines the contravariant functor $A\Theta : \mathbb{I} \to \mathbb{SS}$ as follows. For an object i in $\mathbb{I}$, let $i \downarrow \mathbb{I}$ be the category over i and $i \downarrow \mathbb{I} \to \mathbb{I}$ the natural projection functor. Then, the simplicial set $(A\Theta)(i)$ is given by the pull-back

$$\begin{tikzcd}
(A\Theta)(i) \arrow[r] \arrow[d] & N(i \downarrow \mathbb{I}) \arrow[d] \\
\mathbb{X} \arrow[r, "\Theta"] & N\mathbb{I}.
\end{tikzcd}$$

Moreover, for a map $i \to i'$ in $\mathbb{I}$, we get the induced simplicial map $(A\Theta)(i') \to (A\Theta)(i)$. Given an object F in the category $\mathbb{I}$-$\mathbb{SS}$, one defines BF as follows. Let $\operatorname{hocolim} F$ be the homotopy colimit of F and define BF as the simplicial map $\operatorname{hocolim} F \to N\mathbb{I}$ given by the natural projection.

The adjunction of a map $f : A\Theta \to F$ in the category $\mathbb{I}$-$\mathbb{SS}$ is determined as follows. Associate, with the induced map $\bar{f} : A\Theta \to F \times N(\text{-} \downarrow \mathbb{I})$ by the maps f and $A\Theta \to N(- \downarrow \mathbb{I})$ given by the pull-back above, the simplicial map $\tilde{f} = \operatorname{colim} \bar{f} : \operatorname{colim} A\Theta \to \operatorname{colim} F \times$

$N(\text{-}\downarrow \mathbb{I})$ over the nerve $N\mathbb{I}$. But, $\operatorname{colim} F \times N(\text{-}\downarrow \mathbb{I}) = \operatorname{hocolim} F$ by [3, Chap. XII] and $\operatorname{colim} A\Theta = \coprod_i (A\Theta)(i)/\sim$, where i runs over all objects in $\mathbb{I}$ and $A\Theta(\alpha)x \sim x$ for $\alpha : i' \to i$ and $x \in A\Theta(i)$. An n-simplex in $\mathbb{X}$ determines an simplex $\Theta(x) = (i_0 \to \cdots \to i_n)$ in the nerve $N\mathbb{I}$. Then, $\widetilde{\Theta(x)} = (i_0 = i_0 \to \cdots \to i_n)$ is a simplex in $N(i_0 \downarrow \mathbb{I})$ and we get an injection $\eta : \mathbb{X} \to \operatorname{colim} A\Theta$ such that $\eta(x) = (x, \widetilde{\Theta(x)})$. On the other hand, for any $(x,\sigma) \in (A\Theta)(i)$ we have $(x,\sigma) \sim (x, \widetilde{\Theta(x)})$. Thus, the map $\eta : \mathbb{X} \to \operatorname{colim} A\Theta$ is an isomorphism. Finally, we have a map $\tilde{f} : \mathbb{X} \to \operatorname{hocolim} F$ such that the diagram

$$\begin{tikzcd} \mathbb{X} \arrow[rr, "\tilde{f}"] \arrow[dr, "\Theta"'] & & \operatorname{hocolim} F \arrow[dl] \\ & N\mathbb{I} & \end{tikzcd}$$

commutes.

