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A TENSOR PRODUCT FOR **Gray**-CATEGORIES

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ABSTRACT. In this paper I extend Gray's tensor product of 2-categories to a new tensor product of **Gray**-categories. I give a description in terms of generators and relations, one of the relations being an "interchange" relation, and a description similar to Gray's description of his tensor product of 2-categories. I show that this tensor product of **Gray**-categories satisfies a universal property with respect to quasi-functors of two variables, which are defined in terms of lax-natural transformations between **Gray**-categories. The main result is that this tensor product is part of a monoidal structure on **Gray-Cat**, the proof requiring interchange in an essential way. However, this does not give a monoidal (bi)closed structure, precisely because of interchange. And although I define composition of lax-natural transformations, this composite need not be a lax-natural transformation again, making **Gray-Cat** only a partial $(\mathbf{Gray\text{-}Cat})_\otimes$-CATegory.

1. Introduction

In the cartesian product of 2-categories $\mathbb{C} \times \mathbb{D}$ there is for every $f : C \to C'$ in $\mathbb{C}$ and $g : D \to D'$ in $\mathbb{D}$ an arrow $(f, g) : (C, D) \to (C', D')$, and both triangles in the diagram

$$\begin{tikzcd}
 & (C,D') \arrow[dr, "{(f,\mathrm{id}_{D'})}"] & \\
(C,D) \arrow[ur, "{(\mathrm{id}_C,g)}"] \arrow[rr, "{(f,g)}"] \arrow[dr, "{(f,\mathrm{id}_D)}"'] & & (C',D') \\
 & (C',D) \arrow[ur, "{(\mathrm{id}_{C'},g)}"'] &
\end{tikzcd}$$

commute. There is a corresponding internal hom, the 2-category $[\mathbb{C},\mathbb{D}]$ having as objects 2-functors $\mathbb{C} \to \mathbb{D}$, as arrows 2-natural transformations, and as 2-arrows modifications. This way, 2-**Cat**, with objects (*small*) 2-categories, becomes, for standard reasons from enriched category theory [25], a (large) (2-**Cat**)-category, also known as a 3-category.

In Gray's tensor product of 2-categories [16] there is, instead, for every $f : C \to C'$ in $\mathbb{C}$ and $g : D \to D'$ in $\mathbb{D}$ a 2-arrow

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$$\begin{tikzcd}
 & C \otimes D' \arrow[dr, "f\otimes D'"] & \\
C \otimes D \arrow[ur, "C\otimes g"] \arrow[dr, "f\otimes D"'] & \Downarrow f \otimes g & C' \otimes D' \\
 & C' \otimes D \arrow[ur, "C'\otimes g"'] &
\end{tikzcd}.$$

There is a corresponding internal hom as well, the 2-category $\mathrm{Fun}(\mathbb{C},\mathbb{D})$ having as objects 2-functors $\mathbb{C} \to \mathbb{D}$, as arrows *lax-natural* transformations, and as 2-arrows modifications. This way, 2-**Cat** becomes a (large) $(2\text{-}\mathbf{Cat})_\otimes$-category. A slightly different tensor product is obtained if one requires $f \otimes g$ to be an isomorphism. The corresponding internal hom $\mathrm{Pseud}(\mathbb{C},\mathbb{D})$ has as arrows *pseudo*-natural transformations. 2-**Cat** with this monoidal closed structure, which is in fact symmetric, is, following Gordon, Power and Street [15], denoted by **Gray**, and this way **Gray** itself becomes a **Gray**-category.

Gray introduced his tensor product of 2-categories in order to describe lax-natural transformations and their composition properly. Since then, Gray's tensor product has gained wider significance, particularly in the form of **Gray**-categories. The first instance of this is in topology, where 3-groupoids are insufficient to classify homotopy 3-types [6], while **Gray**-groupoids do suffice [21, 5] for this. The second instance is in category theory, where there is a coherence theorem stating that every tricategory is equivalent, in some precise sense, to a **Gray**-category, but not necessarily to a 3-category [15]. It should be noted here that the pseudo-version of the tensor product seems to be more important than the lax-version, at least from the evidence from tricategories, as the proof of the coherence theorem does not work for lax tricategories.

The difference between the cartesian product and Gray's tensor product, and between 2-natural and lax- (or pseudo-) natural transformations, and between 3-categories and **Gray**-categories, is not just the difference between a commuting square and a square commuting up to a 2-arrow. I.e., this difference is not due to some MAIN PRINCIPLE OF CATEGORY THEORY, that in any category it is unnatural and undesirable to speak about equality of two objects [22, p. 179]. If that were the case, it would first of all be a tensor product of bicategories, the lax-natural transformations would be between (homo)morphisms instead of between 2-functors, and one would probably get tricategories via some theory of "weak enrichment". And in category theory, equality often is important, as can be seen from Kelly's body of work [24, 23, 4], and from the abundance of coherence theorems [27, 29, 30, 15].

No, the conceptual difference lies in the treatment of dimension. The cartesian product of 2-categories, and of ω-categories, is basically set-theoretical: $\mathbb{C} \times \mathbb{D}$ has as basic ingredient pairs (x, y) of dimension p for $x \in C_p$ and $y \in D_p$, and functoriality then gives, more generally, pairs (x, y) of dimension $\max\{p, q\}$ for $x \in C_p$ and $y \in D_q$. The tensor product of 2-categories, and of ω-categories [10, Section 3-7], is basically topological:

$\mathbb{C} \otimes \mathbb{D}$ has as basic ingredient expressions $c \otimes d$ of dimension $p+q$ for $c \in C_p$ and $d \in D_q$, and functoriality then needs to be imposed separately.

This difference in viewpoint has profound implications. Firstly, lax-natural transformations and modifications of 2-categories, and, more generally, lax-q-transformations of ω-categories [10, Section 3-9], are 2- or ω-functors $\mathbb{C} \otimes 2_q \to \mathbb{D}$, where 2_q denotes the ω-category free on one q-dimensional element. Because of the dimension-raising aspect of the tensor product, they become maps $\mathbb{C} \to \mathbb{D}$ sending a p-arrow to a $(p+q)$-arrow, i.e., *degree q maps*, satisfying some conditions with respect to faces. This is very much like in topology, where degree q maps between chain complexes satisfying some condition with respect to the boundary are known as *q-homotopies*. In fact, there is a very precise correspondence between q-homotopies and lax-q-transformations, the latter being the directed, functorial form of the former [20].

Secondly, horizontal composition in a **Gray**-category, and, more generally, in an $(\boldsymbol{\omega}\text{-}\mathbf{Cat})_\otimes$-category [10, Section 3-12], is a 2- or ω-functor $\mathrm{Hom}(C,D) \otimes \mathrm{Hom}(D,E) \to \mathrm{Hom}(C,E)$. Because of the dimension-raising aspect of the tensor product, the 0-composite of a p-arrow, i.e., a $(p-1)$-dimensional element of $\mathrm{Hom}(C,D)$, with a q-arrow, i.e., a $(q-1)$-dimensional element of $\mathrm{Hom}(D,E)$, becomes a $(p-1+q-1)$-dimensional element of $\mathrm{Hom}(C,E)$, i.e., a $(p+q-1)$-arrow. This reflects the topological importance of **Gray**-groupoids, as a topological space has trivial Whitehead products precisely when its homotopy type can be represented by an ω-groupoid, which has non-dimension-raising horizontal composition [6] (see also [7] for the connected case).

I use the idea of dimension raising as the basis for the extension of Gray's tensor product of 2-categories to a tensor product of **Gray**-categories. Such an extension should be a useful one, the corresponding 4-dimensional generalization of **Gray**-categories being particularly important. Firstly, the Whitehead product has precisely this dimension-raising property, as can be seen from [7] for example, so the groupoid-version of these 4-dimensional categorical structures is a good candidate for classification of homotopy 4-types. Secondly, I expect such structures to feature in a coherence theorem for weak 4-categories.[1] Thirdly, there is a possible application in 4-dimensional quantum field theory [8, 9].

Street has conjectured that it should be possible to use denseness of the (strict) n-cubes in $(\boldsymbol{\omega}\text{-}\mathbf{Cat})_\otimes\text{-}\mathbf{Cat}$ as a basis for generalizing the monoidal biclosed structure on ω-categories [32]. Dolan has claimed to have found a contradiction in Street's argument [14]. I will return to Dolan's comments, and to my own objection to Street's approach, later.

As said before, I take the dimension raising principle as basic. So for **Gray**-categories $\mathbb{C}$ and $\mathbb{D}$, their tensor product $\mathbb{C} \otimes \mathbb{D}$ has as generators expressions $c \otimes d$ of dimension $p+q$ for $c \in C_p$ and $d \in D_q$, for $p+q \leq 3$. The faces of such a generator $c \otimes d$ are

[1] Because weak n-categories [1, 3, 18] more or less correspond to Grothendieck's stacks [17] I have started to refer to —hypothetical!— higher-dimensional categorical structures with dimension-raising horizontal composition as *ω-teisi* (*Tas*, plural teisi (pronounced TAY-see), is Welsh for "stack".) [11].

composites of generators $c' \otimes d'$ for some specific faces c' and d' of c and d respectively, with (denoting p-source and p-target by d_p^- and d_p^+ respectively) $d_{p'}^{\alpha}(c) \otimes d_{q'}^{\beta}(d)$ occurring in the source (target) of $c \otimes d$ if and only if $\alpha = -$ $(+)$ and $\beta = (-)^{p'+1}$ $(\beta = (-)^{p'})$, just as for the tensor product of ω-categories.

For $c \in C_p$ and $d \in D_q$ with $p + q = 4$ there is not a generator of dimension 4, because **Gray**-categories don't have 4-dimensional elements, but instead an axiom requiring its would-be faces to be equal. On the other hand, thinking of a **Gray**-category as an infinite-dimensional structure in which all elements of dimension 4 and above are identities, this axiom can be seen as an identity 4-arrow $c \otimes d$. I will refer to axioms which come from identity generators, and more generally, from dimension raising, as *naturality axioms*.

In one of these naturality axioms there is a new feature, namely that a composite can involve a horizontal composition of 2-arrows, which in a **Gray**-category results in a 3-arrow. Not only need these 3-arrows be taken into account, but 3-arrows in *both* directions are needed to make all composites that occur sensible, see the diagram on page 24. This is why I restrict myself to **Gray**-categories as opposed to $(\mathbf{2\text{-}Cat})_{\otimes}$-categories, i.e., horizontal composition of 2-arrows is required to result in an *iso*-3-arrow. But the tensor product of **Gray**-categories I describe is the lax one; the pseudo version is an easy modification.

Taking this viewpoint back to Gray's tensor product, the 3-dimensional generators here are the non-identity version of Gray's naturality axioms. This is reminiscent of Baez and Dolan's plus-construction [1], which also replaces relations ("reduction laws") by generators ("operations") and rewrites between relations by new relations. Their construction is a "universal" one, though, whereas here only a few of the rewrites are considered.

The next basic thing is the behaviour of the tensor product with respect to composition (denoted by $\#$) and identities. Just as for Gray's tensor product, I require, for composable c and $c' \in \mathbb{C}$ and $d \in \mathbb{D}$, the generator $(c' \# c) \otimes d$ to be equal to some specific composite involving $c \otimes d$ and $c' \otimes d$, and similarly for $c \in \mathbb{C}$ and composable d and $d' \in \mathbb{D}$, see the diagrams on page 25, and similarly for identities. As with naturality, the composites occurring here can involve dimension-raising composites which need to be taken into account, as needs the direction of their result. I will refer to axioms which describe behaviour with respect to composition and identities as *functoriality axioms*.

For Gray's tensor product the functoriality axioms in each variable separately imply "functoriality in both variables at the same time". Here they do not. The problem is that, for composable 1-arrows f and $f' \in \mathbb{C}$ and composable 1-arrows g and $g' \in \mathbb{D}$, the functoriality axioms imply that $(f' \#_0 f) \otimes (g' \#_0 g)$ is equal to both possible vertical composites of the diagram

$$\begin{tikzcd}
 & & \bullet \arrow[dr] & & \\
 & \bullet \arrow[ur] \arrow[dr] \arrow[rr, phantom, "\Downarrow f\otimes g'"] \arrow[dd, phantom, "f\otimes g\Downarrow"] & & \bullet \arrow[dr] \arrow[dd, phantom, "\Downarrow f'\otimes g'"] & \\
\bullet \arrow[ur] \arrow[dr] & & \bullet \arrow[ur] \arrow[dr] & & \bullet \\
 & \bullet \arrow[ur] \arrow[dr] \arrow[rr, phantom, "\Downarrow f'\otimes g"] & & \bullet \arrow[ur] & \\
 & & \bullet \arrow[ur] & &
\end{tikzcd},$$

but that this diagram can also be composed using a horizontal composition of 2-arrows, resulting in a 3-arrow between the vertical composites, see the diagram on page 31. Hence $(f' \#_0 f) \otimes (g' \#_0 g)$ is not equal to "the" composite of $f \otimes g$, $f' \otimes g$, $f \otimes g'$ and $f' \otimes g'$, as would be required for "functoriality in both variables at the same time". Therefore, I impose an extra axiom, requiring the 3-arrow referred to above to be equal to the identity 3-arrow between its source and target (which are indeed equal). I will refer to this axiom as the *interchange axiom*.

Functoriality in both variables at the same time, and hence interchange, also ensures that the tensor product of **Gray**-categories satisfies an appropriate universal property. To express this universal property properly, I introduce the notion of *transfor*, which is the extension of the notions of lax-natural transformation and modification to **Gray**-categories, and *quasi-functors of two variables*, which should be thought of as "bi-functorial mappings". All this is similar to Gray's treatment of the universal property of the tensor product of 2-categories, except that here quasi-functors of two variables involve interchange explicitly. Actually, Gray's implicit treatment of interchange has some minor errors, which I correct. Another difference with Gray is that he starts with lax-natural transformations and looks at the tensor product afterwards, while I do it the other way around.

Transfors are also the "semistrict" version of the tritransformations, trimodifications and perturbations of Gordon, Power and Street [15]. I will return to some implications of this later as well.

One could argue that functoriality in both variables at the same time is not an essential feature of a tensor product, and hence question the introduction of the interchange axiom. But regardless of any conceptual or aesthetic reasons whether to include it or not, there is an overriding mathematical reason: it is needed in the proof of associativity of the tensor product, see the diagram on page 57. And from that proof it is clear that it is really functoriality in both variables at the same time that is used.

A perhaps unwanted consequence of interchange is that the monoidal structure on **Gray-Cat** of which the tensor product of **Gray**-categories is a part, is not monoidal *(bi-)closed*. The point is that interchange spoils the preservation of colimits in each variable of the tensor product. But it does come close: the obvious candidate for an internal hom has as i-dimensional elements i-transfors, and this fails to be a **Gray**-category only in that composition of 1-transfors need not always result in a 1-transfor again. Also, this is the only reason that **Gray-Cat** fails to be enriched over $(\mathbf{Gray\text{-}Cat})_{\otimes}$.

Dolan's objection to Street's files essentially amounts to the the statement that if you insist on biclosedness, which you do if the monoidal structure is to be induced from a (pro)monoidal structure on a dense subcategory [13], then interchange should hold, but it doesn't. He goes on to say that he thinks that "the meaning of the problem is something like that [...] monoidal biclosed categories [...] are not going to provide a basis for a good coherence theorem for weak n-categories". The results here show that biclosedness is indeed too much to hope for, but I think that, with interchange, such monoidal structures are a good candidate for providing a basis for "semistrict" n-categories. Finally, Dolan suspects that "the fact that [weak] n-categories are not completely comfortable until they can be treated as objects of a [weak] $(n+1)$-category asserts itself strongly", but, as shown here, this fact is not true for "semistrict 3-categories" so its relevance to a coherence result for weak n-categories is doubtful.

Having given up on biclosedness, this still leaves the question whether strict n-cubes can possibly be used as a basis for "semistrict" n-categories, thereby preserving part of the guiding principle of files. I think the answer to this question is negative, because in order to take the dimension-raising into account one then needs some sort of lax functors, which quickly leads outside the realm of "semistrict" n-categories.

Gordon, Power and Street claim that, for tricategories $\mathbb{C}$ and $\mathbb{D}$, $\mathbf{Tricat}(\mathbb{C},\mathbb{D})$ is a tricategory, and that if ($\mathbb{C}$ and) $\mathbb{D}$ are **Gray**-categories, so too is $\mathbf{Tricat}(\mathbb{C},\mathbb{D})$ [15, Corollary 8.3]. Now tritransformations etc. between **Gray**-categories are "weaker" than transfors, so the obstruction for their composition need not (and does not) occur. But for **Gray**-categories $\mathbb{C}$ and $\mathbb{D}$ this weakness does imply that the strict unit axiom for composition in $\mathbf{Tricat}(\mathbb{C},\mathbb{D})$ does not hold, and hence that in this case $\mathbf{Tricat}(\mathbb{C},\mathbb{D})$ is *not* a **Gray**-category. In fact, this error, and the underlying misconception about transformations, is *exactly the same* one as made by Baez and Neuchl in [2], which I observed and corrected in [12]. Nonetheless, the general result, that for tricategories $\mathbb{C}$ and $\mathbb{D}$, $\mathbf{Tricat}(\mathbb{C},\mathbb{D})$ is a tricategory, still holds, with a virtually identical proof, the only difference being that, for $\mathbb{D}$ a **Gray**-category, one shows that $\mathbf{Tricat}(\mathbb{C},\mathbb{D})$ is a (particularly simple sort of) tricategory, which is all that is needed.

This paper is organized as follows. Section 2 gives preliminaries on **Gray**-categories from the dimension-raising viewpoint. In section 3 I give a presentation for the tensor product of **Gray**-categories, and in section 4 a description of the **Gray**-category $\mathbb{C}\otimes\mathbb{D}$ *à la Gray.* I also correct a few minor errors in Gray's description of his tensor product. In section 5 I define transfors and quasi-functors of two variables, and give the universal property of the tensor product of **Gray**-categories in terms of these. In section 6 I give a presentation for the triple tensor product of **Gray**-categories, and use this in section 7 to prove that the tensor product is part of a monoidal structure. In section 8 I establish some further universal properties of the tensor product and the triple tensor product. In section 9 I look at composition of transfors, and at the extent to which **Gray-Cat** fails to be enriched over itself.

2. Preliminaries on **Gray**-categories

2-categories are usually introduced as double categories with an extra condition, after which one observes that 2-categories are exactly **Cat**-enriched categories [26, 28, 31]. Both these viewpoints, however, ignore, to a greater or lesser extent, the difference between horizontal and vertical composition. From the dimension-raising viewpoint, vertical composition and "whiskering", which don't raise dimension, come first, then horizontal composition, and it is only because there are no 3-dimensional elements in a 2-category that horizontal and vertical composition appear to be similar. More formally:

2.1. Proposition. *A 2-category is a sesqui-category* [33] $\mathbb{C}$ *in which for every* $\gamma : f \to f'$ *and* $\delta : g \to g'$ *in* C_2 *with* $t_0(\gamma) = s_0(\delta)$,

$$(g' \#_0 \gamma) \#_1 (\delta \#_0 f) \quad = \quad (\delta \#_0 f') \#_1 (g \#_0 \gamma), \tag{1}$$

the common value being denoted by $\delta \#_0 \gamma$. ■

Gray-categories are usually introduced via Gray's tensor product of 2-categories, which is dimension-raising, as follows.

2.2. Definition. **Gray** is the monoidal category of 2-categories and 2-functors with tensor product the pseudo-version of Gray's tensor product of 2-categories [16, 15]. ◇

Gray is in fact a (symmetric) monoidal closed category [16, 15, 10].

I need not go into Gray's tensor product any further, because it is just the reflection to 2-categories of the tensor product of **Gray**-categories which I will give in the next two sections. Also, at this point I only need **Gray** in order to define **Gray**-categories.

