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FREENESS CONDITIONS FOR 2-CROSSED MODULES AND COMPLEXES.

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ABSTRACT. Using free simplicial groups, it is shown how to construct a free or totally free 2-crossed module on suitable construction data. 2-crossed complexes are introduced and similar freeness results for these are discussed.

Introduction

Crossed modules were introduced by Whitehead in [23] with a view to capturing the relationship between π_1 and π_2 of a space. Homotopy systems (which would now be called free crossed complexes [5] or totally free crossed chain complexes (cf. Baues [3, 4]) were introduced, again by Whitehead, to incorporate the action of π_1 on the higher relative homotopy groups of a CW-complex. They consist of a crossed module at the base and a chain complex of modules over π_1 further up.

Conduché [9] defined 2-crossed modules as a model of connected 3-types and showed how to obtain a 2-crossed module from a simplicial group. A variant of 2-crossed modules are the quadratic modules of Baues [3, 4] and he also defines a notion of quadratic complex bearing a similar relationship to quadratic modules that crossed complexes have to crossed modules. It is a logical step to introduce an intermediate concept, namely 2-crossed complexes. These use a 2-crossed module plus a chain complex of modules and we will show how to derive such a thing from a simplicial group or groupoid. This is not really new as Baues derives quadratic complexes in a similar way from simplicial groups in Appendix B to Chapter IV of [3]. His treatment is thorough, but much remains to be said about the explicit structure of such objects and in particular about freeness conditions for them. We have therefore included a purely algebraic treatment of 2-crossed complexes giving explicit formulae for the structure involved in the passage from simplicial groups to 2-crossed complexes and an explicit direct proof of a freeness result essentially due to Baues in [3]. We also discuss the relationship between crossed complexes and 2-crossed complexes, especially with regards to freeness.

There is an alternative way of storing the information from a 2-crossed complex, namely as a 'squared' complex as introduced by Ellis [14]. Discussion of freeness in that

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context together with a comparison of the two approaches will be postponed to another article.

There is increasing evidence that crossed homological or homotopical methods can play a useful role in non-abelian homological algebra and hence potentially in such diverse fields as non-abelian homology of groups and the algebraic K-theory of C^*-algebras, however much of this crossed homotopical algebra is at present available only in forms that rely heavily on the corresponding machinery of algebraic topology such as the van Kampen theorem arguments used by Brown, Ellis, Higgins and Loday, or the algebraic homotopy results of Baues. Such machinery will not be available away from the crossed homotopical algebra of groups or groupoids and this paper is one of a series, which tries to lay down an alternative more purely algebraic and combinatorial route to these results. Clearly discussion of freeness in such a context is central to applications.

The choice as to whether to use groups or groupoids as a setting is difficult. Simplicial groups model reduced CW-complexes and thus connected homotopy types. Simplicial groupoids model all CW-complexes and thus one can remove the connectedness assumption on the homotopy types. The extra generality of the groupoid case is however often felt not to outweigh the fact that for many readers groupoids are less familiar than are groups. There is a middle way and that is to use groups as the basic case except when the extra generality of groupoids is needed. We try to follow this middle way and use groups to start with but later we will go over to groupoids when the links with crossed complexes (usually defined over groupoids) are discussed. The way in which the group based results can be generalised to the wider context will be discussed in detail later.

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1. Preliminaries

In this section we will concentrate on the reduced case and hence on simplicial groups rather than simplicial groupoids. This is for ease of exposition only and all the results do go through for simplicially enriched groupoids.

Notation: If X is a set, $F(X)$ will denote the free group on X. If Y is a subset of $F(X)$, $< Y >$ will denote the normal subgroup generated by Y within $F(X)$.

1.1. TRUNCATED SIMPLICIAL GROUPS. Denoting the usual category of finite ordinals by Δ, we obtain for each $k \geq 0$, a subcategory $\Delta_{\leq k}$ determined by the objects $[j]$ of Δ with $j \leq k$. A simplicial group is a functor from the opposite category Δ^{op} to $\mathfrak{Grp}$; a k-truncated simplicial group is a functor from $\Delta^{op}_{\leq k}$ to $\mathfrak{Grp}$. We will denote the category of simplicial groups by $\mathfrak{SimpGrp}$ and the category of k-truncated simplicial groups by $\mathfrak{Tr}_k\mathfrak{SimpGrp}$. By a *k-truncation of a simplicial group*, we mean a k-truncated simplicial group $\mathbf{tr}_k\mathbf{G}$ obtained by forgetting dimensions of order $> k$ in a simplicial group $\mathbf{G}$. This gives a truncation functor $\mathfrak{tr}_\mathfrak{k} : \mathfrak{SimpGrp} \longrightarrow \mathfrak{Tr}_\mathfrak{k}\mathfrak{SimpGrp}$ which admits a right adjoint $\mathfrak{cosk}_\mathfrak{k} : \mathfrak{Tr}_\mathfrak{k}\mathfrak{SimpGrp} \longrightarrow \mathfrak{SimpGrp}$ called the *k-coskeleton functor*, and a left adjoint $\mathfrak{sk}_\mathfrak{k} : \mathfrak{Tr}_\mathfrak{k}\mathfrak{SimpGrp} \longrightarrow \mathfrak{SimpGrp}$, called the *k-skeleton functor.* For explicit constructions of these see [11].

Recall that given a simplicial group $\mathbf{G}$, *the Moore complex* $(\mathbf{NG}, \partial)$ *of* $\mathbf{G}$ is the normal chain complex defined by

$$(\mathbf{NG})_n = \bigcap_{i=0}^{n-1} \mathrm{Ker} d_i^n$$

with $\partial_n : NG_n \to NG_{n-1}$ induced from d_n^n by restriction. There is an alternative form of Moore complex given by the convention of taking

$$\bigcap_{i=1}^{n} \mathrm{Ker} d_i^n$$

and using d_0 instead of d_n as the boundary. One convention is used by Curtis [10] (the d_0 convention) and the other by May [17] (the d_n convention). They lead to equivalent theories.

The n^{th} *homotopy group* $\pi_n(\mathbf{G})$ of $\mathbf{G}$ is the n^{th} homology of the Moore complex of $\mathbf{G}$, i.e.

$$\begin{aligned} \pi_n(\mathbf{G}) &\cong H_n(\mathbf{NG}, \partial) \\ &= \bigcap_{i=0}^{n} \mathrm{Ker} d_i^n / d_{n+1}^{n+1}\left(\bigcap_{i=0}^{n} \mathrm{Ker} d_i^{n+1}\right). \end{aligned}$$

We say that the Moore complex $\mathbf{NG}$ of a simplicial group is of *length* k if $NG_n = 1$ for all $n \geq k+1$, so that a Moore complex of length k is also of length l for $l \geq k$.

1.2. Free Simplicial Groups. Recall from [10] and [15] the definitions of free simplicial group and of a $CW-basis$ for a free simplicial group.

1.3. Definition. *A simplicial group* $\mathbf{F}$ *is called free if*
(a) F_n *is a free group with a given basis, for every integer* $n \geq 0$,
(b) *The bases are stable under all degeneracy operators, i.e., for every pair of integers* (i, n) *with* $0 \leq i \leq n$ *and every generator* $x \in F_n$ *the element* $s_i(x)$ *is a generator of* F_{n+1}.