Given a map $g : \Theta \to BF$ in the category $\mathbb{SS} \downarrow N\mathbb{I}$ one can also get an adjunction $\tilde{g} : A\Theta \to F$. For an object i in $\mathbb{I}$, we define a simplicial map $\tilde{g}(i) : A\Theta(i) \to F(i)$. By [1, p.338], the simplicial set $\operatorname{hocolim} F$ is isomorphic to the diagonal of the double simplicial set $\coprod_* F$ which in dimension n consists of the union $\coprod_n F = \coprod_{i_0 \to \cdots \to i_n} F(i_n)$. Thus, for an n-simplex x in $\mathbb{X}$ we have $g(x) = y$ with $y \in F(i_n)$ and $\Theta(x) = (i_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} i_n)$. Hence, for $(x, (i \xrightarrow{\alpha} i_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} i_n))$ in $(A\Theta)(i)$ we may define $\tilde{g}(i)(x, (i \xrightarrow{\alpha} i_0 \xrightarrow{\alpha_0} \cdots \xrightarrow{\alpha_n} i_n)) = F(\alpha_n \cdots \alpha_0 \alpha)(y)$.

1) Let now $\Theta : \mathbb{X} \to N\mathbb{I}$ and $\Theta' : \mathbb{X}' \to N\mathbb{I}$ be two objects in the category $\mathbb{SS} \downarrow \mathbb{I}$ and $f : \Theta \to \Theta'$ a map such that the associated simplicial map $\mathbb{X} \to \mathbb{X}'$ is a weak homotopy equivalence. Then, for an object i in $\mathbb{I}$ the induced maps $A\Theta(i) \to \mathbb{X}$ and $A\Theta'(I) \to \mathbb{X}'$ have the same homotopy fibre (up to homotopy). Hence, from the long homotopy exact sequences determined by $A\Theta(i) \to \mathbb{X}$ and $A\Theta'(i) \to \mathbb{X}'$, it follows that the map $Af(i) : A\Theta(i) \to A\Theta'(i)$ is a weak homotopy equivalence. Thus, the induced map $Af : A\Theta \to A\Theta'$ in the category $\mathbb{I}$-$\mathbb{SS}$ is a weak homotopy equivalence.

On the other hand, if $f : F \to F'$ is a map in the category $\mathbb{I}$-$\mathbb{SS}$ such that the simplicial maps $f(i) : F(i) \to F'(i)$ are weak homotopy equivalences for any object i in $\mathbb{I}$ then by [3, p.335], one gets that the induced map $\operatorname{hocolim} f : \operatorname{hocolim} F \to \operatorname{hocolim} F'$ is a weak homotopy equivalence. Thus, the map $Bf : BF \to BF'$ is a weak homotopy equivalence. The proof of the preservation of homotopic maps by functors A and B is straightforward.

2) Let $|{-}|$ be the geometric realization functor. For an object $\Theta : \mathbb{X} \to N\mathbb{I}$ in the category $\mathbb{SS} \downarrow N\mathbb{I}$, the adjunction map $\tau_\Theta : \Theta \to BA\Theta$ is defined as follows. For n-simplex x in $\mathbb{X}$ let $\Theta(x) = (i_0 \to \cdots \to i_n)$ be an n-simplex in the nerve $N\mathbb{I}$ and $\widetilde{\Theta(x)} = (i_0 = i_0 \to \cdots \to i_n)$ an associated n-simplex in $N(i_0 \downarrow \mathbb{I})$. Then, $(x, \widetilde{\Theta(x)})$ is an n-simplex in $A\Theta$ and $\tau_\Theta(x) = (x, \widetilde{\Theta(x)})$. Moreover, there is a natural projection $\pi_\Theta : \operatorname{hocolim} A\Theta \to \mathbb{X}$ such that $\pi_\Theta \tau_\Theta = \mathrm{id}_\Theta$ and it is a routine to check that the map $|\tau_\Theta||\pi_\Theta|$ is homotopic to the identity map $\mathrm{id}_{|\operatorname{hocolim} A\Theta|}$.