2.3. Definition. A **Gray**-category is a category enriched in the monoidal category **Gray**. ◇

Separating out the dimension-raising information, a **Gray**-category $\mathbb{C}$ can be described as consisting of collections C_0 of objects, C_1 of arrows, C_2 of 2-arrows and C_3 of 3-arrows, together with

- functions $s_n, t_n : C_i \to C_n$ for all $0 \leq n < i \leq 3$, also denoted d_n^- and d_n^+ and called *n-source* and *n-target*,
- functions $\#_n : C_{n+1} \; {}_{s_n}\!\times_{t_n} C_{n+1} \to C_{n+1}$ for all $0 \leq n < 3$, called *vertical composition*,
- functions $\#_n : C_i \; {}_{s_n}\!\times_{t_n} C_{n+1} \to C_i$ and $\#_n : C_{n+1} \; {}_{s_n}\!\times_{t_n} C_i \to C_i$ for all $0 \leq n \leq 1$, $n+1 < i \leq 3$, called *whiskering*,
- a function $\#_0 : C_2 \; {}_{s_0}\!\times_{t_0} C_2 \to C_3$, called *horizontal composition*, and
- functions $\mathrm{id}_- : C_i \to C_{i+1}$ for all $0 \leq i \leq 2$, called *identity*,

such that:

(i) $\mathbb{C}$ is a 3-skeletal reflexive globular set [34, p. 2],

(ii) for every $C, C' \in C_0$, the collection of elements of $\mathbb{C}$ with 0-source C and 0-target C' forms a 2-category $\mathbb{C}(C, C')$, with n-composition in $\mathbb{C}(C, C')$ given by $\#_{n+1}$ and identities given by id_{-},

(iii) for every $g : C' \to C''$ in C_1 and every C and $C''' \in C_0$, $- \#_0\, g$ is a 2-functor $\mathbb{C}(C'', C''') \to \mathbb{C}(C', C''')$ and $g \#_0 -$ is a 2-functor $\mathbb{C}(C, C') \to \mathbb{C}(C, C'')$,

(iv) for every $C' \in C_0$ and every C and $C'' \in C_0$, $-\#_0 \mathrm{id}_{C'}$ is equal to the identity functor $\mathbb{C}(C', C'') \to \mathbb{C}(C', C'')$ and $\mathrm{id}_{C'} \#_0 -$ is equal to the identity functor $\mathbb{C}(C, C') \to \mathbb{C}(C, C')$,

(v) for every $\gamma :$ $\begin{tikzcd} C \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow"] & C' \end{tikzcd}$ in C_2 and $\delta :$ $\begin{tikzcd} C' \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow"] & C'' \end{tikzcd}$ in C_2,

$$\begin{aligned} s_1(\delta \#_0 \gamma) &= (g' \#_0 \gamma) \#_1 (\delta \#_0 f) \\ t_1(\delta \#_0 \gamma) &= (\delta \#_0 f') \#_1 (g \#_0 \gamma) \end{aligned}$$

(compare equation (1)), and $\delta \#_0 \gamma$ is an iso-3-arrow,

(vi) for every $\varphi :$ $\begin{tikzcd} C \arrow[r, bend left=60, "f"] \arrow[r, bend right=60, "f'"'] \arrow[r, phantom, "\gamma' \Lleftarrow_3 \gamma"] & C' \end{tikzcd}$ in C_3 and $\delta :$ $\begin{tikzcd} C' \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow"] & C'' \end{tikzcd}$ in C_2,

$$((\delta \#_0 f') \#_1 (g \#_0 \varphi)) \#_2 (\delta \#_0 \gamma) \;=\; (\delta \#_0 \gamma') \#_2 ((g' \#_0 \varphi) \#_1 (\delta \#_0 f))$$

(compare the diagram for $\gamma \otimes g$ in section 3), and for every $\gamma :$ $\begin{tikzcd} C \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow"] & C' \end{tikzcd}$ in C_2 and

$\psi :$ $\begin{tikzcd} C' \arrow[r, bend left=60, "g"] \arrow[r, bend right=60, "g'"'] \arrow[r, phantom, "\delta' \Lleftarrow_3 \delta"] & C'' \end{tikzcd}$ in C_3,

$$(\delta' \#_0 \gamma) \#_2 ((g' \#_0 \gamma) \#_1 (\psi \#_0 f)) \;=\; ((\psi \#_0 f') \#_1 (g \#_0 \gamma)) \#_2 (\delta \#_0 \gamma)$$

(compare the diagram for $f \otimes \delta$ in section 3),

(vii) for every $\begin{tikzcd} C \arrow[r, bend left=60, "f"] \arrow[r, "f'" description] \arrow[r, bend right=60, "f''"'] \arrow[r, phantom, shift left=3, "\Downarrow\gamma"] \arrow[r, phantom, shift right=3, "\Downarrow\gamma'"] & C' \end{tikzcd}$ and $\delta :$ $\begin{tikzcd} C' \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow"] & C'' \end{tikzcd}$ in $\mathbb{C}$,

$$\delta \#_0 (\gamma' \#_1 \gamma) \;=\; ((\delta \#_0 \gamma') \#_1 (g \#_0 \gamma)) \#_2 ((g' \#_0 \gamma') \#_1 (\delta \#_0 \gamma)),$$

and for every $\gamma : \begin{tikzcd} C \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow"] & C' \end{tikzcd}$ and $\begin{tikzcd} C' \arrow[r, bend left=60, "g"] \arrow[r, "g'" description] \arrow[r, bend right=60, "g''"'] \arrow[r, phantom, shift left=3, "\Downarrow\delta"] \arrow[r, phantom, shift right=3, "\Downarrow\delta'"] & C'' \end{tikzcd}$ in $\mathbb{C}$,

$$(\delta' \#_1 \delta) \#_0 \gamma \;\; = \;\; ((\delta' \#_0 f') \#_1 (\delta \#_0 \gamma)) \#_2 ((\delta' \#_0 \gamma) \#_1 (\delta \#_0 f)),$$

(viii) for every $f : C \to C'$ in C_1 and $\delta : \begin{tikzcd} C' \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow"] & C'' \end{tikzcd}$ in C_2,

$$\delta \#_0 \mathrm{id}_f \;\; = \;\; \mathrm{id}_{\delta \#_0 f},$$

and for every $\gamma : \begin{tikzcd} C \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow"] & C' \end{tikzcd}$ in C_2 and $g : C' \to C''$ in C_1,

$$\mathrm{id}_g \#_0 \gamma \;\; = \;\; \mathrm{id}_{g \#_0 \gamma},$$

(ix) for every $c \in \mathbb{C}(C, C')_p$, $c' \in \mathbb{C}(C', C'')_q$ and $c'' \in \mathbb{C}(C'', C''')_r$ with $p + q + r \leq 2$,

$$(c'' \#_0 c') \#_0 c = c'' \#_0 (c' \#_0 c).$$

In this description, condition (ii) gives the vertical structure, conditions (iii) and (iv) give the behaviour of whiskering, condition (v) gives the faces of a horizontal composite, condition (vi) gives the *naturality axioms*, which describe behaviour of horizontal composition with respect to higher dimensional cells, conditions (vii) and (viii) give the *functoriality* axioms, which describe behaviour of horizontal composition with respect to composition and identity, and condition (ix) gives that $\#_0$ is *associative*.

The reader is strongly advised to draw the necessary diagrams, which I omit because there will be enough diagrams later on.

This description of a **Gray**-category shows that **Gray**-categories are algebraic structures, being given by some data with operations that have to satisfy certain relations. One implication of this is that it is possible to define a **Gray**-category by means of a *presentation*. That is exactly what I will do in the next section.

3. A presentation for the tensor product

I will now give a presentation for the tensor product of **Gray**-categories $\mathbb{C}$ and $\mathbb{D}$. The generators will be expressions $c \otimes d$ of dimension $p + q$ for $c \in C_p$ and $d \in D_q$, for $p + q \leq 3$. The faces of such a generator $c \otimes d$ will be composites of generators $c' \otimes d'$ for some specific faces c' and d' of c and d respectively. These generators are to satisfy relations, of which there are three kinds: *naturality* relations, which come from dimension raising, *functoriality* relations, which describe behaviour with respect to composition and identities, and an *interchange* relation.

The description I will give involves quite a lot of diagrams. I could have economized somewhat, but giving them all will be instructive, and they will all be re-used in the rest of the paper. Also, drawing a diagram is the most transparent way to give the generators and relations, and to show that all composites that occur are legitimate. It should be remembered, though, that a diagram is a *combinatorial* structure codifying combinatorial data [19].

Let $\mathbb{C}$ and $\mathbb{D}$ be **Gray**-categories. Define a **Gray**-category $\mathbb{C}\otimes\mathbb{D}$ by the following presentation.

3.1. Generators

Generators are expressions $c\otimes d$, with $c \in C_p$ and $d \in D_q$, of dimension $p+q$, for $p+q \leq 3$. Faces of these are (compare the tensor product of globes [10, Section 3-5]):

- for $p \leq 3$ and $q = 0$, if $\varphi :$ $\begin{tikzcd} C \arrow[r, bend left=60, "f"] \arrow[r, bend right=60, "f'"'] \arrow[r, phantom, "\gamma' \overset{3}{\Lleftarrow} \gamma"] & C' \end{tikzcd}$ in $\mathbb{C}$ and $D \in \mathbb{D}$, then $\varphi\otimes D$ is given by the diagram

$$\begin{tikzcd} C\otimes D \arrow[rr, bend left, "f\otimes D"] \arrow[rr, bend right, "f'\otimes D"'] \arrow[rr, phantom, "\Downarrow \gamma\otimes D"] & & C'\otimes D \end{tikzcd} \xrightarrow[3]{\varphi\otimes D} \begin{tikzcd} C\otimes D \arrow[rr, bend left, "f\otimes D"] \arrow[rr, bend right, "f'\otimes D"'] \arrow[rr, phantom, "\Downarrow \gamma'\otimes D"] & & C'\otimes D \end{tikzcd} ,$$

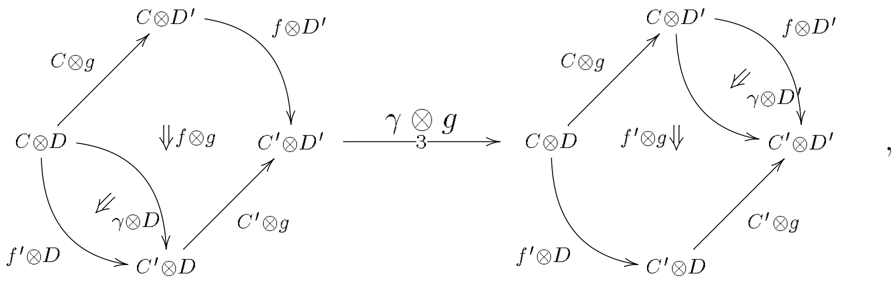
- for $p = 0$ and $q \leq 3$, if $C \in \mathbb{C}$ and $\psi :$ $\begin{tikzcd} D \arrow[r, bend left=60, "g"] \arrow[r, bend right=60, "g'"'] \arrow[r, phantom, "\delta' \overset{3}{\Lleftarrow} \delta"] & D' \end{tikzcd}$ in $\mathbb{D}$, then $C\otimes\psi$ is given by the diagram

$$\begin{tikzcd} C\otimes D \arrow[rr, bend left, "C\otimes g"] \arrow[rr, bend right, "C\otimes g'"'] \arrow[rr, phantom, "\Downarrow C\otimes\delta"] & & C\otimes D' \end{tikzcd} \xrightarrow[3]{C\otimes\psi} \begin{tikzcd} C\otimes D \arrow[rr, bend left, "C\otimes g"] \arrow[rr, bend right, "C\otimes g'"'] \arrow[rr, phantom, "\Downarrow C\otimes\delta'"] & & C\otimes D' \end{tikzcd} ,$$

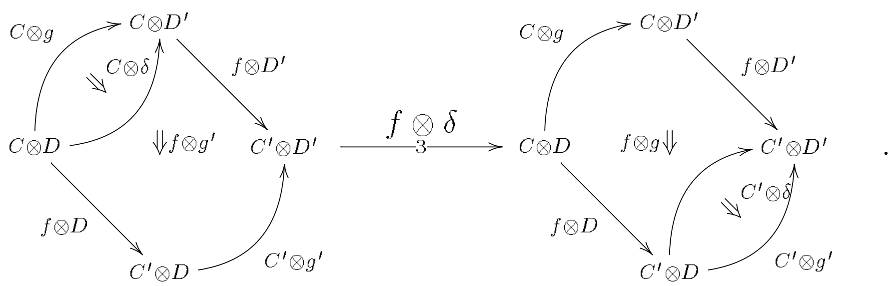
- for $p, q = 1$, if $f : C \to C'$ in $\mathbb{C}$ and $g : D \to D'$ in $\mathbb{D}$, then $f\otimes g$ is given by the diagram

$$\begin{tikzcd} & C\otimes D' \arrow[dr, "f\otimes D'"] & \\ C\otimes D \arrow[ur, "C\otimes g"] \arrow[dr, "f\otimes D"'] \arrow[rr, phantom, "\Downarrow f\otimes g"] & & C'\otimes D' \\ & C'\otimes D \arrow[ur, "C'\otimes g"'] & \end{tikzcd} ,$$

- for $p = 2$, $q = 1$, if $\gamma : \begin{tikzcd} C \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow"] & C' \end{tikzcd}$ in $\mathbb{C}$ and $g : D \to D'$ in $\mathbb{D}$, then $\gamma \otimes g$ is given by the diagram



- for $p = 1$, $q = 2$, if $f : C \to C'$ in $\mathbb{C}$ and $\delta : \begin{tikzcd} D \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow"] & D' \end{tikzcd}$ in $\mathbb{D}$, then $f \otimes \delta$ is given by the diagram

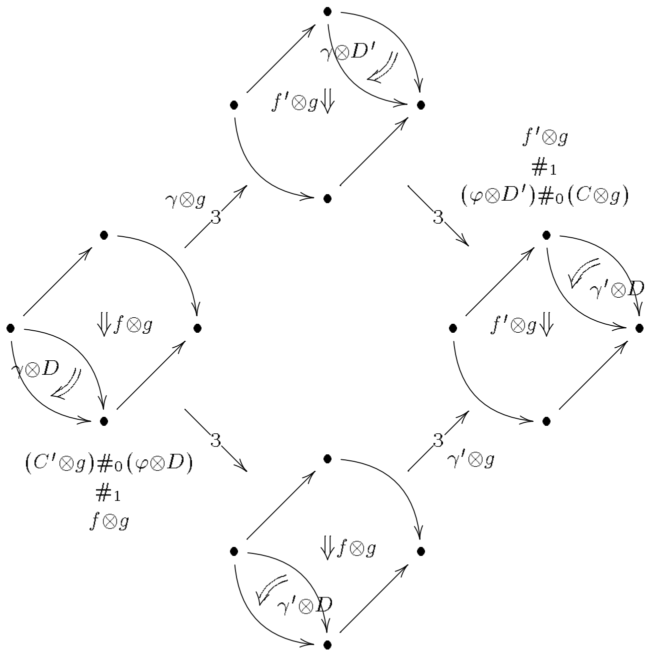


The generator $f \otimes g$ also occurs in Gray's tensor product of 2-categories. The generators $\gamma \otimes g$ and $f \otimes \delta$ also occur in Gray's tensor product, in the form of *relations* between their source and target.

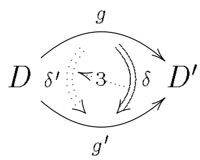
3.2. Naturality relations

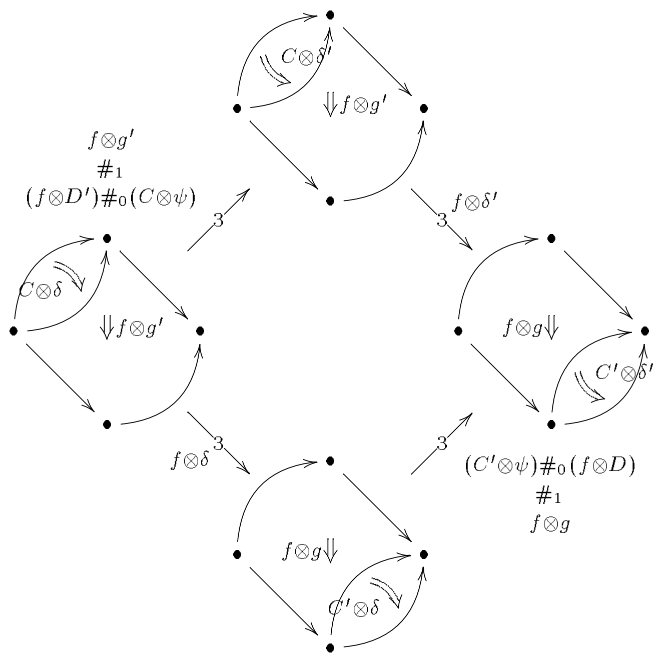
The naturality relations are:

- for $p = 3$ and $q = 1$, if $\varphi : \begin{tikzcd} C \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\gamma' \overset{3}{\Lleftarrow} \gamma"] & C' \end{tikzcd}$ in $\mathbb{C}$ and $g : D \to D'$ in $\mathbb{D}$, then the diagram



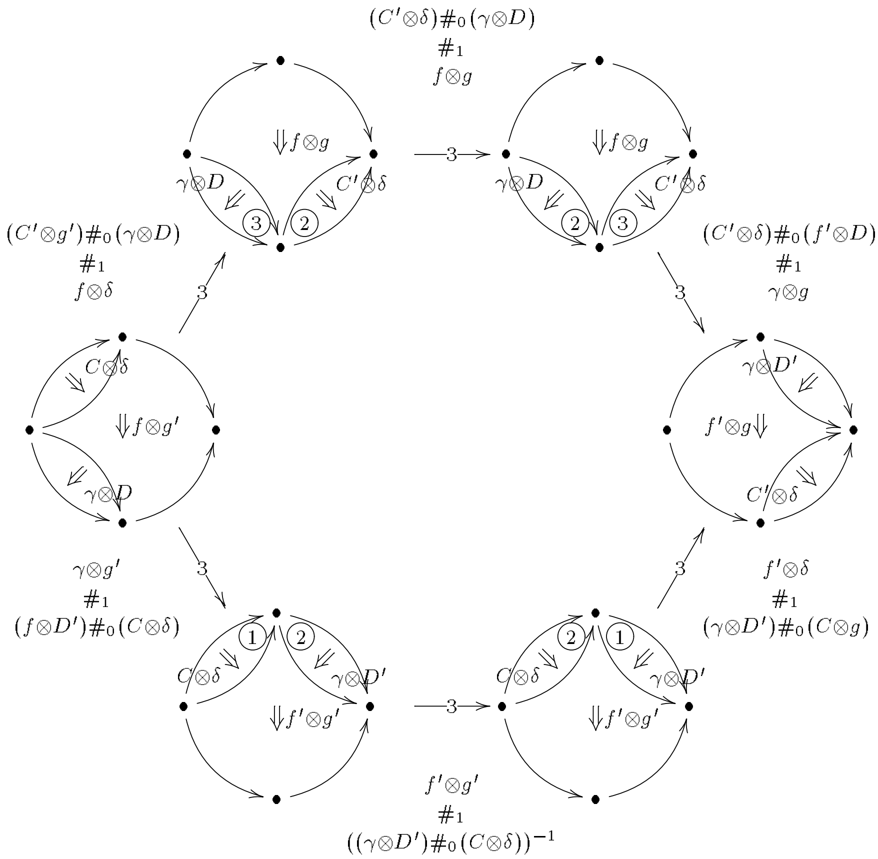
commutes,

- for $p = 1$ and $q = 3$, if $f : C \to C'$ in $\mathbb{C}$ and $\psi :$  in $\mathbb{D}$ then the diagram



commutes,

- for $p = 2$ and $q = 2$, if $\gamma : C \underset{f'}{\overset{f}{\Downarrow}} C'$ in $\mathbb{C}$ and $\delta : D \underset{g'}{\overset{g}{\Downarrow}} D'$ in $\mathbb{D}$, then the diagram



commutes.

In this final naturality relation there are two horizontal compositions of 2-arrows which need to be taken into account. The one in the top half of the diagram gives no problems, but it is necessary to take the *inverse* of the bottom one in order to make the bottom half of the diagram composable.

