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THE SEPARATED EXTENSIONAL CHU CATEGORY

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ABSTRACT. This paper shows that, given a factorization system, $\mathcal{E}/\mathcal{M}$ on a closed symmetric monoidal category, the full subcategory of separated extensional objects of the Chu category is also $*$-autonomous under weaker conditions than had been given previously ([Barr, 1991)]. In the process we find conditions under which the intersection of a full reflective subcategory and its coreflective dual in a Chu category is $*$-autonomous.

1. Introduction

1.1. Chu categories. An appendix to [Barr, 1979] was an extract from the master's thesis of P.-H. Chu that described what seemed at the time a too-simple-to-be-interesting construction of $*$-autonomous categories [Chu, 1979]. In fact, this construction, now called the Chu construction has turned out to be surprisingly interesting, both as a way of providing models of Girard's linear logic [Seely, 1988], in theoretical computer science [Pratt, 1993a, 1993b, 1995] and as a general approach to duality [Barr and Kleisli, to apear] and [Schläpfer, 1998]..

Given an autonomous (symmetric, closed monoidal) category $\mathcal{A}$ and an object $\perp$ of $\mathcal{A}$, the category we denote $\mathrm{Chu}(\mathcal{A},\perp)$ has as objects pairs (A_1, A_2) of objects equipped with a pairing $A_1 \otimes A_2 \longrightarrow \perp$. An arrow $(f_1, f_2) : (A_1, A_2) \longrightarrow (B_1, B_2)$ consists of arrows of $\mathcal{A}$, $f_1 : A_1 \longrightarrow B_1$ and $f_2 : B_2 \longrightarrow A_2$ such that the square

$$\begin{tikzcd}
A_1 \otimes B_2 \arrow[r, "f_1 \otimes B_2"] \arrow[d, "A_1 \otimes f_2"'] & B_1 \otimes B_2 \arrow[d] \\
A_1 \otimes A_2 \arrow[r] & \perp
\end{tikzcd}$$

commutes. It is evident that the endofunctor that interchanges the two components is a contravariant equivalence, so that $\mathrm{Chu}(\mathcal{A},\perp)$ is a self-dual category. Less obvious, but true is that, provided $\mathcal{A}$ has pullbacks, $\mathrm{Chu}(\mathcal{A},\perp)$ is also an autonomous category so that it is, in fact, a $*$-autonomous category. The object $\perp$ will be called the *dualizing object* of the Chu construction. Details can all be found in [Chu, 1979].

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1.2. Separated and extensional objects. Now suppose $\mathcal{A}$ is an autonomous category with pullbacks and $\mathcal{E}/\mathcal{M}$ is a factorization system in $\mathcal{A}$. See Section 5 for definitions and elementary properties of factorization systems. Let $\perp$ be an arbitrary object of $\mathcal{A}$, which we call our dualizing object. We will write Chu for $\mathrm{Chu}(\mathcal{A}, \perp)$ and write $A^{\perp} = A \multimap \perp$ for an object A of $\mathcal{A}$. We say that an object (A_1, A_2) is $\mathcal{M}$-*separated*, or simply separated if the transpose $A_1 \longrightarrow {A_2}^{\perp}$ of the structure arrow $A_1 \otimes A_2 \longrightarrow \perp$ belongs to $\mathcal{M}$ and $\mathcal{M}$-*extensional*, or simply extensional, if the other transpose $A_2 \longrightarrow {A_1}^{\perp}$ does.

We denote the full subcategories of separated objects by $\mathrm{Chu}_s = \mathrm{Chu}_s(\mathcal{A}, \perp)$ and of extensional objects by $\mathrm{Chu}_e = \mathrm{Chu}_e(\mathcal{A}, \perp)$. We follow Pratt in denoting the full subcategory of objects that are both separated and extensional by $\mathrm{chu} = \mathrm{chu}(\mathcal{A}, \perp)$. Since Chu_s is evidently the dual of Chu_e, it is immediate that chu is self-dual. It is useful to ask if chu is also $*$-autonomous. In [Barr, 1991], I showed the following result (more or less). Suppose $\mathcal{A}$ is an autonomous category with pullbacks, D is an object of $\mathcal{A}$ and $\mathcal{E}/\mathcal{M}$ is a factorization system on $\mathcal{A}$ that satisfies the following conditions

FS–1. Every arrow in $\mathcal{E}$ is an epimorphism;

FS–2. if $m \in \mathcal{M}$, then for any object A of $\mathcal{A}$, the induced $A \multimap m$ is in $\mathcal{M}$;

FS–3. if $e \in \mathcal{E}$ then for any object A of $\mathcal{A}$, the induced $e \multimap A$ is in $\mathcal{M}$.

