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A 2-CATEGORICAL APPROACH TO CHANGE OF BASE AND GEOMETRIC MORPHISMS II

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ABSTRACT. We introduce a notion of *equipment* which generalizes the earlier notion of *pro-arrow equipment* and includes such familiar constructs as $\mathbf{rel}\mathcal{K}$, $\mathbf{spn}\mathcal{K}$, $\mathbf{par}\mathcal{K}$, and $\mathbf{pro}\mathcal{K}$ for a suitable category $\mathcal{K}$, along with related constructs such as the $\mathcal{V}$-$\mathbf{pro}$ arising from a suitable monoidal category $\mathcal{V}$. We further exhibit the equipments as the objects of a 2-category, in such a way that *arbitrary* functors $F : \mathcal{L} \longrightarrow \mathcal{K}$ induce equipment arrows $\mathbf{rel}F : \mathbf{rel}\mathcal{L} \longrightarrow \mathbf{rel}\mathcal{K}$, $\mathbf{spn}F : \mathbf{spn}\mathcal{L} \longrightarrow \mathbf{spn}\mathcal{K}$, and so on, and similarly for arbitrary monoidal functors $\mathcal{V} \longrightarrow \mathcal{W}$. The article I with the title above dealt with those equipments $\mathcal{M}$ having each $\mathcal{M}(A, B)$ only an ordered set, and contained a detailed analysis of the case $\mathcal{M} = \mathbf{rel}\mathcal{K}$; in the present article we allow the $\mathcal{M}(A, B)$ to be general categories, and illustrate our results by a detailed study of the case $\mathcal{M} = \mathbf{spn}\mathcal{K}$. We show in particular that $\mathbf{spn}$ is a locally-fully-faithful 2-functor to the 2-category of equipments, and determine its image on arrows. After analyzing the nature of adjunctions in the 2-category of equipments, we are able to give a simple characterization of those $\mathbf{spn}G$ which arise from a *geometric morphism* G.

1. Introduction

1.1. Given a regular category $\mathcal{K}$ we have the bicategory $\mathbf{rel}\mathcal{K}$ whose objects are those of $\mathcal{K}$, whose arrows are the relations in $\mathcal{K}$, and whose transformations (2-cells) are containments. Evidently, a functor $G : \mathcal{K} \longrightarrow \mathcal{L}$ between regular categories which is left exact and preserves regular epimorphisms (called a *regular* functor) gives rise to a homomorphism of bicategories $\mathbf{rel}\mathcal{K} \longrightarrow \mathbf{rel}\mathcal{L}$ that warrants the name $\mathbf{rel}G$. This simple observation does not suffice for a detailed examination of "change of base" problems — by which we mean generally the effect of functors such as $G : \mathcal{K} \longrightarrow \mathcal{L}$ on constructs such as $\mathbf{rel}\mathcal{K}$. Rather, one is led, notably when considering adjunctions, to consider what can be put forward as an arrow $\mathbf{rel}G : \mathbf{rel}\mathcal{K} \longrightarrow \mathbf{rel}\mathcal{L}$ when $G : \mathcal{K} \longrightarrow \mathcal{L}$ is an *arbitrary* functor between regular categories. The rationale was discussed fully in [CKW] and an answer provided there. In general, $\mathbf{rel}G$ is not a morphism of bicategories between those in question (nor any of their duals) but something different in kind. In [CKW] a general theory was explored that is applicable not only to $\mathbf{rel}$ but also to other constructions that yield bicategories whose hom categories are only ordered sets (also called *locally ordered* bicategories), such as those of order ideals and of partial maps. A sequel was promised that would extend

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the considerations of [CKW] to constructions such as those of spans and profunctors, for which the resulting bicategories are usually not locally ordered. Independently, the more general problem had also been considered in [VTY] and the present article builds on both earlier works.

1.2. To continue for a moment with relations and a regular category $\mathcal{K}$, consider $(-)_* : \mathcal{K} \longrightarrow \mathbf{rel}\mathcal{K}$, defined by sending an arrow $f : X \longrightarrow A$ of $\mathcal{K}$ to its graph $f_* : X \longrightarrow\!\!\!\!\!\!+\;\, A$. A naive analogy with ring-homomorphisms allows us to consider $\mathbf{rel}\mathcal{K}$ as a "$\mathcal{K}$-algebra" via $(-)_*$. For a regular functor $G : \mathcal{K} \longrightarrow \mathcal{L}$, the homomorphism of bicategories $\mathbf{rel}G : \mathbf{rel}\mathcal{K} \longrightarrow \mathbf{rel}\mathcal{L}$ is then seen as a "homomorphism" of "algebras" that takes account of the variation of "scalars" provided by G. The key idea we build on here is that a "$\mathcal{K}$-algebra" is also a "$\mathcal{K},\mathcal{K}$-bimodule" together with a "multiplication". It transpires that if we forget "multiplication" and concentrate instead on the "actions" giving the bimodule structure, then an *arbitrary* functor $G : \mathcal{K} \longrightarrow \mathcal{L}$ gives rise to a "morphism" of "bimodules", compatible with the variation of "scalars". While this analogy was not explicitly expressed in [CKW] and [VTY], the results there are now easily seen in this context.

1.3. Just as for rings, it turns out be useful to study $\mathcal{K},\mathcal{L}$-(bi)modules and see the $\mathcal{K},\mathcal{K}$-modules alluded to above, which we call *equipments*, as a special case. Loosely speaking, a $\mathcal{K},\mathcal{L}$-module $\mathcal{M}$ is a $\mathbf{CAT}$-valued profunctor $\mathcal{M} : \mathcal{L} \longrightarrow\!\!\!\!\!\!+\;\, \mathcal{K}$, so our use of the word "module" is in fact well-established. Section 2 provides the precise definition of modules in this context and of the 2-category $\mathbf{MOD}$ of all modules. Its goal is a characterization of adjunctions in $\mathbf{MOD}$ in terms of the data that arise in change-of-base considerations. Many, but not all, of the $\mathcal{K},\mathcal{L}$-modules $\mathcal{M}$ arising in change-of-base problems have the property that for each $l : L \longrightarrow L'$ in $\mathcal{L}$, the action functor $l(-) = \mathcal{M}(K,l) : \mathcal{M}(K,L) \longrightarrow \mathcal{M}(K,L')$ has a right adjoint $l^*(-)$; while for each k in $\mathcal{K}$, the action $(-)k$ has a left adjoint $(-)k^*$; and the resulting families of adjunctions satisfy conditions of the Beck-Chevalley type. We speak of *starred modules* and of *starred equipments* in these cases; and throughout Section 2 we conduct a parallel study of the simplifications, particularly with respect to adjunctions, that ensue in the *starred* case. We emphasize that being starred is a *property*, the "starring" being essentially unique when it exists.

1.4. In this paper we have no need to consider a full "$\mathcal{K}$-algebra" stucture on a $\mathcal{K},\mathcal{K}$-module $\mathcal{M}$; yet, by analogy with the identity element of an algebra, the additional stucture of a "base point" for such an equipment provides an important tightening of the ideas that are encountered in change of base. For any category $\mathcal{K}$, the hom-functor provides a simple, canonical $\mathcal{K},\mathcal{K}$-module structure on $\mathcal{K}$; and a *pointing* for $\mathcal{M}$ is but a certain kind of arrow $(-)_* : \mathcal{K} \longrightarrow \mathcal{M}$ in the 2-category $\mathbf{MOD}$ of all modules. Section 3 is devoted to the study of *pointed equipments*, and a *starred pointed equipment* is defined to be a pointed and starred equipment for which the pointing structure is suitably compatible with the starring. Once again, the goal of this section is a characterization of adjunctions, this time in the 2-category $\mathbf{EQT}_*$ of pointed equipments, in terms of the data that tend to arise in change of base. Here too, we keep track of the simplifications provided by the starred property; and the result is a theorem, 3.19, that fully extends Theorem 3.5 of

[CKW] from the locally-ordered to the general case.

1.5. In [CKW] the theory was illustrated in detail with a section devoted to **rel** seen as a colax functor from the 2-category **REG** of regular categories, arbitrary functors, and natural transformations that, in the context of this paper, takes values in the 2-category ${}^*\mathbf{EQT}_*$ of starred pointed equipments. In Section 4 we carry out a similar analysis for spans, beginning with the observation that the span construction, denoted by **spn**, provides a locally-fully-faithful 2-functor to the 2-category of starred pointed equipments. Here the domain of **spn** is the 2-category of categories with pullbacks, *arbitrary* functors, and natural transformations. We determine its image on arrows, and give a simple characterization of those $\mathbf{spn}G$ which arise from a geometric morphism G.

2. Modules and Equipments

2.1. Informally, an *equipment* is a 4-sorted structure having *objects* $\cdots, K, L, \cdots$, *scalar arrows* $f : K \longrightarrow L$, *vector arrows* $\mu : K \longrightarrow\!\!\!\!\!\!+\;\; L$ and *vector transformations* $\Phi : \mu \longrightarrow \nu : K \longrightarrow\!\!\!\!\!\!+\;\; L$. The objects and scalar arrows carry the structure of a category; for each pair of objects K, L, the vector arrows from K to L and the vector transformations between these carry the structure of a category; finally, there are actions of the scalars on the vectors, as suggested by

$$\begin{tikzcd} K' \arrow[r, "k"] & K \arrow[r, shift left=2, "\mu", "\shortmid"{marking}] \arrow[r, shift right=2, "\nu"', "\shortmid"{marking}] \arrow[r, phantom, "\Downarrow\Phi"] & L \end{tikzcd} \quad = \quad \begin{tikzcd} K' \arrow[r, shift left=2, "\mu k", "\shortmid"{marking}] \arrow[r, shift right=2, "\nu k"', "\shortmid"{marking}] \arrow[r, phantom, "\Downarrow\Phi k"] & L \end{tikzcd}$$

and

$$\begin{tikzcd} K \arrow[r, shift left=2, "\mu", "\shortmid"{marking}] \arrow[r, shift right=2, "\nu"', "\shortmid"{marking}] \arrow[r, phantom, "\Downarrow\Phi"] & L \arrow[r, "l"] & L' \end{tikzcd} \quad = \quad \begin{tikzcd} K \arrow[r, shift left=2, "l\mu", "\shortmid"{marking}] \arrow[r, shift right=2, "l\nu"', "\shortmid"{marking}] \arrow[r, phantom, "\Downarrow l\Phi"] & L' \end{tikzcd}$$

which are functorial in Φ, strictly unitary, and coherently associative in the three possible senses. (We will make this precise in the next subsection.)

To see $\mathbf{spn}\mathcal{K}$ as such a structure, begin by taking the category of scalars to be $\mathcal{K}$ itself and, for each pair of objects K, L in $\mathcal{K}$, taking the category of vectors from K to L to be the usual category $\mathbf{spn}\mathcal{K}(K, L)$ of spans from K to L, a typical object of which we denote by $x : K \longleftarrow S \longrightarrow L : a$, or just (x, S, a). For $k : K' \longrightarrow K$ in $\mathcal{K}$, $(x, S, a)k$ is the span $p : K' \longleftarrow P \longrightarrow L : aq$ where

$$\begin{tikzcd} P \arrow[r, "q"] \arrow[d, "p"'] & S \arrow[d, "x"] \\ K' \arrow[r, "k"'] & K \end{tikzcd}$$

is a (chosen) pullback; while for $l : L \longrightarrow L'$, $l(x, S, a) = (x, S, la)$.

An equipment may be *starred*: which is to say that for each scalar k the action $(-)k$ has a left adjoint, denoted by $(-)k^*$, and for each scalar l the action $l(-)$ has a right adjoint, denoted by $l^*(-)$, with the adjunctions satisfying conditions of the Beck-Chevalley type to ensure that these further actions satisfy associative laws with each other and with the original actions.

The equipments of the form $\mathbf{spn}\mathcal{K}$, for $\mathcal{K}$ with pullbacks, are starred. For $K' \longleftarrow K : k$ in $\mathcal{K}$, $(x, S, a)k^*$ is given by composition; and for $L \longleftarrow L' : l$ in $\mathcal{K}$, $l^*(x, S, a)$ is given by a pullback.

An equipment may carry the further structure of a *pointing*, which provides a distinguished vector arrow $\iota_K : K \longrightarrow\!\!\!\!\!+ K$ for each object K, and isomorphisms $f\iota_K \xrightarrow{\simeq} \iota_L f$ for each scalar $f : K \longrightarrow L$, these satisfying $(1\iota_K \xrightarrow{\simeq} \iota_K 1) = 1_{\iota_K}$ and the familiar hexagonal coherence condition. In the case of $\mathbf{spn}\mathcal{K}$, such "identity vectors" are provided by the spans $1 : K \longleftarrow K \longrightarrow K : 1$.

Finally, a *starred pointed equipment* is a starred equipment together with a pointing for which the mates (in the sense of [K&S]) $\iota_K f^* \longrightarrow f^* \iota_L$ of the isomorphisms $f\iota_K \xrightarrow{\simeq} \iota_L f$ are again isomorphisms. In fact, $\mathbf{spn}\mathcal{K}$ is a starred pointed equipment.

Our *formal* definition is the following. (Recall that a homomorphism of bicategories is said to be *normal* when it preserves identities strictly.)

2.2. Definition. *An* equipment *with scalar category* $\mathcal{K}$ *is a normal homomorphism of bicategories* $\mathcal{M} : \mathcal{K} \longrightarrow \mathrm{Hom}(\mathcal{K}^{op}, \mathbf{CAT})$*, where* $\mathrm{Hom}(-,-)$ *denotes normal homomorphisms of bicategories, strong transformations, and modifications.* ∎

All our homomorphisms of bicategories will be assumed to be normal, so for brevity we drop the word "normal" when speaking of such homomorphisms. It is evident in our displays of actions in 2.1 that the scalars acting from the left need bear no relationship to those acting from the right. We will need this extra generality to explore equipments fully, and so pose:

2.3. Definition. *For categories* $\mathcal{K}$ *and* $\mathcal{L}$*, by a* $\mathcal{K},\mathcal{L}$-module $\mathcal{M}$ *is meant a homomorphism* $\mathcal{M} : \mathcal{L} \longrightarrow \mathrm{Hom}(\mathcal{K}^{op}, \mathbf{CAT})$ *of bicategories. We also write* $\mathcal{M} : \mathcal{L} \longrightarrow\!\!\!\!\!+ \mathcal{K}$. ∎

To give a homomorphism $\mathcal{K}^{op} \longrightarrow \mathrm{Hom}(\mathcal{L}, \mathbf{CAT})$ amounts to the very same thing. Accordingly, we often write such a module $\mathcal{M}$ as $\mathcal{M} : \mathcal{K}^{op}, \mathcal{L} \longrightarrow \mathbf{CAT}$, looking upon it as being compatibly a homomorphism in each variable separately, sometimes called a "bi-homomorphism".

For each K in $\mathcal{K}$ and for each L in $\mathcal{L}$, then, we have a category $\mathcal{M}(K, L)$, the vector arrows and vector transformations of our less formal definition in 2.1. (We use this "vector" terminology for general modules as well as equipments.) For each $k : K' \longrightarrow K$ in $\mathcal{K}$ and each L in $\mathcal{L}$, the functor $\mathcal{M}(k, L) : \mathcal{M}(K, L) \longrightarrow \mathcal{M}(K', L)$ provides pre-action by k as in the first display of 2.1 and for each K in $\mathcal{K}$ and each $l : L \longrightarrow L'$ in $\mathcal{L}$, the functor $\mathcal{M}(K, l) : \mathcal{M}(K, L) \longrightarrow \mathcal{M}(K, L')$ provides post-action by l as in the second display. When we need to mention them explicitly we write

$$\xi = \xi_{l',l,\mu} : (l'l)\mu \xrightarrow{\simeq} l'(l\mu)$$

$$\eta = \eta_{l,\mu,k} : l(\mu k) \xrightarrow{\simeq} (l\mu)k$$
$$\zeta = \zeta_{\mu,k,k'} : (\mu k)k' \xrightarrow{\simeq} \mu(kk')$$

for the structural isomorphisms of $\mathcal{M}$ (which are natural in μ). Of course our normality assumptions on homomorphisms are equivalent to the actions being strictly unitary. As for the conditions on ξ, η, ζ arising from the homomorphism assumptions, see 2.5 below.

2.4. Before further consideration of modules we give some simple examples.

(i) **spn′** For *any* category $\mathcal{C}$, we define a module $\mathbf{spn}'\mathcal{C} : \mathcal{C} \dashrightarrow \mathcal{C}^{op}$. Here $\mathbf{spn}'\mathcal{C}(X, A)$ is the usual category of spans. For an arrow $g : A \longrightarrow B$ in $\mathcal{C}$ we define $g(x, S, a) : X \dashrightarrow B$ to be the span (x, S, ga). In dealing with actions of the opposite of a given category it is convenient to write $f^* : Y \longrightarrow X$ for the arrow in $\mathcal{C}^{op}$ that is determined by $Y \longleftarrow X : f$ in $\mathcal{C}$, and now for such an f and for $(x, S, a) : X \dashrightarrow A$ we define $(x, S, a)f^* : Y \dashrightarrow A$ to be the span (fx, S, a). In this example the associativities ξ, η and ζ are identities. *If* $\mathcal{C}$ *has pullbacks* then we can define *further* actions: for $A \longleftarrow C : h$ we can define $h^*(x, S, a) : X \dashrightarrow C$ to be (xp, P, q) where

$$\begin{tikzcd} P \arrow[r, "q"] \arrow[d, "p"'] & C \arrow[d, "h"] \\ S \arrow[r, "a"'] & A \end{tikzcd}$$

is a (chosen) pullback, and for $k : Z \longrightarrow X$ we can define $(x, S, a)k : Z \dashrightarrow A$ using a similar pullback. These last two actions on the categories $\mathbf{spn}'\mathcal{C}(X, A)$ produce a new module $\mathbf{spn}''\mathcal{C} : \mathcal{C}^{op} \dashrightarrow \mathcal{C}$, with $\mathbf{spn}''\mathcal{C}(X, A) = \mathbf{spn}'\mathcal{C}(X, A)$; for this module, of course, the ξ, η and ζ are not identities.

(ii) **rel″** Write $\mathbf{rel}'\mathcal{C}(X, A)$ for the full subcategory of $\mathbf{spn}'\mathcal{C}(X, A)$ determined by the monic spans. The module $\mathbf{spn}'\mathcal{C}$ does not restrict to a module $\mathbf{rel}'\mathcal{C}$; but if $\mathcal{C}$ has pullbacks then the module structure of $\mathbf{spn}''\mathcal{C} : \mathcal{C}^{op} \dashrightarrow \mathcal{C}$ restricts to give a module $\mathbf{rel}''\mathcal{C} : \mathcal{C}^{op} \dashrightarrow \mathcal{C}$.

(iii) **par′** For a category $\mathcal{C}$, let $\mathcal{C}_{\mathbf{mono}}$ be the category whose objects are those of $\mathcal{C}$ and whose arrows are the monomorphisms of $\mathcal{C}$. Define $\mathbf{par}'\mathcal{C}(X, A)$ to be the full subcategory of $\mathbf{spn}'\mathcal{C}(X, A)$ determined by the spans $x : X \longleftarrow S \longrightarrow A : a$ with x in $\mathcal{C}_{\mathbf{mono}}$. The module structure of $\mathbf{spn}'\mathcal{C} : \mathcal{C} \dashrightarrow \mathcal{C}^{op}$ restricts to give a module $\mathbf{par}'\mathcal{C} : \mathcal{C} \dashrightarrow \mathcal{C}^{op}_{\mathbf{mono}}$. If $\mathcal{C}$ has pullbacks of arrows in $\mathcal{C}_{\mathbf{mono}}$ along arbitrary arrows then the module structure of $\mathbf{spn}''\mathcal{C} : \mathcal{C}^{op} \dashrightarrow \mathcal{C}$ restricts to a module $\mathcal{C}^{op}_{\mathbf{mono}} \dashrightarrow \mathcal{C}$ with underlying categories the $\mathbf{par}'\mathcal{C}(X, A)$.

(iv) **pro″** We write **cat** for the 2-category of small categories, $\mathbf{cat}_0$ for the underlying ordinary category of small categories and all functors, and **set** for the category of small sets. For small categories $\mathbf{X}$ and $\mathbf{A}$, we set $\mathbf{pro}''(\mathbf{X}, \mathbf{A}) = \mathbf{set}^{\mathbf{A}^{op} \times \mathbf{X}}$ and write

$\Phi : \mathbf{X} \longrightarrow\!\!\!\!\!+\!\!\!\!\rightarrow \mathbf{A}$ for a typical object of $\mathbf{pro}''(\mathbf{X}, \mathbf{A})$, which we call a *profunctor* from $\mathbf{X}$ to $\mathbf{A}$. For a functor $F : \mathbf{Y} \longrightarrow \mathbf{X}$ define $\Phi F : \mathbf{Y} \longrightarrow\!\!\!\!\!+\!\!\!\!\rightarrow \mathbf{A}$ by $(\Phi F)(A, Y) = \Phi(A, FY)$. For a functor $\mathbf{A} \longleftarrow \mathbf{B} : G$, define $G^*\Phi : \mathbf{X} \longrightarrow\!\!\!\!\!+\!\!\!\!\rightarrow \mathbf{B}$ by $(G^*\Phi)(B, X) = \Phi(GB, X)$. In this way we get a module $\mathbf{pro}'' : \mathbf{cat}_0^{op} \longrightarrow\!\!\!\!\!+\!\!\!\!\rightarrow \mathbf{cat}_0$. Observe too that this example can be "parametrized" in at least two ways. For a category $\mathcal{K}$ with pullbacks, there is an evident $\mathbf{pro}''\mathcal{K} : (\mathbf{cat}\mathcal{K})_0^{op} \longrightarrow\!\!\!\!\!+\!\!\!\!\rightarrow (\mathbf{cat}\mathcal{K})_0$. For a monoidal category $\mathcal{V}$ (more generally a bicategory $\mathcal{V}$) there is $\mathcal{V}\text{-}\mathbf{pro}'' : (\mathcal{V}\text{-}\mathbf{cat})_0^{op} \longrightarrow\!\!\!\!\!+\!\!\!\!\rightarrow (\mathcal{V}\text{-}\mathbf{cat})_0$. In the case of $\mathbf{cat}$ there are also the actions $H\Phi$ for $\Phi : \mathbf{X} \longrightarrow\!\!\!\!\!+\!\!\!\!\rightarrow \mathbf{A}$ and $H : \mathbf{A} \longrightarrow \mathbf{B}$, given by $(H\Phi)(B, X) = \int^A \mathbf{B}(B, HA) \times \Phi(A, X)$; which, together with ΦK^* for $\mathbf{Y} \longleftarrow \mathbf{X} : K$ given by a similar formula, give rise to a module $\mathbf{pro}' : \mathbf{cat}_0 \longrightarrow\!\!\!\!\!+\!\!\!\!\rightarrow \mathbf{cat}_0^{op}$, with $\mathbf{pro}'(\mathbf{X}, \mathbf{A}) = \mathbf{pro}''(\mathbf{X}, \mathbf{A})$. With reasonable assumptions on $\mathcal{K}$ or $\mathcal{V}$, this last remark also applies to the parametrized versions.

(v) **hty** Let **top** denote the category of small topological spaces. For r a non-negative real number, write I_r for the closed interval $[0, r]$. For spaces X and A define $\mathbf{hty}(X, A)$ to be the category whose objects are continuous functions $f : X \longrightarrow A$ and whose arrows are homotopies with duration, as in Moore's definition of the path space. Explicitly, an arrow from f to g is a pair (r, H) where r is a non-negative real and $H : I_r \times X \longrightarrow A$ is a continuous function with $H(0, x) = fx$ and $H(r, x) = gx$ for all x in X. Composition is given by pasting homotopies, the first component of the composite $(s, K)(r, H)$ being $s + r$. With the evident actions we have a module $\mathbf{hty} : \mathbf{top} \longrightarrow\!\!\!\!\!+\!\!\!\!\rightarrow \mathbf{top}$. This example is interesting in that there is not an obvious "horizontal" composition of homotopies with duration, so that it is is a genuine example of a "module" rather than a bicategory. Were we to restrict to homotopies in the usual sense (with duration 1), we would gain horizontal composition but at the expense of associativity of "vertical" composition.

(vi) **comod′** For R a commutative ring, write $\mathbf{coalg}R$ for the category of cocommutative R-coalgebras and coalgebra homomorphisms. For such coalgebras X and A, write $\mathbf{comod}'R(X, A)$ for the category of X, A-bicomodules. An object $M : X \longrightarrow\!\!\!\!\!+\!\!\!\!\rightarrow A$ of this is an R-module M together with compatible coactions $x : M \longrightarrow X \otimes_R M$ and $a : M \longrightarrow M \otimes_R A$. An arrow $\Phi : M \longrightarrow N$ is an R-module homomorphism $M \longrightarrow N$ which is also a homomorphism with respect to the coactions. For a coalgebra homomorphism $g : A \longrightarrow B$, we define $gM : X \longrightarrow\!\!\!\!\!+\!\!\!\!\rightarrow B$ to be M together with $x : M \longrightarrow X \otimes_R M$ and $(M \otimes_R g)a : M \longrightarrow M \otimes_R B$. Similarly, for a coalgebra homomorphism $Y \longleftarrow X : f$, we define $Mf^* : Y \longrightarrow\!\!\!\!\!+\!\!\!\!\rightarrow A$ by composition. The result is a module $\mathbf{comod}R : \mathbf{coalg}R \longrightarrow\!\!\!\!\!+\!\!\!\!\rightarrow (\mathbf{coalg}R)^{op}$. It should be compared with the coalgebra-indexed category of vector spaces in [G&P]. We have given this example independently of example (iv) so as to make it more readily accessible; but observe that $(R\text{-}\mathbf{mod})^{op}$ is a symmetric monoidal category for which the one-object enriched categories are the R-coalgebras, so that the bicomodules can be seen as profunctors.

(vii) Any $\mathcal{S}$-indexed category as in [P&S] is a module $\mathcal{S}^{op} \longrightarrow\!\!\!\!\!+\!\!\!\!\rightarrow \mathbf{1}$ or, equivalently, a module $\mathbf{1} \longrightarrow\!\!\!\!\!+\!\!\!\!\rightarrow \mathcal{S}$.

2.5. Given a $\mathcal{K},\mathcal{L}$-module $\mathcal{M} : \mathcal{L} \mathrel{\rightarrow\!\!\!\!\!\!+}\ \mathcal{K}$, the action displays in 2.1 suggest that the data for $\mathcal{M}$ can also be considered to constitute those of a bicategory obtained via a glueing construction. This is the case. We write $\mathbf{gl}\mathcal{M}$ for the bicategory with objects those of $\mathcal{K}$ and $\mathcal{L}$ disjointly; with $\mathbf{gl}\mathcal{M}(K', K) = \mathcal{K}(K', K)$, $\mathbf{gl}\mathcal{M}(K, L) = \mathcal{M}(K, L)$, $\mathbf{gl}\mathcal{M}(L, L') = \mathcal{L}(L, L')$, $\mathbf{gl}\mathcal{M}(L, K) = \mathbf{0}$ (the empty category); with horizontal composition given by the actions; and with structural isomorphisms given by the ξ, η and ζ above. The coherence conditions on the latter, resulting from the definition of $\mathcal{M}$ as a homomorphism, are precisely those required to make $\mathbf{gl}\mathcal{M}$ a bicategory. Of course $\mathcal{K}$ and $\mathcal{L}$ are literally contained in $\mathbf{gl}\mathcal{M}$, so that we have the cospan $\mathcal{K} \longrightarrow \mathbf{gl}\mathcal{M} \longleftarrow \mathcal{L}$. Central to our work are squares in $\mathbf{gl}\mathcal{M}$ of the following kind:

$$
\begin{tikzcd}
K \arrow[r, "k"] \arrow[d, "\mu"', "|" marking] \arrow[dr, phantom, "\xrightarrow{\Phi}"] & K' \arrow[d, "\mu'", "|" marking] \\
L \arrow[r, "l"'] & L'
\end{tikzcd}
\quad ,
$$

and it is convenient to speak of this as a *square* in $\mathcal{M}$. The Grothendieck construction applied to the homomorphism $\mathcal{M} : \mathcal{K}^{op}, \mathcal{L} \longrightarrow \mathbf{CAT}$ yields a span

$$
\mathcal{K} \xleftarrow{\partial_0} \mathbf{gr}\mathcal{M} \xrightarrow{\partial_1} \mathcal{L},
$$

with $\mathbf{gr}\mathcal{M}$ a *category* whose objects are triples (K, μ, L), where $\mu : K \mathrel{\rightarrow\!\!\!\!\!\!+}\ L$ is a vector arrow of $\mathcal{M}$, and whose arrows are triples $(k, \Phi, l) : (K, \mu, L) \longrightarrow (K', \mu', L')$, where the data constitute a square in $\mathcal{M}$. Composition is given by (horizontal) pasting of squares. That this composition is well defined and associative follows from Appendix A of Verity's thesis [VTY], which provides an extension to bicategories of Power's "pasting theorem" [PAJ] for 2-categories. (A direct proof of the associativity of this particular pasting is interesting too, because it shows that the result does not depend on the invertibility of the ξ, η, and ζ in 2.3.) Clearly an identity for (K, μ, L) is provided by

$$
\begin{tikzcd}
K \arrow[r, "1_K"] \arrow[d, "\mu"', "|" marking] \arrow[dr, phantom, "\xrightarrow{1_\mu}"] & K \arrow[d, "\mu", "|" marking] \\
L \arrow[r, "1_L"'] & L
\end{tikzcd}
\quad .
$$

The functors ∂_0 and ∂_1 are the evident projections, regarded as domain and codomain with respect to the squares. In the terminology of [ST1], $\mathcal{K} \longleftarrow \mathbf{gr}\mathcal{M} \longrightarrow \mathcal{L}$ is a *fibration* from $\mathcal{L}$ to $\mathcal{K}$. The module $\mathcal{M}$ can be recovered from this fibration; in particular, for each

K and L we have in **CAT** a pullback

$$\begin{tikzcd} \mathcal{M}(K,L) \arrow[r] \arrow[d] & \mathbf{gr}\mathcal{M} \arrow[d] \\ \mathbf{1} \arrow[r, "{(K,L)}"'] & \mathcal{K}\times\mathcal{L} \end{tikzcd} .$$

Observe that the typical arrow $(k,\Phi,l):(K,\mu,L)\longrightarrow(K',\mu',L')$ in $\mathbf{gr}\mathcal{M}$ has a canonical factorization as the composite

$$\begin{tikzcd} K \arrow[r, "1_K"] \arrow[d, "\mu"'] \arrow[dr, phantom, "\xrightarrow{1_{l\mu}}"] & K \arrow[r, "1_K"] \arrow[d, "l\mu"'] \arrow[dr, phantom, "\xrightarrow{\Phi}"] & K \arrow[r, "k"] \arrow[d, "\mu' k"] \arrow[dr, phantom, "\xrightarrow{1_{\mu' k}}"] & K' \arrow[d, "\mu'"] \\ L \arrow[r, "l"'] & L' \arrow[r, "1_{L'}"'] & L' \arrow[r, "1_{L'}"'] & L' \end{tikzcd} .$$

In the terminology of [ST1], $(1_K, 1_{l\mu}, l)$ is right cartesian and $(k, 1_{\mu' k}, 1_{L'})$ is left cartesian, while $(1_K, \Phi, 1_{L'})$ is an arrow in the fibre $\mathcal{M}(K,L)$. The arrows of these three basic kinds *generate* those of $\mathbf{gr}\mathcal{M}$. Note that the first basic arrow is an identity if l is an identity, the second if Φ is an identity, and the third if k is an identity. The general right cartesian and left cartesian arrows for this fibration are easy to describe. It is convenient to record:

2.6. Lemma. *An arrow $(k,\Phi,l):(K,\mu,L)\longrightarrow(K',\mu',L')$ in $\mathbf{gr}\mathcal{M}$ is*

i) right cartesian if and only if k and Φ are invertible

ii) left cartesian if and only if Φ and l are invertible

iii) invertible if and only if each of k, Φ and l is invertible. ■

2.7. The starred equipments introduced informally in 2.1 are a special case of *starred modules*. In discussing these we use the language of *liftings* and *extensions*, introduced for 2-categories in [S&W], but equally applicable to bicategories.