3.3. Functoriality relations

The functoriality relations are:

- for $p \leq 3$ and $q = 0$, if $c' \#_n c$ is defined in $\mathbb{C}$ and $D \in \mathbb{D}$, then

$$(c' \#_n c) \otimes D \quad = \quad (c' \otimes D) \#_n (c \otimes D),$$

- for $p = 0$ and $q \leq 3$, if $C \in \mathbb{C}$ and $d' \#_n d$ is defined in $\mathbb{D}$, then

$$C \otimes (d' \#_n d) \quad = \quad (C \otimes d') \#_n (C \otimes d),$$

- for $p, q = 1$, if $C \xrightarrow{f} C' \xrightarrow{f'} C''$ in $\mathbb{C}$ and $g : D \to D'$ in $\mathbb{D}$, then

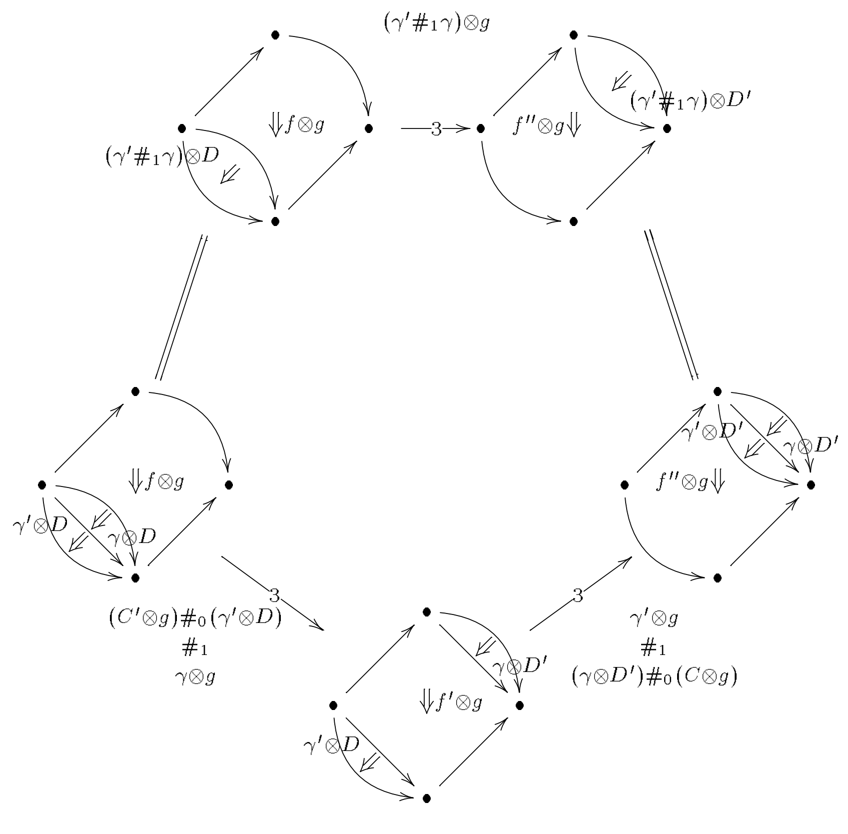
$$\begin{tikzcd} & \bullet \arrow[dr] & & \\ \bullet \arrow[ur] \arrow[dr] & & \bullet \arrow[dr] & \\ & \bullet \arrow[dr] \arrow[r, phantom, "\Downarrow (f' \#_0 f) \otimes g"] & & \bullet \\ & & \bullet \arrow[ur] & \end{tikzcd} \quad = \quad \begin{tikzcd} & \bullet \arrow[dr] \arrow[d, phantom, "\Downarrow f \otimes g"] & & \\ \bullet \arrow[ur] \arrow[dr] & {} & \bullet \arrow[dr] \arrow[d, phantom, "\Downarrow f' \otimes g"] & \\ & \bullet \arrow[ur] \arrow[dr] & {} & \bullet \\ & & \bullet \arrow[ur] & \end{tikzcd} ,$$

and if $f : C \to C'$ in $\mathbb{C}$ and $D \xrightarrow{g} D' \xrightarrow{g'} D''$ in $\mathbb{D}$, then

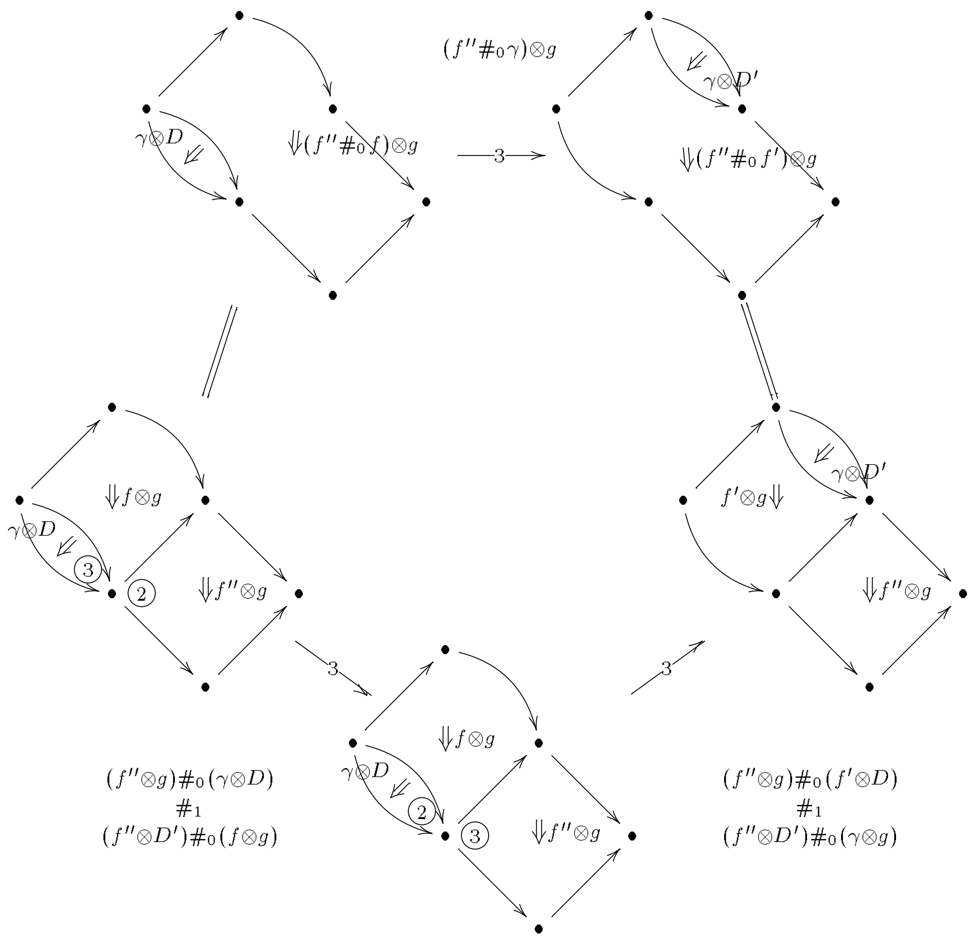
$$\begin{tikzcd} & & \bullet \arrow[dr] & \\ & \bullet \arrow[ur] \arrow[r, phantom, "\Downarrow f \otimes (g' \#_0 g)"] & & \bullet \\ \bullet \arrow[ur] \arrow[dr] & & \bullet \arrow[ur] & \\ & \bullet \arrow[ur] & & \end{tikzcd} \quad = \quad \begin{tikzcd} & & \bullet \arrow[dr] \arrow[d, phantom, "\Downarrow f \otimes g'"] & \\ & \bullet \arrow[ur] \arrow[dr] \arrow[d, phantom, "\Downarrow f \otimes g"] & {} & \bullet \\ \bullet \arrow[ur] \arrow[dr] & {} & \bullet \arrow[ur] & \\ & \bullet \arrow[ur] & & \end{tikzcd} ,$$

- for $p = 2$, $q = 1$, composition in left factor, if $\begin{tikzcd} C \arrow[r, bend left=50, "f"] \arrow[r, "f'" description] \arrow[r, bend right=50, "f''"'] \arrow[r, phantom, bend left=25, "\Downarrow\gamma"] \arrow[r, phantom, bend right=25, "\Downarrow\gamma'"] & C' \end{tikzcd}$ in $\mathbb{C}$ and $g : D \to D'$

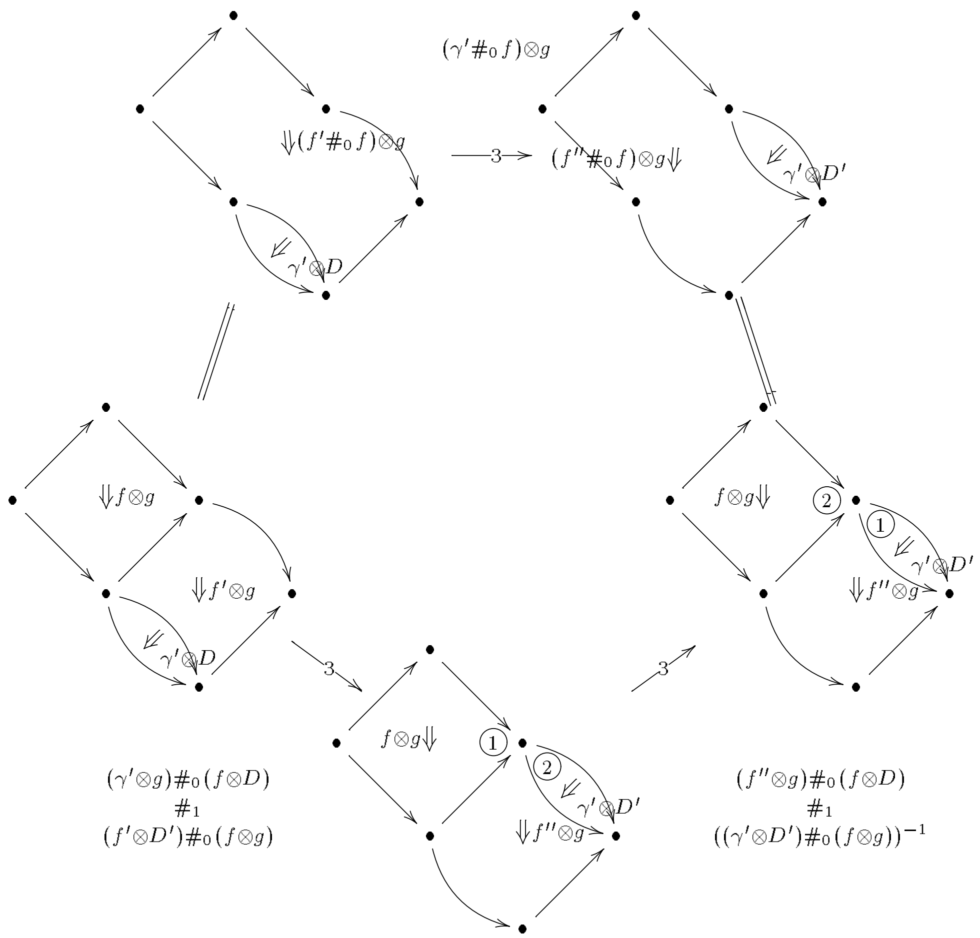
in $\mathbb{D}$, then the diagram



commutes, if $C \underset{f'}{\overset{f}{\underset{\Downarrow\gamma}{\rightrightarrows}}} C' \xrightarrow{f''} C''$ in $\mathbb{C}$ and $g : D \to D'$ in $\mathbb{D}$, then the diagram

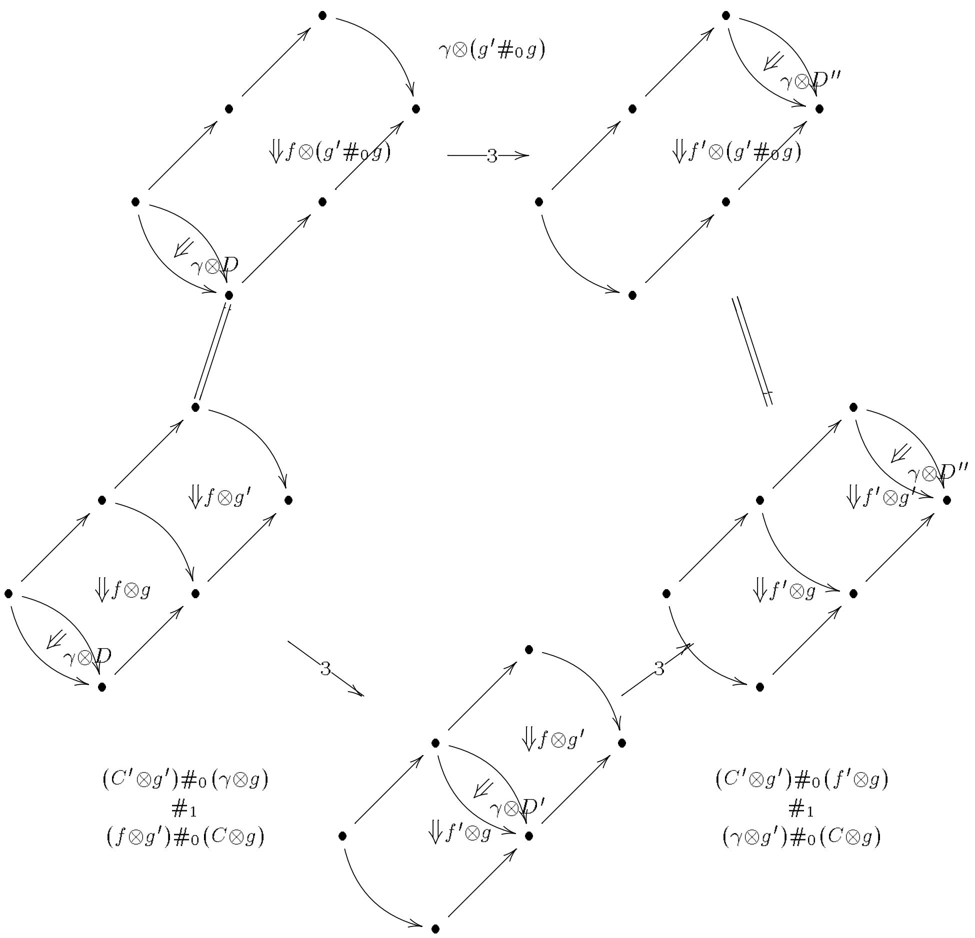


commutes, if $C \xrightarrow{f} C' \underset{f''}{\overset{f'}{\Downarrow\gamma'}} C'$ in $\mathbb{C}$ and $g : D \to D'$ in $\mathbb{D}$, then the diagram



commutes,

- for $p = 2$, $q = 1$, composition in right factor, if $\gamma : \begin{tikzcd} C \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow"] & C' \end{tikzcd}$ in $\mathbb{C}$ and $D \xrightarrow{g} D' \xrightarrow{g'} D''$ in $\mathbb{D}$, then the diagram



commutes,

- for $p = 1$, $q = 2$, composition in left factor, analogous to $p = 2$, $q = 1$, composition in right factor, and left to the reader,
- for $p = 1$, $q = 2$, composition in right factor, analogous, but slightly different, to $p = 2$, $q = 1$, composition in left factor, and left to the reader.

The conditions for $q = 0$ and for $p = 0$ can be summarized by saying that $- \otimes D$ is a **Gray**-functor $\mathbb{C} \to \mathbb{C} \otimes \mathbb{D}$ and that $C \otimes -$ is a **Gray**-functor $\mathbb{D} \to \mathbb{C} \otimes \mathbb{D}$.

The functoriality relations not involving 3-arrows, either in $\mathbb{C}$ or $\mathbb{D}$, or in $\mathbb{C} \otimes \mathbb{D}$, also occur in Gray's tensor product of 2-categories.

In two of the 3-dimensional functoriality relations there again occur horizontal compositions of 2-arrows, and they are taken into account as indicated.

Note that the well-definedness of the functoriality relations for $p \neq 0$, $q \neq 0$, i.e., that the sources and targets of the composites that are to be related are equal, relies on the lower-dimensional functoriality relations.

3.4. Identity relations

The identity relations are:

- for $p + q \leq 2$, if id_c is defined in $\mathbb{C}$ and $d \in \mathbb{D}$, then

$$\mathrm{id}_c \otimes d \quad = \quad \mathrm{id}_{c\otimes d},$$

  and if $c \in \mathbb{C}$ and id_d is defined in $\mathbb{D}$, then

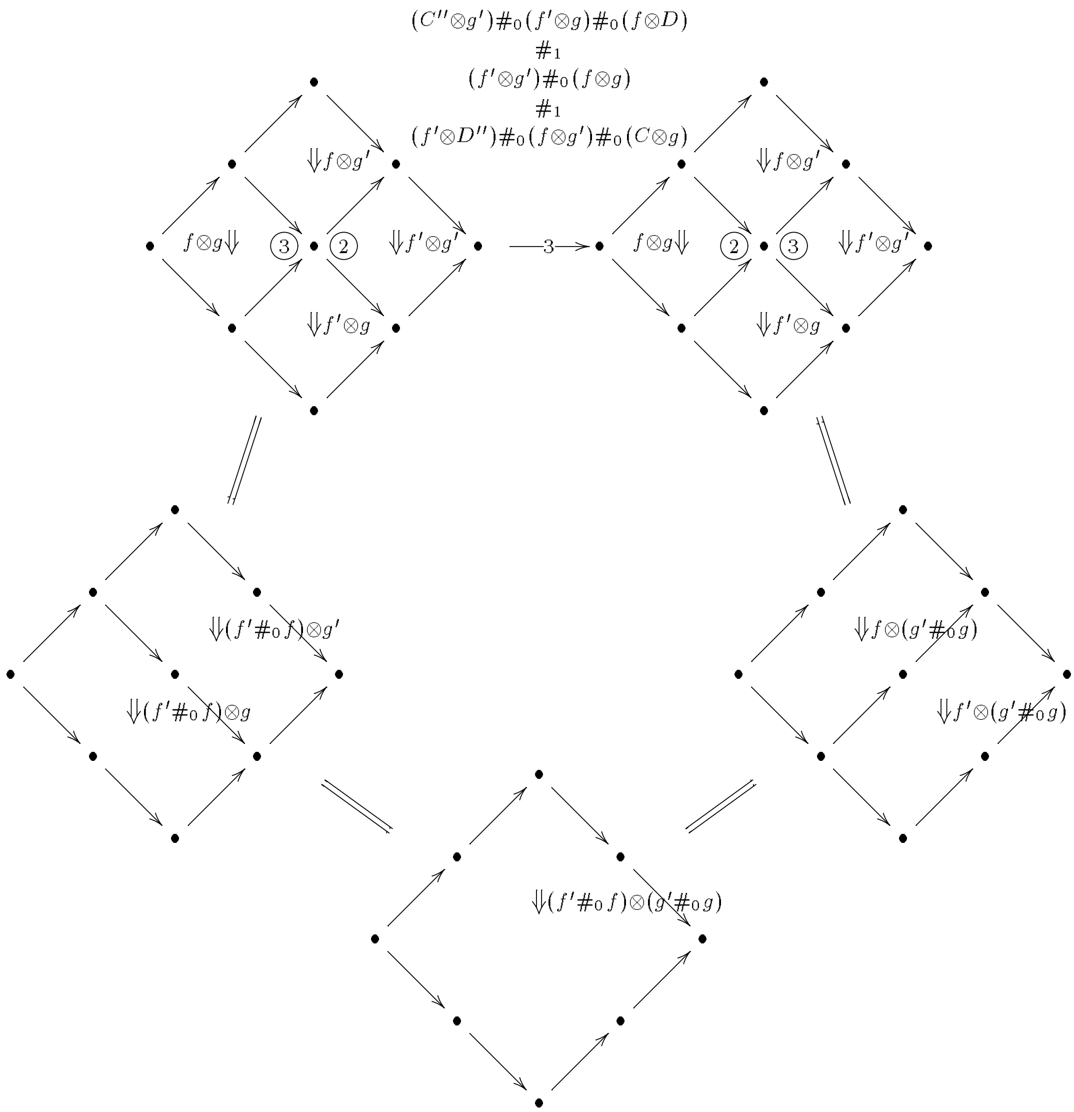
$$c \otimes \mathrm{id}_d \quad = \quad \mathrm{id}_{c\otimes d}\,.$$

Note that the well-definedness of the higher-dimensional identity relations, i.e., that the source and target of an element that is to be related to an identity are equal, relies on the lower-dimensional identity relations.