Then the category $\mathrm{chu}(\mathcal{A}, D)$ of separated, extensional objects (with respect to $\mathcal{M}$) in $\mathrm{Chu}(\mathcal{A}, D)$ is a $*$-autonomous category.

The main purpose of this note is to show that FS–3 is unnecessary. We also show two conditions that are equivalent to FS–3 and draw a consequence. However, FS–1 and FS–2 still appear to be needed. It should be observed that although FS–2 and FS–3 seem in some sense to be dual, they are actually of a quite different character. For example, both extremal monics and all monics are automatically preserved by right adjoint functors, so that FS–2 is automatic in those cases. But, for example, when $\mathcal{M}$ is the extremal monics, $\mathcal{E}$ consists of all epics. While it might be reasonable that the duality take epics to monics, it seems unlikely that they take them to extremal monics.

2. Reflections in $*$-autonomous categories

2.1. We begin by treating a somewhat common situation in a $*$-autonomous category. Suppose $\mathcal{A}$ is a such a category and $\mathcal{A}_l$ is a reflective full subcategory with reflector l. Assume that $\mathcal{A}_l$ is isomorphism closed, although that is no fundamental importance, but simplifies the treatment. The full subcategory $\mathcal{A}_r = {\mathcal{A}_l}^{\perp}$ is quite evidently a coreflective subcategory with $rA = (lA^{\perp})^{\perp}$ as coreflector. Let $\mathcal{A}_{rl} = \mathcal{A}_r \cap \mathcal{A}_l$. Assume that when A is an object of $\mathcal{A}_r$, then so lA, which is then an object of $\mathcal{A}_{rl}$. This is readily seen to imply that when A is an object of $\mathcal{A}_l$, so is rA. It is also immediate that under these conditions, l and r induce left and right adjoints, resp., to the inclusions of $\mathcal{A}_{rl} \longrightarrow \mathcal{A}_r$ and $\mathcal{A}_{rl} \longrightarrow \mathcal{A}_l$.

2.2. Proposition. *Under the above assumptions* $\mathcal{A}_r \otimes \mathcal{A}_r \subseteq \mathcal{A}_r$ *if and only if* $\mathcal{A}_r \multimap \mathcal{A}_l \subseteq \mathcal{A}_l$.

PROOF. These follow immediately from the identities, valid in any $*$-autonomous category

$$A \multimap B \cong (A \otimes B^{\perp})^{\perp}$$

$$A \otimes B = (A \multimap B^{\perp})^{\perp} \qquad \square$$

2.3. THEOREM. *Suppose $\mathcal{A}$ is a $*$-autonomous category and $\mathcal{A}_l$ is a reflective subcategory whose reflector l leaves ${\mathcal{A}_l}^{\perp}$ invariant as in 2.1. Suppose that the unit object belongs to $\mathcal{A}_r$ and that the equivalent conditions of 2.2 are satisfied. Then $\mathcal{A}_{rl}$ is a $*$-autonomous category with unit object $l\top$, dualizing object $r\perp$, tensor $A \boxtimes B = l(A \otimes B)$ and $A \multimap\!\!\square\, B = r(A \multimap B)$.*

PROOF. We have, for any objects A and B of $\mathcal{A}_{rl}$, using the facts that B and $A \multimap B$ belong to $\mathcal{A}_l$,

$$\begin{aligned} \mathrm{Hom}(l\top \boxtimes A, B) &\cong \mathrm{Hom}(l(l\top \otimes A), B) \cong \mathrm{Hom}(l\top \otimes A, B) \\ &\cong \mathrm{Hom}(l\top, A \multimap B) \cong \mathrm{Hom}(\top, A \multimap B) \cong \mathrm{Hom}(A, B) \end{aligned}$$

and so, by Yoneda, we conclude that $l\top \boxtimes A \cong A$.