2.8. Definition. *A module $\mathcal{M}:\mathcal{L}\longrightarrow\!\!\!\!\!\!\mapsto\mathcal{K}$ is said to be* starred *if*

i) for each vector arrow $\mu: K'\longrightarrow\!\!\!\!\!\!\mapsto L'$ in $\mathcal{M}$ and each scalar arrow $l: L\longrightarrow L'$ in $\mathcal{L}$, the bicategory $\mathbf{gl}\mathcal{M}$ admits a right lifting

$$\begin{tikzcd} & K' \arrow[dl, "l^*\mu"'] \arrow[d, "\mu"] \\ L \arrow[r, "l"'] \arrow[ur, phantom, "\xrightarrow{\hat{l}\mu}"] & L' \end{tikzcd}$$

of μ through l, whose right-lifting property moreover is respected by all arrows k : $K\longrightarrow K'$ in $\mathcal{K}$;

ii) similarly, for each $\nu : K \mathrel{-\!\!\!+\!\!\!\rightarrow} L$ in $\mathcal{M}$ and each $k : K \longrightarrow K'$ in $\mathcal{K}$, the bicategory $\mathbf{gl}\mathcal{M}$ *admits a left extension*

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d, "\nu"', "+" marking] \arrow[r, phantom, shift right=3, "\xrightarrow{\nu\check{k}}"] & K' \arrow[dl, "\nu k^*", "\times" marking] \\ L & \end{tikzcd}$$

of ν along k, whose left extension property is respected by all arrows $l : L \longrightarrow L'$ in $\mathcal{L}$.

■

In other words, $\mathcal{M} : \mathcal{L} \mathrel{-\!\!\!+\!\!\!\rightarrow} \mathcal{K}$ is starred if and only if

(a) each $\mathcal{M}(K', l) : \mathcal{M}(K', L) \longrightarrow \mathcal{M}(K', L')$ admits a right adjoint $\mu \mapsto l^*\mu$, which we may denote by $\mathcal{M}(K', l^*) : \mathcal{M}(K', L') \longrightarrow \mathcal{M}(K', L)$;

(b) each $\mathcal{M}(k, L) : \mathcal{M}(K', L) \longrightarrow \mathcal{M}(K, L)$ admits a left adjoint $\nu \mapsto \nu k^*$, which we may denote by $\mathcal{M}(k^*, L) : \mathcal{M}(K, L) \longrightarrow \mathcal{M}(K', L)$;

(c) the mate

$$\tilde{\eta} : \mathcal{M}(k, L)\mathcal{M}(K', l^*) \longrightarrow \mathcal{M}(K, l^*)\mathcal{M}(k, L') : \mathcal{M}(K, L) \longrightarrow \mathcal{M}(K', L')$$

of the isomorphism

$$\eta : \mathcal{M}(K, l)\mathcal{M}(k, L) \longrightarrow \mathcal{M}(k, L')\mathcal{M}(K', l) : \mathcal{M}(K', L) \longrightarrow \mathcal{M}(K, L')$$

under the adjunctions $\mathcal{M}(K, l) \dashv \mathcal{M}(K, l^*)$ and $\mathcal{M}(K', l) \dashv \mathcal{M}(K', l^*)$, is itself invertible; and

(d) the mate

$$\eta' : \mathcal{M}(k^*, L')\mathcal{M}(K', l) \longrightarrow \mathcal{M}(K', l)\mathcal{M}(k^*, L) : \mathcal{M}(K, L) \longrightarrow \mathcal{M}(K', L')$$

under the adjunctions $\mathcal{M}(k^*, L) \dashv \mathcal{M}(k, L)$ and $\mathcal{M}(k^*, L') \dashv \mathcal{M}(k, L')$ of the isomorphism η above is also invertible. (Recall that conditions such as (c) and (d) are said to be of the *Beck-Chevalley* type.)

Let us suppose that a choice has been made of the liftings and extensions above; we may always suppose it is so made that $1_L^*\mu = \mu$ and $\nu 1_K^* = \nu$. Because composites of adjunctions are adjunctions, the structural isomorphisms $\xi : (l'l)\mu \xrightarrow{\cong} l'(l\mu)$, $\eta : l(\mu k) \xrightarrow{\cong} (l\mu)k$, and $\zeta : (\mu k)k' \xrightarrow{\cong} \mu(kk')$ of $\mathcal{M}$ give rise to isomorphisms $\bar{\xi} : (l^*l'^*)\mu \xrightarrow{\cong} l^*(l'^*\mu)$, $\bar{\eta} : l^*(\mu k^*) \xrightarrow{\cong} (l^*\mu)k^*$, and $\bar{\zeta} : (\mu k'^*)k^* \xrightarrow{\cong} \mu(k'^*k^*)$; and besides the six isomorphisms ξ, η, ζ, $\bar{\xi}$, $\bar{\eta}$, $\bar{\zeta}$, we have as above the two Beck-Chevalley type isomorphisms $\tilde{\eta}^{-1} : l^*(\mu k) \xrightarrow{\cong} (l^*\mu)k$ and $\eta'^{-1} : l(\mu k^*) \xrightarrow{\cong} (l\mu)k^*$. These eight isomorphisms provide between them the structural isomorphisms for three more modules in addition to $\mathcal{M} : \mathcal{L} \longrightarrow\!\!\!\!\!\!\!+\ \mathcal{K}$, to be denoted by $\mathcal{M}' : \mathcal{L} \longrightarrow\!\!\!\!\!\!\!+\ \mathcal{K}^{op}$, $\mathcal{M}'' : \mathcal{L}^{op} \longrightarrow\!\!\!\!\!\!\!+\ \mathcal{K}$, and $\mathcal{M}^* : \mathcal{L}^{op} \longrightarrow\!\!\!\!\!\!\!+\ \mathcal{K}^{op}$, and having $\mathcal{M}'(K, L) = \mathcal{M}''(K, L) = \mathcal{M}^*(K, L) = \mathcal{M}(K, L)$: in the case of $\mathcal{M}'$, for instance, the action of $l \in \mathcal{L}(L, L')$ on $\mu \in \mathcal{M}(K, L)$ sends it to $l\mu$, while the action of $k^* : K' \longrightarrow K$ in $\mathcal{K}^{op}$ (which is just $k : K \longrightarrow K'$ in $\mathcal{K}$) on $\mu \in \mathcal{M}(K, L)$ sends it to μk^*; the coherence conditions for $\mathcal{M}'$ and the others follow from those for $\mathcal{M}$ by a simple application of the "naturality of mates" established in [K&S, Section 2].

Although we have obtained $\mathcal{M}'$, $\mathcal{M}''$, and $\mathcal{M}^*$ by starting with $\mathcal{M} : \mathcal{L} \longrightarrow\!\!\!\!\!\!\!+\ \mathcal{K}$ and requiring $\mathcal{M}(K', l)$ and $\mathcal{M}(k, L)$ to have respectively a right adjoint and a left adjoint satisfying Beck-Chevalley type conditions, observe that we might equally have begun with with any of the four modules $\mathcal{M}$, $\mathcal{M}'$,$\mathcal{M}''$, $\mathcal{M}^*$. For instance, we might begin with a module $\mathcal{M}' : \mathcal{L} \longrightarrow\!\!\!\!\!\!\!+\ \mathcal{K}^{op}$, writing its actions as $l, \mu \mapsto l\mu$ and $\mu, k^* \mapsto \mu k^*$, where now $k^* : K' \longrightarrow K$ again denotes the arrow of $\mathcal{K}^{op}$ corresponding to the arrow $k : K \longrightarrow K'$ of $\mathcal{K}$; and then require the existence of the right adjoints in $l(-) \dashv l^*(-)$ and $(-)k^* \dashv (-)k$, again subject to Beck-Chevalley conditions: the resulting $\mathcal{M} : \mathcal{L} \longrightarrow\!\!\!\!\!\!\!+\ \mathcal{K}$ is a starred module, whose $\mathcal{M}'$ (which is in any case determined only to within isomorphism) is isomorphic to that we began with. In practical examples, it is often $\mathcal{M}'$ that is most simply described: as we saw with $\mathbf{spn}'\mathcal{C}$ in 2.4, where the corresponding $\mathcal{M} : \mathcal{C} \longrightarrow\!\!\!\!\!\!\!+\ \mathcal{C}$ is the starred module — indeed the equipment — $\mathbf{spn}\mathcal{C} : \mathcal{C} \longrightarrow\!\!\!\!\!\!\!+\ \mathcal{C}$ of 2.1.

To reconcile our approach to these matters with others, suppose that we are given a category $\mathcal{K}$ and a bicategory $\mathcal{M}$ whose objects are those of $\mathcal{K}$. Any homomorphism of bicategories $\mathcal{K} \longrightarrow \mathcal{M}$ which is the identity on objects determines a module $\mathcal{M} : \mathcal{K} \longrightarrow\!\!\!\!\!\!\!+\ \mathcal{K}$ — and hence an equipment. In particular, when $\mathcal{K}$ is a regular category and $\mathcal{M}$ is the bicategory $\mathbf{rel}\mathcal{K}$, the inclusion $\mathcal{K} \longrightarrow \mathbf{rel}\mathcal{K}$ gives in this way the equipment $\mathcal{K} \longrightarrow\!\!\!\!\!\!\!+\ \mathcal{K}$ which is also called $\mathbf{rel}\mathcal{K}$. For such a homomorphism to be proarrow equipment in the sense of [WRJ] one requires that $\mathcal{K} \longrightarrow \mathcal{M}$ be locally fully faithful and that for each arrow k in $\mathcal{K}$ there be given an adjunction $k \dashv k^*$ in $\mathcal{M}$. Such adjunctions make the resulting equipment $\mathcal{M} : \mathcal{K} \longrightarrow\!\!\!\!\!\!\!+\ \mathcal{K}$ starred. The objects of the 3-category $\mathbf{F}$ introduced in [CKW] are proarrow equipments with the bicategory $\mathcal{M}$ (and hence also $\mathcal{K}$) merely locally ordered. The definition of proarrow equipment did not require that $\mathcal{K}$ be locally discrete. Had this condition been imposed in [CKW], a 2-category of starred locally ordered equipments would have resulted, rather than a 3-category. Only minor adjustments to the treatment in [CKW] are necessary to carry the results there into the present context.

2.9. We now define the 2-category $\mathbf{MOD}$ of all modules. An object $(\mathcal{K}, \mathcal{M}, \mathcal{L})$ of $\mathbf{MOD}$ consists of categories $\mathcal{K}$ and $\mathcal{L}$, along with a $\mathcal{K}, \mathcal{L}$-module $\mathcal{M}$; we may often call it $\mathcal{M}$ for short. It determines as in 2.5 a span $\partial_0 : \mathcal{K} \longleftarrow \mathbf{gr}\mathcal{M} \longrightarrow \mathcal{L} : \partial_1$ in $\mathbf{CAT}$, called $\widehat{\mathbf{gr}}\mathcal{M}$. This span is an object of the functor-2-category $[\Lambda, \mathbf{CAT}]$, where Λ denotes the three-object category

$$0 \longleftarrow \star \longrightarrow 1.$$

An arrow $(\mathcal{K},\mathcal{M},\mathcal{L})\longrightarrow(\mathcal{R},\mathcal{N},\mathcal{S})$ is by definition an arrow $\widehat{\mathbf{gr}}\mathcal{M}\longrightarrow\widehat{\mathbf{gr}}\mathcal{N}$ in $[\Lambda,\mathbf{CAT}]$ and is thus a triple (G,T,H) of functors making commutative

$$\begin{tikzcd}
\mathcal{K} \arrow[d, "G"'] & \mathbf{gr}\mathcal{M} \arrow[l, "\partial_0"'] \arrow[r, "\partial_1"] \arrow[d, "T"] & \mathcal{L} \arrow[d, "H"] \\
\mathcal{R} & \mathbf{gr}\mathcal{N} \arrow[l, "\partial_0"] \arrow[r, "\partial_1"'] & \mathcal{S}
\end{tikzcd} \quad ;$$

while a transformation $(G,T,H)\longrightarrow(F,S,J)$ in **MOD** is by definition a transformation $(G,T,H)\longrightarrow(F,S,J)$ in $[\Lambda,\mathbf{CAT}]$, consisting therefore of transformations $t : G\longrightarrow F$, $u : T\longrightarrow S$, and $s : H\longrightarrow J$ for which $\partial_0.u = t.\partial_0$ and $\partial_1.u = s.\partial_1$. It is immediate that **MOD** so defined is a 2-category, with a fully-faithful 2-functor $\widehat{\mathbf{gr}} : \mathbf{MOD}\longrightarrow[\Lambda,\mathbf{CAT}]$. When using the one-letter notation $\mathcal{M}$ for a module it is convenient to write $\widehat{\mathbf{gr}}\mathcal{M}$ as

$$\mathcal{M}_0 \xleftarrow{\partial_0} \mathbf{gr}\mathcal{M} \xrightarrow{\partial_1} \mathcal{M}_1,$$

with $T = (T_0,\mathbf{gr}T,T_1)$ for a typical arrow and $u = (u_0,\mathbf{gr}u,u_1)$ for a typical transformation. Thus we have 2-functors $(-)_0,(-)_1,\mathbf{gr} : \mathbf{MOD}\longrightarrow\mathbf{CAT}$, all of which are in fact representable (since one can easily exhibit modules $\mathcal{M}$ for which $\widehat{\mathbf{gr}}\mathcal{M}$ is any of the representables $1\longleftarrow1\longrightarrow1$, $1\longleftarrow0\longrightarrow0$, and $0\longleftarrow0\longrightarrow1$), along with 2-natural transformations $\partial_0 : \mathbf{gr}\longrightarrow(-)_0$ and $\partial_1 : \mathbf{gr}\longrightarrow(-)_1$. We further define the 2-category ${}^*\mathbf{MOD}$ of all starred modules: it is just the full sub-2-category of **MOD** given by those objects $(\mathcal{K},\mathcal{M},\mathcal{L})$ for which $\mathcal{M}$ is a *starred* module. Since an equipment is merely a module for which the scalars acting from the left are the same as those acting from the right, we define the 2-categories of equipments and of starred equipments by the following pullback diagrams (in which Δ denotes the diagonal 2-functor):

$$\begin{tikzcd}
\mathbf{EQT} \arrow[r] \arrow[d, "(-)_\#"'] & \mathbf{MOD} \arrow[d, "((-)_0,(-)_1)"] \\
\mathbf{CAT} \arrow[r, "\Delta"'] & \mathbf{CAT}\times\mathbf{CAT}
\end{tikzcd} \quad \text{and} \quad
\begin{tikzcd}
{}^*\mathbf{EQT} \arrow[r] \arrow[d, "(-)_\#"'] & {}^*\mathbf{MOD} \arrow[d, "((-)_0,(-)_1)"] \\
\mathbf{CAT} \arrow[r, "\Delta"'] & \mathbf{CAT}\times\mathbf{CAT}
\end{tikzcd} \quad .$$

Both 2-functors named $(-)_\#$ above are represented by the starred equipment $\mathcal{M}$ with $\mathcal{M}_\# = 1$ and $\mathbf{gl}\mathcal{M} = 1$, while the restrictions of $\mathbf{gr}$ to **EQT** and to ${}^*\mathbf{EQT}$ are represented by the starred equipment $\mathcal{M}$ with $\mathcal{M}_\#$ equal to the discrete category 2 and $\mathbf{gl}\mathcal{M}$ the arrow category $\mathbf{2}$.

2.10. For a $\mathcal{K},\mathcal{L}$-module $\mathcal{M} : \mathcal{L}\longrightarrow\!\!\!\!\!\!\mapsto\mathcal{K}$, the bicategory $\mathbf{gl}\mathcal{M}$ has the dual $(\mathbf{gl}\mathcal{M})^{coop}$ given by reversing both the arrows $[(-)^{op}]$ and the transformations $[(-)^{co}]$. It is apparent from the first two diagrams in 2.1 that $(\mathbf{gl}\mathcal{M})^{coop}$ results from glueing an $\mathcal{L}^{op},\mathcal{K}^{op}$-module that we may as well call $\mathcal{M}^{coop} : \mathcal{K}^{op}\longrightarrow\!\!\!\!\!\!\mapsto\mathcal{L}^{op}$, giving $(\mathbf{gl}\mathcal{M})^{coop} = \mathbf{gl}(\mathcal{M}^{coop})$. Thus $\mathcal{M}^{coop}(L,K) = \mathcal{M}(K,L)^{op}$, while for the actions we have $k\mu$ in $\mathcal{M}^{coop}$ given by μk in

$\mathcal{M}$, and so on. One might think of defining $\mathcal{M}^{op}$ and $\mathcal{M}^{co}$ similarly; but it is only $\mathcal{M}^{coop}$, in this generality, that is well-behaved with respect to the Grothendieck construction, in terms of which we defined **MOD** in 2.9. An arrow $(k, \Phi, l) : (K, \mu, L) \longrightarrow (K', \mu', L')$ in $\mathbf{gr}\mathcal{M}$ is equally an arrow $(l, \Phi, k) : (L', \mu', K') \longrightarrow (L, \mu, K)$ in $\mathbf{gr}(\mathcal{M}^{coop})$; this gives an isomorphism

$$(\mathbf{gr}\mathcal{M})^{op} \xrightarrow{\cong} \mathbf{gr}(\mathcal{M}^{coop}),$$

which in fact constitutes an isomorphism

$$(\widehat{\mathbf{gr}}\mathcal{M})^{op} \xrightarrow{\cong} \widehat{\mathbf{gr}}(\mathcal{M}^{coop})$$

of spans, since it clearly takes $\partial_1^{op} : (\mathbf{gr}\mathcal{M})^{op} \longrightarrow \mathcal{L}^{op}$ to the $\partial_0 : \mathbf{gr}(\mathcal{M}^{coop}) \longrightarrow \mathcal{L}^{op}$ for $\mathcal{M}^{coop}$.

It follows that an arrow $T : \mathcal{M} \longrightarrow \mathcal{N}$ in **MOD**, being the same thing as an arrow $\widehat{\mathbf{gr}}\mathcal{M} \longrightarrow \widehat{\mathbf{gr}}\mathcal{N}$ of spans, is in effect the same thing as an arrow $T^{coop} : \mathcal{M}^{coop} \longrightarrow \mathcal{N}^{coop}$ in **MOD**. Again, a transformation $u : T \longrightarrow S : \mathcal{M} \longrightarrow \mathcal{N}$ in **MOD**, being a transformation $u : T \longrightarrow S : \widehat{\mathbf{gr}}\mathcal{M} \longrightarrow \widehat{\mathbf{gr}}\mathcal{N}$ of spans, and hence a transformation $u^{op} : S^{op} \longrightarrow T^{op} : (\widehat{\mathbf{gr}}\mathcal{M})^{op} \longrightarrow (\widehat{\mathbf{gr}}\mathcal{N})^{op}$, is the same thing as a transformation

$$u^{coop} : S^{coop} \longrightarrow T^{coop} : \mathcal{M}^{coop} \longrightarrow \mathcal{N}^{coop}.$$

We conclude that $(-)^{coop}$ constitutes an isomorphism

$$(-)^{coop} : \mathbf{MOD} \xrightarrow{\cong} (\mathbf{MOD})^{co}$$

of 2-categories. Moreover, $\mathcal{M}^{coop}$ is starred when $\mathcal{M}$ is so; and similarly for the property of being an equipment; hence we have also, by restriction, similar dualities ${}^*\mathbf{MOD} \xrightarrow{\cong} {}^*\mathbf{MOD}^{co}$, $\mathbf{EQT} \xrightarrow{\cong} \mathbf{EQT}^{co}$, and ${}^*\mathbf{EQT} \xrightarrow{\cong} {}^*\mathbf{EQT}^{co}$.

Since the other operations $(-)^{co}$ and $(-)^{op}$ suggested above by consideration of $\mathbf{gl}\mathcal{M}$ play no significant role for *general* modules we can henceforth abandon such meanings of $(-)^{co}$ and $(-)^{op}$, thus freeing these symbols for a new role. The point is that, for *starred* modules, there are new involutary isomorphisms that it is convenient to denote by

$$(-)^{op} : {}^*\mathbf{MOD} \xrightarrow{\cong} {}^*\mathbf{MOD}$$

and

$$(-)^{co} : {}^*\mathbf{MOD} \xrightarrow{\cong} ({}^*\mathbf{MOD})^{co}.$$

Given the $\mathcal{K}, \mathcal{L}$-module $\mathcal{M} : \mathcal{L} \longrightarrow\!\!\!\!\!+\ \mathcal{K}$, we have an $\mathcal{L}, \mathcal{K}$-module $\mathcal{M}^{op} : \mathcal{K} \longrightarrow\!\!\!\!\!+\ \mathcal{L}$, where $\mathcal{M}^{op}(L, K) = \mathcal{M}(K, L)$, and we may write $\mu : K \longrightarrow\!\!\!\!\!+\ L$ in $\mathcal{M}$ as $\mu^* : L \longrightarrow\!\!\!\!\!+\ K$ in $\mathcal{M}^{op}$. Then the actions for $\mathcal{M}^{op}$ are defined by setting $k\mu^* = (\mu k^*)^*$ and $\mu^* l = (l^*\mu)^*$. Now $\mu^* \mapsto k\mu^*$ has, as required, a right adjoint $\nu^* \mapsto k^*\nu^*$, where $k^*\nu^* = (\nu k)^*$; for to give $k\mu^* \longrightarrow \nu^*$, that is $(\mu k^*)^* \longrightarrow \nu^*$, is just to give $\mu k^* \longrightarrow \nu$, equivalently to give $\mu \longrightarrow \nu k$, that is $\mu^* \longrightarrow k^*\nu^*$. Similarly, $\mu^* \mapsto \mu^* l$ has a left adjoint $\nu^* \mapsto \nu^* l^*$ where $\nu^* l^* = (l\nu)^*$. It is immediate that the Beck-Chevaley conditions of 2.7 are satisfied, so that $\mathcal{M}^{op}$ is indeed a starred $\mathcal{L}, \mathcal{K}$-module. There is an evident isomorphism $\mathbf{gr}(\mathcal{M}^{op}) \xrightarrow{\cong} \mathbf{gr}\mathcal{M}$; so that each

$T : \mathcal{M} \longrightarrow \mathcal{N}$ can be seen as an arrow $T^{op} : \mathcal{M}^{op} \longrightarrow \mathcal{N}^{op}$, and similarly for transformations $u : T \longrightarrow S : \mathcal{M} \longrightarrow \mathcal{N}$, establishing the first of the two isomorphisms displayed above.

We can define the isomorphism $(-)^{co} : {}^*\mathbf{MOD} \xrightarrow{\cong} ({}^*\mathbf{MOD})^{co}$ by composing $(-)^{op} : {}^*\mathbf{MOD} \xrightarrow{\cong} {}^*\mathbf{MOD}$ with $(-)^{coop} : {}^*\mathbf{MOD} \xrightarrow{\cong} ({}^*\mathbf{MOD})^{co}$. Accordingly, we have $\mathcal{M}^{co} : \mathcal{L}^{op} \relbar\joinrel\mapstochar\joinrel\rightarrow \mathcal{K}^{op}$ with $\mathcal{M}^{co}(K,L) = \mathcal{M}(K,L)^{op}$, and the actions of $\mathcal{L}^{op}$ and $\mathcal{K}^{op}$ are $l^*, \mu \mapsto l^*\mu$ and $\mu, k^* \mapsto \mu k^*$. Now this action of $\mathcal{L}^{op}$ has $\nu \mapsto l\nu$ as a *right* adjoint, and so on. Note that $\mathbf{gr}(\mathcal{M}^{co}) \xrightarrow{\cong} (\mathbf{gr}\mathcal{M})^{op}$. Of course the isomorphisms above restrict to dualities $(-)^{op} : {}^*\mathbf{EQT} \xrightarrow{\cong} {}^*\mathbf{EQT}$ and $(-)^{co} : {}^*\mathbf{EQT} \xrightarrow{\cong} ({}^*\mathbf{EQT})^{co}$.

2.11. For a starred module $\mathcal{M} : \mathcal{L} \relbar\joinrel\mapstochar\joinrel\rightarrow \mathcal{K}$, besides the three duals above given by the starred modules $\mathcal{M}^{op} : \mathcal{K} \relbar\joinrel\mapstochar\joinrel\rightarrow \mathcal{L}$, $\mathcal{M}^{co} : \mathcal{L}^{op} \relbar\joinrel\mapstochar\joinrel\rightarrow \mathcal{K}^{op}$, and $\mathcal{M}^{coop} : \mathcal{K}^{op} \relbar\joinrel\mapstochar\joinrel\rightarrow \mathcal{L}^{op}$, we have from 2.7 the three modules $\mathcal{M}' : \mathcal{L} \relbar\joinrel\mapstochar\joinrel\rightarrow \mathcal{K}^{op}$, $\mathcal{M}'' : \mathcal{L}^{op} \relbar\joinrel\mapstochar\joinrel\rightarrow \mathcal{K}$, and $\mathcal{M}^* : \mathcal{L}^{op} \relbar\joinrel\mapstochar\joinrel\rightarrow \mathcal{K}^{op}$. In general these seven modules are distinct: the only pair agreeing in domain and codomain is that given by $\mathcal{M}^*$ and $\mathcal{M}^{co}$, and these differ because $\mathcal{M}^*(K,L) = \mathcal{M}(K,L)$ while $\mathcal{M}^{co}(K,L) = \mathcal{M}(K,L)^{op}$.

Although $\mathcal{M}'$, $\mathcal{M}''$, and $\mathcal{M}^*$ are not in general starred, we can use the actions they involve to *construct* some useful starred modules. Thus $\mathcal{M}' : \mathcal{L} \relbar\joinrel\mapstochar\joinrel\rightarrow \mathcal{K}^{op}$ is given by a bi-homomorphism $\mathcal{M}' : \mathcal{K}, \mathcal{L} \longrightarrow \mathbf{CAT}$, from which we derive a homomorphism $\mathcal{M}^r : \mathcal{K} \times \mathcal{L} \longrightarrow \mathbf{CAT}$ and so a module $\mathcal{M}^r : \mathcal{K} \times \mathcal{L} \relbar\joinrel\mapstochar\joinrel\rightarrow \mathbf{1}$, where $\mathbf{1}$ is the unit category; we merely set $\mathcal{M}^r(0,(K,L)) = \mathcal{M}'(K,L) = \mathcal{M}(K,L)$, and define the actions by taking $(k,l)\mu$ to be — we make here an arbitrary choice of bracketing — the $(l\mu)k^*$ of $\mathcal{M}'$ (and hence of $\mathcal{M}$); then the module $\mathcal{M}^r$ is indeed starred, with $(k,l)^*\mu = l^*(\mu k)$. Similarly the actions of $\mathcal{M}''$ lead to a starred module $\mathcal{M}^l : \mathbf{1} \relbar\joinrel\mapstochar\joinrel\rightarrow \mathcal{K} \times \mathcal{L}$, with $\mathcal{M}((K,L),0) = \mathcal{M}(K,L)$ and with $\mu(k,l) = l^*(\mu k)$. Finally $\mathcal{M}^* : \mathcal{L}^{op} \relbar\joinrel\mapstochar\joinrel\rightarrow \mathcal{K}^{op}$, since it is given by a bi-homomorphism $\mathcal{M}^* : \mathcal{K}, \mathcal{L}^{op} \longrightarrow \mathbf{CAT}$, which is equally a bi-homomorphism $\mathcal{M}^* : \mathcal{L}^{op}, K \longrightarrow \mathbf{CAT}$, gives rise to a module $\mathcal{K} \relbar\joinrel\mapstochar\joinrel\rightarrow \mathcal{L}$ involving the same actions as $\mathcal{M}^*$; however this module needs no new name, since it is clearly nothing but the starred module $\mathcal{M}^{op} : \mathcal{K} \relbar\joinrel\mapstochar\joinrel\rightarrow \mathcal{L}$.