1.4. Definition. *Let* $\mathbf{F}$ *be a free simplicial group (as above). A subset* $\mathfrak{F} \subset \mathbf{F}$ *will be called a* $CW-basis$ *of* $\mathbf{F}$ *if*
(a) $\mathfrak{F}_\mathfrak{n} = \mathfrak{F} \cap F_n$ *freely generates* F_n *for all* $n \geq 0$,
(b) $\mathfrak{F}$ *is closed under degeneracies, i.e.* $x \in \mathfrak{F}_\mathfrak{n}$ *implies* $s_i(x) \in \mathfrak{F}_{\mathfrak{n}+1}$ *for all* $0 \leq i \leq n$,
(c) *if* $x \in \mathfrak{F}_\mathfrak{n}$ *is non-degenerate, then* $d_i(x) = e_{n-1}$, *the identity element of* F_n, *for all* $0 \leq i < n$.

As explained earlier, we have restricted attention so far to simplicial groups and hence to connected homotopy types. This is traditional but a bit unnatural as all the results and definitions so far extend with little or no trouble to simplicial groupoids in the sense of Dwyer and Kan [12] and hence to non-connected homotopy types. It should be noted that such simplicial groupoids have a fixed and constant simplicial set of objects and so are not merely simplicial objects in the category of groupoids. In this context if $\mathbf{G}$ is a simplicial groupoid with set of objects O, the natural form of the Moore complex $\mathbf{NG}$ is given by the same formula as in the reduced case, interpreting $\mathrm{Ker}d_i^n$ as being the subgroupoid of elements in G_n whose i$^{\text{th}}$ face is an identity of G_{n-1}. Of course if $n \geq 1$, the resulting NG_n is a disjoint union of groups, so $\mathbf{NG}$ is a disjoint union of the Moore complexes of the vertex simplicial groups of $\mathbf{G}$ together with the groupoid G_0 providing elements that allow conjugation between (some of) these vertex complexes (cf. Ehlers and Porter [13]).

Crossed modules of, or over, groupoids are well known from the work of Brown and Higgins. The only changes from the definition for groups (cf. [16]) is that one has to handle the conjugation operation slightly more carefully:

A *crossed module* is a morphism of groupoids $\partial : M \longrightarrow N$ where N is a groupoid with object set O say and M is a family of groups $M = \{M(a) : a \in O\}$ together with an action of N on M satisfying (i) if $m \in M(a)$ and $n \in N(a,b)$ for $a, b, \in O$, the result of n acting on m is ${}^n m \in M(b)$; (ii) $\partial({}^n m) = n\partial(m)n^{-1}$ and (iii) ${}^{\partial(m)}m' = mm'm^{-1}$ for all $m, m' \in M,\ \ n \in N$.

The definition of a CW-basis likewise generalises with each $\mathfrak{F}$ a subgraph of the corresponding free simplicial groupoid.

2. 2-Crossed modules of group(oid)s

Crossed modules were initially defined by Whitehead as models for 2-types. D. Conduché, [9], in 1984 described the notion of 2-crossed module as a model for (homotopy) 3-types.

In this section, we give a definition of 2-crossed module and describe a free 2-crossed module of groups by using the second dimensional Peiffer elements.

The following definition of 2-crossed modules is equivalent to that given by D.Conduché, [9].

2.1. Definition. *A 2-crossed module consists of a complex of groups*

$$L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N$$

together with an action of N on L and M so that ∂_2, ∂_1 are morphisms of N-groups, where the group N acts on itself by conjugation, and a N-equivariant function

$$\{\ \ ,\ \ \} : M \times M \to L$$

called a Peiffer lifting, which satisfies the following axioms:

$$\begin{array}{llrcl}
2CM1: & & \partial_2\{m, m'\} & = & ({}^{\partial_1 m}m')\ (m(m')^{-1}(m)^{-1}), \\
2CM2: & & \{\partial_2(l), \partial_2(l')\} & = & [l', l], \\
2CM3: & (i) & \{mm', m''\} & = & {}^{\partial_1 m}\{m', m''\}\{m, m'm''(m')^{-1}\}, \\
 & (ii) & \{m, m'm''\} & = & \{m, m'\}\ {}^{mm'(m)^{-1}}\{m, m''\}, \\
2CM4: & & \{m, \partial_2 l\}\{\partial_2 l, m\} & = & {}^{\partial_1 m}l(l)^{-1}, \\
2CM5 & & {}^n\{m, m'\} & = & \{{}^n m,\ {}^n m'\},
\end{array}$$

for all $l, l' \in L$, $m, m', m'' \in M$ *and* $n \in N$.

Here we have used ${}^m l$ as a shorthand for $\{\partial_2 l, m\} l$ in the condition 2CM3(ii) where l is $\{m, m''\}$ and m is $mm'(m)^{-1}$. This gives a new action of M on L. Using this notation, we can split 2CM4 into two pieces, the first of which is tautologous:

$$\begin{array}{rrcl} 2CM4: \ (a) & \{\partial_2(l), m\} & = & {}^m(l).l^{-1}, \\ (b) & \{m, \partial_2(l)\} & = & ({}^{\partial_1 m} l)({}^m(l)^{-1}). \end{array}$$

The old action of M on L via the $N-$action on L is in general distinct from this second action with $\{m, \partial_2(l)\}$ measuring the difference (by 2CM4(b)). An easy argument using 2CM2 and 2CM4(b) shows that with this action, ${}^m l$, of M on L, (L, M, ∂_2) becomes a crossed module.

We denote such a 2-crossed module by $\{L,\ M,\ N,\ \partial_2,\ \partial_1\}$. A morphism of 2-crossed modules is given by a diagram

$$\begin{tikzcd} L \arrow[r, "\partial_2"] \arrow[d, "f_2"'] & M \arrow[r, "\partial_1"] \arrow[d, "f_1"'] & N \arrow[d, "f_0"'] \\ L' \arrow[r, "\partial_2'"'] & M' \arrow[r, "\partial_1'"'] & N' \end{tikzcd}$$

where $f_0\partial_1 = \partial_1' f_1$, $f_1\partial_2 = \partial_2' f_2$,

$$f_1({}^n m_1) = {}^{f_0(n)} f_1(m_1), \quad f_2({}^n l) = {}^{f_0(n)} f_2(l),$$

and

$$\{\ \ ,\ \ \} f_1 \times f_1 = f_2 \{\ \ ,\ \ \},$$

for all $l \in L$, $m_1 \in M$, $n \in N$. These compose in an obvious way.

The groupoid analogues of these definitions are left to the reader. We will concentrate on the reduced case i.e. with groups rather than groupoids.

We thus can consider the category of 2-crossed modules denoting it as $\mathfrak{X}_2\mathfrak{Mod}$. Conduché [9] proved that 2-crossed modules give algebraic models of connected homotopy 3-types.

The following theorems, in some sense, are well known. We do not give all details of the proofs as analogous proofs can be found in the literature, [9, 16], but do need some indication of the proofs for later use.

We denote the category of simplicial groups with Moore complex of length n by $\mathfrak{SimpGrp}_{\leq n}$ in the following.

2.2. Theorem. ([9, 16].) *The category of crossed modules is equivalent to the category of simplicial groups with Moore complex of length* 1.