3.5. Interchange relations

The interchange relation is:

- for $p, q = 1$, if $C \xrightarrow{f} C' \xrightarrow{f'} C''$ in $\mathbb{C}$ and $D \xrightarrow{g} D' \xrightarrow{g'} D''$ in $\mathbb{D}$ then the diagram



commutes.

The interchange relation is necessitated by the occurrence of the horizontal composition of 2-arrows in the diagram above.

Note that the well-definedness of the interchange relation relies on the lower-dimensional functoriality relations.

3.6. Naturality

Let $F : \mathbb{C} \to \mathbb{C}'$ and $G : \mathbb{D} \to \mathbb{D}'$ be **Gray**-functors between **Gray**-categories. Define a **Gray**-functor $F \otimes G : \mathbb{C} \otimes \mathbb{D} \to \mathbb{C}' \otimes \mathbb{D}'$ using the presentation for the tensor product given above.

On generators, $(F \otimes G)(c \otimes d) = F(c) \otimes G(d)$, which is a generator for $\mathbb{C}' \otimes \mathbb{D}'$. That this preserves the relations is immediate.

3.1. LEMMA. *$\otimes$ is a functor* **Gray-Cat** $\times$ **Gray-Cat** $\to$ **Gray-Cat**.

PROOF. Indeed, it preserves composition of functors in both variables. ∎

4. The tensor product in terms of whiskers

In his original treatment of his tensor product of 2-categories [16, p. 73–77], Gray gives the elements of (the 2-category) $\mathbb{C}\otimes\mathbb{D}$ as equivalence classes of well-formed sequences of cells, with composition given by juxtaposition. I will give a description of the **Gray**-category $\mathbb{C}\otimes\mathbb{D}$, defined in the previous section, in the same spirit. This will also involve "normal forms" for the elements of dimension less than 3.

Comparing the description of the tensor product of **Gray**-categories given here with Gray's description of his tensor product of 2-categories, it will turn out that there are some minor inaccuracies in Gray's treatment, and that, because of the dimension-raising aspect of horizontal composition of 2-arrows, horizontal composition is dealt with in a different, more appropriate and more interesting, way.

Let $\mathbb{C}$ and $\mathbb{D}$ be **Gray**-categories. I will use the notational conventions from the previous section, i.e., objects of $\mathbb{C}$ will be denoted by C, C', arrows of $\mathbb{C}$ by f, f', which have source and target C and C', and so on, and similarly for $\mathbb{D}$. Generic elements of $\mathbb{C}$ and $\mathbb{D}$ will be denoted by c, c' and d, d' respectively.

4.1. Dimension 0

A 0-*cell* is a 0-dimensional generator $a = C\otimes D$.

A 0-*whisker* is a 0-cell.

Two 0-whiskers are *equivalent* if they are equal.

An *object* is an equivalence class of 0-whiskers.

These definitions are quite trivial, but are made this way for consistency with higher dimensions.

4.2. Dimension 1

A 1-*cell* is a 1-dimensional generator $a = f\otimes D$ or $a = C\otimes g$.

The faces of a 1-cell are given by:

$$\begin{array}{rcl} s_0(f\otimes D) & = & C\otimes D \\ s_0(C\otimes g) & = & C\otimes D \\ t_0(f\otimes D) & = & C'\otimes D \\ t_0(C\otimes g) & = & C\otimes D'. \end{array}$$

A 1-*dimensional* 1-*whisker* is a, possibly empty, sequence $(a_1,\ldots,a_m)$ of 1-cells with $s_0(a_{i+1})\sim t_0(a_i)$ for all $0<i<m$. Empty sequences come together with a 0-whisker.

1-whiskers can be thought of as zigzags, or, in Gray's terminology, as "approximations to the diagonal".

The faces of a 1-dimensional 1-whisker are given by:

$$\begin{array}{rcl} s_0(a_1,\ldots,a_m) & = & s_0(a_1) \\ t_0(a_1,\ldots,a_m) & = & t_0(a_m) \\ s_0((),C\otimes D) & = & C\otimes D \\ t_0((),C\otimes D) & = & C\otimes D. \end{array}$$

Two 1-dimensional 1-whiskers are *equivalent* if they are so in the smallest equivalence relation compatible with juxtaposition generated by:

$$\begin{array}{rcl} ((f' \#_0 f)\otimes D) & \sim & (f\otimes D, f'\otimes D) \\ (C\otimes (g' \#_0 g)) & \sim & (C\otimes g, C\otimes g') \\ (\mathrm{id}_C \otimes D) & \sim & ((),C\otimes D) \\ (C\otimes \mathrm{id}_D) & \sim & ((),C\otimes D). \end{array}$$

An *arrow* is an equivalence class of 1-dimensional 1-whiskers. I will not make a notational distinction between 1-dimensional 1-whiskers and the arrow they represent. This is all right as long as I take care that all operations on arrows are well-defined.

Composition of arrows and identity on objects is given by:

$$\begin{array}{rcl} (b_1,\ldots,b_{m'}) \#_0 (a_1,\ldots,a_m) & = & (a_1,\ldots,a_m,b_1,\ldots,b_{m'}) \\ \mathrm{id}_{C\otimes D} & = & ((),C\otimes D). \end{array}$$

A 1-dimensional 1-whisker $(a_1,\ldots,a_m)$ is *in normal form* if

- no two consecutive a_i, a_{i+1} are $f\otimes D, f'\otimes D$ or $C\otimes g, C\otimes g'$,
- no a_i is $\mathrm{id}_C \otimes D$ or $C\otimes \mathrm{id}_D$.

4.1. Proposition. (Normal form theorem for arrows) *Every arrow has a unique representative which is in normal form.*

Proof. Given any 1-dimensional 1-whisker, "contract" it by using the equivalences in the direction of the shorter 1-whiskers. The only instances of overlapping contractions are $(a_1,\ldots,a_{j-1}, f\otimes D, f'\otimes D, f''\otimes D, a_{j+1},\ldots,a_m)$ and $(a_1,\ldots,a_{j-1}, C\otimes g, C\otimes g', C\otimes g, a_{j+1},\ldots,a_m)$, where associativity of $\#_0$ in $\mathbb{C}$ and $\mathbb{D}$ respectively gives the required common contraction, and $(a_1,\ldots,a_{j-1}, f\otimes D, \mathrm{id}_{C'}\otimes D, a_{j+1},\ldots,a_m)$ and obvious variations thereof, where the axioms for identity give that both contractions are actually equal. ∎

In terms of normal forms of arrows, composition is concatenation followed by (possibly repeated) contraction. In general, the description of composition in terms of normal forms might not be very useful, as contraction might take many steps. But in the special case that $\mathbb{C}$ and $\mathbb{D}$ have no non-trivial compositions resulting in identities, for example when $\mathbb{C}$ and $\mathbb{D}$ are free, contraction takes only one step, and the description might have practical, computational, relevance.

4.3. Dimension 2

A 2-*cell* is a 2-dimensional generator $a = \gamma \otimes D$ or $a = C \otimes \delta$ or $a = f \otimes g$.

The faces of a 2-cell are given by:

$$\begin{array}{rcl} s_1(\gamma \otimes D) & = & f \otimes D \\ s_1(C \otimes \delta) & = & C \otimes g \\ s_1(f \otimes g) & = & C \otimes g, f \otimes D' \\ t_1(\gamma \otimes D) & = & f' \otimes D \\ t_1(C \otimes \delta) & = & C \otimes g' \\ t_1(f \otimes g) & = & f \otimes D, C' \otimes g. \end{array}$$

A 2-*dimensional* 1-*whisker* is a, possibly empty, sequence $(a_1, \ldots, a_m)$ of one 2-cell and 1-cells with $s_0(a_{i+1}) \sim t_0(a_i)$ for all $0 < i < m$. Empty sequences come together with a 0-whisker.

2-dimensional 1-whiskers can be thought of as zigzags with one blob, which is either square or a 2-dimensional glob. The ordinary categorical use of the word “whisker” is a blob, i.e., 2-arrow, with protruding hairs, i.e., arrows, here the blob is a 2-cell and the hairs are made up of 1-cells.

The faces of a 2-dimensional 1-whisker are given by:

$$\begin{array}{rcl} s_1(a_1, \ldots, a_m) & = & (s_1(a_1), \ldots, s_1(a_m)) \\ t_1(a_1, \ldots, a_m) & = & (t_1(a_1), \ldots, t_1(a_m)) \\ s_1((), C \otimes D) & = & ((), C \otimes D) \\ t_1((), C \otimes D) & = & ((), C \otimes D). \end{array}$$

Two 2-dimensional 1-whiskers are *equivalent* if they are so in the smallest equivalence relation compatible with juxtaposition generated by:

$$\begin{array}{rcl} ((c' \#_0 c) \otimes D) & \sim & (c \otimes D, c' \otimes D) \\ (C \otimes (d' \#_0 d)) & \sim & (C \otimes d, C \otimes d') \\ (\mathrm{id}_C \otimes D) & \sim & ((), C \otimes D) \\ (C \otimes \mathrm{id}_D) & \sim & ((), C \otimes D). \end{array}$$

There is a normal form for 2-dimensional 1-whiskers, just as for 1-dimensional 1-whiskers.

A 2-*dimensional* 2-*whisker* is a, possibly empty, sequence $[\Lambda_1, \ldots, \Lambda_n]$ of 2-dimensional 1-whiskers with $s_1(\Lambda_{i+1}) \sim t_1(\Lambda_i)$ for all $0 < i < n$. Empty sequences come together with a 1-dimensional 1-whisker.

The faces of a 2-dimensional 2-whisker are given by:

$$\begin{array}{rcl} s_1[\Lambda_1, \ldots, \Lambda_n] & = & s_1(\Lambda_1) \\ t_1[\Lambda_1, \ldots, \Lambda_n] & = & t_1(\Lambda_n) \\ s_1([], (a_1, \ldots, a_m)) & = & (a_1, \ldots, a_m) \\ t_1([], (a_1, \ldots, a_m)) & = & (a_1, \ldots, a_m). \end{array}$$

Two 2-dimensional 2-whiskers are *equivalent* if they are so in the smallest equivalence relation compatible with juxtaposition with 1-cells and with 2-dimensional 1-whiskers generated by:

$$\begin{array}{rcl}
[(a_1,\ldots,a_m)] & \sim & [(b_1,\ldots,b_{m'})] \qquad \text{if } (a_1,\ldots,a_m)\sim(b_1,\ldots,b_{m'}) \\
[((c' \#_1 c)\otimes D)] & \sim & [(c\otimes D),(c'\otimes D)] \\
[(C\otimes(d' \#_1 d))] & \sim & [(C\otimes d),(C\otimes d')] \\
[(\mathrm{id}_f\otimes D)] & \sim & ([],(f\otimes D)) \\
[(C\otimes \mathrm{id}_g)] & \sim & ([],(C\otimes g)) \\
[((f' \#_0 f)\otimes g)] & \sim & [(f\otimes g, f'\otimes D'),(f\otimes D, f'\otimes g)] \\
[(f\otimes(g' \#_0 g))] & \sim & [(C\otimes g, f\otimes g'),(f\otimes g, C'\otimes g')] \\
[(f\otimes \mathrm{id}_D)] & \sim & ([],(f\otimes D)) \\
[(\mathrm{id}_C\otimes g)] & \sim & ([],(C\otimes g)).
\end{array}$$

A 2-*arrow* is an equivalence class of 2-dimensional 2-whiskers.

1-composition of 2-arrows, identity on arrows, 0-composition of a 2-arrow with an arrow and 0-composition of an arrow with a 2-arrow are given (with juxtaposition denoting juxtaposition of sequences) by:

$$\begin{array}{rcl}
[\Xi_1,\ldots,\Xi_{n'}] \#_1 [\Lambda_1,\ldots,\Lambda_n] & = & [\Lambda_1,\ldots,\Lambda_n,\Xi_1,\ldots,\Xi_{n'}] \\
\mathrm{id}_{(a_1,\ldots,a_m)} & = & ([],(a_1,\ldots,a_m)) \\
\Xi \#_0 [\Lambda_1,\ldots,\Lambda_n] & = & [\Lambda_1\Xi,\ldots,\Lambda_n\Xi] \\
[\Xi_1,\ldots,\Xi_{n'}] \#_0 \Lambda & = & [\Lambda\Xi_1,\ldots,\Lambda\Xi_{n'}].
\end{array}$$

Note that I cannot define 0-composition of 2-arrows yet, because I haven't defined 3-arrows yet.

A 2-dimensional 2-whisker $[\Lambda_1,\ldots,\Lambda_n]$ is *in normal form* if

- each Λ_i is in normal form,
- no two consecutive Λ_i,Λ_{i+1} are $(a_1,\ldots,a_{j-1},c\otimes D,a_{j+1},\ldots,a_m),(a_1,\ldots,a_{j-1},c'\otimes D,a_{j+1},\ldots,a_m)$ or $(a_1,\ldots,a_{j-1},C\otimes d,a_{j+1},\ldots,a_m),(a_1,\ldots,a_{j-1},C\otimes d',a_{j+1},\ldots,a_m)$,
- no Λ_i is $(a_1,\ldots,a_{j-1},\mathrm{id}_f\otimes D,a_{j+1},\ldots,a_m)$ or $(a_1,\ldots,a_{j-1},C\otimes\mathrm{id}_g,a_{j+1},\ldots,a_m)$,
- no two consecutive Λ_i,Λ_{i+1} are $(a_1,\ldots,a_{j-1},f\otimes g,f'\otimes D,a_{j+1},\ldots,a_m),(a_1,\ldots,a_{j-1},f\otimes D,f'\otimes g,a_{j+1},\ldots,a_m)$ or $(a_1,\ldots,a_{j-1},C\otimes g,f\otimes g',a_{j+1},\ldots,a_m),(a_1,\ldots,a_{j-1},f\otimes g,C'\otimes g',a_{j+1},\ldots,a_m)$,
- no Λ_i is $(a_1,\ldots,a_{j-1},f\otimes\mathrm{id}_D,a_{j+1},\ldots,a_m)$ or $(a_1,\ldots,a_{j-1},\mathrm{id}_C\otimes g,a_{j+1},\ldots,a_m)$

4.2. Proposition. (Normal form theorem for 2-arrows) *Every 2-arrow has a unique representative which is in normal form.* ■

4.4. Dimension 3

A 3-*cell* is a 3-dimensional generator $a = \varphi \otimes D$ or $a = C \otimes \psi$ or $a = \gamma \otimes g$ or $a = f \otimes \delta$.

The faces of a 3-cell are given by:

$$\begin{array}{rcl} s_2(\varphi \otimes D) & = & \gamma \otimes D \\ s_2(C \otimes \psi) & = & C \otimes \delta \\ s_2(\gamma \otimes g) & = & (f \otimes g), (\gamma \otimes D, C' \otimes g) \\ s_2(f \otimes \delta) & = & (C \otimes \delta, f \otimes D'), (f \otimes g') \\ t_2(\varphi \otimes D) & = & \gamma' \otimes D \\ t_2(C \otimes \psi) & = & C \otimes \delta' \\ t_2(\gamma \otimes g) & = & (C \otimes g, \gamma \otimes D'), (f' \otimes g) \\ t_2(f \otimes \delta) & = & (f \otimes g), (f \otimes D, C' \otimes \delta). \end{array}$$

A 3-*dimensional* 1-*whisker* is a, possibly empty, sequence $(a_1, \ldots, a_m)$ of one 3-cell and 1-cells *or two* 2-*cells and* 1-*cells* with $s_0(a_{i+1}) \sim t_0(a_i)$ for all $0 < i < m$, in the case of two 2-cells and 1-cells together with a specification "$\mathfrak{r}$" or "$\mathfrak{l}$". Empty sequences come together with a 0-whisker.

The faces of a 3-dimensional 1-whisker are given by:

$$\begin{array}{rcl} s_1(a_1, \ldots, a_m) & = & (s_1(a_1), \ldots, s_1(a_m)) \\ t_1(a_1, \ldots, a_m) & = & (t_1(a_1), \ldots, t_1(a_m)) \\ s_1((), C \otimes D) & = & ((), C \otimes D) \\ t_1((), C \otimes D) & = & ((), C \otimes D). \end{array}$$

Two 3-dimensional 1-whiskers are *equivalent* if they are so in the smallest equivalence relation compatible with juxtaposition generated by:

$$\begin{array}{rcl} ((c' \#_0 c) \otimes D) & \sim & (c \otimes D, c' \otimes D) \\ (C \otimes (d' \#_0 d)) & \sim & (C \otimes d, C \otimes d') \\ (\mathrm{id}_C \otimes D) & \sim & ((), C \otimes D) \\ (C \otimes \mathrm{id}_D) & \sim & ((), C \otimes D). \end{array}$$

In particular, the first two relations for horizontal composition of 2-cells are to be interpreted as stating that $(\gamma \otimes D, \gamma' \otimes D)^{\mathfrak{r}}$ is equivalent to $(\gamma' \#_0 \gamma) \otimes D$ and $(\gamma \otimes D, \gamma' \otimes D)^{\mathfrak{l}}$ is equivalent to $(\gamma' \#_0 \gamma)^{-1} \otimes D$, and so on. Note also that in these relations, at most one of c and c' is a 3-cell.

There is a normal form for 3-dimensional 1-whiskers, just as for 1- and 2-dimensional 1-whiskers.

A 3-*dimensional* 2-*whisker* is a, possibly empty, sequence $[\Lambda_1, \ldots, \Lambda_n]$ of one 3-dimensional 1-whisker and 2-dimensional 1-whiskers with $s_1(\Lambda_{i+1}) \sim t_1(\Lambda_i)$ for all $0 < i < n$. Empty sequences come together with a 1-dimensional 1-whisker.