For an object F in the category $\mathbb{I}\text{-}\mathbb{SS}$, we define the adjunction map $\theta_F : ABF \to F$. Let Θ : $\operatorname{hocolim} F \to N\mathbb{I}$ be the natural projection and consider the simplicial map $\theta_F(i) : A\Theta(i) \to F(i)$, for an object i in $\mathbb{I}$ defined as follows. If $(x, (i \xrightarrow{\alpha} i_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} i_n))$ is an n-simplex in $A\Theta(i)$ then $(x, (i_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} i_n))$ is an n-simplex in $\operatorname{hocolim} F$ with x in $F(i_n)$. Then, $\theta_F(i)(x, (i \xrightarrow{\alpha} i_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} i_n)) = F(\alpha_n \cdots \alpha_0 \alpha)(x)$. Moreover, there is an injection $\sigma(i) : F(i) \to A\Theta(i)$ given by $\sigma(i)(x) = (x, \widetilde{\mathrm{id}_i})$, for an n-simplex x in $F(i)$ with $\widetilde{\mathrm{id}_i}$ the degenerated n-simplex determined by an object i in $\mathbb{I}$. Of course $\theta_F(i)\sigma(i) = \mathrm{id}_{F(i)}$ and it is also easy to check that the map $|\sigma(i)||\theta_F(i)|$ is homotopic to the identity map $\mathrm{id}_{|A\Theta(i)|}$ for any object i in $\mathbb{I}$, and the proof is completed. ∎

The nerve functor N : $\mathbb{C}at \to \mathbb{SS}$ determines functors $\overline{N} : \mathbb{C}at \downarrow \mathbb{I} \to \mathbb{SS} \downarrow N\mathbb{I}$ and $\widetilde{N} : \mathbb{I}\text{-}\mathbb{C}at \to \mathbb{I}\text{-}\mathbb{SS}$, respectively. Then, the diagram

$$\begin{tikzcd}
\mathbb{C}at \downarrow \mathbb{I} \arrow[r, "D"] \arrow[d, "\overline{N}"'] & \mathbb{I}\text{-}\mathbb{C}at \arrow[d, "\widetilde{N}"] \\
\mathbb{SS} \downarrow N\mathbb{I} \arrow[r, "A"] & \mathbb{I}\text{-}\mathbb{SS}
\end{tikzcd}$$

commutes and by [13], the diagram

$$\begin{tikzcd}
\mathbb{C}at \downarrow \mathbb{I} \arrow[d, "\overline{N}"'] & \mathbb{I}\text{-}\mathbb{C}at \arrow[l, "\mathbb{I}\int -"'] \arrow[d, "\widetilde{N}"] \\
\mathbb{SS} \downarrow N\mathbb{I} & \mathbb{I}\text{-}\mathbb{SS} \arrow[l, "B"]
\end{tikzcd}$$

commutes up to homotopy. Moreover, by [7, 9] there are homotopy inverses $\Gamma : \mathbb{SS} \to \mathbb{C}at$, for the nerve functor N : $\mathbb{C}at \to \mathbb{SS}$ such that for any small category $\mathbb{I}$ there is a weak equivalence $\Gamma N\mathbb{I} \to \mathbb{I}$. Thus, we get induced functors $\overline{\Gamma}$: $\mathbb{SS} \downarrow N\mathbb{I} \to \mathbb{C}at \downarrow \mathbb{I}$ and $\widetilde{\Gamma} : \mathbb{I}\text{-}\mathbb{SS} \to \mathbb{I}\text{-}\mathbb{C}at$, respectively. Consequently, we may generalize the result presented in [8] and summarize this section with the following

1.5. Theorem. *For any small category* $\mathbb{I}$*, the diagram of above functors*

$$\begin{tikzcd}
\mathbb{C}at \downarrow \mathbb{I} \arrow[r, shift left, "D"] \arrow[d, shift left, "\bar{N}"] & \mathbb{I}\text{-}\mathbb{C}at \arrow[l, shift left, "\mathbb{I}\int -"] \arrow[d, shift left, "\widetilde{N}"] \\
\mathbb{SS} \downarrow N\mathbb{I} \arrow[r, shift left, "A"] \arrow[u, shift left, "\bar{\Gamma}"] & \mathbb{I}\text{-}\mathbb{SS} \arrow[l, shift left, "B"] \arrow[u, shift left, "\widetilde{\Gamma}"]
\end{tikzcd}$$

determines equivalences of respective homotopy categories.