Proof. (Sketch) Let $\mathbf{G}$ be a simplicial group with Moore complex of length 1. Put

$$M = NG_1,\ N = NG_0 \text{ and } \partial_1 = d_1 \text{ (restricted to } M).$$

Then NG_0 acts on NG_1 by conjugation via s_0. Since the Moore complex is of length 1, we have

$$[\mathrm{Ker}d_1, \mathrm{Ker}d_0] = \partial_2 NG_2 = 1$$

and the elements of the form $[x^{-1}s_0d_1x, y]$ with $x, y \in NG_1$ generate this normal subgroup (see [18]). It then follows that for all $m, m' \in M$,

$$\begin{aligned} {}^{\partial_1(m)}m' &= s_0d_1(m)(m')s_0d_1m^{-1} && \text{by the action,} \\ &= mm'(m)^{-1} && \text{since } \partial_2 NG_2 = 1. \end{aligned}$$

Thus $\partial_1 : M \to N$ is a crossed module. This yields a functor

$$\mathcal{F}_1 \ :\ \mathfrak{SimpGrp}_{\leq 1} \longrightarrow \mathfrak{XMod}.$$

Conversely, let $\partial_1 : M \to N$ be a crossed module. By using the action of N on M, one forms the semidirect product $M \rtimes N$ together with homomorphisms

$$d_0(m, n) = n,\ d_1(m, n) = \partial_1(m)n,\ s_0(n) = (1, n).$$

Define

$$H_0 = N, \qquad H_1 = M \rtimes N,$$

then we have a 1-truncated simplicial group $\mathbf{H} = \{H_0, H_1\}$. Applying the right adjoint to 1-truncation gives us a simplicial group which will again be denoted $\mathbf{H}$. A calculation due to Conduché [9] shows that this has Moore complex

$$Ker\partial_1 \to M \stackrel{\partial_1}{\to} N.$$

We set $\mathbf{H}' = \mathfrak{sk}_1\mathbf{H}$ and note $NH'_p = D_p \cap NH_p$, where D_p is the subgroup of H_p generated by the degenerate elements, and so NH'_p is trivial if $p > 2$. We claim $NH'_2 = 1$ and clearly $\mathbf{NH}'$ is then the given crossed module (M, N, ∂_1). Now $\partial_2(D_2 \cap NH_2)$ is $[Kerd_0, Kerd_1]$ by the Brown-Loday lemma and a direct calculation using the descriptions of d_0 and d_1 above shows that $[Kerd_0, Kerd_1] = 1$, however $\partial_2^{\mathbf{H}}$ is a monomorphism so $D_2 \cap NH_2 = 1$ as required. ∎

Remark. In general we will use D_n as above to denote the subgroup or subgroupoid generated by degenerate elements in dimension n. The context will determine in which group or groupoid it lives.

We recall from [20] the following result.

2.3. Proposition. *Let* $\mathbf{G}$ *be a simplicial group with Moore complex* $\mathbf{NG}$. *Then the complex of groups*

$$NG_2/\partial_3(NG_3 \cap D_3) \xrightarrow{\overline{\partial}_2} NG_1 \xrightarrow{\partial_1} NG_0$$

is a 2-crossed module of groups, where the Peiffer map is defined as follows:

$$\begin{array}{rccl} \{\ \ ,\ \ \}: & NG_1 \times NG_1 & \longrightarrow & NG_2/\partial_3(NG_3 \cap D_3) \\ & (x_0, x_1) & \longmapsto & \overline{s_0x_0s_1x_1s_0(x_0)^{-1}s_1x_0s_1(x_1)^{-1}s_1(x_0)^{-1}}. \end{array}$$

(Here the overbar on the right hand side denotes a coset in $NG_2/\partial_3(NG_3 \cap D_3)$ *represented by the corresponding element in* NG_2.*)* ■

The two actions of NG_1 on $NG_2/\partial_3(NG_3 \cap D_3)$ are given by
(i) ${}^{\partial_1 m}l$ corresponds to the action $s_0(m)ls_0(m)^{-1}$ via s_0 and conjugation;
(ii) ${}^{m}l$ corresponds to the action $s_1(m)ls_1(m)^{-1}$ via s_1 and conjugation.

This proposition leads to the generalisation of Theorem 2.2 as follows. The methods we use for proving the next result are based on ideas of Conduché [9]. A slightly different proof of this result is given in [9].

2.4. Theorem. [9] *The category of* 2*-crossed modules is equivalent to the category of simplicial groups with Moore complex of length* 2.

Proof. (Sketch) Let $\mathbf{G}$ be a simplicial group with Moore complex of length 2. In the previous proposition, a 2-crossed module

$$NG_2 \xrightarrow{\partial_2} NG_1 \xrightarrow{\partial_1} NG_0$$

has already been constructed. This defines a functor

$$\mathcal{F}_2 \ : \ \mathfrak{SimpGrp}_{\leq 2} \longrightarrow \mathfrak{X}_2\mathfrak{Mod}.$$

Conversely suppose given a 2-crossed module

$$L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N.$$

Define $H_0 = N$. We construct the semidirect product $H_1 = M \rtimes N$ by using the action of N on M with homomorphisms

$$d_0(m,n) = n,\ \ d_1(m,n) = \partial_1(m)n,\ \ s_0(n) = (1,n).$$

There is an action ${}^{m}l$ of $m \in M$ on $l \in L$ given as above by ${}^{m}l = \{\partial_2 l, m\}l$. Using this action we form the semidirect product $L \rtimes M$. An action of $(m,n) \in M \rtimes N$ on $(l,m') \in L \rtimes M$ is given by

$$^{(1,n)}(l,m') = (^{n}l,\ ^{n}m'),$$

$$^{(m,1)}(l,m') = (^{\partial m}l\{m,m'\}, mm'm^{-1}).$$

Using this action we get the semidirect product

$$H_2 = (L \rtimes M) \rtimes (M \rtimes N).$$

We have homomorphisms

$$\begin{array}{rclcrcl} d_0(l,m,m',n) & = & (m',n), & & s_0(m',n) & = & (1,1,m',n), \\ d_1(l,m,m',n) & = & (m\,m',n), & & s_1(m',n) & = & (1,m',1,n), \\ d_2(l,m,m',n) & = & (\partial_2 lm, \partial_1 m'n). & & & & \end{array}$$

We thus have a 2-truncated simplicial group $\mathbf{H} = \{H_0, H_1, H_2\}$. Again the right adjoint to a truncation (at level 2 this time) extends this $\mathbf{H}$ and the resullt has Moore complex

$$Ker\partial_2 \to L \to M \to N.$$

We set $\mathbf{H}' = \mathfrak{sk}_2\mathbf{H}$ and claim $NH'_3 = 1$. The argument follows the same plan as above. By the extension of the Brown-Loday lemma to dimension 3 (given as Theorem B in [20]), we have that $\partial_3(NH_3 \cap D_3)$ is the product of $[Kerd_0, Kerd_1 \cap Kerd_2]$, $[Kerd_1, Kerd_0 \cap Kerd_2]$ and $[Kerd_2, Kerd_0 \cap Kerd_1]$. A direct calculation using the descriptions of the actions and the face maps above shows that these are all trivial, so $\partial_3(NH_3 \cap D_3) = 1$, but again ∂_3 is a monomorphism so $NH'_3 = 1$ as required. ■

2.5. Free 2-Crossed Modules. The definition of a free 2-crossed module is similar in some ways to the corresponding definition of a free crossed module. However, the construction of a free 2-crossed module is a bit more complicated and will be given by means of the 2-skeleton of a free simplicial group with given CW-basis.