Consider now the four categories $\mathbf{gr}\mathcal{M}$, $\mathbf{gr}\mathcal{M}^r$, $\mathbf{gr}\mathcal{M}^{op}$, and $\mathbf{gr}\mathcal{M}^l$. An object in any of these is in effect a triple (K, μ, L) where $\mu : K \relbar\joinrel\mapstochar\joinrel\rightarrow L$ in $\mathcal{M}$. An arrow from (K,μ,L) to (K',μ',L') has in $\mathbf{gr}\mathcal{M}$ the form (k, Φ, l) where $\Phi : l\mu \longrightarrow \mu' k$; it has in $\mathbf{gr}\mathcal{M}^r$ the form (k, Φ^r, l) where $\Phi^r : (l\mu)k^* \longrightarrow \mu'$; it has in $\mathbf{gr}\mathcal{M}^{op}$ the form (k, Φ^*, l) where $\Phi^* : \mu k^* \longrightarrow l^*\mu'$, and it has in $\mathbf{gr}\mathcal{M}^l$ the form (k, Φ^l, l) where $\Phi^l : \mu \longrightarrow l^*(\mu' k)$. We may represent such arrows by their respective "squares"

$$\begin{tikzcd}
K \arrow[r, "k"] \arrow[d, "\mu"', "|" marking] \arrow[dr, phantom, "\xrightarrow{\Phi}"] & K' \arrow[d, "\mu'", "|" marking] \\
L \arrow[r, "l"'] & L'
\end{tikzcd}
,\quad
\begin{tikzcd}
K \arrow[d, "\mu"', "|" marking] \arrow[dr, phantom, "\xrightarrow{\Phi^r}"] & K' \arrow[l, "k^*"'] \arrow[d, "\mu'", "|" marking] \\
L \arrow[r, "l"'] & L'
\end{tikzcd}
,\quad
\begin{tikzcd}
K \arrow[d, "\mu"', "|" marking] \arrow[dr, phantom, "\xrightarrow{\Phi^*}"] & K' \arrow[l, "k^*"'] \arrow[d, "\mu'", "|" marking] \\
L & L' \arrow[l, "l^*"]
\end{tikzcd}
,\quad
\begin{tikzcd}
K \arrow[r, "k"] \arrow[d, "\mu"', "|" marking] \arrow[dr, phantom, "\xrightarrow{\Phi^l}"] & K' \arrow[d, "\mu'", "|" marking] \\
L & L' \arrow[l, "l^*"]
\end{tikzcd}
,$$

which are unambiguous once we fix on the bracketings $(l\mu)k^*$ for the domain of Φ^r and $l^*(\mu' k)$ for the codomain of Φ^l.

However the adjunctions $l(-) \dashv l^*(-)$ and $(-)k^* \dashv (-)k$ provide a *bijection* between such Φ and such Φ^r; and equally between such Φ and such Φ^l, or between such Φ and such

Φ^*; for we are just replacing Φ by one of its *mates* in the sense of [K&S], under one or both of these adjunctions. It follows moreover, from the results in [K&S] on the *naturality* of mates, that these bijections respect composition in the various categories, thus providing isomorphisms of categories $\mathbf{gr}\mathcal{M} \cong \mathbf{gr}\mathcal{M}^r \cong \mathbf{gr}\mathcal{M}^{op} \cong \mathbf{gr}\mathcal{M}^l$. These isomorphisms, moreover, are clearly compatible with the projections in the spans $\mathcal{K} \longleftarrow \mathbf{gr}\mathcal{M} \longrightarrow \mathcal{L}$, $\mathbf{1} \longleftarrow \mathbf{gr}\mathcal{M}^r \longrightarrow \mathcal{K} \times \mathcal{L}$, $\mathcal{L} \longleftarrow \mathbf{gr}\mathcal{M}^{op} \longrightarrow \mathcal{K}$, $\mathcal{K} \times \mathcal{L} \longleftarrow \mathbf{gr}\mathcal{M}^l \longrightarrow \mathbf{1}$, which constitute $\widehat{\mathbf{gr}}\mathcal{M}$, $\widehat{\mathbf{gr}}\mathcal{M}^r$, $\widehat{\mathbf{gr}}\mathcal{M}^{op}$, and $\widehat{\mathbf{gr}}\mathcal{M}^l$. It follows that an arrow $T : \mathcal{M} \longrightarrow \mathcal{N}$ in ${}^*\mathbf{MOD}$ can be seen not only as a functor $\widehat{\mathbf{gr}}\mathcal{M} \longrightarrow \widehat{\mathbf{gr}}\mathcal{N}$, but equally as a functor $\widehat{\mathbf{gr}}\mathcal{M}^r \longrightarrow \widehat{\mathbf{gr}}\mathcal{N}^r$, and so on; similarly for transformations $u : T \longrightarrow S$ in ${}^*\mathbf{MOD}$.

Invertibility of any one of Φ, Φ^r, Φ^*, and Φ^l above provides in general four distinct conditions on Φ; and such invertibilities will be important to us:

2.12. Definition. *The square $\Phi : l\mu \longrightarrow \mu' k$ in $\mathcal{M}$ is said to be*

(i) a commutative *square if Φ is invertible;*

(ii) a right *square if Φ^r is invertible;*

(iii) an exact *square if Φ^* is invertible;*

(iv) a left *square if Φ^l is invertible.* ■

2.13. It will be necessary to have descriptions of the arrows and transformations of $\mathbf{MOD}$ in more primitive terms. For an arrow $T : \mathcal{M} \longrightarrow \mathcal{N}$ in $\mathbf{MOD}$ it follows from the considerations of 2.5 that T is determined by its values on the objects of $\mathbf{gr}\mathcal{M}$ and by its values on the the three basic kinds of generating arrows of $\mathbf{gr}\mathcal{M}$. For its value on the object (K, μ, L), the definitions in 2.9 oblige us to write $T(K, \mu, L) = (T_0 K, (\mathbf{gr}T)\mu, T_1 L)$; but in cases where there is no danger of confusion we may safely write this as $(T_0 K, T\mu, T_1 L)$, or even as $(TK, T\mu, TL)$. Similarly, 2.9 prescribes that application of T to the canonical factorization in 2.5 of a square in $\mathcal{M}$ produce a diagram of the form:

$$\begin{tikzcd}
T_0K \arrow[r, "1_{T_0K}"] \arrow[d, "T\mu"'] \arrow[dr, phantom, "\xrightarrow{T_{l,\mu}}"] & T_0K \arrow[r, "1_{T_0K}"] \arrow[d, "T(l\mu)"'] \arrow[dr, phantom, "\xrightarrow{T\Phi}"] & T_0K \arrow[r, "T_0k"] \arrow[d, "T(\mu'k)"] \arrow[dr, phantom, "\xrightarrow{T^{\mu',k}}"] & T_0K' \arrow[d, "T\mu'"] \\
T_1L \arrow[r, "T_1l"'] & T_1L' \arrow[r, "1_{T_1L'}"'] & T_1L' \arrow[r, "1_{T_1L'}"'] & T_1L'
\end{tikzcd} .$$

From 2.9 it follows that the restriction of $\mathbf{gr}T$ to the fibre $\mathcal{M}(K, L)$ determines, for each K in $\mathcal{K}$ and L in $\mathcal{L}$, a functor $\mathcal{M}(K, L) \longrightarrow \mathcal{N}(T_0K, T_1L)$ that we call $T_{K,L}$. Thus in the middle square we have $T\Phi = (\mathbf{gr}T)\Phi = T_{K,L'}\Phi$; moreover in this notation the effect of T on objects is given by $T(K, \mu, L) = (T_0K, T_{K,L}\mu, T_1L)$. For the effect of T on $(1_K, 1_{l\mu}, l)$ and on $(k, 1_{\mu'k}, 1_{L'})$ we have written $(1_{T_0K}, T_{l,\mu}, T_1 l)$ and $(T_0k, T^{\mu',k}, 1_{T_1L'})$ respectively. It is convenient to refer to the $T_{l,\mu}$ and $T^{\mu',k}$ as *action comparisons*. To determine the conditions to be satisfied by these derived data, one has only to determine the relations satisfied by the generating arrows of $\mathbf{gr}\mathcal{M}$, by examining the canonical factorization of 2.5 for each binary composite of generating arrows. We leave this task to the reader; completing it leads to the following:

2.14. Proposition. *Giving an arrow $T : \mathcal{M} \longrightarrow \mathcal{N}$ in* **MOD** *is equivalent to giving*

(i) functors $T_0 : \mathcal{M}_0 \longrightarrow \mathcal{N}_0$ and $T_1 : \mathcal{M}_1 \longrightarrow \mathcal{N}_1$ (where we frequently write just T for both T_0 and T_1);

(ii) for each K in $\mathcal{M}_0$ and L in $\mathcal{M}_1$, a functor $T_{K,L} : \mathcal{M}(K,L) \longrightarrow \mathcal{N}(T_0K, T_1L)$ (whose value at $\Phi : \mu \longrightarrow \mu'$ we write for short as $T\Phi : T\mu \longrightarrow T\mu'$);

(iii) arrows $T_{l,\mu} : Tl.T\mu \longrightarrow T(l\mu)$, natural in $\mu \in \mathcal{M}(K,L)$;

(iv) arrows $T^{\mu,k} : T(\mu k) \longrightarrow T\mu.Tk$, natural in $\mu \in \mathcal{M}(K,L)$;

where these data are to satisfy the conditions

$$(T_{1_L,\mu} : T1_L.T\mu \longrightarrow T(1_L\mu)) = (1_{T\mu} : T\mu \longrightarrow T\mu),$$

$$(T^{\mu,1_K} : T(\mu 1_K) \longrightarrow T\mu.T1_K) = (1_{T\mu} : T\mu \longrightarrow T\mu),$$

and to make commutative the diagrams

$$\begin{tikzcd}
(Tl'.Tl).T\mu \arrow[r, "\xi"] \arrow[d, equal] & Tl'.(Tl.T\mu) \arrow[r, "Tl'.T_{l,\mu}"] & Tl'.T(l\mu) \arrow[d, "T_{l',l\mu}"] \\
T(l'l).T\mu \arrow[r, "T_{l'l,\mu}"'] & T((l'l)\mu) \arrow[r, "T\xi"'] & T(l'(l\mu))
\end{tikzcd} ,$$

$$\begin{tikzcd}
Tl.T(\mu k) \arrow[r, "Tl.T^{\mu,k}"] \arrow[d, "T_{l,\mu k}"'] & Tl.(T\mu.Tk) \arrow[r, "\eta"] & (Tl.T\mu).Tk \arrow[d, "T_{l,\mu}.Tk"] \\
T(l(\mu k)) \arrow[r, "T\eta"'] & T((l\mu)k) \arrow[r, "T^{l\mu,k}"'] & T(l\mu).Tk
\end{tikzcd} ,$$

$$\begin{tikzcd}
T((\mu k)k') \arrow[r, "T\zeta"] \arrow[d, "T^{\mu k,k'}"'] & T(\mu(kk')) \arrow[r, "T^{\mu,kk'}"] & T\mu.T(kk') \arrow[d, equal] \\
T(\mu k).Tk' \arrow[r, "T^{\mu,k}.Tk'"'] & (T\mu.Tk).Tk' \arrow[r, "\zeta"'] & T\mu.(Tk.Tk')
\end{tikzcd} .$$

■

2.15. The *right* fibrations from $\mathcal{L}$ to $\mathcal{K}$ are described in [ST2] as the algebras for a KZ-doctrine $\mathbf{R}_{\mathcal{L},\mathcal{K}}$ on $\mathbf{spnCAT}(\mathcal{L},\mathcal{K})$. It takes but a simple extension of that account to show that those objects in $[\Lambda, \mathbf{CAT}]$ which are right fibrations are the algebras for a single KZ-doctrine $\mathbf{R}$ on $[\Lambda, \mathbf{CAT}]$ — with $\mathbf{R}$ restricting to the $\mathbf{R}_{\mathcal{L},\mathcal{K}}$. Left fibrations are algebras for a dual KZ-doctrine $\mathbf{L}$ on $[\Lambda, \mathbf{CAT}]$; while fibrations are algebras for a doctrine $\mathbf{M}$ obtained as the composite of $\mathbf{L}$ and $\mathbf{R}$ via a canonical isomorphic distributive law $\mathbf{RL} \cong \mathbf{LR}$. Thus the objects of $\mathbf{MOD}$ are those $\mathbf{M}$-algebras of the form $\widehat{\mathbf{gr}}\mathcal{M}$ for a module $\mathcal{M}$, but the arrows (and transformations) of $\mathbf{MOD}$ are just those of $[\Lambda, \mathbf{CAT}]$. Again, a straightforward extension of [ST1] shows that a $[\Lambda, \mathbf{CAT}]$ arrow $T : \widehat{\mathbf{gr}}\mathcal{M} \longrightarrow \widehat{\mathbf{gr}}\mathcal{N}$ is an $\mathbf{R}$-homomorphism precisely when it takes right cartesian arrows to right cartesian arrows and it follows easily that this is the case precisely when all the $T_{l,\mu}$ are isomorphisms. Dually, T is an $\mathbf{L}$-homomorphism precisely when all the $T^{\mu,k}$ are isomorphisms. From the general theory of KZ-doctrines (see [KCK]) we have:

2.16. Proposition. *For any adjunction $S \dashv T : \mathcal{M} \longrightarrow \mathcal{N}$ in $\mathbf{MOD}$, S is an $\mathbf{R}$-homomorphism and T is an $\mathbf{L}$-homomorphism; that is, the $S_{l,\nu}$ and the $T^{\mu,k}$ are all invertible.* ∎

2.17. We now expand on the discussion begun in 2.13 for the case where $T : \mathcal{M} \longrightarrow \mathcal{N}$ is in $^*\mathbf{MOD}$. Evidently the discussion then applies equally to $T^{op} : \mathcal{M}^{op} \longrightarrow \mathcal{N}^{op}$. Consider the following squares:

$$
\begin{tikzcd}
K \arrow[d, "\mu"'] \arrow[r, phantom, ""] & K' \arrow[l, "k^*"'] \arrow[d, "\mu k^*"] \\
L \arrow[r, phantom, "1_{\mu k^*}"] & L \arrow[l, "1^*"]
\end{tikzcd}
\quad
\begin{tikzcd}
K \arrow[r, "k"] \arrow[d, "\mu"'] & K' \arrow[d, "\mu k^*"] \\
L \arrow[r, "1"'] & L
\end{tikzcd}
\quad
\begin{tikzcd}
TK \arrow[r, "Tk"] \arrow[d, "T\mu"'] & TK' \arrow[d, "T(\mu k^*)"] \\
TL \arrow[r, "1"'] & TL
\end{tikzcd}
\quad
\begin{tikzcd}
TK \arrow[d, "T\mu"'] & TK' \arrow[l, "(Tk)^*"'] \arrow[d, "T(\mu k^*)"] \\
TL & TL \arrow[l, "1"]
\end{tikzcd}
\; .
$$

with middle arrows $1_{\mu k^*}$, $\widetilde{\mu k}$, (unlabelled) and $T^{op}_{k,\mu}$ respectively.

The first displays a basic right cartesian arrow in $\mathbf{gr}\mathcal{M}^{op}$ and the second is its mate in $\mathbf{gr}\mathcal{M}$, where $\widetilde{\mu k}$ is the μ-component of the unit of the adjunction $(-)k^* \dashv (-)k$ for $\mathcal{M}$. Application of T to the second square yields the third square in $\mathcal{N}$, wherein the unlabelled arrow is necessarily $T^{\mu k^*,k}.T(\widetilde{\mu k})$, and finally the fourth square is the mate in $\mathbf{gr}\mathcal{N}^{op}$ of the third square — which describes the effect of T^{op} on the first. Thus, explicitly,

$$
\begin{tikzcd}
T\mu.(Tk)^* \arrow[rr, "T^{op}_{k,\mu}"] \arrow[d, "T(\widetilde{\mu k}).(Tk)^*"'] & & T(\mu k^*) \\
T((\mu k^*)k).(Tk)^* \arrow[rr, "T^{\mu k^*,k}.(Tk)^*"'] & & (T(\mu k^*).Tk).(Tk)^* \arrow[u, "T(\mu k^*).(Tk)\hat{}"']
\end{tikzcd}
\quad ,
$$

where $(Tk)\hat{}$ is the counit of the adjunction $(-)(Tk)^* \dashv (-)Tk$. Dually, if the exercise above is carried out for a basic left cartesian arrow in $\mathbf{gr}\mathcal{M}^{op}$, we arrive at a description of $T^{op}_{\mu,l}$. Expressed in terms of actions in $\mathcal{M}$ and $\mathcal{N}$ it is convenient to label $T^{op}_{k,\mu}$ and $(T^{op})^{\mu,l}$ as

$$T_{\mu,k^*} : T\mu.Tk^* \longrightarrow T(\mu k^*) \qquad \text{and} \qquad T^{l^*,\mu} : T(l^*\mu) \longrightarrow Tl^*.T\mu$$

and refer to these also as *action comparisons*; we are now for convenience writing Tk^* for $(Tk)^*$ and Tl^* for $(Tl)^*$, which are unambiguous in that $T(k^*)$ and $T(l^*)$ make no sense. Comparing the explicit formula above for $T^{op}_{k,\mu} = T_{\mu,k^*}$ with [K&S, section 2] reveals that

$$T_{-,k^*} : \mathcal{N}(k^*,L)T_{K,L} \longrightarrow T_{K',L}\mathcal{M}(k^*,L) : \mathcal{M}(K,L) \longrightarrow \mathcal{M}(K',L)$$

is the mate under the adjunctions $\mathcal{N}(k^*,L) \dashv \mathcal{N}(k,L)$ and $\mathcal{M}(k^*,L) \dashv \mathcal{M}(k,L)$ in **CAT** of

$$T^{-,k} : T_{K,L}\mathcal{M}(k,L) \longrightarrow \mathcal{N}(k,L)T_{K',L} : \mathcal{M}(K',L) \longrightarrow \mathcal{N}(K,L);$$

with a dual result for $T^{l^*,-}$.

In $\mathcal{M}^r$ and $\mathcal{M}^l$ the only non-trivial cartesian arrows are respectively the right ones and the left ones, the typical such being displayed below:

$$\begin{tikzcd} K \arrow[d, "\mu"'] & K' \arrow[l, "k^*"'] \arrow[d, "(l\mu)k^*"] \\ L \arrow[r, "l"'] \arrow[u, phantom, ""] & L' \end{tikzcd} \quad \begin{tikzcd} {} \arrow[r, phantom, "\xrightarrow{1_{(l\mu)k^*}}"] & {} \end{tikzcd} \quad , \qquad \begin{tikzcd} K \arrow[r, "k"] \arrow[d, "l^*(\mu k)"'] \arrow[dr, phantom, "\xrightarrow{1_{l^*(\mu k)}}"] & K' \arrow[d, "\mu"] \\ L & L' \arrow[l, "l^*"] \end{tikzcd} \quad .$$

Proceeding as we did for $\mathcal{M}^{op}$ we find the further action comparisons

$$T_{l,\mu,k^*} : Tl.(T\mu.Tk^*) \longrightarrow T((l\mu)k^*) \qquad \text{and} \qquad T^{l^*,\mu,k} : T(l^*(\mu k)) \longrightarrow (Tl^*.T\mu).Tk.$$

However, the factorization $(k,1)(1,l) = (k,l) = (1,l)(k,1)$ in $\mathcal{K} \times \mathcal{L}$ and the observations $T_{1,\mu,k^*} = T_{\mu,k^*}$ et cetera together with Proposition 2.14 provide:

2.18. Lemma. *The following diagrams commute:*

$$\begin{tikzcd} Tl.(T\mu.k^*) \arrow[r, "\eta"] \arrow[d, "Tl.T_{\mu,k^*}"'] \arrow[ddr, "T_{l,\mu,k^*}"] & (Tl.T\mu).Tk^* \arrow[d, "T_{l,\mu}.Tk^*"] \\ Tl.T(\mu k^*) \arrow[d, "T_{l,\mu k^*}"'] & T(l\mu).Tk^* \arrow[d, "T_{l\mu,k^*}"] \\ T(l(\mu k^*)) \arrow[r, "T\eta"'] & T((l\mu)k^*) \end{tikzcd} \quad , \quad \begin{tikzcd} T(l^*(\mu k)) \arrow[r, "T\eta"] \arrow[d, "T^{l^*,\mu k}"'] \arrow[ddr, "T^{l^*,\mu,k}"] & T((l^*\mu)k) \arrow[d, "T^{l^*\mu,k}"] \\ Tl^*.T(\mu k) \arrow[d, "Tl^*.T^{\mu,k}"'] & T(l^*\mu).Tk \arrow[d, "T^{l^*,\mu}.Tk"] \\ Tl^*.(T\mu.Tk) \arrow[r, "\eta"'] & (Tl^*.T\mu)Tk \end{tikzcd} \;;$$

whence we easily conclude that

(i) each T_{l,μ,k^*} is invertible if and only if each $T_{l,\mu}$ and each T_{μ,k^*} are invertible,

(ii) each $T^{l^*,\mu,k}$ is invertible if and only if each $T^{\mu,k}$ and each $T^{l^*,\mu}$ are invertible. ∎

2.19. Definition. *An arrow $T : \mathcal{M} \longrightarrow \mathcal{N}$ in* ${}^*\mathbf{MOD}$ *is said to be*

(i) strong *if each $T_{l,\mu}$ and each $T^{\mu,k}$ are invertible;*

(ii) *strong *if each T_{μ,k^*} and each $T^{l^*,\mu}$ are invertible;*

(iii) a right homomorphism *if each T_{l,μ,k^*} is invertible; that is, if each $T_{l,\mu}$ and each T_{μ,k^*} are invertible;*

(iv) a left homomorphism *if each $T^{l^*,\mu,k}$ is invertible; that is, if each $T^{l^*,\mu}$ and each $T^{\mu,k}$ are invertible;*

(v) a homomorphism *if it is both a right and a left homomorphism; that is, if it is both strong and *strong.* ∎

Of course the term "strong" is also applicable to arrows in $\mathbf{MOD}$. Preservation of the properties named in Definition 2.12 for a square in $\mathcal{M}$ being important properties for an arrow $T : \mathcal{M} \longrightarrow \mathcal{N}$ in ${}^*\mathbf{MOD}$, it is convenient to record:

2.20. Proposition. *An arrow $T : \mathcal{M} \longrightarrow \mathcal{N}$ in* ${}^*\mathbf{MOD}$

(i) preserves commutative squares if and only if it is strong;

(ii) preserves right squares if and only if it is a right homomorphism;

(iii) preserves exact squares if and only if it is *strong;

(iv) preserves left squares if and only if it is a left homomorphism;

(v) preserves commutative squares and exact squares if and only if it preserves right squares and left squares if and only it is a homomorphism. ∎

Because an adjunction $S \dashv T$ in ${}^*\mathbf{MOD}$ gives rise to further adjunctions $S^r \dashv T^r$ and $S^l \dashv T^l$, Proposition 2.16 for $\mathbf{MOD}$ admits an improved form for starred modules:

2.21. Proposition. *For any adjunction $S \dashv T : \mathcal{M} \longrightarrow \mathcal{N}$ in* ${}^*\mathbf{MOD}$, *S is a right homomorphism and T is a left homomorphism.* ∎

2.22. We now complete the task set in 2.13 of also describing the *transformations* of $\mathbf{MOD}$ in more primitive terms. Certainly a transformation $u : T \longrightarrow S : \mathcal{M} \longrightarrow \mathcal{N}$ in $\mathbf{MOD}$ involves natural transformations $u_i : T_i \longrightarrow S_i : \mathcal{M}_i \longrightarrow \mathcal{N}_i$, for $i = 0, 1$, and for each object (K, μ, L) of $\mathbf{gr}\mathcal{M}$, an arrow in $\mathbf{gr}\mathcal{N}$ of the particular form

$$\begin{tikzcd} T_0K \arrow[r, "u_0K"] \arrow[d, "T\mu"'] \arrow[dr, phantom, "\xrightarrow{u_\mu}"] & S_0K \arrow[d, "S\mu"] \\ T_1L \arrow[r, "u_1L"'] & S_1L \end{tikzcd} .$$

Since it suffices to require naturality only with respect to the basic generating arrows of 2.5 we have:

2.23. Proposition. *Giving a transformation $u : T \longrightarrow S : \mathcal{M} \longrightarrow \mathcal{N}$ in* **MOD** *is equivalent to giving natural transformations $u_0 : T_0 \longrightarrow S_0$ and $u_1 : T_1 \longrightarrow S_1$ and the squares u_μ above, natural with respect to transformations $\Phi : \mu \longrightarrow \mu'$ in $\mathcal{M}(K,L)$, and satisfying the following two conditions:*

$$
\begin{tikzcd}
T_0K \arrow[r, "1"] \arrow[d, "T\mu"'] \arrow[dr, phantom, "\xrightarrow{T_{l,\mu}}"] & T_0K \arrow[r, "u_0K"] \arrow[d, "T(l\mu)"] \arrow[dr, phantom, "\xrightarrow{u_{l\mu}}"] & S_0K \arrow[d, "S(l\mu)"] \\
T_1L \arrow[r, "T_1l"'] & T_1L' \arrow[r, "u_1L'"'] & S_1L'
\end{tikzcd}
=
\begin{tikzcd}
T_0K \arrow[r, "u_0K"] \arrow[d, "T\mu"'] \arrow[dr, phantom, "\xrightarrow{u_{\mu}}"] & S_0K \arrow[r, "1"] \arrow[d, "S\mu"] \arrow[dr, phantom, "\xrightarrow{S_{l,\mu}}"] & S_0K \arrow[d, "S(l\mu)"] \\
T_1L \arrow[r, "u_1L"'] & S_1L \arrow[r, "S_1l"'] & S_1L'
\end{tikzcd}
,
$$

$$
\begin{tikzcd}
T_0K' \arrow[r, "u_0K'"] \arrow[d, "T(\mu k)"'] \arrow[dr, phantom, "\xrightarrow{u_{\mu k}}"] & S_0K' \arrow[r, "S_0k"] \arrow[d, "S(\mu k)"] \arrow[dr, phantom, "\xrightarrow{S^{\mu,k}}"] & S_0K \arrow[d, "S\mu"] \\
T_1L \arrow[r, "u_1L"'] & S_1L \arrow[r, "1"'] & S_1L
\end{tikzcd}
=
\begin{tikzcd}
T_0K' \arrow[r, "T_0k"] \arrow[d, "T(\mu k)"'] \arrow[dr, phantom, "\xrightarrow{T^{\mu,k}}"] & T_0K \arrow[r, "u_0K"] \arrow[d, "T\mu"] \arrow[dr, phantom, "\xrightarrow{u_{\mu}}"] & S_0K \arrow[d, "S\mu"] \\
T_1L \arrow[r, "1"'] & T_1L \arrow[r, "u_1L"'] & S_1L
\end{tikzcd}
.
$$

■

2.24. We will find it convenient to use "fibrational" properties of the 2-functors

$$((-)_0, (-)_1) : \mathbf{MOD} \longrightarrow \mathbf{CAT} \times \mathbf{CAT} \text{ and } ((-)_0, (-)_1) : {}^*\mathbf{MOD} \longrightarrow \mathbf{CAT} \times \mathbf{CAT}.$$

We note that *2-fibrations* have been defined and studied in [HRM], along with a characterization of adjunctions in the domain 2-category of a particular 2-fibration. Here we do no not need the full general theory of 2-fibrations, but on the other hand we require a very detailed analysis of adjunctions in **MOD** and ***MOD**. Accordingly, we provide an independent account of the case at hand. We will see $((-)_0, (-)_1) : \mathbf{MOD} \longrightarrow \mathbf{CAT} \times \mathbf{CAT}$ as a fibrational structure *from* **1** *to* **CAT** × **CAT** (in the terminology of [ST1]).