In particular, if $\mathbb{I}$ is the category associated with a discrete group G then the nerve $N\mathbb{I}$ is the classifying complex $\overline{W}G$ and $\mathbb{I}\text{-}\mathbb{C}at$ (resp. $\mathbb{I}\text{-}\mathbb{SS}$) is the category of right G-small categories (resp. right G-simplicial sets). Thus from the above, we may also deduce an equivalence of respective homotopy categories (cf. [5]).

2. Equivariant applications

Let G be a discrete group and $\mathcal{O}(G)$ the associated category of canonical orbits; its objects are the left cosets G/H as H ranges over all subgroups of G and morphisms are G-maps $G/H \to G/K$ with respect to left translation. If G-$\mathbb{SS}$ denotes the category of G-simplicial sets then there is a canonical functor $\Phi : G\text{-}\mathbb{SS} \to \mathcal{O}(G)\text{-}\mathbb{SS}$ defined on objects by $\Phi(\mathbb{X})(G/H) = \mathbb{X}^H$ and in the obvious way on morphisms in G-$\mathbb{SS}$, where $\mathbb{X}^H$ is the H-fixed point simplicial subset of $\mathbb{X}$. Then (cf. [1]), the category G-$\mathbb{SS}$ is a closed model category; a map $f : \mathbb{X} \to \mathbb{Y}$ is a *fibration* (resp. *weak equivalence*) if the induced map $\Phi(f) : \Phi(\mathbb{X}) \to \Phi(\mathbb{Y})$ is a fibration (resp. weak equivalence) in $\mathcal{O}(G)$-$\mathbb{SS}$. *Cofibrations* in G-$\mathbb{SS}$ are determined by means of the left lifting property with respect to maps which are simultaneously fibrations and weak equivalences. From [4, 6], one may derive that the functor $\Phi : G\text{-}\mathbb{SS} \to \mathcal{O}(G)\text{-}\mathbb{SS}$ establishes an equivalence of respective homotopy categories. Therefore, Proposition 1.4 yields

2.1. Corollary. *If G is a discrete group then the homotopy categories of G-$\mathbb{SS}$ and $\mathbb{SS}\downarrow N\mathcal{O}(G)$ are equivalent.*

Recall that a G-simplicial set $\mathbb{X}$ is called *fibrant* (resp. *cofibrant*) if the unique map $\mathbb{X} \to *$ (resp. $\emptyset \to \mathbb{X}$) is a fibration (resp. cofibration) in the category G-$\mathbb{SS}$, where $\emptyset$ is the empty simplicial set and $*$ a single point simplicial set with the trivial action of G.

2.2. Lemma. *Every G-simplicial set $\mathbb{X}$ is cofibrant in the category G-$\mathbb{SS}$.*

Proof. Let $p : \mathbb{E} \to \mathbb{B}$ be a fibration and a weak equivalence in the category G-$\mathbb{SS}$, and $f : \mathbb{X} \to \mathbb{B}$ a G-map. The n-skeleton $\mathbb{X}^{(n)}$, for $n \geq 0$ is a G-simplicial subset of $\mathbb{X}$ and $\mathbb{X} = \operatorname{colim}_n \mathbb{X}^{(n)}$. Let $f^{(n)}$ denote the restriction of f to $\mathbb{X}^{(n)}$; we proceed by induction to show an existence of a G-map $\tilde{f}^{(n)} : \mathbb{X}^{(n)} \to \mathbb{E}$ with $p\tilde{f}^{(n)} = f^{(n)}$, for $n \geq 0$.