We have, for any objects A, B, C, and D of $\mathcal{A}_{rl}$,

$$\begin{aligned} \mathrm{Hom}(A, (B \boxtimes C) \multimap\!\!\square\, D) &\cong \mathrm{Hom}(A, r((B \boxtimes C) \multimap D)) \cong \mathrm{Hom}(A, (B \boxtimes C) \multimap D) \\ &\cong \mathrm{Hom}(A \otimes (B \boxtimes C), D) \cong \mathrm{Hom}((B \boxtimes C), A \multimap D) \\ &\cong \mathrm{Hom}(l(B \otimes C), A \multimap D) \cong \mathrm{Hom}(B \otimes C, A \multimap D) \\ &\cong \mathrm{Hom}(A \otimes B \otimes C, D) \end{aligned}$$

and

$$\begin{aligned} \mathrm{Hom}(A, B \multimap\!\!\square\, (C \multimap\!\!\square\, D)) &\cong \mathrm{Hom}(A, r(B \multimap (C \multimap\!\!\square\, D))) \cong \mathrm{Hom}(A, B \multimap (C \multimap\!\!\square\, D)) \\ &\cong \mathrm{Hom}(A \otimes B, C \multimap\!\!\square\, D) \cong \mathrm{Hom}(A \otimes B, r(C \multimap D)) \\ &\cong \mathrm{Hom}(A \otimes B, C \multimap D) \cong \mathrm{Hom}(A \otimes B \otimes C, D) \end{aligned}$$

By letting $A = l\top$, we see that $\mathrm{Hom}(B \boxtimes C, D) \cong \mathrm{Hom}(B, C \multimap\!\!\square\, D)$ and by Yoneda we get $(B \boxtimes C) \multimap\!\!\square\, D \cong B \multimap\!\!\square\, (C \multimap\!\!\square\, D)$, which is the internal version and implies the associativity of the tensor. Thus we have a $*$-autonomous category.

3. The separated extensional subcategory

We apply the above construction to show that under the conditions FS–1 and FS–2 on a factorization system $\mathcal{E}/\mathcal{M}$, the full subcategory of $\mathcal{M}$-separated and $\mathcal{M}$-extensional objects forms a $*$-autonomous category.

3.1. THEOREM. *Suppose the category $\mathcal{A}$ has pullbacks and $\mathcal{E}/\mathcal{M}$ is a factorization system that satisfies FS–1 and FS–2. Then for any object D, the category* chu$(\mathcal{A}, D)$ *of $\mathcal{M}$-separated, and $\mathcal{M}$-extensional objects of* Chu *is $*$-autonomous.*

We give the proof as a series of propositions.

3.2. PROPOSITION. *The inclusion of* $\mathrm{Chu}_s \longrightarrow \mathrm{Chu}$ *has a left adjoint.*

PROOF. Let $\mathbf{A} = (A_1, A_2)$ be an object of Chu. Factor $A_1 \longrightarrow {A_2}^{\perp}$ as $A_1 \twoheadrightarrow \widetilde{A}_1 \rightarrowtail {A_2}^{\perp}$, where we adopt the usual notation of writing $\twoheadrightarrow$ for an arrow of $\mathcal{E}$ and $\rightarrowtail$ for an arrow of $\mathcal{M}$. Then $s\mathbf{A} = (\widetilde{A}_1, A_2)$ is a separated object and we have an obvious map $\mathbf{A} \longrightarrow s\mathbf{A}$. If $(f_1, f_2) : (A_1, A_2) \longrightarrow (B_1, B_2)$ is a map in $\mathrm{Chu}(\mathcal{A}, K)$ and (B_1, B_2) is separated, the unique diagonal fill-in in the square

$$\begin{tikzcd}
A_1 \arrow[r, two heads] \arrow[dd] & \widetilde{A}_1 \arrow[d] \\
 & {A_2}^{\perp} \arrow[d] \\
B_1 \arrow[r, tail] & {B_2}^{\perp}
\end{tikzcd}$$

gives the required map $s\mathbf{A} \longrightarrow \mathbf{B}$. ∎

It follows that the inclusion $\mathrm{Chu}_e \longrightarrow \mathrm{Chu}$ has a right adjoint, which we denote e.