The first of the 2-categories defined by the following pullbacks in 2-**CAT**

$$
\begin{tikzcd}
\mathcal{K},\mathcal{L}\text{-}\mathbf{MOD} \arrow[r] \arrow[d] & \mathbf{MOD} \arrow[d, "((-)_0,(-)_1)"] \\
\mathbf{1} \arrow[r, "(\mathcal{K},\mathcal{L})"'] & \mathbf{CAT} \times \mathbf{CAT}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathcal{K},\mathcal{L}\text{-}{}^*\mathbf{MOD} \arrow[r] \arrow[d] & {}^*\mathbf{MOD} \arrow[d, "((-)_0,(-)_1)"] \\
\mathbf{1} \arrow[r, "(\mathcal{K},\mathcal{L})"'] & \mathbf{CAT} \times \mathbf{CAT}
\end{tikzcd}
$$

provides the "fibres" for $((-)_0, (-)_1)$. Of course, $\mathcal{K},\mathcal{L}$-***MOD** is then the full sub-2-category of $\mathcal{K},\mathcal{L}$-**MOD** determined by the starred modules, while $\mathcal{K},\mathcal{L}$-**MOD** itself is the (non-full) sub-2-category of **MOD** obtained by fixing $\mathcal{K}$ and $\mathcal{L}$, and by requiring the T_0, T_1, u_0, and u_1 of 2.9 to be identities. Thus to give an arrow $T : \mathcal{M} \longrightarrow \mathcal{N}$ in $\mathcal{K},\mathcal{L}$-**MOD** (or in $\mathcal{K},\mathcal{L}$-***MOD**) is to give functors $T_{K,L} : \mathcal{M}(K,L) \longrightarrow \mathcal{N}(K,L)$ and to give arrows $T_{l,\mu} : l.T\mu \longrightarrow T(l\mu)$ and $T^{\mu,k} : T(\mu k) \longrightarrow T\mu.k$ natural in μ, with these data satisfying what the conditions of Proposition 2.14 become when when we take T_0 and T_1

to be the identities of $\mathcal{K}$ and of $\mathcal{L}$; and to give a transformation $u : T \longrightarrow S : \mathcal{M} \longrightarrow \mathcal{N}$ in $\mathcal{K}, \mathcal{L}$-**MOD** (or in $\mathcal{K}, \mathcal{L}$-***MOD**) is then to give *natural transformations*

$$u_{K,L} : T_{K,L} \longrightarrow S_{K,L} : \mathcal{M}(K, L) \longrightarrow \mathcal{N}(K, L)$$

satisfying what the conditions of Proposition 2.23 become when not only T_0 and T_1 but also u_0 and u_1 are identities: namely the equations $u_{l\mu}.T_{l,\mu} = S_{l,\mu}.lu_\mu$ and $S^{\mu,k}u_{\mu k} = u_\mu k.T^{\mu,k}$.

2.25. From a module $\mathcal{N} : \mathcal{S} \relbar\joinrel\mapstochar\joinrel\rightarrow \mathcal{R}$ together with functors $G : \mathcal{K} \longrightarrow \mathcal{R}$ and $H : \mathcal{L} \longrightarrow \mathcal{S}$, we may construct by *substitution* a new module $G^+\mathcal{N}H : \mathcal{L} \relbar\joinrel\mapstochar\joinrel\rightarrow \mathcal{K}$, for which it is often more convenient to use the alternative notation $\mathcal{N}(G, H) : \mathcal{L} \relbar\joinrel\mapstochar\joinrel\rightarrow \mathcal{K}$. As a 2-variable homomorphism, $\mathcal{N}(G, H) : \mathcal{K}^{op}, \mathcal{L} \longrightarrow \mathbf{CAT}$ is defined to be the result of composing $\mathcal{N} : \mathcal{R}^{op}, \mathcal{S} \longrightarrow \mathbf{CAT}$ with $G^{op}, H : \mathcal{K}^{op}, \mathcal{L} \longrightarrow \mathcal{R}^{op}, \mathcal{S}$. Thus $\mathcal{N}(G, H)(K, L) = \mathcal{N}(GK, HL)$, and so on, from which it is clear that

$$\begin{tikzcd} \mathbf{gr}\mathcal{N}(G, H) \arrow[r] \arrow[d] & \mathbf{gr}\mathcal{N} \arrow[d] \\ \mathcal{K} \times \mathcal{L} \arrow[r, "G \times H"'] & \mathcal{R} \times \mathcal{S} \end{tikzcd}$$

is a pullback. Moreover, $\mathcal{N}(G, H)$ is starred if $\mathcal{N}$ is so. It follows that to give an arrow $(G, T, H) : (\mathcal{K}, \mathcal{M}, \mathcal{L}) \longrightarrow (\mathcal{R}, \mathcal{N}, \mathcal{S})$ in **MOD** (respectively ***MOD**) is to give an arrow $\tilde{T} : \mathcal{M} \longrightarrow \mathcal{N}(G, H)$ in $\mathcal{K}, \mathcal{L}$-**MOD** (respectively $\mathcal{K}, \mathcal{L}$-***MOD**). Of course T and $\tilde{T}$ are related as follows: if (G, T, H) sends the arrow $(k, \Psi, l) : (K, \mu, L) \longrightarrow (K', \mu', L')$ of $\mathbf{gr}\mathcal{M}$ to the arrow $(Gk, \Phi, Hl) : (GK, T\mu, HL) \longrightarrow (GK', T\mu', HL')$ of $\mathbf{gr}\mathcal{N}$, then $\tilde{T}$ sends it to the arrow $(k, \Phi, l) : (K, T\mu, L) \longrightarrow (K', T\mu', L')$ of $\mathbf{gr}\mathcal{N}(G, H)$. An evident notation allows us to regard $\tilde{T}$ as a "square'

$$\begin{tikzcd} \mathcal{L} \arrow[r, "H"] \arrow[d, "\mathcal{M}"', "\shortmid" marking] \arrow[dr, phantom, "\xrightarrow{\tilde{T}}"] & \mathcal{S} \arrow[d, "\mathcal{N}", "\shortmid" marking] \\ \mathcal{K} & \mathcal{R} \arrow[l, "G^+"] \end{tikzcd} \quad ;$$

compare this with the Φ^l of 2.11. The description of $\mathcal{N}(G, H)$ as a pullback also facilitates the observation that an arrow $S : \bar{\mathcal{N}} \longrightarrow \mathcal{N}$ in $\mathcal{R}, \mathcal{S}$-**MOD** (or in $\mathcal{R}, \mathcal{S}$-***MOD**) provides an arrow

$$S_{G,H} : \bar{\mathcal{N}}(G, H) \longrightarrow \mathcal{N}(G, H)$$

in $\mathcal{K}, \mathcal{L}$-**MOD** (or in $\mathcal{K}, \mathcal{L}$-***MOD**) with $S_{G,H}(K, \nu, L) = (K, S\nu, L)$, and so on. Similarly, a transformation $u : \bar{S} \longrightarrow S : \bar{\mathcal{N}} \longrightarrow \mathcal{N}$ in $\mathcal{R}, \mathcal{S}$-**MOD** (or in $\mathcal{R}, \mathcal{S}$-***MOD**) provides a transformation $u_{G,H} : \bar{S}_{G,H} \longrightarrow S_{G,H} : \bar{\mathcal{N}}(G, H) \longrightarrow \mathcal{N}(G, H)$ in $\mathcal{K}, \mathcal{L}$-**MOD** (or in $\mathcal{K}, \mathcal{L}$-***MOD**) with $(u_{G,H})_{K,L} = u_{GK,HL}$. Observe first that these assignments define a 2-functor $G^+(-)H : \mathcal{R}, \mathcal{S}$-**MOD** $\longrightarrow \mathcal{K}, \mathcal{L}$-**MOD** and similarly in the starred case. Next observe that the assignments $(\mathcal{K}, \mathcal{L}) \mapsto \mathcal{K}, \mathcal{L}$-**MOD**, together with the $G^+(-)H$, are *strictly functorial* in $(G,H):(\mathcal{K},\mathcal{L})\longrightarrow(\mathcal{R},\mathcal{S})$ in $(\mathbf{CAT}\times\mathbf{CAT})_0^{op}$ because on *objects* the $G^+(-)H$ are given by *composition* (rather than pullback).. Finally in this context, observe that the case of $(K,L):(\mathbf{1},\mathbf{1})\longrightarrow(\mathcal{K},\mathcal{L})$ shows that all notations of this subsection are consistent with our usage above for components, and that we have "(K,L)-component 2-functors" $\mathcal{K},\mathcal{L}\text{-}\mathbf{MOD}\longrightarrow\mathbf{1},\mathbf{1}\text{-}\mathbf{MOD}=\mathbf{CAT}$ and $\mathcal{K},\mathcal{L}\text{-}^*\mathbf{MOD}\longrightarrow\mathbf{1},\mathbf{1}\text{-}^*\mathbf{MOD}=\mathbf{CAT}$.

2.26. Continuing the considerations above, suppose next that, besides the module $\mathcal{N}:\mathcal{S}\to\!\!\!\!\!+\,\mathcal{R}$ and the functor $G:\mathcal{K}\longrightarrow\mathcal{R}$, we have a natural transformation $s:J\longrightarrow H:\mathcal{L}\longrightarrow\mathcal{S}$. We show there is then an induced arrow $\mathcal{N}(G,s):\mathcal{N}(G,J)\longrightarrow\mathcal{N}(G,H)$ of $\mathcal{K},\mathcal{L}$-modules which sends the object $(K,\nu:GK\to\!\!\!\!\!+\,JL,L)$ of $\mathbf{gr}\mathcal{N}(G,J)$ to the object $(K,sL.\nu:GK\to\!\!\!\!\!+\,HL,L)$ of $\mathbf{gr}\mathcal{N}(G,H)$. Once again, we use the description of $\mathbf{gr}\mathcal{N}(G,H)$ as a pullback, at the beginning of 2.25, to exhibit the requisite functor $\widehat{\mathbf{gr}}\mathcal{N}(G,J)\longrightarrow\widehat{\mathbf{gr}}\mathcal{N}(G,H)$. For this we need a functor $\mathbf{gr}\mathcal{N}(G,J)\longrightarrow\mathbf{gr}\mathcal{N}$ whose effect on $(K,\nu:GK\to\!\!\!\!\!+\,JL,L)$ is $(GK,sL.\nu,HL)$. Let $(k,\Phi,l):(K,\nu,L)\longrightarrow(K',\nu',L')$ be an arrow in $\mathbf{gr}\mathcal{N}(G,J)$ and consider the following diagram in the bicategory $\mathbf{gl}\mathcal{N}$:

$$\begin{tikzcd}
GK \arrow[r, "Gk"] \arrow[d, "\nu"'] \arrow[r, phantom, shift right=6ex, "\Phi"] & GK' \arrow[r, "Gk'"] \arrow[d, "\nu'"] \arrow[r, phantom, shift right=6ex, "\Phi'"] & GK'' \arrow[d, "\nu''"] \\
JL \arrow[r, "Jl"] \arrow[d, "sL"'] & JL' \arrow[r, "Jl'"] \arrow[d, "sL'"] & JL'' \arrow[d, "sL''"] \\
HL \arrow[r, "Hl"'] & HL' \arrow[r, "Hl'"'] & HL''
\end{tikzcd}$$
.

The value at (k,Φ,l) of the required functor is (Gk,Ψ,Hl) where Ψ is the vertical pasting composite of the two left-most squares, seen as a square $Hl.(sL.\nu)\longrightarrow(sL'.\nu').Gk$. Given a further arrow (k',Φ',l') in $\mathbf{gr}\mathcal{N}(G,J)$ as above, we see that the assignment is functorial by appealing to Verity's pasting theorem; for in the diagram above, it makes no difference whether we first paste horizontally and then vertically, or first vertically and then horizontally. So we do have the desired $\mathcal{N}(g,s)$, and its value at the arrow (k,Φ,l) of $\mathbf{gr}\mathcal{N}(G,J)$ is the arrow (k,Ψ,l) of $\mathbf{gr}\mathcal{N}(G,H)$ with Ψ as above. The diagram above also shows that $\mathcal{N}(G,s):\mathcal{N}(G,J)\longrightarrow\mathcal{N}(G,H)$ preserves commutative squares and is thus a strong arrow in $\mathbf{MOD}$. Moreover if $\mathcal{N}$, and hence $\mathcal{N}(G,J)$ and $\mathcal{N}(G,H)$, are starred, it follows from the naturality of mates in [K&S, Section 2] that the arrow Ψ^r in $\mathbf{gr}\mathcal{N}^r$

corresponding to the vertical pasting of the left-most squares above is

$$\begin{tikzcd} GK \arrow[d, "\nu"'] \arrow[r, phantom, "\xrightarrow{\Phi^r}"] & GK' \arrow[l, "Gk^*"'] \arrow[d, "\nu'"] \\ JL \arrow[r, "Jl"] \arrow[d, "sL"'] & JL' \arrow[d, "sL'"] \\ HL \arrow[r, "Hl"'] & HL' \end{tikzcd} .$$

Thus, when $\mathcal{N}$ is starred, $\mathcal{N}(G,s)$ also preserves right squares, and accordingly is a strong right homomorphism. Dually, given $t : G \longrightarrow F$ we define $\mathcal{N}(t,H) : \mathcal{N}(F,H) \longrightarrow \mathcal{N}(G,H)$ by pasting naturality squares on top of squares in $\mathcal{N}$; and $\mathcal{N}(t,H)$ is a strong arrow in **MOD**, and a strong left homomorphism in ***MOD** when $\mathcal{N}$ is starred.

Of course the $\mathcal{N}(G,s)$ are not functorial in s — there are rather isomorphisms

$$\mathcal{N}(G,s's) \xrightarrow{\cong} \mathcal{N}(G,s')\mathcal{N}(G,s),$$

constructed from the ξ for $\mathcal{N}$. A similar remark holds for the $\mathcal{N}(t,H)$ and the ζ, while the η provide isomorphisms

$$\mathcal{N}(G,s)\mathcal{N}(t,H) \xrightarrow{\cong} \mathcal{N}(t,H)\mathcal{N}(G,s).$$

At the end of 2.25 we could have summarized by saying that we have a functor

$$(-,-)\text{-}\mathbf{MOD} : (\mathbf{CAT} \times \mathbf{CAT})_0^{op} \longrightarrow (2\text{-}\mathbf{CAT})_0.$$

We leave it as an exercise for the interested reader to formulate the nature of the arrow with domain $(\mathbf{CAT}^{co} \times \mathbf{CAT})^{op}$, enriching the functor displayed above, which arises from the considerations of 2.26.

2.27. Remark. For the 2-functor $((-)_0,(-)_1) : \mathbf{MOD} \longrightarrow \mathbf{CAT} \times \mathbf{CAT}$ and fixed modules $\mathcal{M}$ and $\mathcal{N}$, consider the effect-on-homs functor

$$\mathbf{MOD}(\mathcal{M},\mathcal{N}) \longrightarrow \mathbf{CAT} \times \mathbf{CAT}((\mathcal{M}_0,\mathcal{M}_1),(\mathcal{N}_0,\mathcal{N}_1)).$$

This can be seen as a span $\mathbf{CAT}(\mathcal{M}_0,\mathcal{N}_0) \longleftarrow \mathbf{MOD}(\mathcal{M},\mathcal{N}) \longrightarrow \mathbf{CAT}(\mathcal{M}_1,\mathcal{N}_1)$, and the considerations of 2.26 can be adapted to show that it is a fibration *from* $\mathbf{CAT}(\mathcal{M}_1,\mathcal{N}_1)$ *to* $\mathbf{CAT}(\mathcal{M}_0,\mathcal{N}_0)$. This is a "two variable" version of the idea that appears in the "local characterization" theorem of 2-fibrations given in [HRM].

2.28. Note that $\mathcal{N}(G,s)_{K,L} : \mathcal{N}(GK,JL) \longrightarrow \mathcal{N}(GK,HL)$ is $\mathcal{N}(GK,sL)$, which for a starred $\mathcal{N}$ has in **CAT** the right adjoint $\mathcal{N}(GK,(sL)^*)$, while we have already noted that $\mathcal{N}(G,s)$ is a right homomorphism. It will follow by Theorem 2.39 that for starred $\mathcal{N}$ the $\mathcal{N}(G,s)$ have right adjoints in ***MOD** — in fact in $\mathcal{K},\mathcal{L}$-***MOD**. However, even now, we can define an arrow $\mathcal{N}(G,s^*) : \mathcal{N}(G,H) \longrightarrow \mathcal{N}(G,J)$ of $\mathcal{K},\mathcal{L}$-modules by prescribing its effect on an arrow (k,Φ,l) in $\mathcal{N}(G,H)$ to be the arrow in $\mathcal{N}(G,J)$ suggested by the *symbolic* diagram:

$$
\begin{tikzcd}
GK \arrow[r, "Gk"] \arrow[d, "\nu"'] \arrow[dr, phantom, "\xrightarrow{\Phi}"] & GK' \arrow[d, "\nu'"] \\
HL \arrow[r, "Hl"] \arrow[d, "(sL)^*"'] \arrow[dr, phantom, "\longrightarrow"] & HL' \arrow[d, "(sL')^*"] \\
JL \arrow[r, "Jl"'] & JL'
\end{tikzcd}
\quad .
$$

We say that this diagram is "symbolic" because it is not a diagram in any of the "**gl**" bicategories that have been introduced. (It involves composites of the form $x.\nu$ and $y^*.\nu$ for x and y arrows of $\mathcal{S}$. For this to make sense in our framework we would require a *category* containing both $\mathcal{S}$ and $\mathcal{S}^{op}$ which acts on the $\mathcal{N}(R,S)$ — but what is one to make then of the un-named transformation in the lower square?) The point here is that the equality natural transformation $sL'.Jl(-) \xrightarrow{=} Hl.sL(-)$ between the action-functors has the mate $Jl.((sL)^*(-) \longrightarrow (sL')^*.Hl(-)$ whose ν-component enables us to give a well-defined meaning to the displayed pasting composite. Using the mate calculus of [K&S] it is straightforward to establish a four-square interchange law, similar to that in 2.26, showing the assignment to be functorial. Moreover, contemplation of the squares in 2.11 and their relationship via mating shows that the $\mathcal{N}(G,s^*)$ preserve exact squares and left squares and are thus by 2.20 *strong left homomorphisms in $\mathcal{K},\mathcal{L}$-***MOD**. Dually, given $t : G \longrightarrow F$, we obtain a *strong right homomorphism $\mathcal{N}(t^*,H) : \mathcal{N}(G,H) \longrightarrow \mathcal{N}(F,H)$ in $\mathcal{K},\mathcal{L}$-***MOD**.

2.29. We turn now to the study of adjunctions in **MOD** and in the 2-categories derived from it. An adjunction $x,y : S \dashv T : \mathcal{M} \longrightarrow \mathcal{N}$ in **MOD**, as in any 2-category, consists of arrows $T : \mathcal{M} \longrightarrow \mathcal{N}$ and $S : \mathcal{N} \longrightarrow \mathcal{M}$, along with transformations $x : 1 \longrightarrow TS$ and $y : ST \longrightarrow 1$ satisfying the triangular equations $Ty \cdot xT = 1$ and $yS \cdot Sx = 1$. A direct expression of these triangular equations in terms of the primitive data for T and for S as given in Proposition 2.14 and the primitive data for x and for y given in Proposition 2.23 would clearly be rather complicated; yet in suitable cases we can often study adjunctions without explicit reference to the triangular equations. It is classical that adjunctions in the 2-category **CAT** are particularly simple to study: to find T given S, one has only to find for each M a representation $\mathcal{N}(-,TM)$ of the presheaf $\mathcal{M}(S-,M)$, whereupon the funtoriality of T, along with the existence and naturality of the unit and the counit and their satisfaction of the triangular equations, is automatic. Next, a similar simplification is available for adjunctions in $[\Lambda, \mathbf{CAT}]$, which reduce pointwise to adjunctions in $\mathbf{CAT}$; so that, finally, we have a corresponding simplification for adjunctions in the full sub-2-categories $\mathbf{MOD}$ and $^*\mathbf{MOD}$ of $[\Lambda, \mathbf{CAT}]$. It is our aim in the rest of this section to use these and other simplifications to provide, in terms of the primitive data for an arrow S in $\mathbf{MOD}$ or in $^*\mathbf{MOD}$, simple necessary and sufficient conditions for it to admit a right adjoint.

Observe first though that adjunctions in $\mathbf{MOD}$ which lie in some $\mathcal{K}, \mathcal{L}$-$\mathbf{MOD}$ are considerably simpler than general ones, because as we noted in 2.24 the data for a transformation in $\mathcal{K}, \mathcal{L}$-$\mathbf{MOD}$ consist merely of a family of natural transformations. In particular, for starred $\mathcal{N}$ and with the definitions given in 2.26 and 2.28, the reader will have no difficulty in supplying the unit and the counit for an adjunction $\mathcal{N}(G, s) \dashv \mathcal{N}(G, s^*)$ in $\mathcal{K}, \mathcal{L}$-$^*\mathbf{MOD}$ and in verifying the triangular equations in this case. Duality then provides a further adjunction $\mathcal{N}(t^*, H) \dashv \mathcal{N}(t, H)$ in $\mathcal{K}, \mathcal{L}$-$^*\mathbf{MOD}$.

Our goal in the next five numbered items will be to reduce an adjunction in $\mathbf{MOD}$ to a pair of adjunctions in $\mathbf{CAT}$ and an adjunction in a suitable fibre of $((-)_0, (-)_1) : \mathbf{MOD} \longrightarrow \mathbf{CAT} \times \mathbf{CAT}$. (This should be compared with Lemma 2.6.)

2.30. Because $((-)_0, (-)_1) : \mathbf{MOD} \longrightarrow \mathbf{CAT} \times \mathbf{CAT}$ is a 2-functor it follows that an adjunction $x, y : S \dashv T : (\mathcal{K}, \mathcal{M}, \mathcal{L}) \longrightarrow (\mathcal{R}, \mathcal{N}, \mathcal{S})$ in $\mathbf{MOD}$ gives rise to adjunctions $x_0, y_0 : S_0 \dashv T_0 : \mathcal{K} \longrightarrow \mathcal{R}$ and $x_1, y_1 : S_1 \dashv T_1 : \mathcal{L} \longrightarrow \mathcal{S}$ in $\mathbf{CAT}$. We keep fixed here these adjunctions in $\mathbf{CAT}$ and consider various functors $T : \mathbf{gr}\mathcal{M} \longrightarrow \mathbf{gr}\mathcal{N}$ for which (T_0, T, T_1) is an arrow $(\mathcal{K}, \mathcal{M}, \mathcal{L}) \longrightarrow (\mathcal{R}, \mathcal{N}, \mathcal{S})$ in $\mathbf{MOD}$, and functors $S : \mathbf{gr}\mathcal{N} \longrightarrow \mathbf{gr}\mathcal{M}$ for which (S_0, S, S_1) is an arrow $(\mathcal{R}, \mathcal{N}, \mathcal{S}) \longrightarrow (\mathcal{K}, \mathcal{M}, \mathcal{L})$ in $\mathbf{MOD}$. To give such a T is by 2.25 to give an arrow $\tilde{T} : \mathcal{M} \longrightarrow \mathcal{N}(T_0, T_1)$ in $\mathcal{K}, \mathcal{L}$-$\mathbf{MOD}$; and this gives rise, also as in 2.25, to an arrow $\tilde{T}_{S_0,1} : \mathcal{M}(S_0, 1) \longrightarrow \mathcal{N}(T_0 S_0, T_1)$ in $\mathcal{R}, \mathcal{L}$-$\mathbf{MOD}$, and hence using 2.26 to an arrow $\tau(T) : \mathcal{M}(T_0, 1) \longrightarrow \mathcal{N}(1, T_1)$ in $\mathcal{R}, \mathcal{L}$-$\mathbf{MOD}$, defined as the first of the composites below. Similarly, such an S gives rise to the second composite $\sigma(S)$ below in $\mathcal{R}, \mathcal{L}$-$\mathbf{MOD}$.

$$\begin{tikzcd}
\mathcal{M}(S_0,1) \arrow[rr, "\tau(T)"] \arrow[dr, "\tilde{T}_{S_0,1}"'] & & \mathcal{N}(1,T_1) \\
& \mathcal{N}(T_0 S_0, T_1) \arrow[ur, "\mathcal{N}(x_0, T_1)"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{N}(1,T_1) \arrow[rr, "\sigma(S)"] \arrow[dr, "\tilde{S}_{1,T_1}"'] & & \mathcal{M}(S_0,1) \\
& \mathcal{M}(S_0, S_1 T_1) \arrow[ur, "\mathcal{M}(S_0, y_1)"'] &
\end{tikzcd}$$

On the other hand, starting with arbitrary arrows $\tau : \mathcal{M}(S_0, 1) \longrightarrow \mathcal{N}(1, T_1)$ and $\sigma : \mathcal{N}(1, T_1) \longrightarrow \mathcal{M}(S_0, 1)$ in $\mathcal{R}, \mathcal{L}$-$\mathbf{MOD}$, we obtain arrows

$$(T_0, T(\tau), T_1) : (\mathcal{K}, \mathcal{M}, \mathcal{L}) \longrightarrow (\mathcal{R}, \mathcal{N}, \mathcal{S}) \quad \text{and} \quad (S_0, S(\sigma), S_1) : (\mathcal{R}, \mathcal{N}, \mathcal{S}) \longrightarrow (\mathcal{K}, \mathcal{M}, \mathcal{L})$$

in **MOD** by taking $(T(\tau))\check{}$ and $(S(\sigma))\check{}$ to be the composites

$$\begin{tikzcd} \mathcal{M} \arrow[rr, "(T(\tau))\check{}"] \arrow[dr, "\mathcal{M}(y_0,1)"'] & & \mathcal{N}(T_0,T_1) \\ & \mathcal{M}(S_0T_0,1) \arrow[ur, "\tau_{T_0,1}"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{N} \arrow[rr, "(S(\sigma))\check{}"] \arrow[dr, "\mathcal{N}(1,x_1)"'] & & \mathcal{M}(S_0,S_1) \\ & \mathcal{N}(1,T_1S_1) \arrow[ur, "\sigma_{1,S_1}"'] & \end{tikzcd}$$

in $\mathcal{K},\mathcal{L}$-**MOD** and $\mathcal{R},\mathcal{S}$-**MOD** respectively. Recall from 2.15 the meanings of **R**-homomorphism and **L**-homomorphism.