Of course, there is a G-map $\tilde{f}^{(0)} \to \mathbb{E}$ such $p\tilde{f}^{(0)} = f^{(0)}$. Suppose there is a G-map $\tilde{f}^{(n-1)} : \mathbb{X}^{(n-1)} \to \mathbb{E}$ with $p\tilde{f}^{(n-1)} = f^{(n-1)}$ and put $\Delta[n]$ for the standard simplicial n-simplex, and $\dot{\Delta}[n]$ for its boundary. Then, there is a pushout diagram

$$\begin{array}{ccc} \coprod_x G/H_x \times \dot{\Delta}[n] & \longrightarrow & \mathbb{X}^{(n-1)} \\ \downarrow & & \downarrow \\ \coprod_x G/H_x \times \Delta[n] & \longrightarrow & \mathbb{X}^{(n)}, \end{array}$$

where x runs over all non-degenerate n-simplexes in $\mathbb{X}$ and H_x is the isotropy subgroup of an n-simplex x. By means of the right lifting property of the map $p : \mathbb{E} \to \mathbb{B}$ (see e.g., [11, p.2.1]) and the map $\tilde{f}^{(n-1)} : \mathbb{X}^{(n-1)} \to \mathbb{E}$, we get a G-map $\tilde{f}^{(n)} : \mathbb{X}^{(n)} \to \mathbb{E}$ such that $p\tilde{f}^{(n)} = f^{(n)}$ and its restriction to $\mathbb{X}^{(n-1)}$ is equal to $\tilde{f}^{(n-1)}$. Therefore, the sequence of G-map $\tilde{f}^{(n)} \to \mathbb{E}$, for $n \geq 0$ determines a G-map $\tilde{f} : \mathbb{X} \to \mathbb{E}$ with $p\tilde{f} = f$ and the proof is completed. ∎

Let now $f : \mathbb{X} \to \mathbb{Y}$ be a weak equivalence in the category G-$\mathbb{SS}$. Then, from [11, p.3.15] it follows that the map $[f, \mathbb{E}]_G : [\mathbb{Y}, \mathbb{E}]_G \to [\mathbb{X}, \mathbb{E}]_G$ of G-homotopy classes is bijective, for any fibrant G-simplicial set $\mathbb{E}$. We are now in a position to state the following G-simplicial version of the Whitehead Theorem (cf.[2]).

2.3. Corollary. *Let $f : \mathbb{X} \to \mathbb{Y}$ be a G-map of fibrant G-simplicial sets. Then the following conditions are equivalent:*

1) *the G-map $f : \mathbb{X} \to \mathbb{Y}$ is a weak equivalence;*
2) *the G-map $f : \mathbb{X} \to \mathbb{Y}$ is a G-homotopy equivalence;*
3) *the map in the category $\mathbb{SS} \downarrow N\mathcal{O}(G)$, induced by the commutative diagram*

$$\begin{tikzcd}
\operatorname{hocolim} \Phi(\mathbb{X}) \arrow[rr, "\operatorname{hocolim} \Phi(f)"] \arrow[dr] & & \operatorname{hocolim} \Phi(\mathbb{Y}) \arrow[dl] \\
& N\mathcal{O}(G) &
\end{tikzcd}$$

is a homotopy equivalence.

Proof. Observe that by Proposition 1.4, we get 2) $\Rightarrow$ 3) and 3) $\Rightarrow$ 1). We show now the implication 1) $\Rightarrow$ 2). By Lemma 2.2 and [11, p.3.15], the induced maps $[f, \mathbb{X}]_G : [\mathbb{Y}, \mathbb{X}]_G \to [\mathbb{X}, \mathbb{X}]_G$ and $[f, \mathbb{Y}]_G : [\mathbb{Y}, \mathbb{Y}]_G \to [\mathbb{X}, \mathbb{Y}]_G$ of G-homotopy classes are bijective. Therefore, the G-map $f : \mathbb{X} \to \mathbb{Y}$ is a G-homotopy equivalence, as claimed. ■

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