It will be helpful to have the notion of a *pre-crossed module*: this is just a homomorphism $\partial : M \longrightarrow N$ with an action satisfying $\partial({}^n m) = n\partial(m)n^{-1}$ for $m \in M,\ n \in N$.

Let (M, N, ∂) be a pre-crossed module, let S be a set and let $\nu : S \longrightarrow M$ be a function, then (M, N, ∂) is said to be a *free pre-crossed N-module on the function $\partial\nu : S \longrightarrow N$* if for any pre-crossed N-module (L', N, ∂') and function $\nu' : S \longrightarrow L'$ such that $\partial'\nu' = \partial\nu$, there is a unique morphism

$$\Phi : (M, N, \partial) \longrightarrow (L', N, \partial')$$

such that $\phi\nu = \nu'$.

The pre-crossed module (M, N, ∂) is *totally free* if N is a free group.

2.6. Definition. *Let $\{L, M, N, \partial_2, \partial_1\}$ be a 2-crossed module, let Y_2 be a set and $\vartheta : Y_2 \to L$ be a function then $\{L, M, N, \partial_2, \partial_1\}$ is said to be a free 2-crossed module on the function $\partial_2\vartheta : Y_2 \to M$ if for a 2-crossed module $\{L', M, N, \delta, \partial_1\}$ and function, $\vartheta' : Y_2 \to L'$ such that $\partial_2\vartheta = \delta\vartheta'$, there is a unique morphism*

$$\Phi : L \longrightarrow L'$$

such that $\delta\Phi = \partial_2$.

The free 2-crossed module $\{L, M, N, \partial_2, \partial_1\}$ is *totally free* if $\partial_1 : M \to N$ is a totally free pre-crossed module.

We shall give an explicit description of the construction of a totally free 2-crossed module. For this, we will need to recall the 2-skeleton of a free simplicial group which is

$$\mathbb{F}^{(2)} : \ F(s_1s_0(X_0) \cup s_0(Y_1) \cup s_1(Y_1) \cup Y_2) \underset{s_1,s_0}{\overset{d_0,d_1,d_2}{\rightrightarrows}} F(s_0(X_0) \cup (Y_1)) \underset{s_0}{\overset{d_1,d_0}{\rightrightarrows}} F(X_0)$$

with the simplicial structure defined as in Section 3 of [21], thus we will assume that X_0 and Y_1 are parts of a CW-basis $\mathfrak{F}$ with $X_0 = \mathfrak{F}_0$, $Y_1 = \mathfrak{F}_1 \setminus s_0(X_0)$, $Y_2 = \mathfrak{F}_2 \setminus (s_0(\mathfrak{F}_1) \cup s_1(\mathfrak{F}_1))$. Analysis of this *2-dimensional construction data* shows that it consists of some *1-dimensional data:* namely the function $\varphi : Y_1 \longrightarrow F(X_0)$ that is used to induce $d_1 : F(s_0(X_0) \cup Y_1) \longrightarrow F(X_0)$, together with strictly 2-dimensional data consisting of the function $\psi : Y_2 \longrightarrow < Y_1 >$ where $< Y_1 >$ is the normal closure of Y_1 in $F(s_0(X_0) \cup Y_1)$. This function induces $d_2 : F(s_1s_0(X_0) \cup s_0(Y_1) \cup s_1(Y_1) \cup Y_2) \longrightarrow F(s_0(X_0) \cup Y_1)$. We will denote this 2-dimensional construction data by $(Y_2, Y_1, \psi, \varphi, F(X_0))$.

2.7. Theorem. *Let $(Y_2, Y_1, F(X_0), \psi, \phi\}$ be 2-dimensional construction data as defined above, then there is a totally free 2-crossed module $\{L, M, F(X_0), \partial_2, \partial_1\}$ defined by the data.*

Proof. Given the construction data we construct a 2-truncated simplicial group as above. Set $M =< Y_1 >$. With the obvious action of $F(X_0)$ on M, the function φ induces a free pre-crossed module $\mathcal{M} = (M, F(X_0), \partial_1)$.

Now set $Z = \{s_1(y)^{-1}s_0(y) : y \in Y_1\}$ and take $D =< s_1(Y_1) \cup Y_2) > \cap < Z \cup Y_2 >$, in $F(s_1s_0(X_0) \cup s_0(Y_1) \cup s_1(Y_1) \cup Y_2)$ so that M acts on D by conjugation via s_1. The function φ induces a morphism

$$\theta : D \longrightarrow < Y_1 >,$$

given by $\theta(y) = \psi(y)$ for $y \in Y_2$, (of course $D = NF_2^{(2)}$, part of the Moore complex of the 2-skeleton of the free simplicial group on the data).

Recall from [21] the second dimension Peiffer normal subgroup P_2 in D. This is the normal subgroup of $NF_2^{(2)}$ that is of D, generated by the elements of the form:

$$\begin{gathered}
[s_0x^{-1}s_1s_0d_1x,\ y_1],\\
[s_1x^{-1}s_0x,\ s_1d_2(y_1){y_1}^{-1}],\\
[xs_1d_2x^{-1}s_0d_2x,\ s_1y_1],\\
[{y_1}^{-1}s_1d_2y_1,\ y_2],\\
[y_1s_1d_2{y_1}^{-1}s_0d_2y_1,\ y_2],\\
[y_1s_1d_2{y_1}^{-1}s_0d_2y_1,\ s_1d_2(y_2){y_2}^{-1}].
\end{gathered}$$

It is easily checked that $\theta(P_2) = 1$ as all generator elements of P_2 are in $\mathrm{Ker} d_2$. Taking the factor module $L = D/P_2$, we get a morphism $\partial_2 : L \to M$ such that the diagram,

$$\begin{tikzcd} D \arrow[rr, "\theta"] \arrow[dr, "q"'] & & M \\ & L \arrow[ur, "\partial_2"'] & \end{tikzcd}$$

commutes, where q is the quotient morphism.

We will show that $\{L, M, F(X_0), \partial_2, \partial_1\}$, i.e. the complex

$$D/P_2 \xrightarrow{\partial_2} < Y_1 > \xrightarrow{\partial_1} F(X_0)$$

is the required free 2-crossed module on $(Y_2, Y_1, F(X_0); \psi, \partial_1)$. The Peiffer lifting map

$$\{\ \ ,\ \ \} : < Y_1 > \times < Y_1 > \longrightarrow D/P_2$$

is induced by the map

$$\omega : < Y_1 > \times < Y_1 > \longrightarrow < s_1(Y_1) \cup Y_2 > \cap < Z \cup Y_2 >$$

given by

$$\omega(b_i,\ b_j) = s_0 b_i s_1 b_j s_0(b_i)^{-1} s_1(b_i {b_j}^{-1} {b_i}^{-1}) \quad \text{with } b_i, b_j \in < Y_1 > .$$

Thus we can define the Peiffer lifting map by

$$\{b_i,\ b_j\} = \overline{\omega(b_i \times b_j)} = \overline{s_0 b_i s_1 b_j s_0(b_i)^{-1} s_1(b_i {b_j}^{-1} b_i)^{-1}}.$$

That $\{L, M, F(X_0), \partial_2, \partial_1\}$, is a 2-crossed module is now easy to check.