2.31. Proposition. *(i) For any such S, there is in* **MOD** *a canonical transformation of the form*

$$(1,m,1) : (S_0,S,S_1) \longrightarrow (S_0,S(\sigma(S)),S_1) : (\mathcal{R},\mathcal{N},\mathcal{S}) \longrightarrow (\mathcal{K},\mathcal{M},\mathcal{L}),$$

and it is invertible when S is an **R**-*homomorphism;*
(ii) For any such T, there is in **MOD** *a canonical transformation of the form*

$$(1,n,1) : (T_0,T(\tau(T)),T_1) \longrightarrow (T_0,T,T_1) : (\mathcal{K},\mathcal{M},\mathcal{L}) \longrightarrow (\mathcal{R},\mathcal{N},\mathcal{S}),$$

and it is invertible when T is an **L**-*homomorphism;*
*(iii) For any such σ, there is in $\mathcal{R},\mathcal{L}$-***MOD** *a canonical transformation*

$$\sigma(S(\sigma)) \longrightarrow \sigma : \mathcal{N}(1,T_1) \longrightarrow \mathcal{M}(S_0,1),$$

and it is invertible when $\sigma$is an **R**-*homomorphism;*
*(iv) For any such τ, there is in $\mathcal{R},\mathcal{L}$-***MOD** *a canonical transformation*

$$\tau \longrightarrow \tau(T(\tau)) : \mathcal{M}(S_0,1) \longrightarrow \mathcal{N}(1,T_1),$$

and it is invertible when τ is an **L**-*homomorphism.*

Proof. The proofs being all of the same kind, it suffices to give that for (i). However $S(\sigma(S))$ sends $\nu : X \longrightarrow\!\!\!\!\!+ A$ to

$$(\sigma(S))(x_1A.\nu) = y_1S_1A.S(x_1A.\nu) : S_0X \longrightarrow\!\!\!\!\!+ S_1A,$$

and we have in $\mathcal{M}(S_0X,S_1A)$ the arrow m_ν given by the composite

$$\begin{tikzcd} S\nu \arrow[r, equal] & 1.S\nu \arrow[r, equal] & (y_1S_1A.S_1x_1A).S\nu \arrow[d, "\xi"] \\ (S(\sigma(S)))\nu & y_1S_1A.S(x_1A.\nu) \arrow[l, equal] & y_1S_1A.(S_1x_1A.S\nu) \arrow[l, "y_1S_1A.S_{x_1A,\nu}"'] \end{tikzcd} ,$$

which is invertible when all the $S_{s,\nu}$ are so. That m_ν is natural with respect to arrows $(r,\Phi,s) : (X,\nu,A) \longrightarrow (X',\nu',A')$ in $\mathbf{gr}\mathcal{N}$ follows immediately from the naturality of ξ and part (iii) of Proposition 2.14. ∎

From 2.26 we have that $\mathcal{N}(x_0, T_1)$, $\mathcal{M}(S_0, y_1)$, $\mathcal{M}(y_0, 1)$, and $\mathcal{N}(1, x_1)$ all preserve commutative squares while substitution as defined in 2.25 evidently takes **R**-homomorphisms to **R**-homomorphisms and **L**-homomorphisms to **L**-homomorphisms. It follows that we may add to the above:

2.32. LEMMA.

*(i) If S is an **R**-homomorphism then so is $\sigma(S)$.*

*(ii) If T is an **L**-homomorphism then so is $\tau(T)$.*

*(iii) If σ is an **R**-homomorphism then so is $S(\sigma)$.*

*(iv) If τ is an **L**-homomorphism then so is $T(\tau)$.* ∎

2.33. PROPOSITION. *(i) Given S and T and an adjunction*

$$(S_0, S, S_1) \dashv (T_0, T, T_1) : (\mathcal{K}, \mathcal{M}, \mathcal{L}) \longrightarrow (\mathcal{R}, \mathcal{N}, \mathcal{S})$$

in **MOD**, *we may construct an adjunction*

$$\sigma \dashv \tau : \mathcal{M}(S_0, 1) \longrightarrow \mathcal{N}(1, T_1)$$

in $\mathcal{R}, \mathcal{L}$-**MOD**, *where $\sigma = \sigma(S)$ and $\tau = \tau(T)$.*
(ii) Given σ and τ and an adjunction

$$\sigma \dashv \tau : \mathcal{M}(S_0, 1) \longrightarrow \mathcal{N}(1, T_1)$$

in $\mathcal{R}, \mathcal{L}$-**MOD**, *we may construct an adjunction*

$$(S_0, S, S_1) \dashv (T_0, T, T_1) : (\mathcal{K}, \mathcal{M}, \mathcal{L}) \longrightarrow (\mathcal{R}, \mathcal{N}, \mathcal{S})$$

in **MOD**, *where $S = S(\sigma)$ and $T = T(\tau)$.*

PROOF. First suppose given an adjunction $(S_0, S, S_1) \dashv (T_0, T, T_1)$ as in (i). This means that, given $\nu : X \longrightarrow\!\!\!\!\!\!+\;\, A$, $\mu : K \longrightarrow\!\!\!\!\!\!+\;\, L$, $r : X \longrightarrow T_0 K$, and $l : S_1 A \longrightarrow L$, there is a natural bijection between transformations Φ in the left diagram, and transformations Ψ in the right diagram, of the pair

$$\begin{tikzcd} S_0 X \arrow[r, "\alpha_0^{-1}(r)"] \arrow[d, "S\nu"', "+" marking] & K \arrow[d, "\mu", "+" marking] \\ S_1 A \arrow[r, "l"'] & L \arrow[from=1-1, to=2-2, phantom, "\xrightarrow{\Phi}"] \end{tikzcd} , \qquad \begin{tikzcd} X \arrow[r, "r"] \arrow[d, "\nu"', "+" marking] & T_0 K \arrow[d, "T\mu", "+" marking] \\ A \arrow[r, "\alpha_1(l)"'] & T_1 L \arrow[from=1-1, to=2-2, phantom, "\xrightarrow{\Psi}"] \end{tikzcd} ,$$

where we have used α_0 and α_1 to denote the hom-set bijections for the adjunctions $S_0 \dashv T_0$ and $S_1 \dashv T_1$ respectively. Now take A here to be of the form T_1L' and K to be of the form S_0X' and take l and r to be the respective composites

$$S_1T_1L' \xrightarrow{y_1L'} L' \xrightarrow{l'} L \quad , \quad X \xrightarrow{r'} X' \xrightarrow{x_0X'} T_0S_0X'.$$

Since $y_1L'.S\nu = \sigma(S)\nu$ and $T\mu.x_0X' = \tau(T)\mu$, while

$$\alpha_1(l'.y_1L') = T_1l' \quad \text{and} \quad \alpha_0^{-1}(x_0X'.r') = S_0r',$$

absorbing isomorphisms of the forms ξ and ζ produces a bijection between transformations Φ and Ψ in the diagrams

$$\begin{tikzcd} S_0X \arrow[r, "S_0r'"] \arrow[d, "\sigma(S)\nu"', "|" marking] & S_0X' \arrow[d, "\mu", "|" marking] \\ L' \arrow[r, "l'"'] & L \arrow[from=1-1, to=2-2, phantom, "\xrightarrow{\Phi}"] \end{tikzcd} \quad , \quad \begin{tikzcd} X \arrow[r, "r'"] \arrow[d, "\nu"', "|" marking] & X' \arrow[d, "\tau(T)\mu", "|" marking] \\ T_1L' \arrow[r, "T_1l'"'] & T_1L' \arrow[from=1-1, to=2-2, phantom, "\xrightarrow{\Psi}"] \end{tikzcd} \quad .$$

Since it is straightforward to verify that this latter bijection is natural for arrows $\mu \longrightarrow \mu'$ in $\mathbf{gr}\mathcal{M}(S_0, 1)$ and $\nu \longrightarrow \nu'$ in $\mathbf{gr}\mathcal{N}(1, T_1)$, given that the former one is natural for arrows $\mu \longrightarrow \mu'$ in $\mathbf{gr}\mathcal{M}$ and $\nu \longrightarrow \nu'$ in $\mathbf{gr}\mathcal{N}$, we have indeed the desired adjunction $\sigma(S) \dashv \tau(T)$. Next suppose instead that we have an adjunction $\sigma \dashv \tau$ as in (ii); so that given $\nu : X \longrightarrow\!\!\!\!\!\!+\ T_1L'$, $\mu : S_0X' \longrightarrow\!\!\!\!\!\!+\ L$, $r' : X \longrightarrow X'$, and $l' : L' \longrightarrow L$, there is a natural bijection between transformations Φ and Ψ as in the diagrams above, but with $\sigma(S)$ replaced by σ and $\tau(T)$ replaced by τ. Now take ν and μ here to be the respective composites

$$X \overset{\bar{\nu}}{\longrightarrow\!\!\!\!\!\!+}\ A \xrightarrow{x_1A} T_1S_1A \quad , \quad S_0T_0K \xrightarrow{y_0K} K \overset{\bar{\mu}}{\longrightarrow\!\!\!\!\!\!+}\ L.$$

Since $\sigma(x_1A.\bar{\nu}) = S(\sigma)\bar{\nu}$ and $\tau(\bar{\mu}.y_0K) = T(\tau)\bar{\mu}$, while $y_0K.S_0r' = \alpha_0^{-1}(r')$ and $T_1l'.x_1A = \alpha_1(l')$, we have a bijection between transformations Φ and Ψ in the diagrams

$$\begin{tikzcd} S_0X \arrow[r, "\alpha_0^{-1}(r')"] \arrow[d, "S(\sigma)\bar{\nu}"', "|" marking] & K \arrow[d, "\bar{\mu}", "|" marking] \\ S_1A \arrow[r, "l'"'] & L \arrow[from=1-1, to=2-2, phantom, "\xrightarrow{\Phi}"] \end{tikzcd} \quad , \quad \begin{tikzcd} X \arrow[r, "r'"] \arrow[d, "\bar{\nu}"', "|" marking] & T_0K \arrow[d, "T(\tau)\bar{\mu}", "|" marking] \\ A \arrow[r, "\alpha_1(l')"'] & T_1L \arrow[from=1-1, to=2-2, phantom, "\xrightarrow{\Psi}"] \end{tikzcd} \quad ;$$

and since once again the bijection is easily verified to be natural in the appropriate sense, we have the desired adjunction $S(\sigma) \dashv T(\tau)$. ∎

2.34. Starting from an adjunction $(S_0, S, S_1) \dashv (T_0, T, T_1)$, we may construct by Proposition 2.33 (i) an adjunction $\sigma(S) \dashv \tau(T)$, and then by Proposition 2.33 (ii) an adjunction $(S_0, S(\sigma(S)), S_1) \dashv (T_0, T(\tau(T)), T_1)$. On the other hand, we have from Propositions 2.31 and 2.16, along with Lemma 2.32 the isomorphisms $m : S \longrightarrow S(\sigma(S))$ and $n : T(\tau(T)) \longrightarrow T$. Comparing the construction of these in the proof of Proposition 2.31 with the constructions in the proof of Proposition 2.33, we easily verify the commutativity of

$$\begin{tikzcd}
\mathbf{gr}\mathcal{M}(S(\sigma(S))\nu, \mu) \arrow[r, "\alpha''"] \arrow[d, "\mathbf{gr}\mathcal{M}(m_\nu, \mu)"'] & \mathbf{gr}\mathcal{N}(\nu, T(\tau(T))\mu) \arrow[d, "\mathbf{gr}\mathcal{N}(\nu, n_\mu)"] \\
\mathbf{gr}\mathcal{M}(S\nu, \mu) \arrow[r, "\alpha"'] & \mathbf{gr}\mathcal{N}(\nu, T\mu)
\end{tikzcd},$$

where α is the hom-set bijection for the adjunction $S \dashv T$ and α'' is its analogue for the adjunction $S(\sigma(S)) \dashv T(\tau(T))$. In other words, n and m are mates under the two adjunctions. There is a corresponding result when we start from an adjunction $\sigma \dashv \tau$ and form in two steps the adjunction $\sigma(S(\sigma)) \dashv \tau(T(\tau))$. What we have therefore, for fixed $x_i, y_i : S_i \dashv T_i$, $i = 0, 1$, is an *equivalence* between the category of those adjunctions of the form

$$(x_0, x, x_1), (y_0, y, y_1) : (S_0, S, S_1) \dashv (T_0, T, T_1) : (\mathcal{K}, \mathcal{M}, \mathcal{L}) \longrightarrow (\mathcal{R}, \mathcal{N}, \mathcal{S})$$

in $\mathbf{MOD}$ and the category of all adjunctions

$$\sigma \dashv \tau : \mathcal{M}(S_0, 1) \longrightarrow \mathcal{N}(1, T_1)$$

in $\mathcal{R}, \mathcal{L}$-$\mathbf{MOD}$. This completes the goal we set at the end of 2.29.

2.35. For adjunctions in $^*\mathbf{MOD}$ the reduction to an adjunction in a fibre can be accomplished in three other ways. In $^*\mathbf{MOD}$, in the constructions of 2.30, $\mathcal{N}(x_0, T_1)$ has by 2.29 a left adjoint $\mathcal{N}(x_0^*, T_1)$ in $\mathcal{R}, \mathcal{L}$-$^*\mathbf{MOD}$, while $\mathcal{M}(S_0, y_1)$ has a right adjoint $\mathcal{M}(S_0, y_1^*)$ there. Similarly, in the second set of diagrams of 2.30, we have $\mathcal{M}(y_0^*, 1) \dashv \mathcal{M}(y_0, 1)$ in $\mathcal{K}, \mathcal{L}$-$^*\mathbf{MOD}$ and $\mathcal{N}(1, x_1) \dashv \mathcal{N}(1, x_1^*)$ in $\mathcal{R}, \mathcal{S}$-$^*\mathbf{MOD}$. Further, if we are given an adjunction $\sigma \dashv \tau : \mathcal{M}(S_0, 1) \longrightarrow \mathcal{N}(1, T_1)$, we also have $\sigma_{T_0,1} \dashv \tau_{T_0,1} : \mathcal{M}(S_0 T_0, 1) \longrightarrow \mathcal{N}(T_0, T_1)$, since the *substitution* $T_0^+(-)1$ is a 2-functor as explained in 2.25. So from the last two diagrams of 2.30 we conclude that:

If $\sigma \dashv \tau : \mathcal{M}(S_0, 1) \longrightarrow \mathcal{N}(1, T_1)$ *in* $\mathcal{R}, \mathcal{L}$-$^*\mathbf{MOD}$*, then*

$$(T(\tau))\hat{} \dashv (T(\tau))\check{} : \mathcal{M} \longrightarrow \mathcal{N}(T_0, T_1) \text{ in } \mathcal{K}, \mathcal{L}\text{-}^*\mathbf{MOD}$$

and

$$(S(\sigma))\check{} \dashv (S(\sigma))\hat{} : \mathcal{M}(S_0, S_1) \longrightarrow \mathcal{N} \text{ in } \mathcal{R}, \mathcal{S}\text{-}^*\mathbf{MOD},$$

where $(T(\tau))\hat{}$ *and* $(S(\sigma))\check{}$ *are the composites*

$$\begin{tikzcd}
\mathcal{M} & & \mathcal{N}(T_0,T_1) \arrow[ll, "(T(\tau))\hat{}"'] \arrow[dl, "\sigma_{T_0,1}"] \\
& \mathcal{M}(S_0T_0,1) \arrow[ul, "\mathcal{M}(y_0^*,1)"] &
\end{tikzcd}
,\qquad
\begin{tikzcd}
\mathcal{N} & & \mathcal{M}(S_0,S_1) \arrow[ll, "(S(\sigma))\check{}"'] \arrow[dl, "\tau_{1,S_1}"] \\
& \mathcal{N}(1,T_1S_1) \arrow[ul, "\mathcal{N}(1,x_1^*)"] &
\end{tikzcd}
.$$

In the other direction, from the first two diagrams of 2.30 we conclude that:

If $\hat{T} \dashv \tilde{T}$ *in* $\mathcal{K},\mathcal{L}$-***MOD** *[respectively, if* $\tilde{S} \dashv \check{S}$ *in* $\mathcal{R},\mathcal{S}$-***MOD***], then* $\tau(T)$ *has the left adjoint* σ *[respectively* $\sigma(S)$ *has the right adjoint* τ*] in* $\mathcal{R},\mathcal{L}$-***MOD***, where* σ *and* τ *are the composites*

$$\begin{tikzcd}
\mathcal{M}(S_0,1) & & \mathcal{N}(1,T_1) \arrow[ll, "\sigma"'] \arrow[dl, "\mathcal{N}(x_0^*,T_1)"] \\
& \mathcal{N}(T_0S_0,T_1) \arrow[ul, "\hat{T}_{S_0,1}"] &
\end{tikzcd}
,\qquad
\begin{tikzcd}
\mathcal{N}(1,T_1) & & \mathcal{M}(S_0,1) \arrow[ll, "\tau"'] \arrow[dl, "\mathcal{M}(S_0,y_1^*)"] \\
& \mathcal{M}(S_0,S_1T_1) \arrow[ul, "\check{S}_{1,T_1}"] &
\end{tikzcd}
.$$

In the starred case we can also use the duality $(-)^{op}$ to get from $(S_0,S,S_1) \dashv (T_0,T,T_1)$ the further adjunction $(S_0,S,S_1)^{op} \dashv (T_0,T,T_1)^{op}$ in ***MOD** which as in 2.30 and the items following it gives rise to $\sigma^*(S) \dashv \tau^*(T) : \mathcal{M}(1,S_1) \longrightarrow \mathcal{N}(T_0,1)$ in $\mathcal{K},\mathcal{S}$-***MOD**, where $\sigma^*(S)$ and $\tau^*(T)$ are given by

$$\begin{tikzcd}
\mathcal{N}(T_0,1) \arrow[rr, "\sigma^*(S)"] \arrow[dr, "\tilde{S}_{T_0,1}"'] & & \mathcal{M}(1,S_1) \\
& \mathcal{M}(S_0T_0,S_1) \arrow[ur, "\mathcal{M}(y_0^*,S_1)"'] &
\end{tikzcd}
,\qquad
\begin{tikzcd}
\mathcal{M}(1,S_1) \arrow[rr, "\tau^*(T)"] \arrow[dr, "\tilde{T}_{1,S_1}"'] & & \mathcal{N}(T_0,1) \\
& \mathcal{N}(T_0,T_1S_1) \arrow[ur, "\mathcal{N}(T_0,x_1^*)"'] &
\end{tikzcd}
,$$

and from which we recover S and T to within isomorphism by

$$\begin{tikzcd}
\mathcal{N} \arrow[rr, "(S(\sigma^*))\check{}"] \arrow[dr, "\mathcal{N}(x_0^*,1)"'] & & \mathcal{M}(S_0,S_1) \\
& \mathcal{N}(T_0S_0,1) \arrow[ur, "\sigma^*_{S_0,1}"'] &
\end{tikzcd}
,\qquad
\begin{tikzcd}
\mathcal{M} \arrow[rr, "(T(\tau^*))\check{}"] \arrow[dr, "\mathcal{M}(1,y_1^*)"'] & & \mathcal{N}(T_0,T_1) \\
& \mathcal{M}(1,S_1T_1) \arrow[ur, "\tau^*_{1,T_1}"'] &
\end{tikzcd}
.$$

Using the isomorphisms m and n of 2.31 that we have in the presence of adjunctions, various pastings of the triangles in 2.30 with those above provide further isomorphisms between the arrows of ***MOD** under consideration. Note too that in the starred context

Lemma 2.32 can be strengthened by the systematic replacement of '**R**-homomorphism' by 'right homomorphism' and '**L**-homomorphism' by 'left homomorphism'.

We now have, in the context of 2.34 and 2.35, *equivalences* between the categories of adjunctions $(S_0, S, S_1) \dashv (T_0, T, T_1)$, $\sigma \dashv \tau$, $\tilde{S} \dashv \check{S}$, $\sigma^* \dashv \tau^*$, and $\hat{T} \dashv \tilde{T}$. It is convenenient to record that starting with $(S_0, S, S_1) \dashv (T_0, T, T_1)$ we can express $\hat{T}$ as the composite:

$$\begin{tikzcd}
\mathcal{N}(T_0, T_1) \arrow[r, "\hat{T}"] \arrow[d, "\tilde{S}_{T_0,T_1}"'] & \mathcal{M} \\
\mathcal{M}(S_0T_0, S_1T_1) \arrow[r, "\mathcal{M}(S_0T_0, y_1)"'] & \mathcal{M}(S_0T_0, 1). \arrow[u, "\mathcal{M}(y_0^*, 1)"']
\end{tikzcd}$$

In particular we have the following:

2.36. THEOREM. *Given adjunctions $x_0, y_0 : S_0 \dashv T_0 : \mathcal{K} \longrightarrow \mathcal{R}$ and $x_1, y_1 : S_1 \dashv T_1 : \mathcal{L} \longrightarrow \mathcal{S}$ in* **CAT**, *an arrow $(S_0, S, S_1) : (\mathcal{R}, \mathcal{N}, \mathcal{S}) \longrightarrow (\mathcal{K}, \mathcal{M}, \mathcal{L})$*

(i) in **MOD** *has a right adjoint if and only if $\sigma(S) : \mathcal{N}(1, T_1) \longrightarrow \mathcal{M}(S_0, 1)$, as defined in the second triangle of 2.30, has a right adjoint τ in $\mathcal{R}, \mathcal{L}$-*__MOD__*; and then $(S_0, S, S_1) \dashv (T_0, T, T_1)$, where $\tilde{T}$ is the $(T(\tau))\check{}$ defined in the third triangle of 2.30;*

(ii) in ***MOD** *has a right adjoint if and only if $\tilde{S} : \mathcal{N} \longrightarrow \mathcal{M}(S_0, S_1)$ has a right adjoint $\check{S}$ in $\mathcal{R}, \mathcal{S}$-**__MOD__*; and then $(S_0, S, S_1) \dashv (T_0, T, T_1)$, where $\tilde{T}$ is given by*

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "\tilde{T}"] \arrow[d, "\mathcal{M}(y_0, 1)"'] & \mathcal{N}(T_0, T_1) \\
\mathcal{M}(S_0T_0, 1) \arrow[r, "\mathcal{M}(S_0T_0, y_1^*)"'] & \mathcal{M}(S_0T_0, S_1T_1) \arrow[u, "\check{S}_{T_0,T_1}"']
\end{tikzcd} .$$

■

2.37. We turn now to adjunctions $x, y : S \dashv T : \mathcal{M} \longrightarrow \mathcal{N}$ in some $\mathcal{K}, \mathcal{L}$-**MOD** or in some $\mathcal{K}, \mathcal{L}$-***MOD**. Given such an adjunction, it follows from the concluding remarks of 2.24 that we have adjunctions $S_{K,L} \dashv T_{K,L} : \mathcal{M}(K, L) \longrightarrow \mathcal{N}(K, L)$ in **CAT**. The right adjoint T in $\mathcal{K}, \mathcal{L}$-**MOD** is an **L**-homomorphism, and a left homomorphism in the starred case. First note that, from the general theory of KZ-doctrines, we can calculate $(T^{\mu,k})^{-1}$ for $k : K' \longrightarrow K$ and $\mu : K \longrightarrow\!\!\!\!\!\!+\;\, L$ explicitly; it turns out to be $t^{\mu,k}$ where

$$t^{\mu,k} = (T\mu.k \xrightarrow{x_{T\mu.k}} TS(T\mu.k) \xrightarrow{T(S^{T\mu,k})} T(ST\mu.k) \xrightarrow{T(y_{\mu k})} T(\mu k)).$$

(To help the reader unfamiliar with the general theory, we recall the essential ideas below in an appendix labelled as Section 5.) Observe now that the natural transformation

$$t^{-,k} : \mathcal{N}(k, L)T_{K,L} \longrightarrow T_{K',L}\mathcal{M}(k, L) : \mathcal{M}(K, L) \longrightarrow \mathcal{N}(K', L)$$

with the $t^{\mu,k}$ as components is the mate, under the adjunctions $S_{K',L} \dashv T_{K',L}$ and $S_{K,L} \dashv T_{K,L}$, of

$$S^{-,k} : S_{K',L}\mathcal{N}(k,L) \longrightarrow \mathcal{M}(k,L)S_{K,L} : \mathcal{N}(K',L) \longrightarrow \mathcal{M}(K,L).$$

Thus given $S : \mathcal{N} \longrightarrow \mathcal{M}$ in $\mathcal{K},\mathcal{L}$-**MOD** and adjunctions $S_{K,L} \dashv T_{K,L}$ in **CAT**, we can, whether or not S has a right adjoint in $\mathcal{K},\mathcal{L}$-**MOD**, define the $t^{-,k}$ to be these mates, and the $t^{\mu,k}$ to be their components.

2.38. LEMMA. *If $S : \mathcal{N} \longrightarrow \mathcal{M}$ in $\mathcal{K},\mathcal{L}$-***MOD** and there are given adjunctions $S_{K,L} \dashv T_{K,L}$ in* **CAT** *then to say that each $t^{\mu,k}$ is invertible is equally to say that each S_{μ,k^*} is invertible. With the same hypotheses, S is a right homomorphism if and only if S is an* **R***-homomorphism and the $t^{\mu,k}$ are invertible.*

PROOF. The second sentence is a trivial consequence of the first. Replacing the domain and codomain of $t^{-,k}$ by their left adjoints gives a transformation

$$\mathcal{M}(k^*,L)S_{K',L} \longrightarrow S_{K',L}\mathcal{N}(k^*,L),$$

which by general principles must be the mate under $\mathcal{M}(k^*,L) \dashv \mathcal{M}(k,L)$ and $\mathcal{N}(k^*,L) \dashv \mathcal{N}(k,L)$ of $S^{-,k}$ and which therefore by 2.17 is S_{-,k^*}. It follows that each $t^{\mu,k}$ is invertible if and only if each S_{μ,k^*} is so. ∎

2.39. THEOREM. *An arrow $S : \mathcal{N} \longrightarrow \mathcal{M}$ in $\mathcal{K},\mathcal{L}$-**MOD** has a right adjoint if and only if the following three conditions are satisfied:*

*(i) S is an **R**-homomorphism;*

(ii) for each K and L, the functor $S_{K,L} : \mathcal{N}(K,L) \longrightarrow \mathcal{M}(K,L)$ has a right adjoint $T_{K,L}$, with counit say $y_{K,L} : S_{K,L}T_{K,L} \longrightarrow 1$ (whose components we write as $y_\mu : ST\mu \longrightarrow \mu$);

(i′) the arrows $t^{\mu,k} : T\mu.k \longrightarrow T(\mu k)$, corresponding under the adjunctions in (ii) to the composite $S(T\mu.k) \xrightarrow{S^{T\mu,k}} ST\mu.k \xrightarrow{y_\mu k} \mu k$, are invertible.

*For $S : \mathcal{N} \longrightarrow \mathcal{M}$ in $\mathcal{K},\mathcal{L}$-***MOD**, conditions (i) and (i′) can be replaced by:*

(i*) S is a right homomorphism.

*Indeed, when these conditions hold, if we define $T\mu$ and y_μ by (ii) and take $T(K,\mu,L)$ to be $(K,T\mu,L)$, the arrows $(1_K,y_\mu,1_L) : S(K,T\mu,L) \longrightarrow (K,\mu,L)$ in **gr**$\mathcal{M}$ are universal among arrows in **gr**$\mathcal{M}$ of the form $(k,\Phi,l) : (K',S\nu,L') \longrightarrow (K,\mu,L)$ and the right adjoint T to which they lead has for its $T_{K,L}$ that of (ii), has for its $T^{\mu,k}$ the inverse of the $t^{\mu,k}$ of (i′), and has for its $T_{l,\mu}$ the vector transformation corresponding under the adjunctions of (ii) to the composite*

$$S(l.T\mu) \xrightarrow{(S_{l,T\mu})^{-1}} l.ST\mu \xrightarrow{ly_\mu} l\mu.$$

PROOF. We have seen that the conditions are necessary. For sufficiency, observe first that for $S : \mathcal{N} \longrightarrow \mathcal{M}$ to have a right adjoint in $\mathcal{K}, \mathcal{L}$-**MOD** is precisely for $\widehat{\mathbf{gr}}S : \widehat{\mathbf{gr}}\mathcal{N} \longrightarrow \widehat{\mathbf{gr}}\mathcal{M}$ to have a right adjoint in $\mathbf{spnCAT}(\mathcal{K}, \mathcal{L})$ (the latter being the pullback of $[\Lambda, \mathbf{CAT}]$ over $\mathbf{CAT} \times \mathbf{CAT}$ along $(\mathcal{K}, \mathcal{L}) : \mathbf{1} \longrightarrow \mathbf{CAT} \times \mathbf{CAT}$). For $\widehat{\mathbf{gr}}S$ to have a right adjoint in $\mathbf{spnCAT}(\mathcal{K}, \mathcal{L})$ is simply for $\mathbf{gr}S : \mathbf{gr}\mathcal{M} \longrightarrow \mathbf{gr}\mathcal{N}$ to have a right adjoint in **CAT** which respects the 'span' constraints, and this can be stated simply as follows: $S : \mathcal{N} \longrightarrow \mathcal{M}$ has a right adjoint if and only if for each object (K, μ, L) in $\mathbf{gr}\mathcal{M}$ there is an object $(K, T\mu, L)$ in $\mathbf{gr}\mathcal{N}$ and an arrow $(1_K, y_\mu, 1_L) : (K, ST\mu, L) \longrightarrow (K, \mu, L)$ in $\mathbf{gr}\mathcal{M}$ such that every arrow $(k, \Phi, l) : (K', S\nu, L') \longrightarrow (K, \mu, L)$ in $\mathbf{gr}\mathcal{M}$ is $(1_K, y_\mu, 1_L).(k, S\Psi, l)$ for a unique arrow $\Psi : l\nu \longrightarrow T\mu.k$ in $\mathcal{N}(K', L)$. Taking $T\mu$ to be $T_{K,L}\mu$ and y_μ to be $y_{K,L}\mu$ as provided by (ii) we have only to verify the universal property.

By 2.13, however, $(k, S\Psi, l)$ is the composite

$$(K', S\nu, L') \xrightarrow{(1_{K'}, S_{l,\nu}, l)} (K', S(l\nu), L) \xrightarrow{(1_{K'}, S\Psi, 1_L)} (K', S(T\mu.k), L) \xrightarrow{(k, S^{T\mu,k}, 1_L)} (K, ST\mu, L),$$

so that, by the meaning of composition in $\mathbf{gr}\mathcal{M}$, the desired equation reduces to

$$y_\mu k.S^{T\mu,k}.S\Psi.S_{l,\nu} = \Phi,$$

which by the invertibility of $S_{l,\nu}$ in (i) and the definition of $t^{\mu,k}$ in (i$'$) is

$$\alpha^{-1}(t^{\mu,k}).S\Psi = \Phi.S_{l,\nu}^{-1},$$

where now

$$\alpha : \mathcal{M}(K, L)(S-, ?) \xrightarrow{\simeq} \mathcal{N}(K, L)(-, T?)$$

denotes the adjunction of (ii). Since the left side here, by the naturality of α^{-1}, is $\alpha^{-1}(t^{\mu,k}\Psi)$, our equation reduces further to

$$t^{\mu,k}\Psi = \alpha(\Phi.S_{l,\nu}^{-1});$$

and this has a unique solution for Ψ because $t^{\mu,k}$ is invertible by (i$'$).

The value $T(a, \Theta, b)$ of T at a general arrow $(a, \Theta, b) : (\bar{K}, \bar{\mu}, \bar{L}) \longrightarrow (K, \mu, L)$ is of course the unique arrow for which

$$(1_K, y_\mu, 1_L).ST(a, \Theta, b) = (a, \Theta, b)(1_{\bar{K}}, y_{\bar{\mu}}, 1_{\bar{L}});$$

we leave to the reader the easy verification that T has the stated values on the basic generating arrows of $\mathbf{gr}\mathcal{M}$. ∎

Combining Theorems 2.36 and 2.39 with the help of Lemma 2.33 and 2.34 and 2.35 we achieve our major goal of this section:

2.40. THEOREM. *An arrow* $(S_0, S, S_1) : (\mathcal{R}, \mathcal{N}, \mathcal{S}) \longrightarrow (\mathcal{K}, \mathcal{M}, \mathcal{L})$ *in* **MOD** *has a right adjoint if and only if the following four conditions are satisfied:*

(o) the S_i *have right adjoints* T_i*, with counits say* $y_i : S_i T_i \longrightarrow 1$*;*

(i) S *is an* **R***-homomorphism;*

(ii) for each X *in* $\mathcal{R}$ *and* L *in* $\mathcal{L}$*, the functor* $\sigma(S)_{X,L} : \mathcal{N}(X, T_1 L) \longrightarrow \mathcal{M}(S_0 X, L)$*, defined as the composite*

$$\mathcal{N}(X, T_1 L) \xrightarrow{S_{X,T_1 L}} \mathcal{M}(S_0 X, S_1 T_1 L) \xrightarrow{\mathcal{M}(S_0 X, y_1 L)} \mathcal{M}(S_0 X, L)$$

has a right adjoint $\tau_{X,L}$*;*

(i′) if $\sigma(S)^{\nu,r} : \sigma(S)(\nu r) \longrightarrow \sigma(S)\nu.S_0 r$ *denotes the composite*

$$y_1 L.S(\nu r) \xrightarrow{y_1 L.S^{\nu,r}} y_1 L.(S\nu.S_0 r) \xrightarrow[\cong]{\eta} (y_1 L.S\nu).S_0 r,$$

and if $\theta^{-,r}$ *denotes the mate of* $\sigma(S)^{-,r}$*, then each* $\theta^{\mu,r}$ *is invertible.*

When these conditions are satisfied, the right adjoint of (S_0, S, S_1) *is* $(T_0, T(\tau), T_1)$*, where the right adjoint* τ *of* $\sigma(S)$ *is constructed from the data above as in Theorem 2.39.*

For $(S_0, S, S_1) : (\mathcal{R}, \mathcal{N}, \mathcal{S}) \longrightarrow (\mathcal{K}, \mathcal{M}, \mathcal{L})$ *in* ***MOD***, conditions (i), (i′), and (ii) can be replaced by:*

(i)* S *is a right homomorphism;*

(ii) for each* X *in* $\mathcal{R}$ *and* A *in* $\mathcal{S}$*, the functor* $S_{X,A} : \mathcal{N}(X, A) \longrightarrow \mathcal{M}(S_0 X, S_1 A)$ *has a right adjoint.* ■

Theorem 2.40 has an obvious specialization to equipments. From the definitions in 2.9 we see that for either **EQT** or ***EQT** we simply apply Theorem 2.40 to an arrow of the form $(S_\#, S, S_\#) : (\mathcal{R}, \mathcal{N}, \mathcal{R}) \longrightarrow (\mathcal{K}, \mathcal{M}, \mathcal{K})$. As we have said before, it is equipments rather than general modules that constitute our main interest; however, inspection of 2.36 now shows that even for $(S_\#, S, S_\#)$ we are led to an adjunction in the 'off-diagonal' $\mathcal{R}, \mathcal{K}$-**MOD**.