3.3. PROPOSITION. *FS–1 implies that when the object (A_1, A_2) of* $\mathrm{Chu}(\mathcal{A}, D)$ *is separated, so is $e(A_1, A_2)$ and similarly if (A_1, A_2) is extensional, so is $s(A_1, A_2)$.*

PROOF. Suppose that $\mathbf{A} = (A_1, A_2)$ is a separated object. Then $e\mathbf{A} = (A_1, \widetilde{A}_2)$ where $A_2 \twoheadrightarrow \widetilde{A}_2 \rightarrowtail {A_1}^{\perp}$. By transposing, this gives $A_1 \longrightarrow (\widetilde{A}_2)^{\perp} \longrightarrow {A_2}^{\perp}$ whose composite belongs to $\mathcal{M}$ and so, by Proposition 5.6, does the first factor $A_1 \longrightarrow (\widetilde{A}_2)^{\perp}$ which means that $e\mathbf{A}$ is still separated. The proof that $s\mathbf{A}$ is extensional when $\mathbf{A}$ is is similar (also dual). ∎

3.4. PROPOSITION. *FS–2 implies that when $\mathbf{A}$ and $\mathbf{B}$ are extensional, so is $\mathbf{A} \otimes \mathbf{B}$.*

PROOF. Let $\mathbf{A} = (A_1, A_2)$ and $\mathbf{B} = (B_1, B_2)$ be extensional. The second component of $\mathbf{A} \otimes \mathbf{B}$ is the pullback

$$\begin{tikzcd}
P \arrow[r] \arrow[d] & A_1 \multimap B_2 \arrow[d] \\
B_1 \multimap A_2 \arrow[r] & B_1 \multimap {A_1}^{\perp} \cong A_1 \multimap {B_1}^{\perp} \cong (A_1 \otimes B_1)^{\perp}
\end{tikzcd}$$

It follows from FS–2 that both $B_1 \multimap A_2 \longrightarrow B_1 \multimap {A_1}^{\perp}$ and $A_1 \multimap B_2 \longrightarrow A_1 \multimap {B_1}^{\perp}$ lie in $\mathcal{M}$. But it is a general property of factorization systems that $\mathcal{M}$ is closed under pullback and composition, whence the arrow $P \longrightarrow (A_1 \otimes B_1)^{\perp}$ also lies in $\mathcal{M}$. ∎

The results of Section 2 now prove the theorem.

4. About FS–3

In this section, we take a brief look at FS–3. Although it is not necessary for our main theorem, it has the consequence that $se = es$. This obviously implies that the separated objects are invariant under the extensional coreflection and vice versa, but, as we have seen, that is true under the hypothesis FS–1. Thus our main results are true under the hypotheses FS–2 and either FS–1 or FS–3.

4.1. PROPOSITION. *the following three conditions are equivalent:*

FS–3. *if $e \in \mathcal{E}$ then for any object A of $\mathcal{A}$, the induced $e \multimap A$ is in $\mathcal{M}$;*

FS–4. *if $e \in \mathcal{E}$, then for any object B, the induced $e \otimes B \in \mathcal{E}$;*

FS–5. *If $m : A \longrightarrow A' \in \mathcal{M}$ and $e : C' \longrightarrow C \in \mathcal{E}$, then*

$$\begin{tikzcd}
C \multimap A \arrow[r, "e \multimap A"] \arrow[d, "C \multimap m"'] & C' \multimap A \arrow[d, "C' \multimap m"] \\
C \multimap A' \arrow[r, "e \multimap A'"'] & C' \multimap A'
\end{tikzcd}$$

is a pullback;

and imply that $es = se$.

PROOF. The filling in of the commutative diagram

$$\begin{tikzcd}
B \arrow[drr] \arrow[dddr] & & \\
& C \multimap A \arrow[r] \arrow[dd] & C' \multimap A \arrow[dd] \\
& & \\
& C \multimap A' \arrow[r] & C' \multimap A'
\end{tikzcd}$$

by a map $B \longrightarrow C \multimap A$ is equivalent, using transpose, to a diagonal fill-in in either of the squares:

$$\begin{tikzcd}
C' \otimes B \arrow[r] \arrow[d] & C \otimes B \arrow[d] \\
A \arrow[r, tail] & A'
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
C' \arrow[r, two heads] \arrow[d] & C \arrow[d] \\
B \multimap A \arrow[r] & B \multimap A'
\end{tikzcd}$$

Thus from FS–5 we can infer both FS–3 and FS–4, while either of FS–3 or FS–4 similarly allows us to conclude FS–5.