The faces of a 3-dimensional 2-whisker are given by:

$$
\begin{aligned}
s_2[\Lambda_1,\ldots,\Lambda_n] &= \begin{bmatrix} s_2(\Lambda_1),\ldots,s_2(\Lambda_{k-1}),\\ (a_1,\ldots,a_{j-1},\gamma\otimes D,a_{j+1},\ldots,a_m),\\ s_2(\Lambda_{k+1}),\ldots,s_2(\Lambda_n) \end{bmatrix}\\
&\qquad\text{if } \Lambda_k=(a_1,\ldots,a_m) \text{ and } a_j=\varphi\otimes D\\
&= \begin{bmatrix} s_2(\Lambda_1),\ldots,s_2(\Lambda_{k-1}),\\ (a_1,\ldots,a_{j-1},C\otimes\delta,a_{j+1},\ldots,a_m),\\ s_2(\Lambda_{k+1}),\ldots,s_2(\Lambda_n) \end{bmatrix}\\
&\qquad\text{if } \Lambda_k=(a_1,\ldots,a_m) \text{ and } a_j=C\otimes\psi\\
&= \begin{bmatrix} s_2(\Lambda_1),\ldots,s_2(\Lambda_{k-1}),\\ (a_1,\ldots,a_{j-1},f\otimes g,a_{j+1},\ldots,a_m),\\ (a_1,\ldots,a_{j-1},\gamma\otimes D,C'\otimes g,a_{j+1},\ldots,a_m),\\ s_2(\Lambda_{k+1}),\ldots,s_2(\Lambda_n) \end{bmatrix}\\
&\qquad\text{if } \Lambda_k=(a_1,\ldots,a_m) \text{ and } a_j=\gamma\otimes g\\
&= \begin{bmatrix} s_2(\Lambda_1),\ldots,s_2(\Lambda_{k-1}),\\ (a_1,\ldots,a_{j-1},C\otimes\delta,f\otimes D',a_{j+1},\ldots,a_m),\\ (a_1,\ldots,a_{j-1},f\otimes g',a_{j+1},\ldots,a_m),\\ s_2(\Lambda_{k+1}),\ldots,s_2(\Lambda_n) \end{bmatrix}\\
&\qquad\text{if } \Lambda_k=(a_1,\ldots,a_m) \text{ and } a_j=f\otimes\delta\\
&= \begin{bmatrix} s_2(\Lambda_1),\ldots,s_2(\Lambda_{k-1}),\\ (a_1,\ldots,a_{j-1},s_1(a_j),a_{j+1},\ldots,a_m),\\ (a_1,\ldots,a_{j'-1},t_1(a_{j'}),a_{j'+1},\ldots,a_m),\\ s_2(\Lambda_{k+1}),\ldots,s_2(\Lambda_n) \end{bmatrix}\\
&\qquad\text{if } \Lambda_k=(a_1,\ldots,a_m)^{\mathfrak{r}} \text{ with } a_j \text{ and } a_{j'} \text{ 2-cells, } j<j'\\
&= \begin{bmatrix} s_2(\Lambda_1),\ldots,s_2(\Lambda_{k-1}),\\ (a_1,\ldots,a_{j-1},t_1(a_j),a_{j+1},\ldots,a_m),\\ (a_1,\ldots,a_{j'-1},s_1(a_{j'}),a_{j'+1},\ldots,a_m),\\ s_2(\Lambda_{k+1}),\ldots,s_2(\Lambda_n) \end{bmatrix}\\
&\qquad\text{if } \Lambda_k=(a_1,\ldots,a_m)^{\mathfrak{l}} \text{ with } a_j \text{ and } a_{j'} \text{ 2-cells, } j<j'\\
t_2[\Lambda_1,\ldots,\Lambda_n] &= \begin{bmatrix} t_2(\Lambda_1),\ldots,t_2(\Lambda_{k-1}),\\ (a_1,\ldots,a_{j-1},\gamma'\otimes D,a_{j+1},\ldots,a_m),\\ t_2(\Lambda_{k+1}),\ldots,t_2(\Lambda_n) \end{bmatrix}\\
&\qquad\text{if } \Lambda_k=(a_1,\ldots,a_m) \text{ and } a_j=\varphi\otimes D\\
&= \begin{bmatrix} t_2(\Lambda_1),\ldots,t_2(\Lambda_{k-1}),\\ (a_1,\ldots,a_{j-1},C\otimes\delta',a_{j+1},\ldots,a_m),\\ t_2(\Lambda_{k+1}),\ldots,t_2(\Lambda_n) \end{bmatrix}\\
&\qquad\text{if } \Lambda_k=(a_1,\ldots,a_m) \text{ and } a_j=C\otimes\psi\\
&= \begin{bmatrix} t_2(\Lambda_1),\ldots,t_2(\Lambda_{k-1}),\\ (a_1,\ldots,a_{j-1},C\otimes g,\gamma\otimes D',a_{j+1},\ldots,a_m),\\ (a_1,\ldots,a_{j-1},f'\otimes g,a_{j+1},\ldots,a_m),\\ t_2(\Lambda_{k+1}),\ldots,t_2(\Lambda_n) \end{bmatrix}\\
&\qquad\text{if } \Lambda_k=(a_1,\ldots,a_m) \text{ and } a_j=\gamma\otimes g
\end{aligned}
$$

$$= \begin{bmatrix} t_2(\Lambda_1), \dots, t_2(\Lambda_{k-1}), \\ (a_1, \dots, a_{j-1}, f \otimes g, a_{j+1}, \dots, a_m), \\ (a_1, \dots, a_{j-1}, f \otimes D, C' \otimes \delta, a_{j+1}, \dots, a_m), \\ t_2(\Lambda_{k+1}), \dots, t_2(\Lambda_n) \end{bmatrix}$$

$$\text{if } \Lambda_k = (a_1, \dots, a_m) \text{ and } a_j = f \otimes \delta$$

$$= \begin{bmatrix} t_2(\Lambda_1), \dots, t_2(\Lambda_{k-1}), \\ (a_1, \dots, a_{j-1}, t_1(a_j), a_{j+1}, \dots, a_m), \\ (a_1, \dots, a_{j'-1}, s_1(a_{j'}), a_{j'+1}, \dots, a_m), \\ t_2(\Lambda_{k+1}), \dots, t_2(\Lambda_n) \end{bmatrix}$$

$$\text{if } \Lambda_k = (a_1, \dots, a_m)^{\mathfrak{r}} \text{ with } a_j \text{ and } a_{j'} \text{ 2-cells, } j < j'$$

$$= \begin{bmatrix} t_2(\Lambda_1), \dots, t_2(\Lambda_{k-1}), \\ (a_1, \dots, a_{j-1}, s_1(a_j), a_{j+1}, \dots, a_m), \\ (a_1, \dots, a_{j'-1}, t_1(a_{j'}), a_{j'+1}, \dots, a_m), \\ t_2(\Lambda_{k+1}), \dots, t_2(\Lambda_n) \end{bmatrix}$$

$$\text{if } \Lambda_k = (a_1, \dots, a_m)^{\mathfrak{l}} \text{ with } a_j \text{ and } a_{j'} \text{ 2-cells, } j < j'.$$

Two 3-dimensional 2-whiskers are *equivalent* if they are so in the smallest equivalence relation compatible with juxtaposition *with* 1-*cells and with* 2- and 3-*dimensional* 1-*whiskers* generated by:

$$[(a_1, \dots, a_m)] \sim [(b_1, \dots, b_{m'})] \qquad \text{if } (a_1, \dots, a_m) \sim (b_1, \dots, b_{m'})$$

functoriality relations:

$$\begin{aligned} [((c' \#_1 c) \otimes D)] &\sim [(c \otimes D), (c' \otimes D)] \\ [(C \otimes (d' \#_1 d))] &\sim [(C \otimes d), (C \otimes d')] \\ [((f' \#_0 f) \otimes g)] &\sim [(f \otimes g, f' \otimes D), (f \otimes D, f' \otimes g)] \\ [(f \otimes (g' \#_0 g))] &\sim [(C \otimes g, f \otimes g'), (f \otimes g, C' \otimes g')] \\ [(\mathrm{id}_f \otimes D)] &\sim ([], (f \otimes D)) \\ [(C \otimes \mathrm{id}_g)] &\sim ([], (C \otimes g)) \\ [(\mathrm{id}_C \otimes g)] &\sim ([], (C \otimes g)) \\ [(f \otimes \mathrm{id}_D)] &\sim ([], (f \otimes D)) \end{aligned}$$

interchange relation:

$$\begin{bmatrix} (C \otimes g, f \otimes g', f' \otimes D''), \\ (f \otimes g, f' \otimes g'), \\ (f \otimes D, f' \otimes g, C'' \otimes g') \end{bmatrix} \sim ([], ((f' \#_0 f) \otimes (g' \#_0 g))).$$

There is a normal form for 3-dimensional 2-whiskers, just as for 2-dimensional 2-whiskers.

A 3-*dimensional* 3-*whisker* is a, possibly empty, sequence $\{\Gamma_1, \dots, \Gamma_p\}$ of 3-dimensional 2-whiskers with $s_2(\Gamma_{i+1}) \sim t_2(\Gamma_i)$ for all $0 < i < p$. Empty sequences come together with a 2-dimensional 2-whisker.

The faces of a 3-dimensional 3-whisker are given by:

$$\begin{array}{rcl}
s_2\{\Gamma_1, \ldots, \Gamma_p\} & = & s_2(\Gamma_1) \\
t_2\{\Gamma_1, \ldots, \Gamma_p\} & = & t_2(\Gamma_p) \\
s_2(\{\}, [\Lambda_1, \ldots, \Lambda_n]) & = & [\Lambda_1, \ldots, \Lambda_n] \\
t_2(\{\}, [\Lambda_1, \ldots, \Lambda_n]) & = & [\Lambda_1, \ldots, \Lambda_n].
\end{array}$$

Two 3-dimensional 3-whiskers are *equivalent* if they are so in the smallest equivalence relation compatible with juxtaposition *with* 1*-cells, with* 2*-dimensional* 1*-whiskers and with* 3*-dimensional* 2*-whiskers* generated by:

$$\{[\Lambda_1, \ldots, \Lambda_n]\} \quad \sim \quad \{[\Xi_1, \ldots, \Xi_{n'}]\} \qquad\qquad \text{if } [\Lambda_1, \ldots, \Lambda_n] \sim [\Xi_1, \ldots, \Xi_{n'}]$$

naturality relations:

$$\begin{array}{ccc}
\left\{\begin{matrix} [(\gamma \otimes g)], \\ [(C \otimes g, \varphi \otimes D'), (f' \otimes g)] \end{matrix}\right\} & \sim & \left\{\begin{matrix} [(f \otimes g), (\varphi \otimes D, C' \otimes g)], \\ [(\gamma' \otimes g)] \end{matrix}\right\} \\
\left\{\begin{matrix} [(C \otimes \psi, f \otimes D'), (f \otimes g')], \\ [(f \otimes \delta')] \end{matrix}\right\} & \sim & \left\{\begin{matrix} [(f \otimes \delta)], \\ [(f \otimes g), (f \otimes D, C' \otimes \psi)] \end{matrix}\right\} \\
\left\{\begin{matrix} [(f \otimes \delta), (\gamma \otimes D, C' \otimes g')], \\ [(f \otimes g), (\gamma \otimes D, C' \otimes \delta)^{\mathfrak{r}}], \\ [(\gamma \otimes g), (f' \otimes D, C' \otimes \delta)] \end{matrix}\right\} & \sim & \left\{\begin{matrix} [(C \otimes \delta, f \otimes D'), (\gamma \otimes g')], \\ [(C \otimes \delta, \gamma \otimes D')^{\mathfrak{l}}, (f' \otimes g')], \\ [(C \otimes g, \gamma \otimes D'), (f' \otimes \delta)] \end{matrix}\right\}
\end{array}$$

functoriality relations:

$$\begin{array}{rcl}
\{[((c' \#_2 c) \otimes D)]\} & \sim & \{[(c \otimes D)], [(c' \otimes D)]\} \\
\{[(C \otimes (d' \#_2 d))]\} & \sim & \{[(C \otimes d)], [(C \otimes d')]\} \\
\{[((\gamma' \#_1 \gamma) \otimes g)]\} & \sim & \left\{\begin{matrix} [(\gamma \otimes g), (\gamma' \otimes D, C' \otimes g)], \\ [(C \otimes g, \gamma \otimes D'), (\gamma' \otimes g)] \end{matrix}\right\} \\
\{[((f'' \#_0 \gamma) \otimes g)]\} & \sim & \left\{\begin{matrix} [(f \otimes g, f'' \otimes D'), (\gamma \otimes D, f'' \otimes g)^{\mathfrak{r}}], \\ [(\gamma \otimes g, f'' \otimes D'), (f' \otimes D, f'' \otimes g)] \end{matrix}\right\} \\
\{[((\gamma' \#_0 f) \otimes g)]\} & \sim & \left\{\begin{matrix} [(f \otimes g, f' \otimes D'), (f \otimes D, \gamma' \otimes g)], \\ [(f \otimes g, \gamma' \otimes D')^{\mathfrak{l}}, (f \otimes D, f'' \otimes g)] \end{matrix}\right\} \\
\{[(\gamma \otimes (g' \#_0 g))]\} & \sim & \left\{\begin{matrix} [(C \otimes g, f \otimes g'), (\gamma \otimes g, C' \otimes g')], \\ [(C \otimes g, \gamma \otimes g'), (f' \otimes g, C' \otimes g')] \end{matrix}\right\} \\
\{[((f' \#_0 f) \otimes \delta)]\} & \sim & \left\{\begin{matrix} [(f \otimes \delta, f' \otimes D'), (f \otimes D, f' \otimes g')], \\ [(f \otimes g, f' \otimes D'), (f \otimes D, f' \otimes \delta)] \end{matrix}\right\} \\
\{[(f \otimes (\delta' \#_1 \delta))]\} & \sim & \left\{\begin{matrix} [(C \otimes \delta, f \otimes D'), (f \otimes \delta')], \\ [(f \otimes \delta), (f \otimes D, C' \otimes \delta')] \end{matrix}\right\} \\
\{[(f \otimes (g'' \#_0 \delta))]\} & \sim & \left\{\begin{matrix} [(C \otimes \delta, f \otimes g'')^{\mathfrak{l}}, (f \otimes g', C' \otimes g'')], \\ [(C \otimes g, f \otimes g''), (f \otimes \delta, C' \otimes g'')] \end{matrix}\right\} \\
\{[(f \otimes (\delta' \#_0 g))]\} & \sim & \left\{\begin{matrix} [(C \otimes g, f \otimes \delta'), (f \otimes g, C' \otimes g'')], \\ [(C \otimes g, f \otimes g'), (f \otimes g, C' \otimes \delta')^{\mathfrak{r}}] \end{matrix}\right\} \\
\{[(\mathrm{id}_\gamma \otimes D)]\} & \sim & (\{\}, [(\gamma \otimes D)]) \\
\{[(C \otimes \mathrm{id}_\delta)]\} & \sim & (\{\}, [(C \otimes \delta)])
\end{array}$$

$$\begin{array}{rcl}
\{[(\mathrm{id}_f \otimes g)]\} & \sim & (\{\},[(f\otimes g)]) \\
\{[(f\otimes \mathrm{id}_g)]\} & \sim & (\{\},[(f\otimes g)]) \\
\{[(\gamma\otimes \mathrm{id}_D)]\} & \sim & (\{\},[(\gamma\otimes D)]) \\
\{[(\mathrm{id}_C \otimes \delta)]\} & \sim & (\{\},[(C\otimes \delta)])
\end{array}$$

Gray-category-axiomatical relations:

$$\left\{\begin{array}{c}[s_2(\Lambda_1),\Lambda_2,\ldots,\Lambda_n],\\ [\Lambda_1,\ldots,\Lambda_{n-1},t_2(\Lambda_n)]\end{array}\right\} \sim \left\{\begin{array}{c}[\Lambda_1,\ldots,\Lambda_{n-1},s_2(\Lambda_n)],\\ [t_2(\Lambda_1),\Lambda_2,\ldots,\Lambda_n]\end{array}\right\}$$

for 3-dimensional 1-whiskers Λ_1 and Λ_n

$$\left\{\begin{array}{c}[(a_1,\ldots,a_m)^{\mathfrak{r}}],\\ [(a_1,\ldots,a_m)^{\mathfrak{l}}]\end{array}\right\} \sim (\{\},[(s_1(a_1),a_2,\ldots,a_m),(a_1,\ldots,a_{m-1},t_1(a_m))])$$

for 2-cells a_1 and a_m

$$\left\{\begin{array}{c}[(a_1,\ldots,a_m)^{\mathfrak{l}}],\\ [(a_1,\ldots,a_m)^{\mathfrak{r}}]\end{array}\right\} \sim (\{\},[(a_1,\ldots,a_{m-1},s_1(a_m)),(t_1(a_1),a_2,\ldots,a_m)])$$

for 2-cells a_1 and a_m

$$\left\{\begin{array}{c}[(s_2(a_1),a_2,\ldots,a_m)],\\ [(a_1,\ldots,a_{m-1},s_1(a_m)),\\ (t_1(a_1),a_2,\ldots,a_m)]\end{array}\right\} \sim \left\{\begin{array}{c}[(s_1(a_1),a_2,\ldots,a_m),\\ (a_1,\ldots,a_{m-1},t_1(a_m))],\\ [(t_2(a_1),a_2,\ldots,a_m)]\end{array}\right\}$$

for 3-cell a_1 and 2-cell a_m

$$\left\{\begin{array}{c}[(s_1(a_1),a_2,\ldots,a_m),\\ (a_1,\ldots,a_{m-1},t_1(a_m))],\\ [(a_1,\ldots,a_{m-1},t_2(a_m))]\end{array}\right\} \sim \left\{\begin{array}{c}[(a_1,\ldots,a_{m-1},s_2(a_m))],\\ [(a_1,\ldots,a_{m-1},s_1(a_m)),\\ (t_1(a_1),a_2,\ldots,a_m)]\end{array}\right\}$$

for 2-cell a_1 and 3-cell a_m.

A 3-*arrow* is an equivalence class of 3-dimensional 3-whiskers.

2-composition of 3-arrows, identity on 2-arrows, 1-composition of a 3-arrow with a 2-arrow, 1-composition of a 2-arrow with a 3-arrow, 0-composition of a 3-arrow with an arrow and 0-composition of an arrow with a 3-arrow are given (with juxtaposition denoting juxtaposition of sequences) by:

$$\begin{array}{rcl}
\{\Delta_1,\ldots,\Delta_{p'}\}\ \#_2\ \{\Gamma_1,\ldots,\Gamma_p\} & = & \{\Gamma_1,\ldots,\Gamma_p,\Delta_1,\ldots,\Delta_{p'}\} \\
\mathrm{id}_{[\Lambda_1,\ldots,\Lambda_n]} & = & (\{\},[\Lambda_1,\ldots,\Lambda_n]) \\
\Delta\ \#_1\ \{\Gamma_1,\ldots,\Gamma_p\} & = & \{\Gamma_1\Delta,\ldots,\Gamma_p\Delta\} \\
\{\Delta_1,\ldots,\Delta_{p'}\}\ \#_1\ \Gamma & = & \{\Gamma\Delta_1,\ldots,\Gamma\Delta_{p'}\} \\
\Xi\ \#_0 \left\{\begin{array}{c}[\Lambda_{11},\ldots,\Lambda_{1n_1}],\\ \ldots,\\ [\Lambda_{p1},\ldots,\Lambda_{pn_p}]\end{array}\right\} & = & \left\{\begin{array}{c}[\Lambda_{11}\Xi,\ldots,\Lambda_{1n_1}\Xi],\\ \ldots,\\ [\Lambda_{p1}\Xi,\ldots,\Lambda_{pn_p}\Xi]\end{array}\right\} \\
\left\{\begin{array}{c}[\Xi_{11},\ldots,\Xi_{1n'_1}],\\ \ldots,\\ [\Xi_{p'1},\ldots,\Xi_{p'n'_{p'}}]\end{array}\right\} \#_0\ \Lambda & = & \left\{\begin{array}{c}[\Lambda\Xi_{11},\ldots,\Lambda\Xi_{1n'_1}],\\ \ldots,\\ [\Lambda\Xi_{p'1},\ldots,\Lambda\Xi_{p'n'_{p'}}]\end{array}\right\}.
\end{array}$$

The most interesting part is 0-composition of 2-arrows. It is given (with juxtaposition again denoting juxtaposition of sequences) by:

$$[\Xi_1, \ldots, \Xi_{n'}] \#_0 [\Lambda_1, \ldots, \Lambda_n] \quad = \\
\left\{ \begin{array}{c}
[s_1(\Lambda_1)\, \Xi_1, \ldots, s_1(\Lambda_1)\, \Xi_{n'-1}, \Lambda_1\, \Xi_{n'}{}^{\mathfrak{r}}, \Lambda_2\, t_1(\Xi_{n'}), \ldots, \Lambda_n\, t_1(\Xi_{n'})], \\
\ldots, \\
[\Lambda_1\, \Xi_1{}^{\mathfrak{r}}, t_1(\Lambda_1)\, \Xi_2, \ldots, t_1(\Lambda_1)\, \Xi_{n'}, \Lambda_2\, t_1(\Xi_{n'}), \ldots, \Lambda_n\, t_1(\Xi_{n'})], \\
\vdots, \\
[\Lambda_1\, s_1(\Xi_1), \ldots, \Lambda_{n-1}\, s_1(\Xi_1), t_1(\Lambda_{n-1})\, \Xi_1, \ldots, t_1(\Lambda_{n-1})\, \Xi_{n'-1}, \Lambda_n\, \Xi_{n'}{}^{\mathfrak{r}}], \\
\ldots, \\
[\Lambda_1\, s_1(\Xi_1), \ldots, \Lambda_{n-1}\, s_1(\Xi_1), \Lambda_n\, \Xi_1{}^{\mathfrak{r}}, t_1(\Lambda_n)\, \Xi_2, \ldots, t_1(\Lambda_n)\, \Xi_{n'}]
\end{array} \right\}.$$

This might look mysterious, but the only thing that happens is that 2-cells in a "horizontal position" with respect to one another pass each other one by one. Here I have chosen a particular order for this passing, but any order would do, because of the first **Gray**-category-axiomatical relation.