2.41. We shall give an application of Theorem 2.40: namely to the question of *limits* in a starred module $\mathcal{M}$. We first need to observe that the 2-category **MOD** admits *powers* (these being what most authors refer to as *cotensor products*), as do ***MOD**, **EQT**, and ***EQT**. That is, for each module $\mathcal{M} = (\mathcal{K}, \mathcal{M}, \mathcal{L})$ and each category $\mathcal{C}$, there is a module $\mathcal{M}^{\mathcal{C}}$ with the universal property expressed by a 2-natural isomorphism

$$\mathbf{MOD}(\mathcal{N}, \mathcal{M}^{\mathcal{C}}) \xrightarrow{\cong} \mathbf{CAT}(\mathcal{C}, \mathbf{MOD}(\mathcal{N}, \mathcal{M})).$$

We give the construction of $\mathcal{M}^{\mathcal{C}}$, and leave the reader to verify that it does indeed have the universal property. The clue to its construction is the observation that, since the 2-functors $\partial_0, \partial_1, \mathbf{gr} : \mathbf{MOD} \longrightarrow \mathbf{CAT}$ are representable as we remarked in 2.9, they preserve any limits that exist, and in particular powers; so that if $\mathcal{M}^{\mathcal{C}}$ does exist, we must have $(\mathcal{M}^{\mathcal{C}})_0 = \mathcal{K}^{\mathcal{C}}$, $(\mathcal{M}^{\mathcal{C}})_1 = \mathcal{L}^{\mathcal{C}}$ and $\mathbf{gr}(\mathcal{M}^{\mathcal{C}}) = (\mathbf{gr}\mathcal{M})^{\mathcal{C}}$, where the right sides are the usual functor-categories.

So $\mathcal{M}^{\mathcal{C}}$ is to be a $\mathcal{K}^{\mathcal{C}}, \mathcal{L}^{\mathcal{C}}$-module. For $P : \mathcal{C} \longrightarrow \mathcal{K}$ and $Q : \mathcal{C} \longrightarrow \mathcal{L}$, an object of $\mathcal{M}^{\mathcal{C}}(P,Q)$ is a functor $\mu : \mathcal{C} \longrightarrow \mathbf{gr}\mathcal{M}$ for which $\partial_0\mu = P$ and $\partial_1\mu = Q$; while an arrow $\Phi : \mu \longrightarrow \nu$ in $\mathcal{M}^{\mathcal{C}}(P,Q)$ is a natural transformation $\Phi : \mu \longrightarrow \nu : \mathcal{C} \longrightarrow \mathbf{gr}\mathcal{M}$ for which $\partial_0\Phi$ and $\partial_1\Phi$ are identities. Natural transformations $q : Q \longrightarrow Q' : \mathcal{C} \longrightarrow \mathcal{L}$ and $p : P' \longrightarrow P : \mathcal{C} \longrightarrow \mathcal{K}$ act on such a μ to give $q\mu$ and μp where, for $C \in \mathcal{C}$, $(q\mu)_C = q_C\mu_C$ and $(\mu p)_C = \mu_C p_C$, with similar definitions on a vector transformation Φ. When $\mathcal{M}$ is starred, so is $\mathcal{M}^{\mathcal{C}}$, and we have a completely analogous description of $\mathcal{M}^{\mathcal{C}}$ in $\mathbf{EQT}$ and in ${}^*\mathbf{EQT}$.

In any 2-category $\mathbf{K}$ with powers, we have the diagonal $D : \mathcal{M} \longrightarrow \mathcal{M}^{\mathcal{C}}$, corresponding to the functor $\mathcal{C} \longrightarrow \mathbf{K}(\mathcal{M},\mathcal{M})$ which is constant at the identity; and $\mathcal{M}$ is said to *admit $\mathcal{C}$-limits* if D has a right adjoint $\lim : \mathcal{M}^{\mathcal{C}} \longrightarrow \mathcal{M}$ in $\mathbf{K}$. In our case where $\mathbf{K} = {}^*\mathbf{MOD}$, the functors $D_i : \mathcal{M}_i \longrightarrow (\mathcal{M}^{\mathcal{C}})_i$ are the usual diagonals $\mathcal{M}_i \longrightarrow (\mathcal{M}_i)^{\mathcal{C}}$. Moreover D is easily seen to be a homomorphism (in the sense of Definition 2.19), and the composite functor

$$\mathcal{M}(K,L) \xrightarrow{D_{K,L}} \mathcal{M}^{\mathcal{C}}(D_0K, D_1L) \xrightarrow{\simeq} \mathcal{M}(K,L)^{\mathcal{C}}$$

is again the diagonal in $\mathbf{CAT}$. Accordingly Theorem 2.40 for starred modules gives:

A starred module $(\mathcal{K},\mathcal{M},\mathcal{L})$ admits $\mathcal{C}$-limits if and only if $\mathcal{K}$ and $\mathcal{L}$ and each $\mathcal{M}(K,L)$ admit them.

3. [Starred] Pointed Equipments

3.1. For each category $\mathcal{K}$, the hom functor $\mathcal{K}(-,-) : \mathcal{K}^{op} \times \mathcal{K} \longrightarrow \mathbf{SET}$ provides a $\mathcal{K},\mathcal{K}$-module $\mathcal{K}(-,-) : \mathcal{K}^{op}, \mathcal{K} \longrightarrow \mathbf{SET} \longrightarrow \mathbf{CAT}$, where $\mathbf{SET} \longrightarrow \mathbf{CAT}$ sends a set to the corresponding discrete category. The structural associativites ξ, η and ζ are identities and we have an equipment $(\mathcal{K},\mathcal{K})$. (If the category $\mathcal{K}$ is a groupoid then the equipment $(\mathcal{K},\mathcal{K})$ is starred. Conversely, if $(\mathcal{K},\mathcal{K})$ is starred then the category $\mathcal{K}$ is a groupoid, for given $k : K' \longrightarrow K$ in $\mathcal{K}$, it follows easily that k^*1_K is its inverse.)

Any functor $G : \mathcal{K} \longrightarrow \mathcal{R}$ gives rise to an equipment arrow $(G,G) : (\mathcal{K},\mathcal{K}) \longrightarrow (\mathcal{R},\mathcal{R})$, with action comparisons given by identities. Similarly, any natural transformation $t : F \longrightarrow G : \mathcal{K} \longrightarrow \mathcal{R}$ gives rise to an equipment transformation $(t,t) : (F,F) \longrightarrow (G,G) : (\mathcal{K},\mathcal{K}) \longrightarrow (\mathcal{R},\mathcal{R})$. These assignments clearly define a 2-functor $\mathbf{CAT} \longrightarrow \mathbf{EQT}$ which, for the moment, it is best to leave un-named. (Similarly, we have a 2-functor $\mathbf{GPD} \longrightarrow {}^*\mathbf{EQT}$, where $\mathbf{GPD}$ is the 2-category of groupoids.)

3.2. Definition. *A* pointed equipment *is an equipment $(\mathcal{K},\mathcal{M})$ together with a strong arrow $(-)_* : \mathcal{K} \longrightarrow \mathcal{M}$ in $\mathcal{K},\mathcal{K}$-$\mathbf{MOD}$.* ∎

In other words we require an arrow in $\mathbf{MOD}$ of the form

$$(1, (-)_*, 1) : (\mathcal{K},\mathcal{K},\mathcal{K}) \longrightarrow (\mathcal{K},\mathcal{M},\mathcal{K})$$

which, by Proposition 2.13 and 2.24 above, admits the following primitive description: For each arrow $f : X \longrightarrow A$ in $\mathcal{K}$ there is a vector arrow $f_* : X \relbar\joinrel\mapstochar\joinrel\rightarrow A$ in $\mathcal{M}$ and for each $X \xrightarrow{f} A \xrightarrow{g} Y$ in $\mathcal{K}$ there are given isomorphisms $gf_* \xrightarrow{\simeq} (gf)_*$ and $(gf)_* \xrightarrow{\simeq} g_*f$ which satisfy

$$1f_* \xrightarrow{\simeq} (1f)_* \text{ is the identity}$$

$$(f1)_* \xrightarrow{\simeq} f_*1 \text{ is the identity}$$

$$\begin{tikzcd}
(fg)h_* \arrow[r, "\xi"] \arrow[d, "\simeq"'] & f(gh_*) \arrow[d, "\simeq"] \\
(fgh)_* & f(gh)_* \arrow[l, "\simeq"]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & f(g_*h) \arrow[r, "\eta"] & (fg_*)h \arrow[dr, "\simeq"] & \\
f(gh)_* \arrow[ur, "\simeq"] \arrow[drr, "\simeq"'] & & & (fg)_*h \\
 & & (fgh)_* \arrow[ur, "\simeq"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
(fgh)_* \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & (fg)_*h \arrow[d, "\simeq"] \\
f_*(gh) & (f_*g)h \arrow[l, "\zeta"]
\end{tikzcd}\;,$$

where ξ, η and ζ are the associativities for $\mathcal{M}$. Observe that for any category $\mathcal{K}$, the equipment $(\mathcal{K}, \mathcal{K})$ is canonically pointed by setting $f_* = f$.

For a pointed equipment $((-)_*, \mathcal{K}, \mathcal{M})$, or $(*, \mathcal{K}, \mathcal{M})$ for short, it is convenient to write $\iota_X : X \relbar\joinrel\mapstochar\joinrel\rightarrow X$ for $(1_X)_* : X \relbar\joinrel\mapstochar\joinrel\rightarrow X$. For any $f : X \longrightarrow A$ in $\mathcal{K}$ we have

$$f\iota_X = f1_* \xrightarrow{\simeq} (f1)_* = f_* = (1f)_* \xrightarrow{\simeq} 1_*f = \iota_A f.$$

3.3. Lemma. *For an equipment $(\mathcal{K}, \mathcal{M})$, let there be given for each object X a vector arrow $\iota_X : X \relbar\joinrel\mapstochar\joinrel\rightarrow X$ and for each scalar arrow $f : X \longrightarrow A$ an isomorphism $\lambda_f : f\iota_X \xrightarrow{\simeq} \iota_A f$, subject to*

$$(\lambda_1 : 1\iota \xrightarrow{\simeq} \iota 1) = (1_\iota : \iota \longrightarrow \iota)$$

$$\begin{tikzcd}
 & f(\iota g) \arrow[r, "\eta"] & (f\iota)g \arrow[dr, "\lambda"] & \\
f(g\iota) \arrow[ur, "\lambda"] \arrow[dr, "\xi^{-1}"'] & & & (\iota f)g \\
 & (fg)\iota \arrow[r, "\lambda"'] & \iota(fg) \arrow[ur, "\zeta^{-1}"'] &
\end{tikzcd}\;,$$

the latter of which is equally the assertion that, in terms of pasting composites, we have

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "\iota_X"'] \arrow[dr, phantom, "\xrightarrow{\lambda_g}"] & Y \arrow[r, "f"] \arrow[d, "\iota_Y"] \arrow[dr, phantom, "\xrightarrow{\lambda_f}"] & Z \arrow[d, "\iota_Z"] \\
X \arrow[r, "g"'] & Y \arrow[r, "f"'] & Z
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "fg"] \arrow[d, "\iota_X"'] \arrow[dr, phantom, "\xrightarrow{\lambda_{fg}}"] & Z \arrow[d, "\iota_Z"] \\
X \arrow[r, "fg"'] & Z
\end{tikzcd}\;.$$

Then, defining $f_\bullet = f\iota_X$ *and*

$$gf_\bullet = g(f\iota) \xrightarrow{\xi^{-1}} (gf)\iota = (gf)_\bullet$$

$$(gf)_\bullet = (gf)\iota \xrightarrow{\xi} g(f\iota) \xrightarrow{g\lambda} g(\iota f) \xrightarrow{\eta} (g\iota)f = g_\bullet f$$

provides a pointing $(-)_\bullet$ *for* $(\mathcal{K}, \mathcal{M})$*. Moreover, if the isomorphisms* $\lambda : f\iota_X \xrightarrow{\simeq} \iota_A f$ *come from a pointing* $(-)_* : \mathcal{K} \longrightarrow \mathcal{M}$*, as in the display preceding the Lemma, then* $f_\bullet = f\iota = f1_* \xrightarrow{\simeq} (f1)_* = f_*$ *describes an isomorphism* $(-)_\bullet \xrightarrow{\simeq} (-)_*$ *in* $\mathcal{K}, \mathcal{K}$-$\mathbf{MOD}(\mathcal{K}, \mathcal{M})$*. Similarly, defining* $f_\circ = \iota_A f$ *and so on provides a pointing with* $(-)_\bullet \xrightarrow{\simeq} (-)_\circ$ *in* $\mathcal{K}, \mathcal{K}$-$\mathbf{MOD}(\mathcal{K}, \mathcal{M})$*; this being the composite* $(-)_\bullet \xrightarrow{\simeq} (-)_* \xrightarrow{\simeq} (-)_\circ$ *if the data come from a given pointing* $(-)_*$*.* ■

3.4. If an equipment $(\mathcal{K}, \mathcal{M})$ has the *property* of being starred and carries the further *structure* of a pointing $(-)_*$ then we have a pointed and starred equipment. But for a starred equipment we have also the equipment $(\mathcal{K}, \mathcal{M}^{op})$ described in the context of modules in 2.10, so that it makes sense to enquire whether or not the ι determined by $(-)_*$ constitute part of the data for a pointing of $(\mathcal{K}, \mathcal{M}^{op})$. The isomorphisms $\lambda : f\iota \xrightarrow{\simeq} \iota f$ have mates $\iota f^* \longrightarrow f^*\iota$. If these are invertible then their inverses satisfy the hypotheses of Lemma 3.3 for the equipment $(\mathcal{K}, \mathcal{M}^{op})$. (The hexagonal coherence condition follows from generalities about mating. The normality condition follows from the normality constraint for starred modules that we imposed in 2.7.) With these observations in mind we express the next definition in the vocabulary of 2.12.

3.5. Definition. *A* starred pointed equipment *is a pointed and starred equipment for which the commutative squares*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\iota"'] & A \arrow[d, "\iota"] \\ X \arrow[r, "f"'] \arrow[ur, phantom, "\overset{\lambda}{\simeq}\!\to"] & A \end{tikzcd}$$

are also exact. ■

In many specific examples of starred pointed equipments, such as $\mathbf{spn}\mathcal{K}$ for a category $\mathcal{K}$ with pullbacks, with the ι_X given by $(1, X, 1) : X \longrightarrow\!\!\!\!\!+ X$, it is consistent with our normality assumptions to take both the $f\iota \xrightarrow{\simeq} \iota f$ and the $f^*\iota \xrightarrow{\simeq} \iota f^*$ to be identities. However, it is not true in general that an identity whose mate is an isomorphism has that mate again an identity. It is for this reason that we did not "normalize" in Definition 3.2.

3.6. Definition. *An* arrow of [starred] pointed equipments

$$(p, G, T) : (*, \mathcal{K}, \mathcal{M}) \longrightarrow (\bullet, \mathcal{R}, \mathcal{N})$$

is an equipment arrow $(G, T) : (\mathcal{K}, \mathcal{M}) \longrightarrow (\mathcal{R}, \mathcal{N})$*, together with an isomorphism* p *of equipment arrows as in*

$$\begin{tikzcd} \mathcal{K} \arrow[r, "G"] \arrow[d, "*"'] & \mathcal{R} \arrow[d, "\bullet"] \\ \mathcal{M} \arrow[r, "T"'] \arrow[ur, phantom, "\overset{p}{\to}\!\simeq"] & \mathcal{N} \end{tikzcd} ,$$

for which $p_\# : (T*)_\# \longrightarrow (\bullet G)_\#$ *is the identity* 1_G. ■

Using Proposition 2.23, we see that this is equivalent to giving isomorphisms

$$p : T(f_*) \xrightarrow{\simeq} (Gf)_\bullet$$

which satisfy

$$\begin{tikzcd}
Gg.T(f_*) \arrow[r, "T_{g,f_*}"] \arrow[d, "Gg.p"', "\simeq"] & T(gf_*) \arrow[r, "\simeq"] & T((gf)_*) \arrow[d, "p", "\simeq"'] \\
Gg(Gf)_\bullet \arrow[r, "\simeq"'] & (Gg.Gf)_\bullet \arrow[r, equal, "1"'] & (G(gf))_\bullet
\end{tikzcd}$$

and

$$\begin{tikzcd}
T((fh)_*) \arrow[r, "\simeq"] \arrow[d, "p"', "\simeq"] & T(f_*h) \arrow[r, "T^{f_*,h}"] & T(f_*).Gh \arrow[d, "p.Gh", "\simeq"'] \\
(G(fh))_\bullet \arrow[r, equal, "1"'] & (Gf.Gh)_\bullet \arrow[r, "\simeq"'] & (Gf)_\bullet Gh
\end{tikzcd},$$

where the unlabelled isomorphisms arise from the structural isomorphisms of the pointings.

3.7. Proposition. *For an arrow*

$$(p, G, T) : (*, \mathcal{K}, \mathcal{M}) \longrightarrow (\bullet, \mathcal{R}, \mathcal{N})$$

of [starred] pointed equipments, all instances of the [T_{f_*,k^*}, T^{l^*,f_*}*,]* T_{g,f_*} *and* $T^{f_*,h}$ *are isomorphisms.*

Proof. The unstarred part follows immediately from the preceding diagrams; it is equally the assertion that $T*$ is a *strong* arrow of equipments, clear since it is isomorphic to the strong arrow $\bullet G$. The starred part then follows from the fact that the data for (p, G, T) determine an arrow between the pointed equipments $(*, \mathcal{K}, \mathcal{M}^{op})$ and $(\bullet, \mathcal{R}, \mathcal{N}^{op})$. ■

3.8. Proposition. *An arrow of pointed equipments*

$$(*, \mathcal{K}, \mathcal{M}) \longrightarrow (\bullet, \mathcal{R}, \mathcal{N})$$

is completely determined by an equipment arrow $(G, T) : (\mathcal{K}, \mathcal{M}) \longrightarrow (\mathcal{R}, \mathcal{N})$, *with either all* $T_{f,\iota}$ *invertible or all* $T^{\iota,f}$ *invertible, together with isomorphisms* $p : T\iota_X \xrightarrow{\simeq} \iota_{GX}$ *such that, for all* $f : X \longrightarrow A$ *in* $\mathcal{K}$,

$$\begin{tikzcd}
Gf.T\iota_X \arrow[r, "T_{f,\iota}"] \arrow[d, "Gf.p"', "\simeq"] & T(f\iota_X) \arrow[r, "T\lambda", "\simeq"'] & T(\iota_A f) \arrow[r, "T^{\iota,f}"] & T\iota_A.Gf \arrow[d, "p.Gf", "\simeq"'] \\
Gf.\iota_{GX} \arrow[rrr, "\lambda", "\simeq"'] & & & \iota_{GA}.Gf
\end{tikzcd}.$$

PROOF. Taking f to be an identity in each of the hexagonal diagrams following Definition 3.6 gives two expressions for a general $p_f : T(f_*) \xrightarrow{\simeq} (Gf)_\bullet$ in terms of $p_1 : T\iota_X \xrightarrow{\simeq} \iota_{GX}$; and the condition for these to agree is the hexagon of the proposition. ∎

3.9. DEFINITION. *A transformation of [starred] pointed equipments*

$$(q, F, S) \longrightarrow (p, G, T) : (*, \mathcal{K}, \mathcal{M}) \longrightarrow (\bullet, \mathcal{R}, \mathcal{N})$$

is an equipment transformation

$$(t, u) : (F, S) \longrightarrow (G, T) : (\mathcal{K}, \mathcal{M}) \longrightarrow (\mathcal{R}, \mathcal{N})$$

such that

$$\begin{tikzcd}[row sep=large] \mathcal{K} \arrow[r, "G"] \arrow[d, "*"'] \arrow[dr, phantom, "\Uparrow p\ \simeq"] & \mathcal{R} \arrow[d, "\bullet"] \\ \mathcal{M} \arrow[r, bend left, "T"] \arrow[r, bend right, "S"'] \arrow[r, phantom, "\Uparrow u"] & \mathcal{N} \end{tikzcd} \quad = \quad \begin{tikzcd}[row sep=large] \mathcal{K} \arrow[r, bend left, "G"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\Uparrow t"] \arrow[d, "*"'] \arrow[dr, phantom, "\Uparrow q\ \simeq"] & \mathcal{R} \arrow[d, "\bullet"] \\ \mathcal{M} \arrow[r, "S"'] & \mathcal{N} \end{tikzcd} .$$

∎

In primitive terms the definition requires

$$\begin{tikzcd} tA.S(f_*) \arrow[rrr, "u_{f_*}"] \arrow[d, "tA.q"', "\simeq"] & & & T(f_*).tX \arrow[d, "p.tX", "\simeq"'] \\ tA(Ff)_\bullet \arrow[r, "\simeq"'] & (tA.Ff)_\bullet \arrow[r, "="', "1"'] & (Gf.tX)_\bullet \arrow[r, "\simeq"'] & (Gf)_\bullet tX \end{tikzcd} ,$$

from which it follows that the u_{f_*} are isomorphisms. We express this observation in the vocabulary of 2.12.

3.10. PROPOSITION. *For a transformation of [starred] pointed equipments*

$$(t, u) : (q, F, S) \longrightarrow (p, G, T) : (*, \mathcal{K}, \mathcal{M}) \longrightarrow (\bullet, \mathcal{R}, \mathcal{N})$$

and each $f : X \longrightarrow A$ in $\mathcal{K}$, the square

$$\begin{tikzcd} FX \arrow[r, "tX"] \arrow[d, "S(f_*)"'] \arrow[dr, phantom, "\xrightarrow{u_{f_*}}"] & GX \arrow[d, "T(f_*)"] \\ FA \arrow[r, "tA"'] & GA \end{tikzcd}$$

is a commutative one. ∎

We can simplify Definition 3.9 in terms of the ι_X.

3.11. PROPOSITION. *To give a transformation of pointed equipments*

$$(q, F, S) \longrightarrow (p, G, T) : (*, \mathcal{K}, \mathcal{M}) \longrightarrow (\bullet, \mathcal{R}, \mathcal{N})$$

is to give an equipment transformation

$$(t, u) : (F, S) \longrightarrow (G, T) : (\mathcal{K}, \mathcal{M}) \longrightarrow (\mathcal{R}, \mathcal{N})$$

satisfying, for each X in $\mathcal{K}$,

$$\begin{tikzcd} tX.S\iota_X \arrow[r, "u_{\iota_X}"] \arrow[d, "tX.q"'] & T\iota_X.tX \arrow[d, "p.tX"] \\ tX.\iota_{FX} \arrow[r, "\lambda"'] & \iota_{GX}.tX. \end{tikzcd}$$

■

3.12. From the form in which we gave Definitions 3.6 and 3.9, it follows immediately that [starred] pointed equipments, arrows of pointed equipments, and pointed transformations, with the evident compositions, form a 2-category, which we denote by [$^*\mathbf{EQT}_*$] $\mathbf{EQT}_*$, that comes with a forgetful 2-functor [$^*\mathbf{EQT}_* \longrightarrow {}^*\mathbf{EQT}$] $\mathbf{EQT}_* \longrightarrow \mathbf{EQT}$. The 2-category $^*\mathbf{EQT}_*$ is a full sub-2-category of $\mathbf{EQT}_*$. We write $(-)_\#$ for the forgetful 2-functor to $\mathbf{CAT}$ from any of the four 2-categories of equipments. It is now clear that the 2-functor $\mathbf{CAT} \longrightarrow \mathbf{EQT}$, which we introduced without a name in 3.1, factorizes canonically through $\mathbf{EQT}_* \longrightarrow \mathbf{EQT}$. For we observed in 3.2 that an equipment of the form $(\mathcal{K}, \mathcal{K})$, with $\mathcal{K}$ a category, is pointed by way of identities; and the same is obviously true of an equipment arrow (G, G) coming from a functor $G : \mathcal{K} \longrightarrow \mathcal{R}$, while an equipment transformation (t, t) coming from a natural transformation $t : F \longrightarrow G : \mathcal{K} \longrightarrow \mathcal{R}$ trivially satisfies the requirement of Proposition 3.11. We now write $D : \mathbf{CAT} \longrightarrow \mathbf{EQT}_*$ for this 2-functor and observe that $(-)_\# D = 1_{\mathbf{CAT}}$. Moreover, for a pointed equipment $(*, \mathcal{K}, \mathcal{M})$,

$$(1, 1, *) : D((*, \mathcal{K}, \mathcal{M})_\#) = (1, \mathcal{K}, \mathcal{K}) \longrightarrow (*, \mathcal{K}, \mathcal{M})$$

becomes an arrow of pointed equipments — the f-component of the stucture isomorphism is the identity vector transformation $1 : f_* \longrightarrow f_*$ in $(\mathcal{K}, \mathcal{M})$.

3.13. PROPOSITION. *The $(1, 1, *) : D((*, \mathcal{K}, \mathcal{M})_\#) \longrightarrow (*, \mathcal{K}, \mathcal{M})$ are pseudo-natural in $(*, \mathcal{K}, \mathcal{M})$ and constitute the counit of a bi-adjunction $D \dashv (-)_\# : \mathbf{EQT}_* \longrightarrow \mathbf{CAT}$.* ■

In 3.2 we observed that a pointed equipment $(*, \mathcal{K}, \mathcal{M})$ gives rise to pointed equipments $(\bullet, \mathcal{K}, \mathcal{M})$ and $(\circ, \mathcal{K}, \mathcal{M})$, where $f_\bullet = f\iota$ and $f_\circ = \iota f$. Observe that $1_\bullet = 1\iota = \iota = \iota 1 = 1_\circ$, so that all three of these pointings for $(\mathcal{K}, \mathcal{M})$ have the same ι. Write $a : (-)_\bullet \xrightarrow{\simeq} (-)_*$ and $b : (-)_* \xrightarrow{\simeq} (-)_\circ$ for the previously given isomorphisms.

3.14. Lemma. *The isomorphisms a and b define arrows of pointed equipments*

$$(a,1,1) : (\bullet,\mathcal{K},\mathcal{M}) \longrightarrow (*,\mathcal{K},\mathcal{M})$$

and

$$(b,1,1) : (*,\mathcal{K},\mathcal{M}) \longrightarrow (\circ,\mathcal{K},\mathcal{M})$$

which exhibit $(\bullet,\mathcal{K},\mathcal{M})$, $(*,\mathcal{K},\mathcal{M})$ and $(\circ,\mathcal{K},\mathcal{M})$ as isomorphic objects of the 2-category $\mathbf{EQT}_*$ ∎

3.15. We turn now to adjunctions in $\mathbf{EQT}_*$ and ${}^*\mathbf{EQT}_*$. For arrows

$$\begin{tikzcd} \mathcal{R} \arrow[r, "F"] \arrow[d, "\bullet"'] \arrow[dr, phantom, "\Uparrow q\ \simeq"] & \mathcal{K} \arrow[d, "*"] \\ \mathcal{N} \arrow[r, "S"'] & \mathcal{M} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{K} \arrow[r, "G"] \arrow[d, "*"'] \arrow[dr, phantom, "\Uparrow p\ \simeq"] & \mathcal{R} \arrow[d, "\bullet"] \\ \mathcal{M} \arrow[r, "T"'] & \mathcal{N} \end{tikzcd}$$

in $\mathbf{EQT}_*$ to be part of an adjunction therein, say $u,v : (q,F,S) \dashv (p,G,T)$, it is necessary and sufficient that there be an adjunction $u,v : S \dashv T$ in $\mathbf{EQT}$ for which u and v are transformations of pointed equipments. Application of $(-)_\#$ to $u,v : S \dashv T$ provides the adjunction $F \dashv G$. Write x for $u_\#$ and y for $v_\#$. The adjunction $x,y : F \dashv G : \mathcal{K} \longrightarrow \mathcal{R}$ can be regarded as a further adjunction in $\mathbf{EQT}$. Pointedness of each of u and v is given by an equation of the kind in Definition 3.9. Explicitly, the unit u is pointed precisely when, in the 2-category $\mathbf{EQT}$, we have

$$\begin{tikzcd} \mathcal{R} \arrow[r, "F"] \arrow[d, "\bullet"'] \arrow[dr, phantom, "\Uparrow q\ \simeq"] & \mathcal{K} \arrow[r, "G"] \arrow[d, "*"] \arrow[dr, phantom, "\Uparrow p\ \simeq"] & \mathcal{R} \arrow[d, "\bullet"] \\ \mathcal{N} \arrow[r, "S"] \arrow[rr, bend right, "1"'] & \mathcal{M} \arrow[r, "T"] & \mathcal{N} \end{tikzcd} \quad = \quad \begin{tikzcd} \mathcal{R} \arrow[r, "F"] \arrow[rr, bend right, "1"'] \arrow[d, "\bullet"'] & \mathcal{K} \arrow[r, "G"] & \mathcal{R} \arrow[d, "\bullet"] \\ \mathcal{N} \arrow[rr, "1"'] & & \mathcal{N} \end{tikzcd} .$$

We need the following purely 2-categorical result, which is closely related to Theorem 1.5 of [K&S]; it could in fact, after some preliminaries, be reduced to that theorem — but it is quicker to give a direct proof.