Let $\{L', M, F(X_0), \delta, \partial_1\}$ be any 2-crossed module on the precrossed module $\mathcal{M}$, and let $\varphi' : Y_2 \to L'$ be such that $\delta\theta' = \psi$. This function $\theta' : Y_2 \longrightarrow L'$ uniquely extends to an M-equivariant homomorphism $D \longrightarrow L'$ sending conjugation via s_1 or s_0 to the corresponding actions. This extension sends elements of P_2 to the trivial element of L', so induces a morphism $\Phi : L \longrightarrow L'$ satisfying the conditions to make (Φ, Id, Id) a morphism of 2-crossed modules. Uniqueness is easily verified. ∎

Remarks. 1) A slight modification of the above will allow the construction of a free 2-crossed module on a function $f : Y_2 \longrightarrow M$ where $\partial : M \longrightarrow N$ is a given pre-crossed module and $\partial f = 1$. One forms a truncated simplicial group from (M, N, ∂) as above, then taking its skeleton one attaches new elements in dimension 2 corresponding to the elements of Y_2 using the given function to get d_2.

2) In [3, 4] Baues introduces a notion of quadratic module and a related quadratic complex. A quadratic module is a 2-crossed module with additional 'nilpotency' conditions, similarly for quadratic complexes. Baues gives a construction of a quadratic module from a simplicial group in Appendix B to Chapter IV of [3] and discusses free and totally free quadratic modules and complexes.

The data needed by his construction can also be used to give a totally free 2-crossed module, which is not surprising given the close relationship between the two concepts. It thus can also be used to produce a simplicial group having that (totally) free 2-crossed module associated to it.

3) Using simple techniques from combinatorial group theory one can obtain more explicit expressions for the intersection

$$< s_1(Y_1) \cup Y_2 > \cap < Z \cup Y_2 >$$

that allow for the manipulation and interpretation of the quotient by P_2. We believe this could be of particular significance when G is a simplicial resolution of a group, say, constructed by a step-by-step method as we outlined in [21] (based on ideas of M. André, see references in [21]). This method not only provides a resolution but, if needed, comes with a CW-basis already chosen. Here we will not explore this in depth as it would take us too far away from our main aims, however some related ideas are explored briefly in the next section.

3. The n-Type of the k-Skeleton

The key invariant in [7] was the module of identities amongst the relations of a presentation $(X : R)$ of a group. Given a way of 'presenting' a group G as a quotient $F(X)/N$ where $N =< R >$, there are often non-trivial identities amongst the elements of N as although free as a group N is not free on the specified relators and their conjugates.

The data $(X : R)$ allows a 1-skeleton of a free simplicial resolution of G to be constructed and the free crossed module used in [7] is isomorphic to that constructed from that 1-skeleton. The various stages of the construction of a resolution, i.e. the k-skeleta, can be observed as they change with k by looking at the way the invariants of the n-type of these k-skeleta change with increasing k. We examine a slightly more general situation namely that of a free simplicial group homotopy equivalent to a given simplicial group. The case of a resolution corresponds to the given simplicial group being a $K(G, 0)$, i.e. constant with value $K(G, 0)_n = G$ and all faces and degeneracies the identity isomorphism. In the lowest dimension cases, this corresponds to going from a 'presentation' to finding a description of the identities between the given 'relations' and then on being given generators for this module of identities, observing the change in the various invariants that results from adding these into dimension 2.

Recall from [17] that a morphism $\mathbf{f} : \mathbf{G} \to \mathbf{H}$ of simplicial groups is called an *n-equivalence* if

$$\pi_i(\mathbf{f}) : \pi_i(\mathbf{G}) \longrightarrow \pi_i(\mathbf{H}),$$

is an isomorphism for all i, $0 \leq i \leq n$. Two simplicial groups $\mathbf{G}$ and $\mathbf{H}$ are said to have the same *n-type* if there is a chain of n-equivalences linking them. A simplicial group $\mathbf{H}$ is called an *n-type* if

$$\pi_i(\mathbf{H}) = 1 \quad \text{for } i > n.$$

Any reduced connected simplicial set X yields a free simplicial group, $G(X)$ via the Kan loop-group construction and the algebraic n-type of $G(X)$ is the same as the topological $(n+1)$-type of X. Any simplicial group is n-equivalent to an n-type, in fact, to a free n-type.

Suppose given a simplicial group $\mathbf{G}$ and $n \geq 1$, then we can use the 'step-by-step' construction (see [21]) to produce a free simplicial group $\mathbf{F}$ and an epimorphism

$$\psi : \mathbf{F} \longrightarrow \mathbf{G}$$

which is an n-equivalence. The construction goes like this:

Take a free group $F(X_0)$ and an epimorphism

$$\psi : F(X_0) \longrightarrow G_0;$$

set $\mathbf{F}^{(0)} = K(F(X_0), 0)$, the constant simplicial group with value $F(X_0)$. We have a morphism

$$\psi^{(0)} : \mathbf{F}^{(0)} \longrightarrow \mathbf{G}.$$

Now take a free $F(X_0)$-group on construction data to enable the kernel of $\psi^{(0)}$ to be killed and G_1 to be 'covered'

$$\begin{tikzcd} F(s_1(X_0) \cup Y_1) \arrow[r, "\psi_1"] \arrow[d, shift left] \arrow[d, shift left=2] & G_1 \arrow[d, shift left] \arrow[d, shift left=2] \\ F(X_0) \arrow[r, "\psi_0"] \arrow[u] & G_0, \arrow[u] \end{tikzcd}$$

to make $\pi_0(\psi^{(1)})$ an isomorphism. Here $\psi^{(1)}$ is the extension of the morphism of 1-truncated simplicial groups given by left adjointness so $\psi^{(1)} : \mathbf{F}^{(1)} \rightarrow \mathbf{G}$. The construction then follows the obvious routine, mirroring the construction of a resolution. You add new generators in dimension $k+1$ to adjust the kernel at level k and to produce an epimorphism onto G_{k+1}. If one is looking only for the invariants of the n-type one need only continue until $\mathbf{F}^{(n+1)}$ has been reached although it is at this stage that interesting things happen! We can therefore represent any n-type of a simplicial group by an n-equivalent free simplicial group constructed using construction data a 'step-by-step' construction of skeleta,

$$\mathbf{F}^{(0)} \hookrightarrow \mathbf{F}^{(1)} \hookrightarrow \ldots \mathbf{F}^{(n)} \hookrightarrow \mathbf{F}^{(n+1)} = \mathbf{F}.$$

Moreover as an n-type does not retain homotopy information in dimensions greater than n, we have that any n-type can be represented by a simplicial group having $NF_k = 1$ if $k > n+1$, and such that each F_k is free, for instance by taking $\mathfrak{cosk}_{n+1}\mathbf{F}^{(n+1)}$ above. The transition from the n-type of the $(n+1)$-skeleton to the $(n+1)$-type outlines the 'lack of fit' of the approximation so far, for example, the non-trivial homotopy groups $\pi_{n+1}(\mathbf{F}^{(n+1)})$, when constructing a resolution.

We look at this transition in more detail for low values of n.