There is *no* normal form for 3-arrows! This is because of the naturality relations, which do not change the "length" of a 3-whisker.

4.3. Theorem. *The objects, arrows,* 2*-arrows and* 3*-arrows above, with sources, targets, composition and identity as described, form a* **Gray***-category. It is precisely the* **Gray***-category* $\mathbb{C} \otimes \mathbb{D}$ *generated by the presentation given in the previous section.*

Proof (sketch). In particular, axiom (vii) follows from the definition of horizontal composition of 2-arrows. ∎

4.5. Comparison with Gray

The tensor product of **Gray**-categories given here extends Gray's tensor product of 2-categories, in the following sense: if the **Gray**-categories $\mathbb{C}$ and $\mathbb{D}$ are both 2-categories, then their tensor product *as* 2*-categories* is precisely the 2-category obtained from *the* **Gray***-category* $\mathbb{C} \otimes \mathbb{D}$ by formally turning all its 3-arrows into identities. In other words, Gray's tensor product is the 2-categorical reflection of the tensor product of **Gray**-categories given here.

Although the description of the tensor product of **Gray**-categories given here is in the spirit of Gray's description of his tensor product of 2-categories, there are, apart from the obvious notational ones, some noticeable, mathematical, differences.

In dimensions 0 and 1 the only difference is that Gray does not allow empty sequences, nor the corresponding relations involving identities. These relations, except where that would result in an empty sequence, do follow implicitly from the functoriality relations. But without empty sequences there would be no normal form, and in order to be able to contract to an empty sequence the relations involving identities need to be included. Gray doesn't have empty sequences nor relations involving identities in dimension 2 either. I do think that the inclusion of these identity relations is conceptually right.

In dimension 2 there is a very big difference in the definition of 2-arrows: Gray has "horizontal" strings comprising an arbitrary number of 2-cells and then looks at "vertical" strings of "horizontal" strings, whereas I have 1-dimensional 2-whiskers which have only one 2-cell and 2-dimensional 2-whiskers which are "vertical" strings of 1-dimensional 2-whiskers. Gray's approach is possible for 2-categories because there horizontal composition is definable in terms of vertical composition, but for **Gray**-categories that is not the case.

Gray's basic equivalences correspond to the basic equivalences here. Specifically, Gray's iv) is the (functoriality) relations for 2-dimensional (and 3-dimensional, actually) 1-whiskers (minus identity relations of course), Gray's viii) is half of the functoriality relations for 2-dimensional 2-whiskers, and Gray's v) is the other half. Gray's vii) is, in my terminology, a naturality relation, and corresponds to the 3-cells here, and Gray's vi) is a 2-category-axiomatical relation. However, vi) is inaccurate, because it should relate both strings mentioned to, in his notation, $\gamma_{f,g} \cdot \gamma_{f',g'}$, and it is incomplete, because it only looks at horizontal juxtaposition of squares, rather than at all possible horizontal juxtapositions of 2-cells, whether they be squares or globes. Even the two cases of two globes, dealing with, in Gray's notation, $(\tau, 1) \cdot (1, \sigma)$ and $(1, \sigma) \cdot (\tau, 1)$, do not follow from his vii).

The difference in the definition of 2-arrows, and the way I define 3-arrows, gives a completely different setup for horizontal composition of 2-arrows. Gray's definition is quite artificial, inserting identity strings in an arbitrary way to make both strings of strings of equal length, and is unsuitable here because it would involve a 1-composite of 3-dimensional 1-whiskers, which is not (immediately) a 3-dimensional 3-whisker. My definition also involves some arbitrariness, in that any order in which to pass the 2-cells past each other would do, but it does avoid 1-composites of 3-dimensional 1-whiskers.

Gray leaves "most of the details of checking that [his tensor product gives] a 2-category to the reader", saying that "there is a non-trivial case of the interchange law [which] is covered by vi)" [16, p. 77]. In fact, there are *six* more instances, which are all covered by the corrected and completed version of vi).

5. Transfors and quasi-functors

Before proving, in the next two sections, that the tensor product of **Gray**-categories defined in section 3 is part of a monoidal structure on **Gray-Cat**, I will show that this tensor product satisfies an appropriate universal property. To express this universal property properly, I introduce the notion of *transfor*, which is the extension of the notions of lax-natural transformation and modification to **Gray**-categories, and *quasi-functors of two variables*, which should be thought of as "bi-functorial mappings". All this is similar to Gray's treatment of the universal property of the tensor product of 2-categories, except that here quasi-functors of two variables involve interchange explicitly.

5.1. Transfors

Lax-q-transformations of ω-categories are maps of degree q satisfying some conditions. q-transfors of **Gray**-categories will be maps of degree q satisfying similar conditions, but taking into account the 3-arrows resulting from the dimension-raising horizontal composition of 2-arrows.

5.1. Definition. Let $\mathbb{C}$ and $\mathbb{D}$ be **Gray**-categories. A 0-*transfor*, or *functor*, $\mathbb{C} \to \mathbb{D}$ is a **Gray**-functor.

Let $F, G : \mathbb{C} \to \mathbb{D}$ be functors. A *(right)* 1-*transfor*, or *lax-natural transformation*, $\alpha : F \to G$ consists of the following data:

- for every object C of $\mathbb{C}$ an arrow $\alpha_C : F(C) \to G(C)$ in $\mathbb{D}$,
- for every arrow $f : C \to C'$ in $\mathbb{C}$ a 2-arrow

$$\begin{tikzcd} & G(C) \arrow[dr, "G(f)"] \arrow[dd, phantom, "\Downarrow\alpha_f"] & \\ F(C) \arrow[ur, "\alpha_C"] \arrow[dr, "F(f)"'] & & G(C') \\ & F(C') \arrow[ur, "\alpha_{C'}"'] & \end{tikzcd}$$

  in $\mathbb{D}$,

- for every 2-arrow $\gamma :$ $\begin{tikzcd} C \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow"] & C' \end{tikzcd}$ in $\mathbb{C}$ a 3-arrow

$$\begin{tikzcd} & G(C) \arrow[dr, bend left] & & & G(C) \arrow[dr, bend left] \arrow[dr, bend right] \arrow[dr, phantom, "\Swarrow G(\gamma)"] & \\ F(C) \arrow[ur, "\alpha_C"] \arrow[dr, bend left] \arrow[dr, bend right] \arrow[dr, phantom, "\Swarrow F(\gamma)"] \arrow[rr, phantom, "\Downarrow\alpha_f"] & & G(C') \arrow[r, "\alpha_\gamma", "3"'] & F(C) \arrow[ur, "\alpha_C"] \arrow[dr, bend right] \arrow[rr, phantom, "\alpha_{f'}\Downarrow"] & & G(C') \\ & F(C') \arrow[ur, "\alpha_{C'}"'] & & & F(C') \arrow[ur, "\alpha_{C'}"'] & \end{tikzcd}$$

  in $\mathbb{D}$,

satisfying the following conditions:

- (naturality) for every $\varphi :$ $\begin{tikzcd} C \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "f'"'] \arrow[r, phantom, "\gamma' \Lleftarrow \gamma"] & C' \end{tikzcd}$ in $\mathbb{C}$,

$$\begin{array}{c}(\alpha_{f'} \#_1 (G(\varphi) \#_0 \alpha_C)) \\ \#_2 \\ \alpha_\gamma\end{array} = \begin{array}{c}\alpha_{\gamma'} \\ \#_2 \\ ((\alpha_{C'} \#_0 F(\varphi)) \#_1 \alpha_f)\end{array},$$

- (functoriality with respect to 0-composition of arrows) for every $C \xrightarrow{f} C' \xrightarrow{f'} C''$ in $\mathbb{C}$,

$$\alpha_{f' \#_0 f} = (\alpha_{f'} \#_0 F(f)) \#_1 (G(f') \#_0 \alpha_f),$$

- (functoriality with respect to 1-composition of 2-arrows) for every $\begin{tikzcd} C \arrow[r, bend left=50, "f"] \arrow[r, "f'" description] \arrow[r, bend right=50, "f''"'] \arrow[r, phantom, shift left=3, "\Downarrow\gamma"] \arrow[r, phantom, shift right=3, "\Downarrow\gamma'"] & C' \end{tikzcd}$ in $\mathbb{C}$,

$$\alpha_{\gamma' \#_1 \gamma} = \begin{array}{c}(\alpha_{\gamma'} \#_1 (G(\gamma) \#_0 \alpha_C)) \\ \#_2 \\ ((\alpha_{C'} \#_0 F(\gamma')) \#_1 \alpha_\gamma)\end{array},$$

- (functoriality with respect to 0-composition of a 2-arrow with an arrow) for every $\begin{tikzcd} C \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "f'"'] \arrow[r, phantom, "\Downarrow\gamma"] & C' \arrow[r, "f''"] & C'' \end{tikzcd}$ in $\mathbb{C}$,

$$\alpha_{f'' \#_0 \gamma} = \begin{array}{c}((\alpha_{f''} \#_0 F(f')) \#_1 (G(f'') \#_0 \alpha_\gamma)) \\ \#_2 \\ ((\alpha_{f''} \#_0 F(\gamma)) \#_1 (G(f'') \#_0 \alpha_f))\end{array},$$

- (functoriality with respect to 0-composition of an arrow with a 2-arrow) for every $\begin{tikzcd} C \arrow[r, "f"] & C' \arrow[r, bend left=50, "f'"] \arrow[r, bend right=50, "f''"'] \arrow[r, phantom, "\Downarrow\gamma'"] & C' \end{tikzcd}$ in $\mathbb{C}$,

$$\alpha_{\gamma' \#_0 f} = \begin{array}{c}((\alpha_{f''} \#_0 F(f)) \#_1 (G(\gamma') \#_0 \alpha_f)^{-1}) \\ \#_2 \\ ((\alpha_{\gamma'} \#_0 F(f)) \#_1 (G(f') \#_0 \alpha_f))\end{array},$$

- (functoriality with respect to identities) for every C in $\mathbb{C}$, $\alpha_{\mathrm{id}_C} = \mathrm{id}_{\alpha_C}$, and for every $f : C \to C'$ in $\mathbb{C}$, $\alpha_{\mathrm{id}_f} = \mathrm{id}_{\alpha_f}$.

Let $\alpha, \beta : F \to G$ be 1-transfors. A *(right) 2-transfor*, or *lax-modification*, $\mu : \alpha \to \beta$ consists of the following data:

- for every object C of $\mathbb{C}$ a 2-arrow $\mu_C :$ $\begin{tikzcd} F(C) \arrow[r, bend left, "\alpha_C"] \arrow[r, bend right, "\beta_C"'] \arrow[r, phantom, "\Downarrow"] & G(C) \end{tikzcd}$ in $\mathbb{D}$,

- for every arrow $f : C \to C'$ in $\mathbb{C}$ a 3-arrow

$$\begin{tikzcd}
& G(C) \arrow[dr] & & & & G(C) \arrow[dr] & \\
F(C) \arrow[ur, bend left, "\alpha_C"] \arrow[ur, bend right] \arrow[ur, phantom, "\Downarrow\mu_C"] \arrow[dr] \arrow[rr, phantom, "\Downarrow\beta_f"] & & G(C') \arrow[r, "\mu_f", "3"'] & {} \arrow[r] & F(C) \arrow[ur, bend left, "\alpha_C"] \arrow[dr] \arrow[rr, phantom, "\alpha_f\Downarrow"] & & G(C') \\
& F(C') \arrow[ur, bend right, "\beta_{C'}"'] & & & & F(C') \arrow[ur, bend left] \arrow[ur, bend right, "\beta_{C'}"'] \arrow[ur, phantom, "\Downarrow\mu_{C'}"] &
\end{tikzcd} .$$

  in $\mathbb{D}$,

satisfying the following conditions:

- (naturality) for every $\gamma :$ $\begin{tikzcd} C \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow"] & C' \end{tikzcd}$ in $\mathbb{C}$,

$$\begin{array}{ccc}
((\mu_{C'} \#_0 F(f')) \#_1 \alpha_\gamma) & & (\mu_{f'} \#_1 (G(\gamma) \#_0 \alpha_C)) \\
\#_2 & & \#_2 \\
((\mu_{C'} \#_0 F(\gamma)) \#_1 \alpha_f) & = & (\beta_{f'} \#_1 (G(\gamma) \#_0 \mu_C)^{-1}) \\
\#_2 & & \#_2 \\
((\beta_{C'} \#_0 F(\gamma)) \#_1 \mu_f) & & (\beta_\gamma \#_1 (G(f) \#_0 \mu_C))
\end{array} ,$$

- (functoriality with respect to 0-composition of arrows) for every $C \xrightarrow{f} C' \xrightarrow{f'} C''$ in $\mathbb{C}$,

$$\mu_{f' \#_0 f} = \begin{array}{c} ((\mu_{f'} \#_0 F(f)) \#_1 (G(f') \#_0 \alpha_f)) \\ \#_2 \\ ((\beta_{f'} \#_0 F(f)) \#_1 (G(f') \#_0 \mu_f)) \end{array} ,$$

- (functoriality with respect to identities) for every C in $\mathbb{C}$, $\mu_{\mathrm{id}_C} = \mathrm{id}_{\mu_C}$.

Let $\mu, \nu : \alpha \to \beta$ be 2-transfors. A *(right) 3-transfor*, or *perturbation*, $u : \mu \to \nu$ consists of the following data:

- for every object C of $\mathbb{C}$ a 3-arrow $u_C :$ $\begin{tikzcd} F(C) \arrow[r, bend left=50, "\alpha_C"] \arrow[r, bend right=50, "\beta_C"'] \arrow[r, phantom, "\nu_C \Leftarrow_3 \mu_C"] & G(C) \end{tikzcd}$ in $\mathbb{D}$, satisfying the following condition:

- (naturality) for every $f : C \to C'$ in $\mathbb{C}$,

$$\begin{array}{c} \nu_f \\ \#_2 \\ (\beta_f \#_1 (G(f) \#_0 u_C)) \end{array} = \begin{array}{c} ((u_{C'} \#_0 F(f)) \#_1 \alpha_f) \\ \#_2 \\ \mu_f \end{array} .$$

$\diamond$

The use of "functor" for "**Gray**-functor" is consistent with Kelly [25].

The axioms for q-transfors are a (lax) naturality condition, which describes behaviour with respect to lower-dimensional data, and functoriality conditions, which describe behaviour with respect to composition[2], and identity. The term "transfor", apart from being a contraction of transformation, combines "natural *transf*ormation" and "funct*or*", and I will use *transforial*[3] to mean being both natural and functorial in this sense.

As for lax-q-transformations, there is a close relation between transfors and the tensor product:

5.2. Proposition. *A (right) q-transfor $\mathbb{C} \to \mathbb{D}$ corresponds to a functor $\mathbb{C} \otimes 2_q \to \mathbb{D}$.* ■

There is also the notion of *left* q-transfor, which is defined in some dual way.

Trivial examples of transfors are given by tensoring:

5.3. Proposition. *For $d \in \mathbb{D}$ of dimension q, $- \otimes d$ is a (right) q-transfor $\mathbb{C} \to \mathbb{C} \otimes \mathbb{D}$, with source and target given by $- \otimes s_{q-1}(d)$ and $- \otimes t_{q-1}(d)$ respectively. For $c \in \mathbb{C}$ of dimension p, $c \otimes -$ is a (left) p-transfor $\mathbb{D} \to \mathbb{C} \otimes \mathbb{D}$, with source and target given by $s_{p-1}(c) \otimes -$ and $t_{p-1}(c) \otimes -$ respectively.* ■

5.2. Quasi-functors of two variables

Quasi-functors of two variables will be defined in terms of lax-natural transformations, as has been done in relation to 2-categories [16, p. 56–58] and ω-categories [10, Section 3-9]. The difference here is the need for an explicit reference to interchange.

5.4. Definition. A *quasi-functor of two variables* $(\mathbb{C}, \mathbb{D}) \to \mathbb{E}$ consists of the following data:

- for every $c \in C_p$ a (left) p-transfor $\chi(c, -) : \mathbb{D} \to \mathbb{E}$,
- for every $d \in D_q$ a (right) q-transfor $\chi(-, d) : \mathbb{C} \to \mathbb{E}$,

satisfying the following conditions:

[2] So I should have said "ω-naturality" in [10, section 3-7.3 and 3-10.5], and I should have shown something about $g' \circ g$ to get ω-functoriality.

[3] The ultimate reason for "transfor" was to be able to use "transforial"!

- $\chi(c,-)(d) = \chi(-,d)(c) \stackrel{\text{def}}{=} \chi(c,d)$,
- (interchange) for every $C \xrightarrow{f} C' \xrightarrow{f'} C''$ in $\mathbb{C}$ and $D \xrightarrow{g} D' \xrightarrow{g'} D''$ in $\mathbb{D}$,

$$\begin{array}{c} (\chi(C'',g') \#_0 \chi(f',g) \#_0 \chi(f,D)) \\ \#_1 \\ (\chi(f',g') \#_0 \chi(f,g)) \\ \#_1 \\ (\chi(f',D'') \#_0 \chi(f,g') \#_0 \chi(C,g)) \end{array} = \mathrm{id}_{\chi(f'\#_0 f, g'\#_0 g)}\,.$$

$\diamond$

The interchange condition is an extra compatibility between the two collections of transfors. It holds automatically for quasi-ω-functors, but here it needs to be included explicitly because the composite on the left involves an extra horizontal composition in $\mathbb{E}$.

5.5. Proposition. *A quasi-functor of two variables* $\chi : (\mathbb{C},\mathbb{D}) \to \mathbb{E}$ *corresponds to a functor* $\mathbb{C} \otimes \mathbb{D} \to \mathbb{E}$. ■

Thus $\mathbb{C} \otimes \mathbb{D}$ is the universal recipient of quasi-functors of two variables from $(\mathbb{C},\mathbb{D})$. To spell this out, define the composite of a quasi-functor $\chi : (\mathbb{C},\mathbb{D}) \to \mathbb{E}$ and a functor $f : \mathbb{E} \to \mathbb{F}$ by $(f \circ \chi)(c,d) = f(\chi(c,d))$, which is a quasi-functor $(\mathbb{C},\mathbb{D}) \to \mathbb{F}$. Now $\mathbb{C} \otimes \mathbb{D}$ is the **Gray**-category characterized by the following property: there is a quasi-functor $(\mathbb{C},\mathbb{D}) \to \mathbb{C} \otimes \mathbb{D}$, and for every quasi-functor $\chi : (\mathbb{C},\mathbb{D}) \to \mathbb{E}$ there is a unique functor $\mathbb{C} \otimes \mathbb{D} \to \mathbb{E}$ whose composite with the quasi-functor $(\mathbb{C},\mathbb{D}) \to \mathbb{C} \otimes \mathbb{D}$ is χ. This property determines $\mathbb{C} \otimes \mathbb{D}$ up to isomorphism.

6. A triple tensor product

As a preparation for the proof of associativity, in the next section, of the tensor product of **Gray**-categories, I will give a presentation for a triple tensor product of **Gray**-categories $\mathbb{C}$, $\mathbb{D}$ and $\mathbb{E}$. The key point of the proof of associativity will be to show that $\mathbb{C} \otimes \mathbb{D} \otimes \mathbb{E}$ is naturally isomorphic to $\mathbb{C} \otimes (\mathbb{D} \otimes \mathbb{E})$.

Let $\mathbb{C}$, $\mathbb{D}$ and $\mathbb{E}$ be **Gray**-categories. Define a **Gray**-category $\mathbb{C} \otimes \mathbb{D} \otimes \mathbb{E}$ by the following presentation.

6.1. Generators

Generators are expressions $c \otimes d \otimes e$, with $c \in C_p$, $d \in D_q$, $e \in E_r$, of dimension $p+q+r$, for $p+q+r \leq 3$. Faces of these are (compare the triple tensor product of globes [10, Section 3-8]):

- for $p+q \leq 3$ and $r = 0$, just as the faces of $c \otimes d$ in $\mathbb{C} \otimes \mathbb{D}$,

- for $p+r \leq 3$ and $q=0$, and for $p=0$ and $q+r \leq 3$, analogous to $p+q \leq 3$ and $r=0$.