3.16. Lemma. *Suppose that in an arbitrary 2-category we are given adjunctions $x,y : F \dashv G : \mathcal{K} \longrightarrow \mathcal{R}$ and $u,v : S \dashv T : \mathcal{M} \longrightarrow \mathcal{N}$, along with arrows $* : \mathcal{K} \longrightarrow \mathcal{M}$ and $\bullet : \mathcal{R} \longrightarrow \mathcal{N}$, and transformations $p : T* \longrightarrow \bullet G$ and $q : S\bullet \longrightarrow *F$ (as in the squares displayed at the beginning of 3.15, except that p and q need not be invertible). Write $\bar{q} : \bullet G \longrightarrow T*$ for the mate of q under the adjunctions. Say that x and u are compatible if $pF.Tq.u\bullet = \bullet x$ (as in the equation displayed before the lemma), and that y and v are compatible if they satisfy the analogous equation $*y.qG.Sp = v*$. Then in fact* *(i) x and u are compatible if and only if $p\bar{q} = 1$;*

(ii) y and v are compatible if and only if $\bar{q}p = 1$.

Thus both pairs are compatible if and only if p and $\bar{q}$ are mutually inverse.

PROOF. By [K&S], the mate $\bar{q}$ of q is the pasting composite

$$\begin{tikzcd}
 & {} \arrow[r, "\bullet"] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow q"] & {} \arrow[r, "1"] \arrow[d, "S"] \arrow[r, phantom, shift right=4, "\Downarrow u"] & {} \\
{} \arrow[ur, "G"] \arrow[r, "1"'] \arrow[r, phantom, shift left=4, "\Downarrow y"] & {} \arrow[r, "*"'] & {} \arrow[ur, "T"'] &
\end{tikzcd},$$

so that $p\bar{q}$ is the composite

$$\begin{tikzcd}
 & {} \arrow[r, "\bullet"] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow q"] & {} \arrow[d, "S"] \arrow[dr, "1"] \arrow[r, phantom, shift right=6, "\Downarrow u"] & \\
{} \arrow[ur, "G"] \arrow[r, "1"'] \arrow[r, phantom, shift left=4, "\Downarrow y"] & {} \arrow[r, "*"'] \arrow[dr, "G"'] & {} \arrow[r, "T"'] \arrow[d, phantom, "\Downarrow p"] & {} \\
 & & {} \arrow[ur, "\bullet"'] &
\end{tikzcd}.$$

Given that the diagrams

$$\begin{tikzcd}
{} \arrow[r, "1"] \arrow[d, "F"'] \arrow[r, phantom, shift right=4, "\Downarrow x"] & {} \\
{} \arrow[ur, "G"'] &
\end{tikzcd}, \qquad
\begin{tikzcd}
 & {} \arrow[d, "F"] \\
{} \arrow[ur, "G"] \arrow[r, "1"'] \arrow[r, phantom, shift left=4, "\Downarrow y"] & {}
\end{tikzcd}$$

are pasting-inverses, to say that $p\bar{q} = 1$ is equally to say that

$$\begin{tikzcd}
{} \arrow[r, "\bullet"] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow q"] & {} \arrow[d, "S"] \arrow[dr, "1"] \arrow[r, phantom, shift right=6, "\Downarrow u"] & \\
{} \arrow[r, "*"'] \arrow[dr, "G"'] & {} \arrow[r, "T"'] \arrow[d, phantom, "\Downarrow p"] & {} \\
 & {} \arrow[ur, "\bullet"'] &
\end{tikzcd}
=
\begin{tikzcd}
{} \arrow[d, "F"'] \arrow[dr, "1"] \arrow[r, phantom, shift right=6, "\Downarrow x"] & & \\
{} \arrow[r, "G"'] & {} \arrow[r, "\bullet"'] & {}
\end{tikzcd};$$

which is the assertion that x and u are compatible. This gives (i), and the proof of (ii) is dual: one uses the expression above for $\bar{q}$ to write $\bar{q}p$ as a pasting composite, and then uses the fact that u and v are pasting-inverses. ■

3.17. Corollary. *(i) If*

$$(x,u),(y,v) : (q,F,S) \dashv (p,G,T) : (*,\mathcal{K},\mathcal{M}) \longrightarrow (\bullet,\mathcal{R},\mathcal{N})$$

is an adjunction in $\mathbf{EQT}_*$ *then* p *and the mate of* q *are inverse to each other.*
(ii) If $(q,F,S) : (\bullet,\mathcal{R},\mathcal{N}) \longrightarrow (*,\mathcal{K},\mathcal{M})$ *is an arrow in* $\mathbf{EQT}_*$ *and*

$$(x,u),(y,v) : (F,S) \dashv (G,T) : (\mathcal{K},\mathcal{M}) \longrightarrow (\mathcal{R},\mathcal{N})$$

is an adjunction in $\mathbf{EQT}$ *and* $\bar{q}$*, the mate of* q*, is invertible then* defining $p = \bar{q}^{-1}$ *provides an arrow* $(p,G,T) : (*,\mathcal{K},\mathcal{M}) \longrightarrow (\bullet,\mathcal{R},\mathcal{N})$ *and an adjunction* $(x,u),(y,v) : (q,F,S) \dashv (p,G,T)$ *in* $\mathbf{EQT}_*$. ■

Corollary 3.17 furnishes us with the following coarse adjunction theorem for $\mathbf{EQT}_*$ (and ${}^*\mathbf{EQT}_*$): *An arrow*

$$(q,S) : (\bullet,\mathcal{N}) \longrightarrow (*,\mathcal{M})$$

in $\mathbf{EQT}_*$ *has a right adjoint if and only if* $S : \mathcal{N} \longrightarrow \mathcal{M}$ *has a right adjoint in* $\mathbf{EQT}$ *and the resulting mate of* q *is invertible.* Our goal in the next section is to refine the "invertible mate" condition so as to relate it to the hypotheses of Theorem 2.40, specialized to the case of equipments.

3.18. We focus now on the hypothesis in Corollary 3.17 (ii) and apply the general theory of 2.34. With $(x, y : F \dashv G) = (u, v : S \dashv T)_\#$ fixed, we have an equivalence between the category of adjunctions $(F,S) \dashv (G,T) : (\mathcal{K},\mathcal{M}) \longrightarrow (\mathcal{R},\mathcal{N})$ in $\mathbf{EQT}$ and the category of all adjunctions $\sigma \dashv \tau : \mathcal{M}(F,1) \longrightarrow \mathcal{N}(1,G)$ in $\mathcal{R},\mathcal{K}$-$\mathbf{MOD}$. In particular, we have an equivalence between the category of adjunctions $(F,F') \dashv (G,G') : (\mathcal{K},\mathcal{K}) \longrightarrow (\mathcal{R},\mathcal{R})$ in $\mathbf{EQT}$ and the category of all adjunctions $\phi' \dashv \gamma' : \mathcal{K}(F,1) \longrightarrow \mathcal{R}(1,G)$ in $\mathcal{R},\mathcal{K}$-$\mathbf{MOD}$; under this, let the particular adjunction $(F,F) \dashv (G,G)$ correspond to $\phi \dashv \gamma : \mathcal{K}(F,1) \longrightarrow \mathcal{R}(1,G)$. This last adjunction is trivial, in the sense that $\gamma : \mathcal{K}(F,1) \longrightarrow \mathcal{R}(1,G)$ is an isomorphism in $\mathcal{R},\mathcal{K}$-$\mathbf{MOD}$, with components the bijections $\mathcal{K}(FX,K) \xrightarrow{\simeq} \mathcal{R}(X,GK)$, and with $\phi : \mathcal{R}(1,G) \longrightarrow \mathcal{K}(F,1)$ for its inverse, the unit and counit for $\phi \dashv \gamma$ being identities.

Now a square q and its mate $\bar{q}$ in $\mathbf{EQT}$,

$$\begin{tikzcd}
\mathcal{R} \arrow[r, "F"] \arrow[d, "\bullet"'] \arrow[dr, phantom, "\Uparrow q"] & \mathcal{K} \arrow[d, "*"] \\
\mathcal{N} \arrow[r, "S"'] & \mathcal{M}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{K} \arrow[r, "G"] \arrow[d, "*"'] \arrow[dr, phantom, "\Downarrow \bar{q}"] & \mathcal{R} \arrow[d, "\bullet"] \\
\mathcal{M} \arrow[r, "T"'] & \mathcal{N}
\end{tikzcd}
\quad ,$$

give rise to mates

$$\begin{tikzcd}
\mathcal{R}(1,G) \arrow[r, "\phi"] \arrow[d, "\bullet(1,G)"'] \arrow[dr, phantom, "\Uparrow Q"] & \mathcal{K}(F,1) \arrow[d, "*(F,1)"] \\
\mathcal{N}(1,G) \arrow[r, "\sigma"'] & \mathcal{M}(F,1)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{K}(F,1) \arrow[r, "\gamma"] \arrow[d, "*(F,1)"'] \arrow[dr, phantom, "\Downarrow \bar{Q}"] & \mathcal{R}(1,G) \arrow[d, "\bullet(1,G)"] \\
\mathcal{M}(F,1) \arrow[r, "\tau"'] & \mathcal{N}(1,G)
\end{tikzcd}$$

in $\mathcal{R},\mathcal{K}$-**MOD**, with q invertible if and only if Q is invertible and $\bar{q}$ invertible if and only if $\bar{Q}$ is invertible. We have $\bar{Q} = (\tau * (F,1)1)(\tau Q\gamma)(a \bullet (1,G)\gamma)$, where the last 1 in $\tau * (F,1)1$ is the counit for the trivial adjunction $\phi \dashv \gamma$ and a is the unit for the adjunction $\sigma \dashv \tau$. Under the hypotheses of Corollary 3.17 (ii), q is an isomorphism. Thus, Q is an isomorphism and it follows that $\bar{q}$ is invertible if and only if $\bar{Q}$ is invertible if and only if $a \bullet (1,G)\gamma$ is invertible. Since the arrow γ is invertible, we conclude that $\bar{q}$ is invertible if and only if $a \bullet (1,G)$ is an isomorphism. In other words, $\bar{q}$ is invertible if and only if, for every X in $\mathcal{R}$, K in $\mathcal{K}$ and $r : X \longrightarrow GK$, the component $a_{r_\bullet}$ is invertible.

Since $r_\bullet \cong \iota_{GK} r$ by the remarks preceding Lemma 3.3, the invertibility of $a_{r_\bullet}$ is equivalent to that of $a_{\iota_{GK}r}$. Recall from 2.37 that the natural transformations $(\tau^{-,r})^{-1}$ are the mates of the $\sigma^{-,r}$; in terms of the unit $a : 1 \longrightarrow \tau\sigma$, this reduces to the commutativity of

$$\begin{tikzcd}
\mu r \arrow[r, "a_\mu r"] \arrow[d, "a_{\mu r}"'] & \tau(\sigma(\mu))r \arrow[d, "(\tau^{\sigma(\mu),r})^{-1}"] \\
\tau(\sigma(\mu r)) \arrow[r, "\tau(\sigma^{\mu,r})"'] & \tau(\sigma(\mu).Fr)
\end{tikzcd}$$

for any vector arrow $\mu : GK \longrightarrow\!\!\!\!\!\!\!+\;\; X$; we are concerned with the case $\mu = \iota_{GK}$. From the description of σ in 2.30 it follows that the $\sigma^{\iota_{GK},r}$ are invertible if the $S^{\iota_{GK},r}$ are so — but the latter is so by Proposition 3.7. It follows that invertibility of the $a_{r_\bullet}$, and hence the invertibility of the mate of q, is equivalent to the invertibility of the $a_{\iota_{GK}}$. It is worth observing that the codomain of $a_{\iota_{GK}}$ is $\tau(\sigma(\iota_{GK})) \xrightarrow{\cong} \tau((\phi(1_{GK}))_*) = \tau((y_K)_*)$, this isomorphism being the 1_{GK}-component of the natural transformation $(\tau Q)_{GK,K}$ (where Q is given above).

Before summarizing this discussion we note the variant of the "invertible mate" condition that is available for starred pointed equipments. Write c for the unit of the adjunction $\tilde{S} \dashv \check{S} : \mathcal{M}(F,F) \longrightarrow \mathcal{N}$ in $\mathcal{R},\mathcal{R}$-***MOD**, as in 2.35, that further corresponds to the adjunction $(F,S) \dashv (G,T) : (\mathcal{K},\mathcal{M}) \longrightarrow (\mathcal{R},\mathcal{N})$ in ***EQT**, and as usual write $\tilde{y}$ for the unit of the adjunction $y- \dashv y^*-$ (where y is the counit for $F \dashv G$). Now the unit a for $\sigma \dashv \tau$ is given by $a = (\check{S}\tilde{y}S)c$. The codomain of $a_{\iota_{GK}}$ is $\check{S}(y_K^*(y_K S\iota_{GK}))$. We have a composite of isomorphisms

$$\check{S}(y_K^*(y_K S\iota_{GK})) \xrightarrow{\cong} \check{S}(y_K^*(y_K\iota_{FGK})) \xrightarrow{\cong} \check{S}(y_K^*(\iota_K y_K)),$$

where the first is $\check{S}(y_K^*(y_K q))$, with q again being the structural isomorphism making S pointed, and where the second is provided by the "commutative square" $y_K\iota_{FGK} \xrightarrow{\cong} \iota_K y_K$. Write $e_K : \iota_{GK} \longrightarrow \check{S}(y_K^*(\iota_K y_K))$ for the composite of $a_{\iota_{GK}}$ with the displayed isomorphism. Clearly, the $a_{\iota_{GK}}$ are invertible if and only if the e_K are invertible.

Combining these observations with the specialization to equipments of Theorem 2.40 provides:

3.19. Theorem. *An arrow* $(q,F,S) : (\bullet,\mathcal{R},\mathcal{N}) \longrightarrow (*,\mathcal{K},\mathcal{M})$ *in* $\mathbf{EQT}_*$ *has a right adjoint if and only if the following five conditions are satisfied:*

(o) the functor F has a right adjoint G, with counit say $y : FG \longrightarrow 1$;

*(i) S is an **R**-homomorphism;*

(ii) for each X in $\mathcal{R}$ and L in $\mathcal{L}$, the functor $\sigma(S)_{X,L} : \mathcal{N}(X, GL) \longrightarrow \mathcal{M}(FX, L)$, defined as the composite

$$\mathcal{N}(X, GL) \xrightarrow{S_{X,GL}} \mathcal{M}(FX, FGL) \xrightarrow{\mathcal{M}(FX,yL)} \mathcal{M}(FX, L),$$

has a right adjoint $\tau_{X,L}$;

(i′) if $\sigma(S)^{\nu,r} : \sigma(S)(\nu r) \longrightarrow \sigma(S)\nu.Fr$ denotes the composite

$$yL.S(\nu r) \xrightarrow{yL.S^{\nu,r}} yL.(S\nu.Fr) \xrightarrow[\simeq]{\eta} (yL.S\nu).Fr,$$

and if $\theta^{-,r}$ denotes the mate of $\sigma(S)^{-,r}$, then each $\theta^{\mu,r}$ is invertible;

(iii) for each object K in $\mathcal{K}$, the component $a_{\iota_{GK}}$ of the unit for the adjunction $\sigma(S)_{GK,K} \dashv \tau_{GK,K}$ of (ii) is an isomorphism.

When these conditions are satisfied, the right adjoint of (q, F, S) is $(p, G, T(\tau))$, where the right adjoint τ of $\sigma(S)$ is constructed from the data above as in Theorem 2.39, and where p is the inverse of the mate of q.

For $(q, F, S) : (\bullet, \mathcal{R}, \mathcal{N}) \longrightarrow (*, \mathcal{K}, \mathcal{M})$ in ${}^*\mathbf{EQT}_*$, conditions (i), (i′), (ii) and (iii) can be replaced by:

(i) S is a right homomorphism;*

(ii) for each X and A in $\mathcal{R}$, the functor $S_{X,A} : \mathcal{N}(X, A) \longrightarrow \mathcal{M}(FX, FA)$ has a right adjoint.*

(iii) for each K in $\mathcal{K}$, the $e_K : \iota_{GK} \longrightarrow \check{S}(y^*_K(\iota_K y_K))$, constructed as above, is invertible.*

■

The reader should compare the codomain of e_K with the diagram in Theorem 2.36 (ii). The inverses of the e_K provide the structure isomorphism for the right adjoint (p, T) of (q, S), where T is the right adjoint of S in ${}^*\mathbf{EQT}$ constructed by Theorem 2.40.

3.20. Quite generally, a pointing for an equipment $(\mathcal{K}, \mathcal{M})$ provides a link between the "global" scalar structure $\mathcal{K}$ and the "local" vector structures $\mathcal{M}(K, L)$. In the presence of adjunctions the link can become very strong. The following typical application of Theorem 3.19, which follows through the example in 2.41 for Theorem 2.40, will illustrate this.

For a pointed equipment $\mathcal{M} = (*, \mathcal{K}, \mathcal{M})$ and a category $\mathcal{C}$, we have $\mathcal{M}^{\mathcal{C}}$ as before, with the pointing given componentwise. Explicitly, if F is an object of $\mathcal{M}^{\mathcal{C}}$ then $\iota_F(C) = \iota_{FC}$, for every C in $\mathcal{C}$, while for every $c : C \longrightarrow C'$ in $\mathcal{C}$ we have $\iota_F(c)$ given by the commutative square $Fc.\iota_{FC} \xrightarrow{\simeq} \iota_{FC'}.Fc$. If $\mathcal{M}$ is a starred pointed equipment then so is $\mathcal{M}^{\mathcal{C}}$. The diagonal $D : \mathcal{M} \longrightarrow \mathcal{M}^{\mathcal{C}}$ is a pointed homomorphism.

3.21. DEFINITION. *A pointed equipment* $\mathcal{M}$ has $\mathcal{C}$-limits *if the diagonal* $D : \mathcal{M} \longrightarrow \mathcal{M}^{\mathcal{C}}$ *has a right adjoint in* $\mathbf{EQT}_*$. ∎

3.22. From Theorem 3.19 it follows that a starred pointed equipment $\mathcal{M}$ has $\mathcal{C}$-limits if and only if $\mathcal{K}$ and each $\mathcal{M}(K, L)$ have $\mathcal{C}$-limits and moreover, for every $F : \mathcal{C} \longrightarrow \mathcal{K}$ with limit cone $p : P \longrightarrow F$ in $\mathcal{K}$, the arrows $\bar{p}_C : \iota_P \longrightarrow p_C^*(\iota_{FC} p_C)$, corresponding by adjunction to the commutative squares $p_C.\iota_P \xrightarrow{\simeq} \iota_{FC}.p_C$, constitute a limit cone in $\mathcal{M}(P, P)$. (The diagram in $\mathcal{M}(P, P)$ has, for each $c : C \longrightarrow C'$ in $\mathcal{C}$, the $p_C^*(\iota_{FC} p_C) \longrightarrow p_{C'}^*(\iota_{FC'} p_{C'}) = (Fc.p_C)^*(\iota_{FC'}(Fc.p_C))$ given by $p_C^*(-)p_C$ applied, with appropriate parenthesization, to $\iota_{FC} \longrightarrow (Fc)^*(\iota_{FC'}.Fc)$, which last also corresponds to an obvious commutative square.)

To give a simple specific case, take $\mathcal{C}$ to be the free-living cospan $a \longrightarrow c \longleftarrow b$. Say that a pointed equipment $\mathcal{M}$ *has pullbacks* if $\mathcal{M}$ has $\mathcal{C}$-limits for this $\mathcal{C}$. It follows that a starred pointed $\mathcal{M}$ has pullbacks if and only $\mathcal{K}$ and each $\mathcal{M}(K, L)$ have pullbacks and moreover, for every pullback

$$\begin{tikzcd} P \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, "r"] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

in $\mathcal{K}$,

$$\begin{tikzcd} \iota_P \arrow[r] \arrow[d] & q^*(\iota_B q) \arrow[d] \\ p^*(\iota_A p) \arrow[r] & r^*(\iota_C r) \end{tikzcd}$$

is a pullback in $\mathcal{M}(P, P)$.

Taking $\mathcal{C}$ to be the discrete category with two objects defines *binary products*, while taking $\mathcal{C}$ to be the empty category defines *terminal objects*, within an equipment. An equipment $\mathcal{M}$ has both if and only if $\mathcal{K}$ and each $\mathcal{M}(K, L)$ have finite products and moreover, for each binary product $p : A \longleftarrow A \times B \longrightarrow B : q$ in $\mathcal{K}$, $p^*(\iota_A p) \longleftarrow \iota_{A\times B} \longrightarrow q^*(\iota_B q)$ is a binary product in $\mathcal{M}(A \times B, A \times B)$, and ι_1 is the terminal object of $\mathcal{M}(1, 1)$, where 1 is the terminal object of $\mathcal{K}$. These conditions (with each ι taken to be an identity) were a crucial part of the definition of cartesian bicategory in [C&W]. The treatment of these in [CKW] can now be seen as the special case of ours, in which all equipments under consideration have each $\mathcal{M}(K, L)$ an ordered set.

4. The 2-functor $\mathbf{spn} : \mathbf{PBK} \longrightarrow {}^*\mathbf{EQT}_*$

4.1. In [CKW] (section 4) it was shown that **rel** can be regarded as a colax functor from the *full* and locally-full sub-2-category of **CAT** determined by the regular categories to a 2-category that we called **F**. While the foundations presented here differ somewhat from those in [CKW], it is a straightforward matter to replace **F** in the earlier treatment by

${}^*\mathbf{EQT}_*$. The starred pointed equipments in question have modules $\mathcal{M} : \mathcal{K}^{op}, \mathcal{K} \longrightarrow \mathbf{CAT}$ which factor through the 2-category of ordered sets. A key goal of the present treatment, as promised in [CKW], is the development of that work beyond such "locally ordered" examples. Accordingly, we give here a treatment of **spn** as an extended example. The following simple result motivates our preoccupation with categories that admit pullbacks.

4.2. Lemma. *For any category $\mathcal{C}$, the module $\mathbf{spn}'\mathcal{C} : \mathcal{C} \mathrel{\longrightarrow\!\!\!\!\!\!\mapsto} \mathcal{C}^{op}$ of 2.4 (i) satisfies the conditions in the penultimate paragraph of 2.8 — namely the existence of right adjoints $l^*(-)$ for $l(-)$ and $(-)k$ for $(-)k^*$, with these satisfying a Beck-Chevalley condition — if and only if $\mathcal{C}$ has pullbacks.*

Proof. The "if" part being clear, suppose that $\mathbf{spn}'\mathcal{C} : \mathcal{C} \mathrel{\longrightarrow\!\!\!\!\!\!\mapsto} \mathcal{C}^{op}$ satisfies the conditions above, and consider a cospan $f : X \longrightarrow A \longleftarrow Y : g$ in $\mathcal{C}$. Write $(1, A, 1)$ for the object $1 : A \longleftarrow A \longrightarrow A : 1$ of $\mathbf{spn}'\mathcal{C}(A, A)$, and $p : X \longleftarrow S \longrightarrow Y : q$ for the object $g^*((1, A, 1)f)$ of $\mathbf{spn}'\mathcal{C}(X, Y)$. Since the identity arrow of the span (p, S, q) corresponds under the adjunctions to an arrow $r : (g(p, S, q))f^* \longrightarrow (1, A, 1)$ of spans, and since $(g(p, S, q))f^* = (fp, S, gq)$, we have in the category $\mathcal{C}$ the commutative square $fp = r = gq$. For a general object (x, T, y) of $\mathbf{spn}'\mathcal{C}(X, Y)$, there is a bijection between arrows $(x, T, y) \longrightarrow (p, S, q) = g^*((1, A, 1)f)$ in $\mathbf{spn}'\mathcal{C}(X, Y)$ and arrows $(fx, T, gy) = (g(x, T, y))f^* \longrightarrow (1, A, 1)$ in $\mathbf{spn}'\mathcal{C}(A, A)$ — of which there is just one if $fx = gy$, and none otherwise. So $p : X \longleftarrow S \longrightarrow Y : q$ is the pullback of $f : X \longrightarrow A \longleftarrow Y : g$. ∎

4.3. We denote by **PBK** the *full* sub-2-category of **CAT** determined by the categories which admit pullbacks. For $\mathcal{K}$ in **PBK** we write $\mathbf{spn}\mathcal{K} : \mathcal{K} \mathrel{\longrightarrow\!\!\!\!\!\!\mapsto} \mathcal{K}$ for the starred equipment obtained from $\mathbf{spn}'\mathcal{K}$ by the use of Lemma 4.2; observe that $(\mathbf{spn}\mathcal{K})_\# = \mathcal{K}$, in the language of 2.9, and that we may sometimes denote the equipment more fully by $(\mathcal{K}, \mathbf{spn}\mathcal{K})$. We recall that, for a span $(x, S, a) : X \mathrel{\longrightarrow\!\!\!\!\!\!\mapsto} A$ and arrows $f : Y \longrightarrow X$ and $g : A \longrightarrow B$ of $\mathcal{K}$, we have $g(x, S, a) = (x, S, ga)$, while $(x, S, a)f$ is obtained by pulling back along f; similarly $(x, S, a)h^*$ is given by composition and $k^*(x, S, a)$ by pulling back. Clearly a square

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "{(x,S,a)}"', "\shortmid" marking] \arrow[dr, phantom, "\xrightarrow{\Sigma}"] & Y \arrow[d, "{(y,T,b)}", "\shortmid" marking] \\
A \arrow[r, "g"'] & B
\end{tikzcd}
$$

in $\mathbf{spn}\mathcal{K}$ is nothing but a commutative diagram

$$
\begin{tikzcd}
X \arrow[r, "f"] & Y \\
S \arrow[u, "x"] \arrow[r, "s"] \arrow[d, "a"'] & T \arrow[u, "y"'] \arrow[d, "b"] \\
A \arrow[r, "g"'] & B
\end{tikzcd}
$$

in $\mathcal{K}$, so that we may as well identify Σ with s; indeed $\mathbf{gr}(\mathbf{spn}\mathcal{K})$ is clearly isomorphic to $\mathcal{K}^{\Lambda}$, where Λ denotes (as in 2.9) the category $0 \longleftarrow \star \longrightarrow 1$. Clearly $\mathbf{spn}\mathcal{K}$ becomes a starred pointed equipment when we define the module-arrow $(-)_* : \mathcal{K} \longrightarrow \mathbf{spn}\mathcal{K}$ of Definition 3.2 by taking h_* for $h \in \mathcal{K}(X, A)$ to be the span $1_X : X \longleftarrow X \longrightarrow A : h$; so that, in particular, ι_X is the span $(1, X, 1)$. Note that, in terms of the Grothendieck categories, $(-)_* : \mathbf{gr}\mathcal{K} \longrightarrow \mathbf{gr}(\mathbf{spn}\mathcal{K})$ is just the evident functor $\mathcal{K}^{\mathbf{2}} \longrightarrow \mathcal{K}^{\Lambda}$ sending the object $h : H \longrightarrow A$ to the span $(1_X, X, h)$ as above.

Any functor $G : \mathcal{K} \longrightarrow \mathcal{L}$ of course induces a functor $G^{\Lambda} : \mathcal{K}^{\Lambda} \longrightarrow \mathcal{L}^{\Lambda}$ — that is, a functor $\mathbf{gr}(\mathbf{spn}\mathcal{K}) \longrightarrow \mathbf{gr}(\mathbf{spn}\mathcal{L})$ — compatible with ∂_0 and ∂_1, thus giving an arrow of equipments $(G, \mathbf{spn}G) : (\mathcal{K}, \mathbf{spn}\mathcal{K}) \longrightarrow (\mathcal{L}, \mathbf{spn}\mathcal{L})$, defined on vector arrows by $\mathbf{spn}G(x, S, a) = (Gx, GS, Ga)$. This arrow is a right homomorphism, since clearly the comparison-arrows $\mathbf{spn}G_{g,(x,S,a)}$ and $\mathbf{spn}G_{(x,S,a),h^*}$ are identities. In particular $\mathbf{spn}G_{g,\iota_X}$ is an identity; whence it follows from Proposition 3.8 that $\mathbf{spn}G$ becomes an arrow of pointed equipments when we augment it with the isomorphism $(\mathbf{spn}G)\iota_X \xrightarrow{\simeq} \iota_{GX}$ given by the identity.