Suppose that $\mathbf{A} = (A_1, A_2)$ is an object of Chu. Then $s\mathbf{A} = (\widetilde{A}_1, A_2)$, where

$$A_1 \twoheadrightarrow \widetilde{A}_1 \rightarrowtail {A_2}^{\perp}$$

is the $\mathcal{E}/\mathcal{M}$ factorization. We compute $e\mathbf{A} = (A_1, \widetilde{A}_2)$ using the factorization

$$A_2 \twoheadrightarrow \widetilde{A}_2 \rightarrowtail {A_1}^{\perp}$$

Assuming FS–3, we have that $\widetilde{A}_1{}^{\perp} \rightarrowtail {A_1}^{\perp}$ which gives us the square

$$\begin{tikzcd} A_2 \arrow[r, two heads] \arrow[d] & \widetilde{A}_2 \arrow[d, tail] \\ \widetilde{A}_1{}^{\perp} \arrow[r, tail] & {A_1}^{\perp} \end{tikzcd}$$

whose diagonal fill-in belongs to $\mathcal{M}$ by 5.6. But then it follows that $A_2 \twoheadrightarrow \widetilde{A}_2 \rightarrowtail \widetilde{A}_1{}^{\perp}$ is also an $\mathcal{E}/\mathcal{M}$ factorization, which implies that $es\mathbf{A} = (\widetilde{A}_1, \widetilde{A}_2)$. Dualizing, we conclude that $se\mathbf{A}$ is exactly the same. ∎

4.2. An example. Here is an example using Banach spaces (see [Kleisli, et al, 1996] for the $*$-autonomous category of Banach spaces) in which FS–3 fails and $es \neq se$, while FS–1 and FS–2 hold so the results of Section 3 are valid. Take the category of Banach spaces and linear contractions. It is a closed monoidal category with bounded linear maps and sup norm on the unit ball as the closed structure. The tensor product is the one with greatest cross-norm. Take as factorization system the epics and regular monics. The epics are actually maps with dense image and the regular monics are the closed isometric inclusions. Let ℓ^p denote the space of absolutely pth power summable sequences. Then there is a Chu object (ℓ^1, ℓ^1) with the inner product $\langle a_i, b_i \rangle = \sum a_i b_i$. Then it is not hard to see that $s(\ell^1, \ell^1) = (\ell^\infty, \ell^1)$ and that is separated *and* extensional, so that $es(\ell^1, \ell^1) = (\ell^\infty, \ell^1)$, while for exactly the same reason, $se(\ell^1, \ell^1) = (\ell^1, \ell^\infty)$.

5. Appendix: A factorization primer

In this section, we collect in one place well known, mainly folkloric, results on factorization systems mainly to have one place to refer to it in the future. We take the minimal hypotheses necessary to derive the standard results.

5.1. Definition. A *factorization system* in a category $\mathcal{C}$ consists of two subclasses $\mathcal{E}$ and $\mathcal{M}$ of the arrows of $\mathcal{C}$ subject to the conditions

FS–1. If $\mathcal{I}$ is the class of isomorphisms, then $\mathcal{M} \circ \mathcal{I} \subseteq \mathcal{M}$ and $\mathcal{I} \circ \mathcal{E} \subseteq \mathcal{E}$.

FS–2. Every arrow f in $\mathcal{C}$ factors as $f = m \circ e$ with $m \in \mathcal{M}$ and $e \in \mathcal{E}$.

FS–3. In any commutative square

$$\begin{tikzcd} A \arrow[r, two heads, "e"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ C \arrow[r, tail, "m"'] & D \end{tikzcd}$$

with $e \in \mathcal{E}$ and $m \in \mathcal{M}$, there is a unique $h : B \longrightarrow C$ such that $h \circ e = f$ and $m \circ h = g$.

The last condition is referred to as the "diagonal fill-in". Note that we are following the usual convention of denoting an element of $\mathcal{M}$ with a tailed arrow and an element of $\mathcal{E}$ with a tailed arrow.

5.2. Proposition. *The classes $\mathcal{E}$ and $\mathcal{M}$ are closed under composition.*

Proof. Suppose $m_1 : A \rightarrowtail B$ and $m_2 : B \rightarrowtail C$ with m_1 and m_2 in $\mathcal{M}$. Factor $m_2 \circ m_1 = m \circ e$ with $m \in \mathcal{M}$ and $e \in \mathcal{E}$. The diagonal fill-in in the square

$$\begin{tikzcd} A \arrow[r, two heads, "e"] \arrow[d, tail, "m_1"'] & B \arrow[d, tail, "m"] \\ D \arrow[r, tail, "m_2"'] & C \end{tikzcd}$$

is an arrow $f : B \longrightarrow D$ such that $f \circ e = m_1$ and $m_2 \circ f = m$. The diagonal fill-in in the square