3.1. From 0-Type to 1-Type. Here we only match $\pi_0(\mathbf{F})$ with $\pi_0(\mathbf{G})$, so assume given $(X_0 : R)$, a presentation of G_0 and then take Y_1 to kill the kernel N of $F(X_0) \to \pi_0(\mathbf{G})$; Y_1 can be chosen to be a disjoint union of R and a set of generators for NG_1. We will assume that $\phi : Y_1 \to F(X_0)$ is given $\phi(y_i) = b_i \in F(X_0)$ say. This gives a free simplicial group

$$\mathbf{F}^{(1)} : \dots\ F(s_1s_0(X_0) \cup s_0(Y_1) \cup s_1(Y_1)) \underset{s_1,s_0}{\overset{d_0,d_1,d_2}{\rightleftarrows}} F(s_0(X_0) \cup Y_1) \underset{s_0}{\overset{d_1,d_0}{\rightleftarrows}} F(X_0)$$

with

$$d_0^1(y) = 1, \quad d_1^1(y) = b, \quad s_0(x) = x \in F(X_0), \quad y \in Y_1,$$

and so that we get a set of degenerate generators in a CW-basis in each dimension greater than 1. There is an isomorphism $\pi_0(\mathbf{F}^{(1)}) \cong F_0^{(1)}/d_1^1(\mathrm{Ker} d_0^1)$ and considering the morphism $d_1^1 : \mathrm{Ker} d_0^1 \longrightarrow F(X_0)$, one readily obtains $\mathrm{Im} d_1^1 = N$ and $F_0^{(1)} = F(X_0)$. Thus

$$\pi_0(\mathbf{F}^{(1)}) \cong F(X_0)/N \cong \pi_0(G)$$

as planned.

The corresponding 1-type is given by the free crossed module,

$$\partial_1 :< Y_1 > /P_1 \longrightarrow F(X_0)$$

where P_1 is the Peiffer subgroup as introduced in [7]. There are the following isomorphisms

$$\begin{array}{rcl} \pi_0(\mathbf{F}^{(1)})) & \cong & F(X_0)/N, \\ \pi_1(\mathbf{F}^{(1)}) & \cong & Ker\partial_1 \end{array}$$

which is the module of identities of the presentation $(X : Y_1)$ of $\pi_0(\mathbf{G})$, and finally

$$\pi_i(\mathbf{F}^{(1)}) \cong 1 \qquad \text{for } i > 1.$$

The 1-type of the 1-skeleton is thus represented by a free crossed module but note this only corresponds to a simplicial group 0-equivalent to $\mathbf{G}$ as here $\mathrm{Ker}\partial_1$ must be free abelian whilst $\pi_1(\mathbf{G})$ has no such restriction..

3.2. From 1-Type to 2-Type. Suppose now given the 2-skeleton $\mathbf{F}^{(2)}$ of a free simplicial group approximating $\mathbf{G}$

$$\mathbf{F}^{(2)} : \dots\ F(s_1s_0(X_0) \cup s_0(Y_1) \cup s_1(Y_1) \cup Y_2) \underset{s_1,s_0}{\overset{d_0,d_1,d_2}{\rightleftarrows}} F(s_0(X_0) \cup (Y_1)) \underset{s_0}{\overset{d_1,d_0}{\rightleftarrows}} F(X_0).$$

As above, one gets the same π for this $\mathbf{F}^{(2)}$:

$$\pi_0(\mathbf{F}^{(2)}) \cong F(X_0)/N,$$

but

$$\pi_1(\mathbf{F}^{(2)}) \cong \mathrm{Ker}(< F(s_1(Y_1)) > /\partial_2 NF_2^{(2)} \longrightarrow F(X_0)),$$

where $\partial_2 NF_2^{(2)}$ is generated by P_1 and the image $d_2(Y_2)$, thus $\pi_1(\mathbf{F}^{(2)}) \cong \pi_1(\mathbf{G})$. There is an isomorphism

$$\pi_2(\mathbf{F}^{(2)}) \cong \mathrm{Ker}(NF_2^{(2)}/\partial_3(NF_3^{(2)}) \longrightarrow F_1^{(2)}).$$

Since $NF_2^{(2)} =< s_1(Y_1) \cup Y_2 > \cap < Z \cup Y_2 >$, the second homotopy group of the 2-skeleton looks like

$$\pi_2(\mathbf{F}^{(2)}) \cong \mathrm{Ker}((< s_1(Y_1) \cup Y_2 > \cap < Z \cup Y_2 >)/P_2 \longrightarrow F(s_0(X_0) \cup Y_1))$$

where P_2 is the second dimension Peiffer normal subgroup as above. This homotopy group is a module over $\pi_0(\mathbf{G})$ and is a measure of the identities among the level 1 elements of the construction data for the homotopy type of $\mathbf{G}$.

Note that free 2-crossed modules correspond to the 2-type of the 2-skeleta of free simplicial groups, and conversely.

There is a neat alternative description of these free 2-crossed modules and, in particular their top terms NF_2/P_2. As this description depends on the corresponding freeness property for free crossed squares, we postpone detailed discussion to a later paper, however the top term of the corresponding crossed square is also NF_2/P_2 and so is isomorphic via an algebraic version of Ellis' results [14] to the coproduct of a tensor product $M \otimes \bar{M}$ of two related free pre-crossed modules with a free crossed module C on Y_2. In particular if $\mathbf{F}$ is a simplicial group model for $\Sigma K(\pi, 1)$, for instance the model proposed by Wu [24] and used by us in [19], one gets NF_2/P_2 is related to $\pi \otimes \pi$, NF_1 similarly related to π and NF_0 is trivial, with $\partial : NF_2/P_2 \to NF_1$ being the commutator map, thus with a little extra work retrieving the Brown-Loday description of $\pi_3(\Sigma K(\pi, 1))$ as $Ker(\pi \otimes \pi \to \pi)$, but without use of the generalised van Kampen theorem.

4. 2-crossed complexes and simplicial groupoids.

Any simplicial group, $\mathbf{G}$, yields a normal chain complex of groups, namely its Moore complex, $(\mathbf{NG}, \partial)$. Carrasco and Cegarra [8] examined the extra structure inherent in a Moore complex that allows the reconstruction of $\mathbf{G}$ from $\mathbf{NG}$. They gave the name hypercrossed complex to the resulting structure. Crossed complexes themselves, (cf. Brown and Higgins, [5]) correspond to a class of hypercrossed complexes in which nearly all of the extra structure is trivial, so the only non-abelian groups occur in dimensions 0 and 1 and are linked by a crossed module structure. The other terms are all modules over $NG_0/\partial NG_1$. Thus a crossed complex looks like a crossed module with a tail that is a chain complex of $\pi_0(G)$-modules. If the original simplicial group is the Kan loop group of a reduced simplicial set, K, it is well known that the corresponding complex has free $\pi_0(\mathbf{G})$-modules analogous to the chains on the universal cover in dimensions greater than 1 and a free crossed module in the bottom two dimensions. (This is implicit in much of the work of Baues on crossed (chain) complexes [2, 3, 4] and was explicitly proved by Ehlers and the second author [13].)

Crossed modules model algebraic 1-types (and hence topological 2-types) and we have recalled from Conduché's work [9] that 2-crossed modules model algebraic 2-types (and hence topological 3-types). It is thus natural to give these latter models also a 'tail' and to consider '2-crossed complexes'. Such gadgets are related to the quadratic complexes of Baues [3, 4] in an obvious way.

4.1. Definition. *A 2-crossed complex of group(oid)s is a sequence of group(oid)s*

$$C: \qquad \ldots \to C_n \stackrel{\partial_n}{\to} C_{n-1} \to \ldots C_2 \stackrel{\partial_2}{\to} C_1 \stackrel{\partial_1}{\to} C_0$$

in which
(i) C_n is abelian for $n \geq 3$;
(ii) C_0 acts on C_n, $n \geq 1$, the action of ∂C_1 being trivial on C_n for $n \geq 3$;
(iii) each ∂_n is a C_0-group(oid) homomorphism and $\partial_i \partial_{i+1} = 1$ for all $i \geq 1$;
and
(iv) $C_2 \stackrel{\partial_2}{\to} C_1 \stackrel{\partial_1}{\to} C_0$ is a 2-crossed module.