- for $p=q=r=1$, if $f : C \to C'$ in $\mathbb{C}$, $g : D \to D'$ in $\mathbb{D}$ and $h : E \to E'$ in $\mathbb{E}$, then $f \otimes g \otimes h$ is given by the diagram

$$
\begin{tikzcd}
 & C{\otimes}D{\otimes}E' \arrow[rr, "C\otimes g\otimes E'"] \arrow[dr, phantom, "\Downarrow C\otimes g\otimes h"] & & C{\otimes}D'{\otimes}E' \arrow[dr, "f\otimes D'\otimes E'"] \arrow[d, phantom, "\Downarrow f\otimes D'\otimes h"] & \\
C{\otimes}D{\otimes}E \arrow[ur, "C\otimes D\otimes h"] \arrow[rr] \arrow[dr, "f\otimes D\otimes E"'] & \arrow[dr, phantom, "\Downarrow f\otimes g\otimes E"] & C{\otimes}D'{\otimes}E \arrow[ur] \arrow[dr] & & C'{\otimes}D'{\otimes}E' \\
 & C'{\otimes}D{\otimes}E \arrow[rr, "C'\otimes g\otimes E"'] & & C'{\otimes}D'{\otimes}E \arrow[ur, "C'\otimes D'\otimes h"'] &
\end{tikzcd}
\xRightarrow[3]{f\otimes g\otimes h}
\begin{tikzcd}
 & C{\otimes}D{\otimes}E' \arrow[rr, "C\otimes g\otimes E'"] \arrow[dr] \arrow[d, phantom, "\Downarrow f\otimes D\otimes h"] & & C{\otimes}D'{\otimes}E' \arrow[dr, "f\otimes D'\otimes E'"] \arrow[dl, phantom, "\Downarrow f\otimes g\otimes E'"] & \\
C{\otimes}D{\otimes}E \arrow[ur, "C\otimes D\otimes h"] \arrow[dr, "f\otimes D\otimes E"'] & & C'{\otimes}D{\otimes}E' \arrow[rr] \arrow[dr, phantom, "\Downarrow C'\otimes g\otimes h"] & & C'{\otimes}D'{\otimes}E' \\
 & C'{\otimes}D{\otimes}E \arrow[ur] \arrow[rr, "C'\otimes g\otimes E"'] & & C'{\otimes}D'{\otimes}E \arrow[ur, "C'\otimes D'\otimes h"'] &
\end{tikzcd}
.
$$

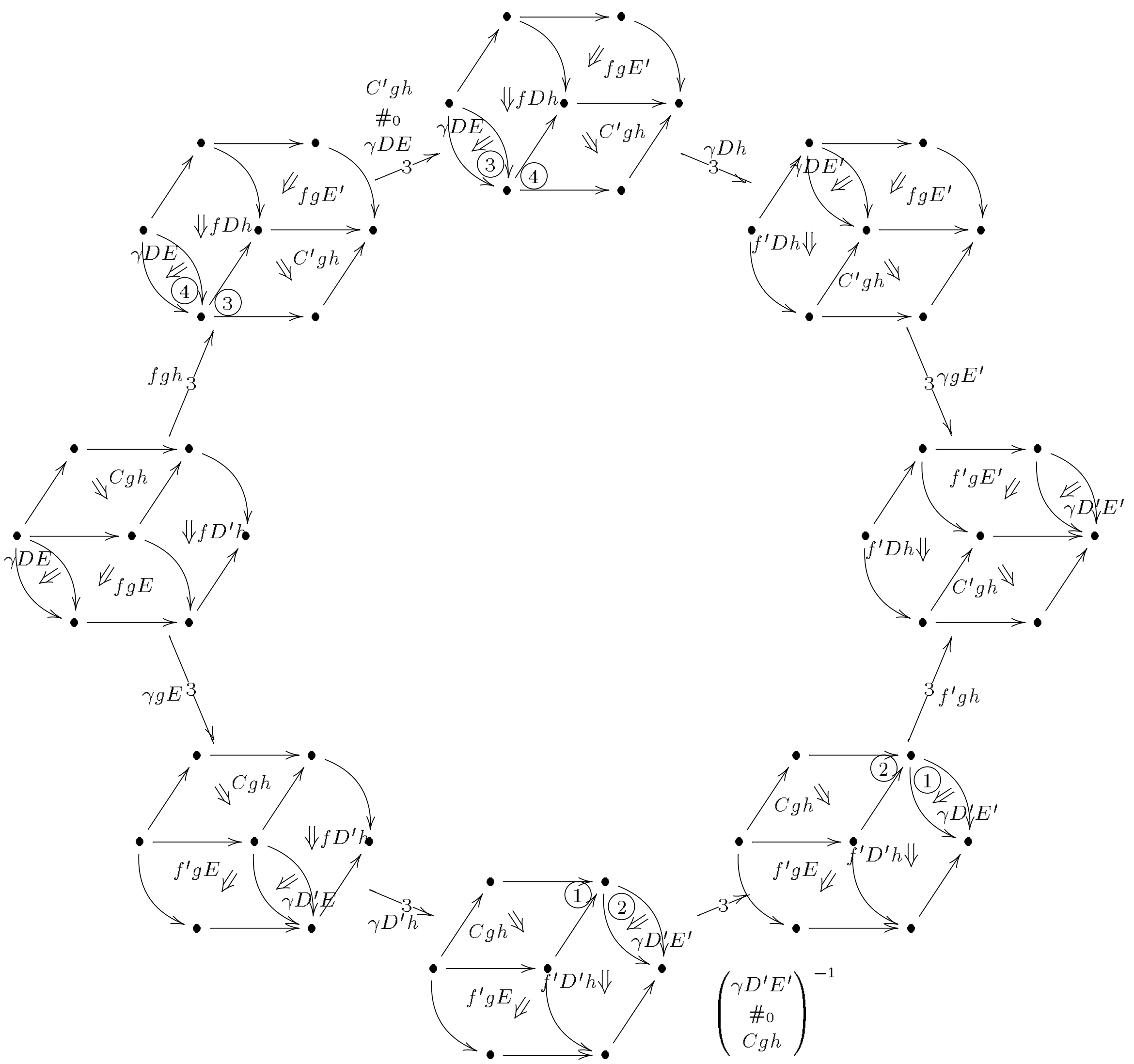
6.2. Naturality relations

From now on I will omit $\otimes$ from the notation. In the following diagrams, in composites of a 3-arrow with lower-dimensional elements, with specified source and target, I will only mention the 3-arrow, the whiskering being clear from the context.

The naturality relations are:

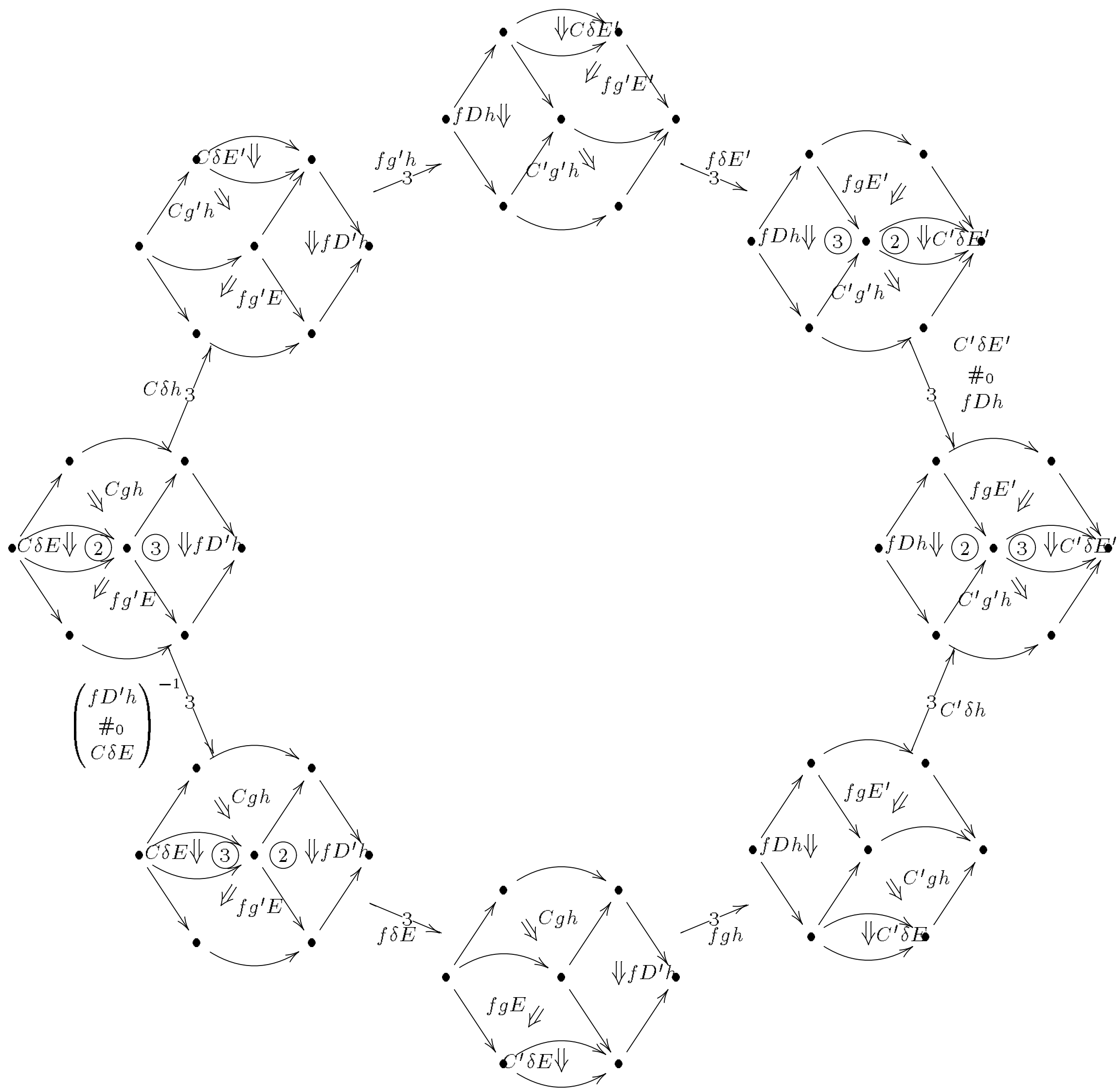
- for p, q or $r=0$, just as the naturality relations in $\mathbb{C} \otimes \mathbb{D}$,

- for $p=2$, $q=r=1$, if $\gamma : \begin{tikzcd} C \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow"] & C' \end{tikzcd}$ in $\mathbb{C}$, $g : D \to D'$ in $\mathbb{D}$ and $h : E \to E'$ in $\mathbb{E}$, then the diagram



commutes,

- for $p = r = 1$, $q = 2$, if $f : C \to C'$ in $\mathbb{C}$, $\delta :\ D \overset{g}{\underset{g'}{\Downarrow}} D'$ in $\mathbb{D}$, and $h : E \to E'$ in $\mathbb{E}$, then the diagram



commutes,

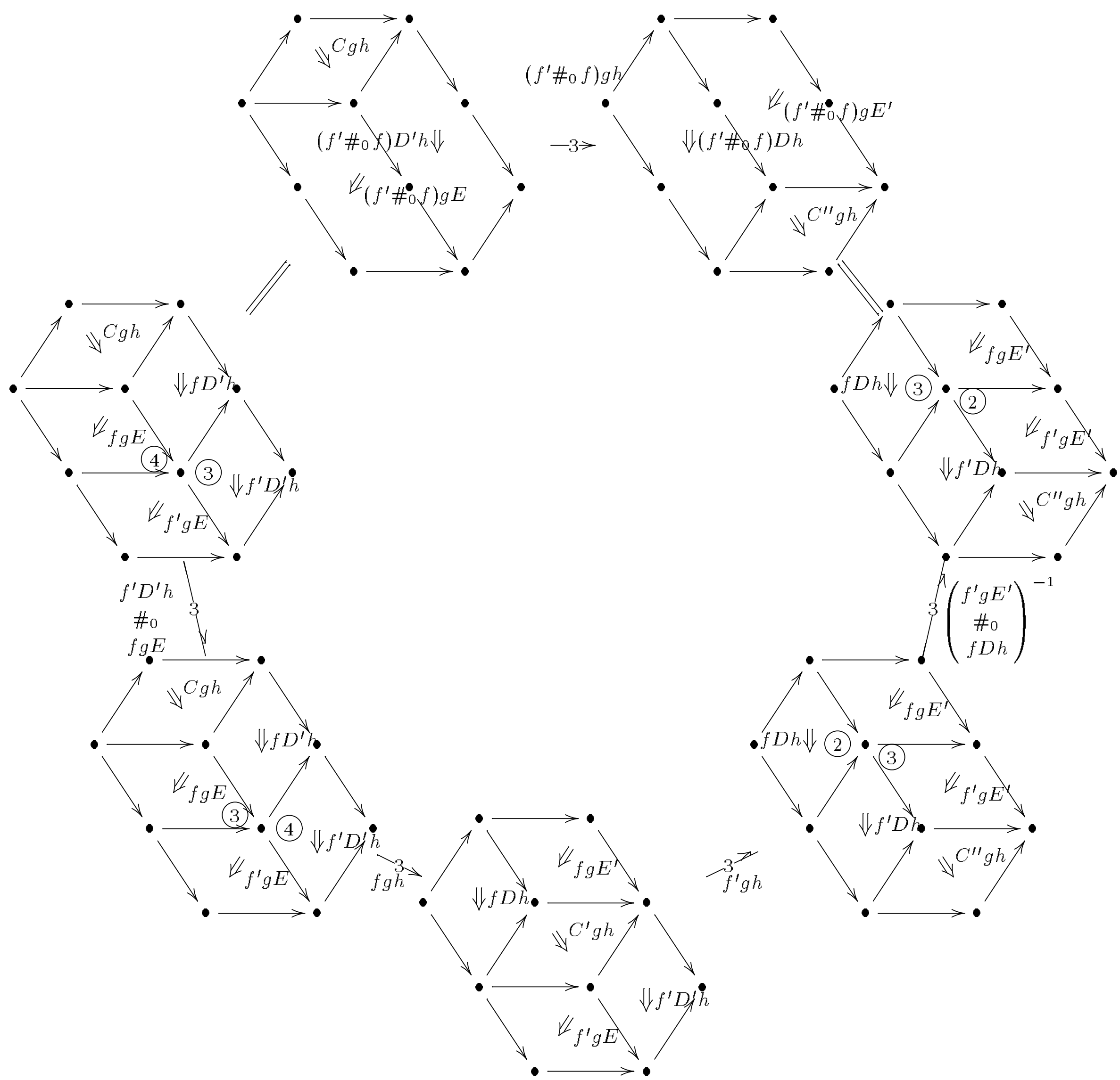
- for $p = q = 1$, $r = 2$, analogous to $p = 2$, $q = r = 1$.

6.3. Functoriality relations

The functoriality relations are:

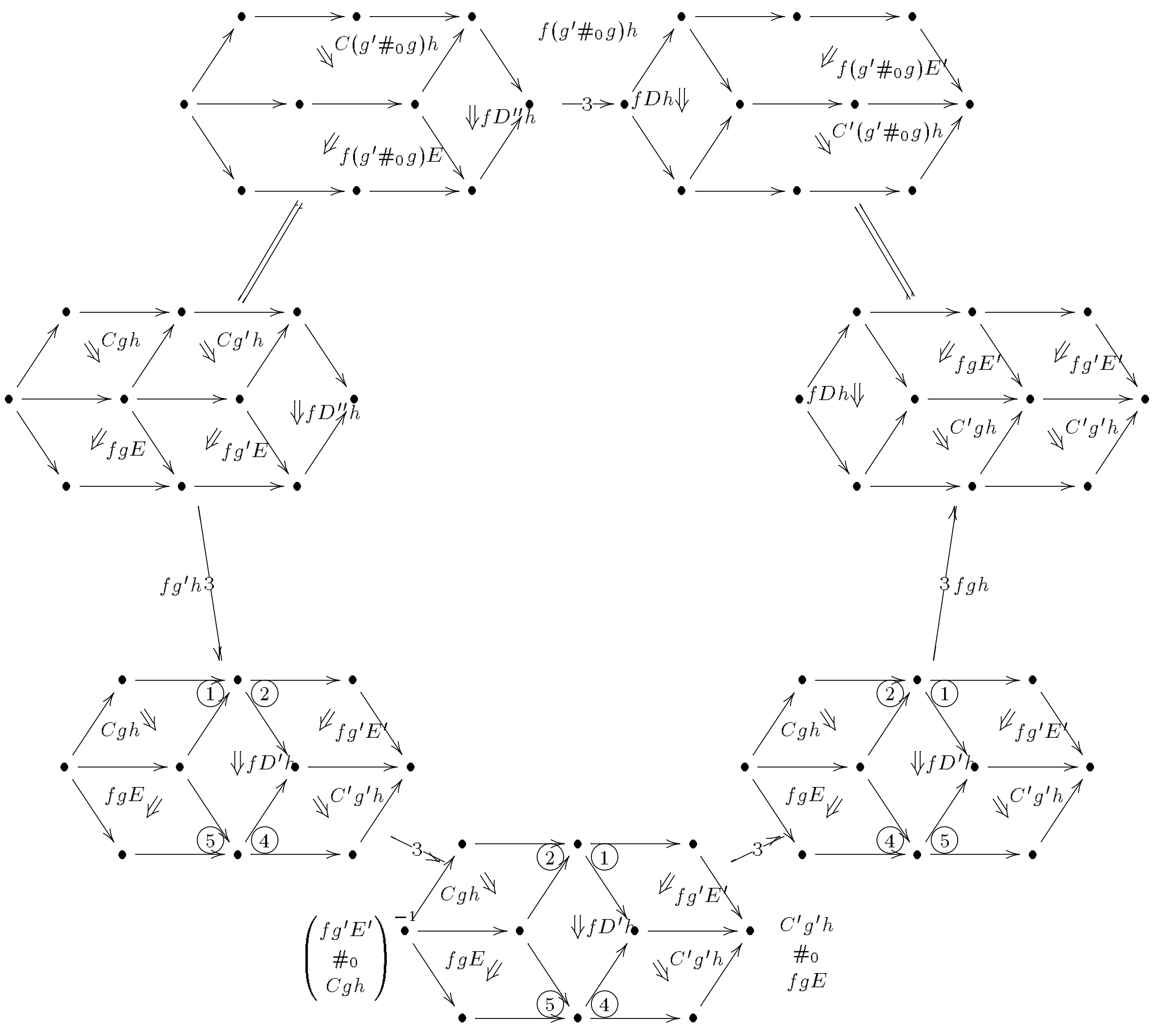
- for $p + q \leq 3$ and $r = 0$, just as the functoriality relations in $\mathbb{C} \otimes \mathbb{D}$,
- for $p + r \leq 3$ and $q = 0$, and for $p = 0$ and $q + r \leq 3$, analogous to $p + q \leq 3$ and $r = 0$.

- for $p = q = r = 1$, composition in left factor, if $C \xrightarrow{f} C' \xrightarrow{f'} C''$ in $\mathbb{C}$, $g : D \to D'$ in $\mathbb{D}$ and $h : E \to E'$ in $\mathbb{E}$, then the diagram



commutes,

- for $p = q = r = 1$, composition in middle factor, if $f : C \to C'$ in $\mathbb{C}$, $D \xrightarrow{g} D' \xrightarrow{g'} D''$ in $\mathbb{D}$ and $h : E \to E'$ in $\mathbb{E}$, then the diagram



commutes,

- for $p = q = r = 1$, composition in right factor, analogous to $p = q = r = 1$, composition in left factor.

6.4. Identity relations

The identity relations are just as the identity relations in $\mathbb{C} \otimes \mathbb{D}$.

6.5. Interchange relations

The interchange relations are:

- for $p = q = 1$ and $r = 0$, just as the interchange relation in $\mathbb{C} \otimes \mathbb{D}$,

- for $p = r = 1$ and $q = 0$, and for $p = 0$ and $q = r = 1$, analogous to $p = q = 1$ and $r = 0$.