$$\begin{tikzcd} A \arrow[r, two heads, "e"] \arrow[d, "\mathrm{id}"'] & B \arrow[d, "f"] \\ A \arrow[r, tail, "m_1"'] & D \end{tikzcd}$$

is a map $g : B \longrightarrow A$ such that $g \circ e = \mathrm{id}$ and $m_1 \circ g = f$. Since $m \circ e \circ g = m_2 \circ m_1 \circ g = m_2 \circ f = m = m \circ \mathrm{id}$ and $e \circ g \circ e = e = \mathrm{id} \circ e$ both the identity and $e \circ g$ supply a diagonal fill-in in the square

$$\begin{tikzcd} A \arrow[r, two heads, "e"] \arrow[d, two heads, "e"'] & B \arrow[d, tail, "m"] \\ B \arrow[r, tail, "m"'] & C \end{tikzcd}$$

and hence, by uniqueness, are equal. This shows that e is an isomorphism and hence that $m_2 \circ m_1 = m \circ e \in \mathcal{M}$. The argument for $\mathcal{E}$ is dual. ∎

5.3. Proposition. *If $f : A \longrightarrow B$ factors as $A \overset{e}{\twoheadrightarrow} C \overset{m}{\rightarrowtail} B$ and also as $A \overset{e'}{\twoheadrightarrow} C' \overset{m'}{\rightarrowtail} B$ then there is a unique arrow $g : C \longrightarrow C'$ such that $g \circ e = e'$ and $m' \circ g = m$; moreover g is an isomorphism.*

Proof. The arrow g is the diagonal fill-in in the square

$$\begin{tikzcd} A \arrow[r, two heads, "e"] \arrow[d, two heads, "e'"'] & C \arrow[d, tail, "m"] \\ C' \arrow[r, tail, "m'"'] & B \end{tikzcd}$$

To see that g is an isomorphism, we transpose the square to get a map $g' : C' \longrightarrow C$ such that $g' \circ e' = e$ and $m \circ g' = m'$. Then we note that these equations imply that both the identity and $g' \circ g$ fill in the square

$$\begin{tikzcd} A \arrow[r, two heads, "e"] \arrow[d, two heads, "e"'] & C \arrow[d, tail, "m"] \\ C \arrow[r, tail, "m"'] & B \end{tikzcd}$$

and the uniqeness of the diagonal fill-in forces $g' \circ g = \mathrm{id}$ and similarly $g \circ g' = \mathrm{id}$. ■

5.4. Proposition. *Suppose $f : C \longrightarrow D$ satisfies the condition that for all $e : A \twoheadrightarrow B$ in $\mathcal{E}$, any commutative square*

$$\begin{tikzcd} A \arrow[r, two heads, "e"] \arrow[d] & B \arrow[d] \\ C \arrow[r, "f"'] & D \end{tikzcd}$$

has a unique diagonal fill-in. Then $f \in \mathcal{M}$.

Proof. Factor f as $C \overset{e}{\twoheadrightarrow} A \overset{m}{\rightarrowtail} D$. From the diagonal fill-in in the square

$$\begin{tikzcd} C \arrow[r, two heads, "e"] \arrow[d, "\mathrm{id}"'] & A \arrow[d, "m"] \\ C \arrow[r, "f"'] & D \end{tikzcd}$$

we get a map $g : A \longrightarrow C$ such that $g \circ e = \mathrm{id}$ and $f \circ g = m$. Then from $e \circ g \circ e = e$ and $m \circ e \circ g = f \circ g = m$ we see that both the identity and $e \circ g$ fill in the diagonal of

$$\begin{tikzcd}
C \arrow[r, two heads, "e"] \arrow[d, two heads, "e"'] & A \arrow[d, tail, "m"] \\
A \arrow[r, tail, "m"'] & C
\end{tikzcd}$$

and uniqueness of the diagonal fill-in implies that e is an isomorphism, whence $f = m \circ e \in \mathcal{M}$. ∎

Of course, the dual statement is true of $\mathcal{E}$.