Note that for any 2-crossed module,

$$L \stackrel{\partial_2}{\to} M \stackrel{\partial_1}{\to} N,$$

$K = Ker\partial_2$ is abelian, since $L \stackrel{\partial_2}{\to} M$ is a crossed module, but more is true. The action of M on L via N restricts to one on K, but by axiom 2CM4, the action is trivial. This implies that the action of N itself on K factors through one of $N/\partial_1 M$. Thus in any 2-crossed complex,

$$\ldots \to C_4 \to C_3 \to \ker \partial_2$$

is a chain complex of $C_0/\partial_1 C_1$-modules and a 2-crossed complex is just a 2-crossed module with a chain complex as tail added on.

Given a simplicial group or groupoid, $\mathbf{G}$, define

$$C_n = \begin{cases} NG_n & \text{for } n = 0, 1 \\ NG_2/d_3(NG_3 \cap D_3) & \text{for } n = 2 \\ NG_n/(NG_n \cap D_n)d_{n+1}(NG_{n+1} \cap D_{n+1}) & \text{for } n \geq 3 \end{cases}$$

with ∂_n induced by the differential of $\mathbf{NG}$. Note that the bottom three terms ($n =$ 0, 1, and 2) form the 2-crossed module already considered in section 2 and that for $n \geq 3$, the groups are all $\pi_0(G)$-modules, since in these dimensions C_n is the same as the corresponding crossed complex term (cf. Ehlers and Porter [13] for instance or use the hypercrossed complex theory of Carrasco and Cegarra [8]).

4.2. Proposition. *With the above structure (C_n, ∂_n) is a 2-crossed complex.*

Proof. The only thing remaining is to check that $\partial_2\partial_3$ is trivial which is straightforward. ∎

If the simplicial group(oid) $\mathbf{G}$ is the loop group(oid) of a simplicial set, K, then for $n \geq 3$, the corresponding 2-crossed complex term C_n is the $n+1^{st}$ module of the chains on the universal cover of K since that is the description of the corresponding (and isomorphic) crossed complex term.

The notion of a morphism for 2-crossed complexes should be clear. Such a morphism will be a morphism of graded group(oid)s restricting to a morphism of 2-crossed modules on the bottom three terms and compatible with the action. This gives the category, $\mathfrak{X}_2\mathfrak{Comp}$ of 2-crossed complexes and morphisms between them. We denote by $\mathfrak{X}_1\mathfrak{Comp}$ the category of crossed complexes together with their morphisms. It is easily seen that the construction C is functorial.

A 2-crossed complex C will be said to be *free* if for $n \geq 3$, the $C_0/\partial C_1$-modules, C_n are free and the 2-crossed module at the base is a free 2-crossed module. It will be *totally free* if in addition the base 2-crossed module is totally free.

Before turning to a detailed examination of freeness in 2-crossed complexes, we will consider the relation between crossed complexes and 2-crossed complexes.

Suppose

$$C: \qquad C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

is a 2-truncated crossed complex, then (C_1, C_0, ∂_1) is a crossed module, C_2 is a module over C_0 on which $\partial_1 C_1$ acts trivially, and $\partial_1\partial_2 = 0$.

4.3. Lemma. *The 2-truncated crossed complex yields a 2-crossed module by taking $\{c, c'\} = 1 \in C_2$ for all $c, c' \in C_1$ and in which the actions of C_1 on C_2 are both trivial.* ∎

The proof is easy.

As a consequence such truncated crossed complexes form a full subcategory of the category of 2-crossed modules, as the two definitions of morphism clearly coincide on this subclass of 2-crossed modules. Passing to the 'complex' version of this one clearly gets:

4.4. Proposition. *There is a full embedding E of the category, $\mathfrak{X}_1\mathfrak{Comp}$, of crossed complexes into that $\mathfrak{X}_2\mathfrak{Comp}$, of 2-crossed complexes.* ∎

We will think of $\mathfrak{X}_1\mathfrak{Comp}$ as a full subcategory of $\mathfrak{X}_2\mathfrak{Comp}$ via this embedding.

4.5. Theorem. *The full subcategory of crossed complexes is a reflexive subcategory of $\mathfrak{X}_2\mathfrak{Comp}$.*

Proof. We have to show that the inclusion functor E has a left adjoint, L. We first look at a slightly simpler situation.

Suppose that D is a 2-truncated crossed complex as above, and

$$C: \quad C_2 \to C_1 \to C_0,$$

with morphisms, ∂_2, ∂_1 and Peiffer lifting, $\{\ ,\ \}$, is a 2-crossed module. If we are given a morphism, $\underline{f} = (f_2, f_1, f_0)$ of 2-crossed modules, $\underline{f}: C \to E(D)$, then if $m_1, m_2 \in C_1$, $f_2\{m_1, m_2\} = 1$ since within $E(D)$ the Peiffer lifting is trivial. This in turn implies that $f_1 < m_1, m_2 >= 1$, where $< m_1, m_2 >= ({}^{\partial m_1}m_2)(m_1 m_2^{-1} m_1^{-1})$ is the Peiffer commutator of m_1 and m_2. Thus any morphism $\underline{f}$ from C to $E(D)$ has a kernel that contains the subgroupoid $\{C_1, C_1\}$ generated by the Peiffer lifts in dimension 2, and the Peiffer subgroup, P_1 of the precrossed module $\partial_1 : C_1 \to C_0$ in dimension 1.

We form $L(C)$ as follows:

$$\begin{aligned} L(C)_0 &= C_0 \\ L(C)_1 &= C_1/P_1 \\ L(C)_2 &= C_2/\{C_1, C_1\}, \end{aligned}$$

with the induced morphisms and actions. The previous discussion makes it clear that $L(C)$ is a 2-truncated crossed complex, and L is clearly functorial. Of course $\underline{f} : C \to E(D)$ yields $L(\underline{f}) : L(C) \to LE(D) \cong D$, so L is the required reflection, at least on this subcategory of truncated objects.

Extending L to all crossed complexes is then simple as we take $L(C)_n = C_n$ if $n \geq 3$ with $L(\partial)_n = \partial_n$ if $n > 3$ and

$$L(\partial)_3 : C_3 \to C_2/\{C_1, C_1\} = L(C)_2$$

given by the composite of ∂_3 and the quotient from C_2 to $L(C)_2$. The details are easy so will be omitted. ∎

We thus have functors from the category of simplicial group(oid)s to both $\mathfrak{X}_1\mathfrak{Comp}$ and $\mathfrak{X}_2\mathfrak{Comp}$ and a relationship between these two categories given by the last result. The first two functors will for greater precision be denoted $C^{(1)}$ and $C^{(2)}$, respectively. The functor $C^{(1)}$ is studied for instance in Ehlers-Porter [13] and Mutlu-Porter [21] whilst $C^{(2)}$ was introduced in 4.1. above. We first note that the three functors have the 'right' sort of interrelationship.