Similarly each natural transformation $t : F \longrightarrow G : \mathcal{K} \longrightarrow \mathcal{L}$ induces a natural transformation $t^{\Lambda} : F^{\Lambda} \longrightarrow G^{\Lambda} : \mathcal{K}^{\Lambda} \longrightarrow \mathcal{L}^{\Lambda}$; and since $\mathcal{K}^{\Lambda}$ and $\mathcal{L}^{\Lambda}$ are $\mathbf{gr}(\mathbf{spn}\mathcal{K})$ and $\mathbf{gr}(\mathbf{spn}\mathcal{L})$, this t^{Λ} is precisely an equipment-transformation $(t, \mathbf{spn}t) : (F, \mathbf{spn}F) \longrightarrow (G, \mathbf{spn}G)$, where $(\mathbf{spn}t)_{(x,S,a)} = t_S$. It is clear from Definition 3.9 that $(t, \mathbf{spn}t)$ is is fact a transformation of *pointed* equipments. Since $(-)^{\Lambda}$ is a 2-functor $\mathbf{PBK} \longrightarrow \mathbf{PBK}$, it is immediate that we have a 2-functor $\mathbf{spn} : \mathbf{PBK} \longrightarrow {}^*\mathbf{EQT}_*$. (Compare this with the case of $\mathbf{rel}$ studied in [CKW]; there we had not a 2-functor but only a colax functor $\mathbf{rel} : \mathbf{REG} \longrightarrow {}^*\mathbf{EQT}_*$.) As usual, we shall denote the functor expressing the "local" aspect of the 2-functor $\mathbf{spn}$ for fixed $\mathcal{K}$ and $\mathcal{L}$ by

$$\mathbf{spn}_{\mathcal{K},\mathcal{L}} : \mathbf{PBK}(\mathcal{K}, \mathcal{L}) \longrightarrow {}^*\mathbf{EQT}_*(\mathbf{spn}\mathcal{K}, \mathbf{spn}\mathcal{L}).$$

It is convenient to observe here that if $(q, S), (p, T) : \mathbf{spn}\mathcal{K} \longrightarrow \mathbf{spn}\mathcal{L}$ are general arrows in ${}^*\mathbf{EQT}_*$ then *any* equipment transformation $u : S \longrightarrow T$ satisfies the condition of Proposition 3.11 and is thus a pointed equipment transformation. This follows directly from the description of squares in $\mathbf{spn}\mathcal{L}$ and is left as an exercise.

4.4. We now consider the image of the functor $\mathbf{spn}_{\mathcal{K},\mathcal{L}}$. If $(p,T):\mathbf{spn}\mathcal{K}\longrightarrow\mathbf{spn}\mathcal{L}$ is an arrow in ${}^*\mathbf{EQT}_*$, the 2-functor $(-)_\#:{}^*\mathbf{EQT}_*\longrightarrow\mathbf{CAT}$ (as in 3.12) sends it to an arrow $T_\#:\mathcal{K}\longrightarrow\mathcal{L}$ in $\mathbf{CAT}$ which, the inclusion $\mathbf{PBK}\longrightarrow\mathbf{CAT}$ being fully faithful, is the same thing as an arrow $T_\#:\mathcal{K}\longrightarrow\mathcal{L}$ in $\mathbf{PBK}$. Similarly a transformation $u:S\longrightarrow T:\mathbf{spn}\mathcal{K}\longrightarrow\mathbf{spn}\mathcal{L}$ gives rise to a transformation $u_\#:S_\#\longrightarrow T_\#:\mathcal{K}\longrightarrow\mathcal{L}$ in $\mathbf{PBK}$. Thus $(-)_\#$ provides a functor

$$(-)_\#:{}^*\mathbf{EQT}_*(\mathbf{spn}\mathcal{K},\mathbf{spn}\mathcal{L})\longrightarrow\mathbf{PBK}(\mathcal{K},\mathcal{L}),$$

which from the definitions in 4.3 is seen to satisfy

$$(-)_\#\mathbf{spn}_{\mathcal{K},\mathcal{L}}=1:\mathbf{PBK}(\mathcal{K},\mathcal{L})\longrightarrow\mathbf{PBK}(\mathcal{K},\mathcal{L}).$$

Now, for $(p,T):\mathbf{spn}\mathcal{K}\longrightarrow\mathbf{spn}\mathcal{L}$ in ${}^*\mathbf{EQT}_*$, we compare $\mathbf{spn}(T_\#)$ and (p,T). Set $G=T_\#$. The typical vector arrow $(x,S,a):X\longrightarrow\!\!\!\!\!+\, A$ in $\mathbf{spn}\mathcal{K}$ can be written as $a\iota_S x^*:X\longrightarrow\!\!\!\!\!+\, A$ (no parentheses being necessary for these actions in this example). We have the comparison $T_{a,\iota_S,x^*}:Ga.T\iota_S.Gx^*\longrightarrow T(a\iota_S x^*)$ and the isomorphism $p:T\iota_S\xrightarrow{\simeq}\iota_{GS}$. Together these give

$$\mathbf{spn}_{\mathcal{K},\mathcal{L}}(T_\#)(x,S,a)=Ga.\iota_{GS}.Gx^*\xrightarrow{\simeq}Ga.T\iota_S Gx^*\longrightarrow T(a\iota_S x^*)=T(x,S,a).$$

This is readily seen to be the (x,S,a)-component of an equipment transformation $e_T:\mathbf{spn}_{\mathcal{K},\mathcal{L}}(T_\#)\longrightarrow T$ with $(e_T)_\#=1_{T_\#}$. It follows that e_T is invertible if and only if all the T_{a,ι_S,x^*} are invertible. From the coherence conditions given in Proposition 2.14 and Lemma 2.18, invertibility of the T_{a,ι_S,x^*} implies invertibility of all the $T_{g,(x,S,a),h^*}$. So e_T is invertible if and only if T is a right homomorphism. Moreover, e_T is natural in T and provides the counit for an adjunction $1,e:\mathbf{spn}_{\mathcal{K},\mathcal{L}}\dashv(-)_\#:{}^*\mathbf{EQT}_*(\mathbf{spn}\mathcal{K},\mathbf{spn}\mathcal{L})\longrightarrow\mathbf{PBK}(\mathcal{K},\mathcal{L})$. We summarize the discussion as follows:

4.5. Proposition. *An arrow $(p,T):\mathbf{spn}\mathcal{K}\longrightarrow\mathbf{spn}\mathcal{L}$ in ${}^*\mathbf{EQT}_*$ is isomorphic to one of the form $\mathbf{spn}G$ for some functor $G:\mathcal{K}\longrightarrow\mathcal{L}$ if and only if T is a right homomorphism, and then G is necessarily isomorphic to $T_\#$. The functor $\mathbf{spn}_{\mathcal{K},\mathcal{L}}$ is fully faithful, every transformation $u:\mathbf{spn}G\longrightarrow\mathbf{spn}H$ being $\mathbf{spn}t$ for a unique $t:G\longrightarrow H:\mathcal{K}\longrightarrow\mathcal{L}$, namely $t=u_\#$.* ∎

From the description of $\mathbf{spn}$ in 4.3, one easily calculates the comparison arrows $\mathbf{spn}G^{(x,S,a),f}$ and $\mathbf{spn}G^{k^*,(x,S,a)}$. For the first, we form the pullbacks

$$\begin{tikzcd} P \arrow[r, "q"] \arrow[d, "p"'] & S \arrow[d, "x"] \\ Y \arrow[r, "f"'] & X \end{tikzcd} \quad , \quad \begin{tikzcd} Q \arrow[r, "s"] \arrow[d, "r"'] & GS \arrow[d, "Gx"] \\ GY \arrow[r, "Gf"'] & GX \end{tikzcd} \quad ,$$

in $\mathcal{K}$ and $\mathcal{L}$ respectively; then

$$\mathbf{spn}G^{(x,S,a),f}:\mathbf{spn}G((x,S,a).f)\longrightarrow\mathbf{spn}G(x,S,a).Gf$$

is the induced arrow of spans given by

$$\begin{tikzcd} GY \arrow[r, "1"] & GY \\ GP \arrow[u, "Gp"] \arrow[r] \arrow[d, "G(aq)"'] & Q \arrow[u, "r"'] \arrow[d, "Ga.s"] \\ GA \arrow[r, "1"'] & GA\,, \end{tikzcd}$$

and $\mathbf{spn}G^{k^*,(x,S,a)}$ is formed similarly. It follows that $\mathbf{spn}G$ is a left homomorphism (and hence a homomorphism) when G preserves pullbacks. Moreover the converse is also true, as we see by taking $a : S \longrightarrow A$ to be $1 : S \longrightarrow S$ in the calculation above. If we start with a general homomorphism of starred pointed equipments $(p, T) : \mathbf{spn}\mathcal{K} \longrightarrow \mathbf{spn}\mathcal{L}$, then $e_T : \mathbf{spn}_{\mathcal{K},\mathcal{L}}(T_\#) \longrightarrow T$ is an isomorphism; and since an isomorph of a homomorphism is again a homomorphism, it follows that $\mathbf{spn}T_\#$ is a homomorphism and hence that $T_\#$ is pullback preserving. We summarize again:

4.6. Proposition. *An arrow $(p, T) : \mathbf{spn}\mathcal{K} \longrightarrow \mathbf{spn}\mathcal{L}$ in ${}^*\mathbf{EQT}_*$ is isomorphic to one of the form $\mathbf{spn}G$ for some pullback-preserving functor $G : \mathcal{K} \longrightarrow \mathcal{L}$ if and only if T is a homomorphism; and then G is necessarily isomorphic to $T_\#$.* ∎

4.7. It is convenient to write $\mathbf{PBK_{pbk}}$ for the locally-full sub-2-category of $\mathbf{PBK}$ determined by the pullback-preserving functors, and ${}^*\mathbf{EQT}_{*\mathbf{hom}}$ [${}^*\mathbf{EQT}_{*\mathbf{rhm}}$, ${}^*\mathbf{EQT}_{*\mathbf{lhm}}$] for the locally-full sub-2-category of ${}^*\mathbf{EQT}_*$ determined by the homomorphisms [right homomorphisms, left homomorphisms]. We may as well regard $\mathbf{spn}$ as a 2-functor $\mathbf{spn} : \mathbf{PBK} \longrightarrow {}^*\mathbf{EQT}_{*\mathbf{rhm}}$, so that the adjunction $1, e : \mathbf{spn}_{\mathcal{K},\mathcal{L}} \dashv (-)_\#$ of 4.4 provides by Proposition 4.5 an (adjoint) equivalence of categories

$$1, e : \mathbf{spn}_{\mathcal{K},\mathcal{L}} \dashv (-)_\# : {}^*\mathbf{EQT}_{*\mathbf{rhm}}(\mathbf{spn}\mathcal{K}, \mathbf{spn}\mathcal{L}) \longrightarrow \mathbf{PBK}(\mathcal{K}, \mathcal{L}).$$

By restriction we have $\mathbf{spn} : \mathbf{PBK_{pbk}} \longrightarrow {}^*\mathbf{EQT}_{*\mathbf{hom}}$ and hence by Proposition 4.6 a further equivalence of categories

$$1, e : \mathbf{spn}_{\mathcal{K},\mathcal{L}} \dashv (-)_\# : {}^*\mathbf{EQT}_{*\mathbf{hom}}(\mathbf{spn}\mathcal{K}, \mathbf{spn}\mathcal{L}) \longrightarrow \mathbf{PBK_{pbk}}(\mathcal{K}, \mathcal{L}).$$

4.8. For any 2-category $\mathbf{K}$ and objects $\mathcal{K}$ and $\mathcal{L}$ therein, we write $\mathbf{LAJ}(\mathbf{K})(\mathcal{L}, \mathcal{K})$ for the evident category whose objects are adjunctions $x, y : F \dashv G : \mathcal{K} \longrightarrow \mathcal{L}$ and whose arrows from $x, y : F \dashv G$ to $x', y' : F' \dashv G'$ are transformations $F \longrightarrow F'$. Since any 2-functor, such as $\mathbf{spn} : \mathbf{PBK} \longrightarrow {}^*\mathbf{EQT}_*$, takes adjunctions to adjunctions, we have a functor

$$\mathbf{spn}_{\mathcal{L},\mathcal{K}} : \mathbf{LAJ}(\mathbf{PBK})(\mathcal{L}, \mathcal{K}) \longrightarrow \mathbf{LAJ}({}^*\mathbf{EQT}_*)(\mathbf{spn}\mathcal{L}, \mathbf{spn}\mathcal{K}).$$

Since $(-)_\#$ is a 2-functor we have also a functor

$$(-)_\# : \mathbf{LAJ}({}^*\mathbf{EQT}_*)(\mathbf{spn}\mathcal{L}, \mathbf{spn}\mathcal{K}) \longrightarrow \mathbf{LAJ}(\mathbf{PBK})(\mathcal{L}, \mathcal{K}),$$

for which the e of 4.4 lifts to provide an adjunction $1, e : \mathbf{spn}_{\mathcal{K},\mathcal{L}} \dashv (-)_\#$, where the $(u, v : S \dashv T)$-component of this new e is the S-component of the former one.

4.9. Proposition. *The adjunction*

$$1, e : \mathbf{spn}_{\mathcal{L},\mathcal{K}} \dashv (-)_\# : \mathbf{LAJ}({}^*\mathbf{EQT}_*)(\mathbf{spn}\mathcal{L}, \mathbf{spn}\mathcal{K}) \longrightarrow \mathbf{LAJ}(\mathbf{PBK})(\mathcal{L}, \mathcal{K})$$

is an equivalence of categories.

Proof. For an object $u, v : S \dashv T$ in $\mathbf{LAJ}({}^*\mathbf{EQT}_*)(\mathbf{spn}\mathcal{L}, \mathbf{spn}\mathcal{K})$, we have only to show that $e_{S \dashv T} = e_S$ is invertible, which is the case if and only if S is a right homomorphism. From Proposition 2.21, however, any left adjoint in ${}^*\mathbf{EQT}_*$ is a right homomorphism. ■

Moreover, the right-adjoint component T of the object $S \dashv T$ in the proof of Proposition 4.9 is clearly isomorphic to $\mathbf{spn}(T_\#)$. So, it is like S a right homomorphism, and the adjunction $T \dashv U$ in ${}^*\mathbf{EQT}_*$ is also an adjunction in ${}^*\mathbf{EQT}_{*\mathbf{rhm}}$. Thus:

4.10. Corollary. *The inclusion of categories*

$$\mathbf{LAJ}({}^*\mathbf{EQT}_{*\mathbf{rhm}})(\mathbf{spn}\mathcal{L}, \mathbf{spn}\mathcal{K}) \longrightarrow \mathbf{LAJ}({}^*\mathbf{EQT}_*)(\mathbf{spn}\mathcal{L}, \mathbf{spn}\mathcal{K})$$

is an equality.

Combining Propositions 4.6 and 4.9 brings us to the image under $\mathbf{spn}$ of adjunctions $F \dashv G : \mathcal{K} \longrightarrow \mathcal{L}$ in $\mathbf{PBK}$ with F pullback-preserving — which is to say adunctions in $\mathbf{PBK}_{\mathbf{pbk}}$. (Such adjunctions were studied in [R&W] from points of view that enabled them to be seen both as *partial geometric morphisms* and *families of geometric morphisms.*) Clearly, they correspond to adjunctions $S \dashv T : \mathbf{spn}\mathcal{K} \longrightarrow \mathbf{spn}\mathcal{L}$ in ${}^*\mathbf{EQT}_*$ with S a homomorphism. On the face of it, such an adjunction is in ${}^*\mathbf{EQT}_{*\mathbf{lhm}}$; but we have already observed that T is also a right homomorphism. We conclude that:

4.11. Corollary. *The inclusion of categories*

$$\mathbf{LAJ}({}^*\mathbf{EQT}_{*\mathbf{hom}})(\mathbf{spn}\mathcal{L}, \mathbf{spn}\mathcal{K}) \longrightarrow \mathbf{LAJ}({}^*\mathbf{EQT}_{*\mathbf{rhm}})(\mathbf{spn}\mathcal{L}, \mathbf{spn}\mathcal{K})$$

is an equality. ■

4.12. Corollary. *The adjunction*

$$\mathbf{spn}_{\mathcal{L},\mathcal{K}} \dashv (-)_\# : \mathbf{LAJ}({}^*\mathbf{EQT}_{*\mathbf{hom}})(\mathbf{spn}\mathcal{L}, \mathbf{spn}\mathcal{K}) \longrightarrow \mathbf{LAJ}(\mathbf{PBK}_{\mathbf{pbk}})(\mathcal{L}, \mathcal{K})$$

is an equivalence of categories. ■

4.13. Suppose that $\mathcal{K}$ is a category with pullbacks and $\mathcal{C}$-limits. The adjunction in **CAT** which provides the $\mathcal{C}$-limits has for its left-adjoint component the diagonal $\mathcal{K} \longrightarrow \mathcal{K}^{\mathcal{C}}$ and can be seen as an adjunction in $\mathbf{PBK_{pbk}}$. The 2-functors $\mathbf{spn} : \mathbf{PBK} \longrightarrow {}^*\mathbf{EQT}_*$ and $\mathbf{spn} : \mathbf{PBK_{pbk}} \longrightarrow {}^*\mathbf{EQT}_{*\mathbf{hom}}$ can be shown to preserve powers. (Powers in **PBK** and $\mathbf{PBK_{pbk}}$ are as in **CAT**. Powers in ${}^*\mathbf{EQT}_{*\mathbf{hom}}$ are as in ${}^*\mathbf{EQT}_*$.) It follows from 3.22 and the above that if $\mathcal{K}$ has $\mathcal{C}$-limits then $\mathbf{spn}\mathcal{K}$ has $\mathcal{C}$-limits as a pointed equipment, and that the resulting $\mathbf{spn}\mathcal{K}^{\mathcal{C}} \longrightarrow \mathbf{spn}\mathcal{K}$ is a homomorphism. For a general starred pointed equipment $\mathcal{M}$ which has $\mathcal{C}$-limits, there is no reason to suppose that $\mathcal{M}^{\mathcal{C}} \longrightarrow \mathcal{M}$ is a *right* homomorphism.

For $T : \mathcal{M} \longrightarrow \mathcal{N}$ in any 2-category with powers, and a category $\mathcal{C}$, we have a commutative square of arrows

$$
\begin{tikzcd}
\mathcal{M} \arrow[r, "D"] \arrow[d, "T"'] \arrow[dr, phantom, "\overset{1}{\Longrightarrow}"] & \mathcal{M}^{\mathcal{C}} \arrow[d, "T^{\mathcal{C}}"] \\
\mathcal{N} \arrow[r, "D"'] & \mathcal{N}^{\mathcal{C}}
\end{tikzcd}
\quad ,
$$

where the D's are diagonals. If each of the diagonals has a right adjoint L, then the identity transformation has a mate

$$
\begin{tikzcd}
\mathcal{M} \arrow[d, "T"'] \arrow[dr, phantom, "\overset{\tilde{T}}{\longrightarrow}"] & \mathcal{M}^{\mathcal{C}} \arrow[l, "L"'] \arrow[d, "T^{\mathcal{C}}"] \\
\mathcal{N} & \mathcal{N}^{\mathcal{C}} \arrow[l, "L"]
\end{tikzcd}
\quad .
$$

4.14. Definition. *If, in a 2-category with $\mathcal{C}$-powers, objects $\mathcal{M}$ and $\mathcal{N}$ have $\mathcal{C}$-limits in the sense above, then an arrow $T : \mathcal{M} \longrightarrow \mathcal{N}$ is said to* preserve $\mathcal{C}$-limits *if $\tilde{T}$ is invertible.* ∎

4.15. In section 5 of [CKW] a very detailed general discussion of preservation is given for finite products. From that discussion it follows easily that *(i) if $G : \mathcal{K} \longrightarrow \mathcal{L}$ in* **PBK** *preserves $\mathcal{C}$-limits then* $\mathbf{spn}G$ *in* ${}^*\mathbf{EQT}_*$ *preserves $\mathcal{C}$-limits* and *(ii) right adjoints in* ${}^*\mathbf{EQT}_*$ *preserve $\mathcal{C}$-limits.*

In the case of pointed equipments, Definition 4.14 requires that, for each vector arrow $\theta : F \longrightarrow G$ in $\mathcal{M}^{\mathcal{C}}$, the square

$$
\begin{tikzcd}
TLF \arrow[r, "\tilde{T}F"] \arrow[d, "TL\theta"'] \arrow[dr, phantom, "\overset{\tilde{T}\theta}{\longrightarrow}"] & LT^{\mathcal{C}}F \arrow[d, "LT^{\mathcal{C}}\theta"] \\
TLG \arrow[r, "\tilde{T}G"'] & LT^{\mathcal{C}}F
\end{tikzcd}
$$

be an isomorphism in $\mathbf{gr}\mathcal{N}$. By Lemma 2.6, this is equivalent to asking for invertibility of each $\tilde{T}F$ and commutativity of each such square. Invertibility of the $\tilde{T}F$ means precisely that $T_\#$ preserves $\mathcal{C}$-limits. In the presence of this condition, for pointed T between starred pointed equipments, commutativity of the squares is equivalent to preservation of $\mathcal{C}$-limits by the $T_{K,L} : \mathcal{M}(K, L) \longrightarrow \mathcal{N}(TK, TL)$ — apparently an additional condition.

However, in the case of starred pointed equipments with $\mathcal{C}$ the empty category, a pointed T for which $T_\#$ preserves $\mathcal{C}$-limits does itself preserve $\mathcal{C}$-limits. For here we are speaking of terminal objects and the only square under consideration is

$$\begin{tikzcd} T_\# 1 \arrow[r] \arrow[d, "T\iota_1"'] \arrow[dr, phantom, "\longrightarrow"] & 1 \arrow[d, "\iota_1"] \\ T_\# 1 \arrow[r] & 1 \end{tikzcd},$$

where each 1 denotes a terminal object in the appropriate category of scalars. Recall from the discussion in 3.22 that ι_1 is the terminal object in $\mathcal{N}(1,1)$. For any $r : X \longrightarrow A$ in $\mathcal{N}_\#$ the (ordinary) functor $(-)r$ preserves limits because it has the left adjoint $(-)r^*$. Thus, in the diagram above, the upper leg is the terminal object in $\mathcal{N}(T_\# 1, 1)$ and the vector transformation is unambiguous. Moreover, the square

$$\begin{tikzcd} T_\# 1 \arrow[r] \arrow[d, "\iota_{T_\# 1}"'] \arrow[dr, phantom, "\longrightarrow"] & 1 \arrow[d, "\iota_1"] \\ T_\# 1 \arrow[r] & 1 \end{tikzcd}$$

is commutative, and the square above is obtained from it by pasting the isomorphism $T\iota_1 \xrightarrow{\simeq} \iota_{T_\# 1}$ at the left edge. Thus an arrow T in ${}^*\mathbf{EQT}_*$ preserves terminal objects if and only if $T_\# 1 \longrightarrow 1$ is invertible.

4.16. For any 2-category $\mathbf{K}$ and objects $\mathcal{K}$ and $\mathcal{L}$ therein admitting terminal objects, let us write $\mathbf{K_1}(\mathcal{K}, \mathcal{L})$ for the full subcategory of $\mathbf{K}(\mathcal{K}, \mathcal{L})$ determined by the arrows $\mathcal{K} \longrightarrow \mathcal{L}$ that preserve terminal objects. Write $\mathbf{LEX}$ for the full sub-2-category of $\mathbf{PBK}$ determined by the categories which have terminal objects. Then the familiar 2-category $\mathbf{LEX_{lex}}$ of finitely complete categories, left exact functors and arbitrary natural transformations has $\mathbf{LEX_{lex}}(\mathcal{K}, \mathcal{L}) = \mathbf{PBK_{pbk,1}}(\mathcal{K}, \mathcal{L})$. The following proposition then follows immediately from the last equivalence in 4.7 and the conclusion of 4.15:

4.17. Proposition. *For categories $\mathcal{K}$ and $\mathcal{L}$ with finite limits, the adjunction*

$$\mathbf{spn}_{\mathcal{K},\mathcal{L}} \dashv (-)_\# : {}^*\mathbf{EQT}_{*\mathbf{hom},\mathbf{1}}(\mathbf{spn}\mathcal{K}, \mathbf{spn}\mathcal{L}) \longrightarrow \mathbf{LEX_{lex}}(\mathcal{K}, \mathcal{L})$$

is an equivalence of categories. ∎

By a *geometric morphism* $\mathcal{K} \longrightarrow \mathcal{L}$ between categories with finite limits we understand an adjunction $x, y : F \dashv G : \mathcal{K} \longrightarrow \mathcal{L}$ with F left exact. Write $\mathbf{LEX_{geo}}$ for the 2-category of finitely-complete categories and geometric morphisms. Now $\mathbf{LEX_{geo}}(\mathcal{K}, \mathcal{L}) = \mathbf{LAJ}(\mathbf{PBK_{pbk,1}})(\mathcal{L}, \mathcal{K})$, and the following proposition follows immediately from Corollary 4.12.

4.18. PROPOSITION. *The adjunction*

$$\mathbf{spn}_{\mathcal{L},\mathcal{K}} \dashv (-)_{\#} : \mathbf{LAJ}({}^{*}\mathbf{EQT_{*hom,1}})(\mathbf{spn}\mathcal{L}, \mathbf{spn}\mathcal{K}) \longrightarrow \mathbf{LEX_{geo}}(\mathcal{K}, \mathcal{L})$$

is an equivalence of categories. ■

5. Appendix

Let $\mathbf{K}$ be a 2-category and $\mathbf{L}$ a dual KZ-doctrine (in the sense of [KCK]) on $\mathbf{K}$, with the 2-natural transformation $\lambda : 1_{\mathbf{K}} \longrightarrow \mathbf{L}$ as its unit. An $\mathbf{L}$-algebra is an object $\mathcal{M}$ of $\mathbf{K}$ together with an arrow $M : \mathbf{L}\mathcal{M} \longrightarrow \mathcal{M}$ which is a reflection right adjoint for $\lambda\mathcal{M} : \mathcal{M} \longrightarrow \mathbf{L}\mathcal{M}$. If $(\mathcal{M}, M)$ and $(\mathcal{N}, N)$ are two such algebras and $T : \mathcal{M} \longrightarrow \mathcal{N}$ is any arrow in $\mathbf{K}$, the identity square $1 : \lambda\mathcal{N}.T \longrightarrow \mathbf{L}T.\lambda\mathcal{M}$ (given by the naturality of λ) has, under the adjunctions $\lambda\mathcal{M} \dashv M$ and $\lambda\mathcal{N} \dashv N$, a mate

$$\begin{tikzcd} \mathbf{L}\mathcal{M} \arrow[r, "\mathbf{L}T"] \arrow[d, "M"'] \arrow[dr, phantom, "\Uparrow T^{-,-}"] & \mathbf{L}\mathcal{N} \arrow[d, "N"] \\ \mathcal{M} \arrow[r, "T"'] & \mathcal{N} \end{tikzcd}$$

called the *canonical comparison* for T; and T is said to be an $\mathbf{L}$-*homomorphism* precisely when $T^{-,-}$ is invertible.

Proposition 2.5 of [KCK] asserts that such a T is certainly an $\mathbf{L}$-homomorphism if it has a left adjoint, and the proof of that proposition describes the inverse of $T^{-,-}$ explicitly in terms of the adjunction $\eta, \epsilon : S \dashv T$. In fact the inverse $t^{-,-} : N.\mathbf{L}T \longrightarrow T.M$ of $T^{-,-}$ is itself the mate, under the adjunctions $S \dashv T$ and $\mathbf{L}S \dashv \mathbf{L}T$, of the canonical comparison $S^{-,-} : S.N \longrightarrow M.\mathbf{L}S$ for S.

Take $\mathbf{K}$ to be $[\Lambda, \mathbf{CAT}]$, or $\mathbf{spnCAT}(\mathcal{L}, \mathcal{K})$ as in 2.15. In either case define

$$\mathbf{L}(\mathcal{K} \xleftarrow{\partial_0} \mathcal{M} \xrightarrow{\partial_1} \mathcal{L})$$

to be the span from $\mathcal{L}$ to $\mathcal{K}$ given by

$$\begin{tikzcd}
& \mathcal{K}/\partial_0 \arrow[dl] \arrow[dr] & & \\
\mathcal{K} \arrow[dr, "1"'] \arrow[rr, phantom, "\longrightarrow"] & & \mathcal{M} \arrow[dl, "\partial_0"] \arrow[dr, "\partial_1"] & \\
& \mathcal{K} & & \mathcal{L}
\end{tikzcd} ,$$

where the square is a comma square. Define $\lambda\mathcal{M} : \mathcal{M} \longrightarrow \mathbf{L}\mathcal{M}$ in $\mathbf{spnCAT}(\mathcal{L}, \mathcal{K})$ to be the functor which sends μ to $(\partial_0\mu, 1_{\partial_0\mu}, \mu)$. It is well known that $\mathbf{L}$ extends to a 2-functor that underlies a dual KZ-doctrine, the algebras for which are the left-fibration spans. Among these are the spans of the form $\widehat{\mathbf{gr}}\mathcal{M}$, for a module $\mathcal{M}$, and these now constitute our sole interest. Consistently with the notation of the paper, let us write $\mathcal{M}$ for $\widehat{\mathbf{gr}}\mathcal{M}$ and

$$K' \xrightarrow{k} K \overset{\mu}{\nrightarrow} L$$

for a typical object of $\mathbf{L}\mathcal{M}$.

The structure arrow $M : \mathbf{L}\mathcal{M} \longrightarrow \mathcal{M}$ is the action functor that sends $K' \xrightarrow{k} K \overset{\mu}{\nrightarrow} L$ to $K' \overset{\mu k}{\nrightarrow} L$. Observe that, for k and L fixed, M restricts to the functor $\mathcal{M}(k, L)$. Moreover if $T : \mathcal{M} \longrightarrow \mathcal{N}$ is an arrow in $\mathbf{MOD}$, and thus an arrow $T : \widehat{\mathbf{gr}}\mathcal{M} \longrightarrow \widehat{\mathbf{gr}}\mathcal{N}$ in $[\Lambda, \mathbf{CAT}]$, then the (k, μ)-component of $T^{-,-}$, as in the first square of this appendix, is the action comparison $T^{\mu,k} : T(\mu k) \longrightarrow T\mu.Tk$. If T is in $\mathcal{K}, \mathcal{L}$-$\mathbf{MOD}$, as in 2.37, then the scalar components of T are identity functors, so that we have $T^{\mu,k} : T(\mu k) \longrightarrow T\mu.k$. This being so, it follows from the general result above that $(T^{\mu,k})^{-1}$ is indeed the $t^{\mu,k}$ of 2.37.

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