5.5. COROLLARY. *Every isomorphism is in $\mathcal{E} \cap \mathcal{M}$.* ∎

5.6. PROPOSITION. *Suppose every arrow in $\mathcal{E}$ is an epimorphism. Then $g \circ f \in \mathcal{M}$ implies that $f \in \mathcal{M}$. Conversely, provided $\mathcal{C}$ has cokernel pairs, the left cancellability of $\mathcal{M}$ implies that every arrow in $\mathcal{E}$ is an epimorphism.*

PROOF. Suppose the composite $C \xrightarrow{f} D \xrightarrow{g} E$ is in $\mathcal{M}$. Suppose we have a commutative square

$$\begin{tikzcd}
A \arrow[r, two heads, "e"] \arrow[d, "h"'] & B \arrow[d, "k"] \\
C \arrow[r, "f"'] & D
\end{tikzcd}$$

The diagonal fill-in in the square

$$\begin{tikzcd}
A \arrow[r, two heads, "e"] \arrow[d, "h"'] & B \arrow[d, "g \circ k"] \\
C \arrow[r, tail, "g \circ f"'] & E
\end{tikzcd}$$

provides a map $l : B \longrightarrow C$ such that $l \circ e = h$. Then e can be cancelled on the right of $f \circ l \circ e = f \circ h = k \circ e$ to conclude that $f \circ l = k$. If l' were another choice, e can be cancelled from $l \circ e = l' \circ e$.

For the converse, suppose $e : A \longrightarrow B$ in $\mathcal{E}$ is not an epimorphism. Then the cokernel pair $B \underset{g}{\overset{f}{\rightrightarrows}} C$ have a common left inverse $h : C \longrightarrow B$ and $h \circ f = \mathrm{id} \in \mathcal{M}$. If f were in

$\mathcal{M}$, the diagonal fill-in in

$$\begin{tikzcd} A \arrow[r, two heads, "e"] \arrow[d, "e"'] & B \arrow[d, "g"] \\ B \arrow[r, "f"'] & C \end{tikzcd}$$

would provide a map $k : B \longrightarrow B$ such that $f \circ k = l$. But then $k = h \circ f \circ k = h \circ g = \mathrm{id}$ so that $f = g$ which contradicts the assumption that e is not epic. ∎

For the following result, we define a category to be *wide complete* if it is complete and if any class of subobjects of an object has an intersection. Of course, a well-powered category is wide complete if and only if it is complete, but it costs no more to state the theorem in this way. We also define an epimorphism $f : A \longrightarrow B$ to be *extremal* if it factors through no subobject of B.

5.7. THEOREM. *Suppose $\mathcal{C}$ is a wide complete category. The there is a factorization system $\mathcal{E}/\mathcal{M}$ in which $\mathcal{E}$ consists of the class of extremal epimorphisms and $\mathcal{M}$ consists of the class of monomorphisms.*

PROOF. Given $f : A \longrightarrow B$, let $C \subseteq B$ denote the intersection of the class of all subobjects of $B' \subseteq B$ for which there is a factorization of f as $A \longrightarrow B' \longrightarrow B$. The universal mapping property of intersection implies that there is a factorization of f as $A \xrightarrow{e} C \xrightarrow{m} B$. If the arrow $e : A \longrightarrow C$ were not epic, there would be two arrows, say $g \neq h : C \longrightarrow D$ with $g \circ e = h \circ e$. But then the equalizer of g and h would be a subobject of B strictly smaller than C that factors e and hence f, a contradiction. The fact that e is extremal is obvious. This gives the factorization. Now suppose

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ C \arrow[r, "m"'] & D \end{tikzcd}$$

is a commutative square with $e \in \mathcal{E}$ and $m \in \mathcal{M}$. Let

$$\begin{tikzcd} A' \arrow[r, "m'"] \arrow[d, "f'"'] & B \arrow[d, "g"] \\ C \arrow[r, "m"'] & D \end{tikzcd}$$

be a pullback. Then m' is monic since a pullback of a monic is always monic. There is a unique arrow $h : A \longrightarrow A'$ such that $m' \circ h = e$ and $f' \circ h = f$. The fact that e is an extremal epic implies that m' is an isomorphism and then $f' \circ m'^{-1}$ is the required diagonal fill-in since $f' \circ m'^{-1} \circ e = f' \circ m'^{-1} \circ m' \circ h = f' \circ h = f$ and $m \circ f' \circ m^{-1} = g \circ m' \circ m'^{-1} = g$. If k were another diagonal fill-in, then from $m \circ k = g$ and $m \circ f' \circ m^{-1} = g$ we can cancel the monic m to conclude that $k = f' \circ m^{-1}$. ∎

Of course, there is a dual theorem with epics and extremal monics in a wide cocomplete category.

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