4.6. Proposition.

$$LC^{(2)} \cong C^{(1)}$$

Proof. The key is to identify $\{C_1, C_1\}$, when $C = C^{(2)}(\mathbf{G})$ for $\mathbf{G}$ a simplicial group(oid), but by the results of [19] and [21] this is $NG_2 \cap D_2$ or rather its image in the quotient $C^{(2)}(\mathbf{G})_2$. The result then follows since

$$C^{(1)}(\mathbf{G})_2 = \frac{NG_2}{(NG_2 \cap D_2)d_3(NG_3 \cap D_3)}.$$

∎

The functor L preserves freeness.

4.7. Proposition. *If C is a (totally) free 2-crossed complex, then $L(C)$ is a (totally) free crossed complex.*

Proof. Above dimension 2, L does nothing and as

$$C_0/\partial C_1 \cong L(C)_0/\partial L(C)_1,$$

the freeness of the modules $L(C)_n$, $n \geq 3$ is not in doubt. In the base 2-crossed module we have merely to check that $L(C)_2$ is a free $L(C)_0/\partial L(C)_1$-module, as the behaviour of L on (C_1, C_0, ∂) is just that of the quotienting operation that turns a pre-crossed module into a crossed module and this preserves freeness, [7].

Suppose therefore that $C_2 \to C_1 \to C_0$ is a free 2-crossed module with basis $\theta : Y_1 \to C_2$. Suppose also given a module M over $G = C_0/\partial C_1$ and a function $\phi : Y_1 \to M$. We need to show that ϕ extends over $L(C)_2$. To do this we construct a 2-crossed complex $\mathcal{D}$ as follows:
The base is the precrossed module (C_1, C_0, ∂). To complete this we put $\mathrm{Ker}\partial_1 \times M$ in dimension 2 with as ∂_2 the inclusion on $\mathrm{Ker}\partial_1$ and the trivial map on M,

$$\mathcal{D} := \{D : Ker\partial_1 \times M \to C_1 \to C_0\}.$$

The Peiffer lifting is just the Peiffer commutator map from $C_1 \times C_1$ to $Ker\partial_1$ and the axioms are easy to check. Now define $\bar{\phi}$ from our given free 2-crossed module to this one $\mathcal{D}$ by defining $\bar{\phi}(y) = (\partial\theta y, \phi y)$ for $y \in Y_2$. Compose $\bar{\phi}$ with the obvious projection from $\mathcal{D}$ to the crossed complex

$$M \to 1 \to G,$$

where as before, $G = C_0/\partial C_1$. The composed map factors through $L(C)$ giving a morphism $L(C)_2 \to M$ extending ϕ. This is the unique extension of ϕ since at each stage uniqueness was a consequence of the conditions. ∎

The functor $C^{(2)}$ has a right adjoint just as does $C^{(1)}$. Given a 2-crossed complex, C, one first constructs the simplicial group (or groupoid) corresponding to the 2-crossed module at the base using Conduché's theorem. We also form the simplicial group from the chain complex given by all C_i, $i \geq 2$. The fact that C_2 may be non-abelian does not cause problems but does force semidirect products to be used rather than products. The two parts are then put together via a semidirect product much as in Ehlers and Porter, [13], Proposition 2.4.

Remark. An alternative but equivalent approach to this result follows a route via hypercrossed complexes (cf. Carrasco and Cegarra, [8]), and their extension of the Dold-Kan theorem.

We can also adapt the methods used in Ashley, [1]. In this case $\mathbf{NG}$ is a crossed complex if and only if $NG_n \cap D_n$ is always trivial. (In fact then $\mathbf{NG} \cong C^{(1)}\mathbf{G}$ since $C^{(1)}\mathbf{G}$ is obtained by dividing NG_n by $(NG_n \cap D_n)d_{n+1}(NG_{n+1} \cap D_{n+1})$ in dimension n, and of course this is assumed to be trivial for all $n \geq 2$.) A similar argument applies if $NG_n \cap D_n$ is trivial for $n \geq 3$. Then $C^{(2)} \cong \mathbf{NG}$, so the Moore complex is a 2-crossed complex.

Similar structures have been studied by Duskin, Glenn and Nan Tie under the name of 'n-hypergroupoids' (here $n = 2$). The simplicial groupoids that give rise to 2-crossed complexes seem to be 2-hypergroupoids internal to the category of groupoids or a slight variant of such things. We have not investigated this connection in any depth.

The category $\mathfrak{X}_2\mathfrak{Comp}$ is clearly equivalent to a reflexive subcategory of the category of hypercrossed complexes of Carrasco and Cegarra [8] and this completes the chain of linked structures, since this implies once again that $\mathfrak{X}_2\mathfrak{Comp}$ is equivalent to a reflexive subcategory (in fact a variety) in the category of simplicial groupoids. (Each of these statements is the result of direct verification using the constructions and structures outlined above.)

Given the linkages between the various categories above, one would expect the following:

4.8. Theorem. *If* $\mathbf{F}$ *is a free simplicial group (or groupoid), then* $C^{(2)}\mathbf{F}$ *is a totally free 2-crossed complex. If* $\mathfrak{F}$ *is a* CW*-basis for* $\mathbf{F}$*, then* $\mathfrak{F}$ *gives construction data for* $C^{(2)}\mathbf{F}$*.*

Proof. We have already seen that the base 2-crossed module of $C^{(2)}\mathbf{F}$ is totally free on construction data derived from the CW-basis. It remains to show that the $C_0/\partial C_1$-modules in higher dimension are free on the corresponding data, but here we can use the case of crossed complexes, and that was proved in [21]. ∎

Remarks. There are various things to note:

(i) The proof given in [21] that if $\mathbf{F}$ is a simplicial resolution of G then $C^{(1)}\mathbf{F}$ is a free crossed resolution of G does not immediately extend to '2-crossed resolutions'. The notion of 2-crossed resolution clearly would make sense and seems to be needed for handling certain problems in group extension theory, however we have not given a construction of a tensor product of a pair of 2-crossed complexes and the result for crossed resolutions used $\pi(1)\otimes\_$ where $\pi(1)$ is the free crossed complex on one generator in dimension 1, and thus is also $\pi(\Delta[1])$ the crossed complex of the 1-simplex, and $\otimes$ is the tensor product of crossed complexes defined by Brown and Higgins in [6]. This construction could be avoided by using enriched tensors $K\overline{\otimes}\_$ in the simplicially enriched category of 2-crossed complexes and then taking $\Delta[1]\overline{\otimes}\_$, which should give the same result, but as we have not yet investigated colimits of 2-crossed complexes that construction must also be put off for a future date. It should be pointed out that Baues has in [3] defined a tensor product of totally free quadratic complexes using a natural construction, so it seems unlikely that the conjectured constructions are technically difficult.

(ii) Although $C^{(2)}\mathbf{F}$ is totally free for $\mathbf{F}$ a free simplicial group, it seems almost certain that not all totally free 2-crossed complexes arise in this way. The difference is that in a CW-basis, any new generators in dimension n influence $\pi_n\mathbf{F}$ or $\pi_{n-1}\mathbf{F}$ either as generators or relations. In a 2-crossed complex, the new generators at each level influence the *relative* homotopy groups, $\pi_n(\mathbf{F}^{(n)}, \mathbf{F}^{(n-1)})$. The differences here are subtle. This is of course more or less equivalent to the realisation problem of Whitehead discussed at length by Baues [3] but occurring here in a purely algebraic context. Clearly this algebraic realisation problem is important for the analysis of the difference in the homotopical information that can be gleaned from crossed or 2-crossed as against simplicial methods.

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