6.6. Naturality

Defining the triple tensor product of functors is done the same way as for the (binary) tensor product of functors, namely by applying the functors in each factor.

6.1. Lemma. *The triple tensor is a functor* **Gray-Cat** $\times$ **Gray-Cat** $\times$ **Gray-Cat** $\to$ **Gray-Cat**. ∎

7. A monoidal structure on **Gray-Cat**

After the preparations of the previous section I will now prove that the tensor product of **Gray**-categories is part of a monoidal structure on **Gray-Cat**. However, it is not part of a monoidal closed structure, because interchange spoils the preservation of colimits in each variable of the tensor product.

7.1. **Gray-Cat** is monoidal

Associativity of the tensor product will be proven via the triple tensor product, as usual. The proof critically depends on the interchange axiom, which therefore cannot be omitted. Coherence will then be proven via a similarly defined quadruple tensor product, and the unit axioms are easy.

7.1. Proposition. *The tensor product of* **Gray**-*categories is associative.*

Proof. As usual, this will be done via the triple tensor product.

Define a functor $\zeta : \mathbb{C} \otimes \mathbb{D} \otimes \mathbb{E} \to \mathbb{C} \otimes (\mathbb{D} \otimes \mathbb{E})$ as follows. On generators, $\zeta(c \otimes d \otimes e) = c \otimes (d \otimes e)$. I have to check that this has the right faces, and that this is well-defined with respect to the relations.

$\zeta(c \otimes d \otimes e)$ has the right faces:

- for $q = 0$ or $r = 0$: trivial,

- for $q = r = 1$, $p = 0$:

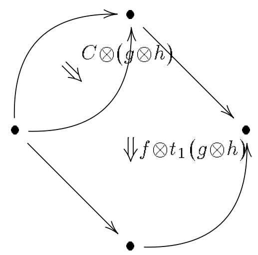
$$\begin{aligned}
s_1(\zeta(C \otimes g \otimes h)) &= s_1(C \otimes (g \otimes h)) \\
&= C \otimes (g \otimes E' \#_0 D \otimes h) \\
&= (C \otimes (g \otimes E')) \#_0 (C \otimes (D \otimes h)) && \text{by a functoriality axiom} \\
&= \zeta(C \otimes g \otimes E') \#_0 \zeta(C \otimes D \otimes h) \\
&= \zeta(C \otimes g \otimes E' \#_0 C \otimes D \otimes h) \\
& && \text{because } \zeta \text{ preserves composition} \\
&= \zeta(s_1(C \otimes g \otimes h)),
\end{aligned}$$

and similarly for t_1.

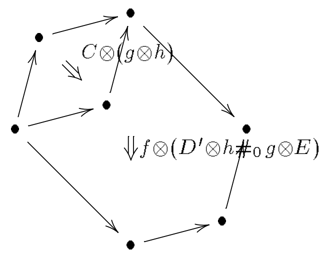
- for $q = r = 1$, $p = 1$:

$$s_2(\zeta(f \otimes g \otimes h)) = s_2(f \otimes (g \otimes h))$$

$=$

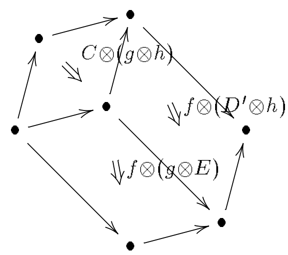


$=$



by the previous case

$=$



by a functoriality axiom

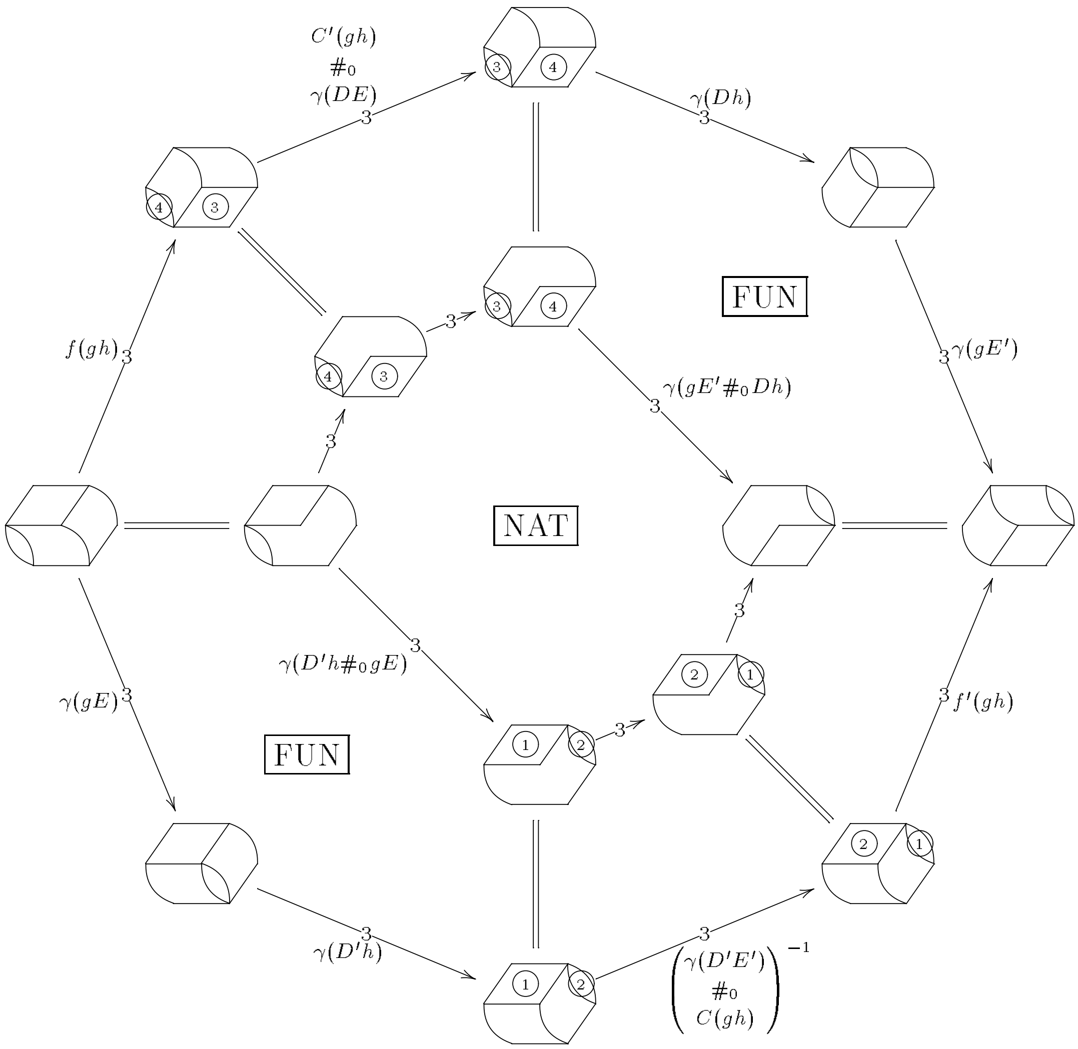
$$= \zeta(s_2(f \otimes g \otimes h)),$$

and similarly for t_2.

ζ is well-defined with respect to the naturality relations:

- for $q = 0$ or $r = 0$: trivial,

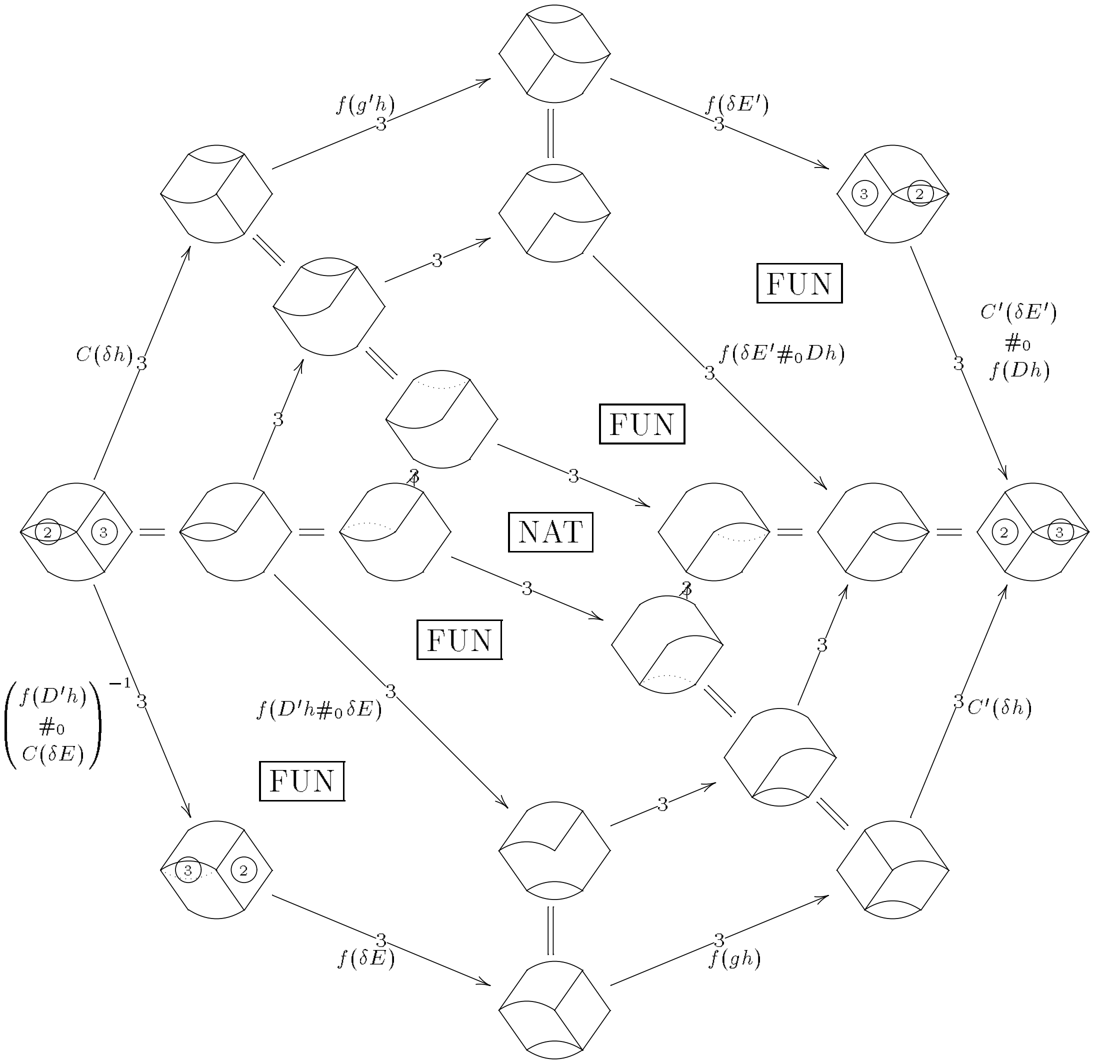
- for $q = r = 1$, $p = 2$: ζ of the diagram on page 49 commutes because it can be decomposed as follows:



,

where the unlabeled regions commute because ζ preserves faces, the regions labeled $\boxed{\text{FUN}}$ commute by functoriality axioms, and the region labeled $\boxed{\text{NAT}}$ commutes by the third naturality axiom,

- for $p = r = 1$, $q = 2$: ζ of the diagram on page 50 commutes because it can be decomposed as follows:



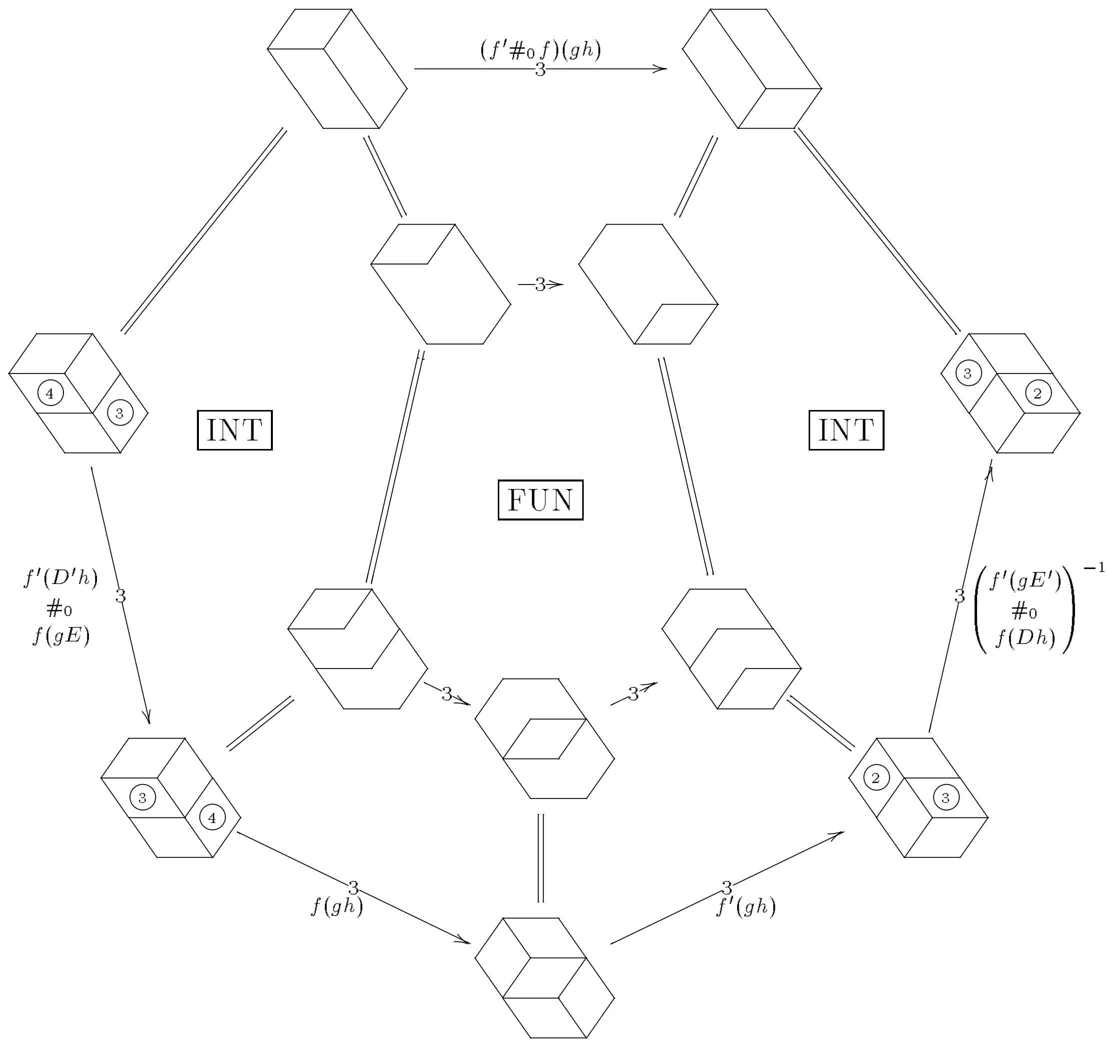
where the unlabeled regions commute because ζ preserves faces, the regions labeled $\boxed{\text{FUN}}$ commute by functoriality axioms, and the region labeled $\boxed{\text{NAT}}$ commutes by the second naturality axiom.

- for $p = q = 1$, $r = 2$, analogous to $p = 2$, $q = r = 1$,
- for $p = 0$: trivial from naturality in $\mathbb{D} \otimes \mathbb{E}$.

ζ is well-defined with respect to the functoriality relations:

- for $q = 0$ or $r = 0$: trivial,
- for $q = r = 1$, $p = 0$: trivial from functoriality in $\mathbb{D} \otimes \mathbb{E}$,

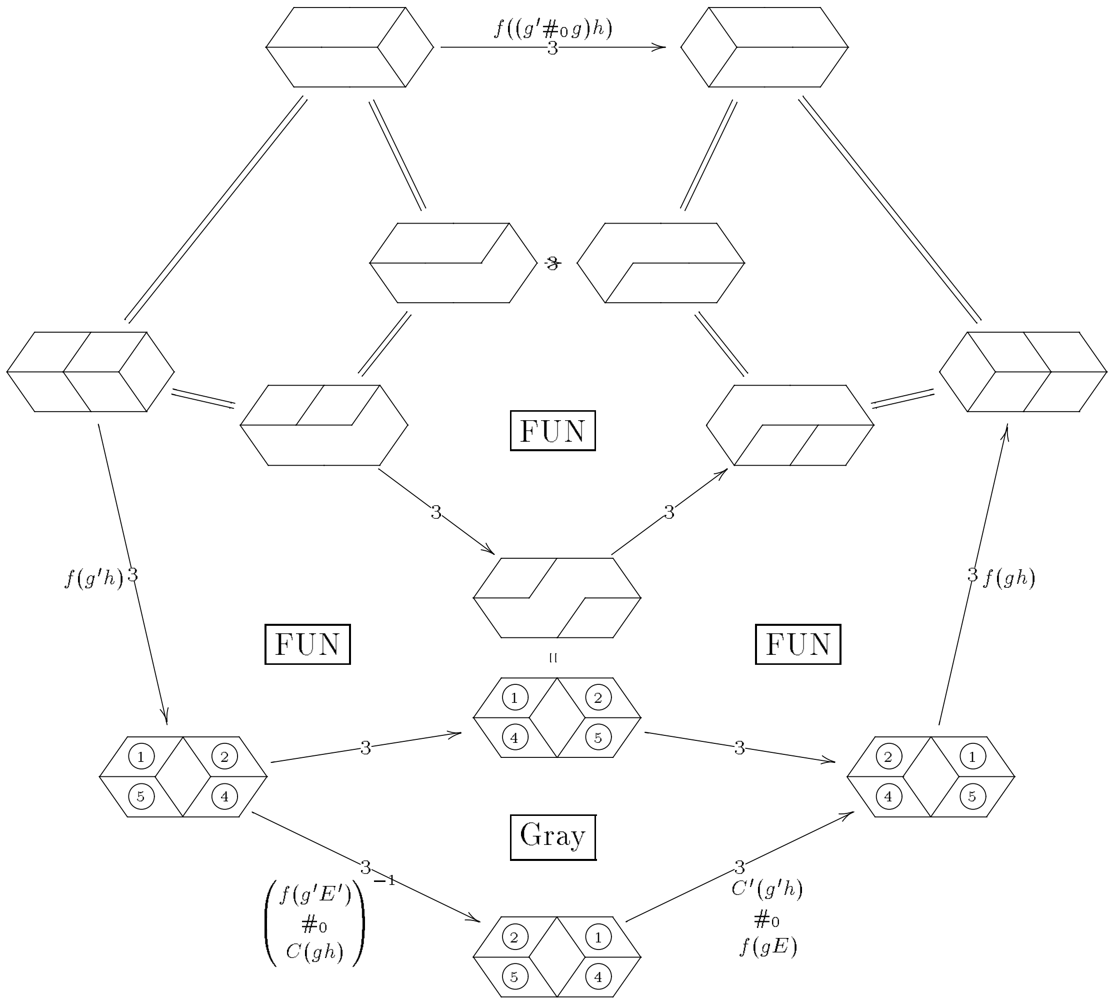
- for $p = q = r = 1$, composition in left factor: ζ of the diagram on page 51 commutes because it can be decomposed as follows:



,

where the unlabeled regions commute because ζ preserves faces, the regions labeled $\boxed{\text{INT}}$ commute by the interchange axiom, and the region labeled $\boxed{\text{FUN}}$ commutes by a functoriality axiom,

- for $p = q = r = 1$, composition in middle factor: ζ of the diagram on page 52 commutes because it can be decomposed as follows:



,

where the unlabeled regions commute because ζ preserves faces, the region labeled $\boxed{\text{Gray}}$ commutes by axiom (v) for a **Gray**-category, and the regions labeled $\boxed{\text{FUN}}$ commute by functoriality axioms,

- for $p = q = r = 1$, composition in right factor, analogous to $p = q = r = 1$, composition in left factor.

ζ is well-defined with respect to the identity relations:

- for identity in first factor: trivial,
- for identity in second or third factor: trivial from identity axiom in $\mathbb{D} \otimes \mathbb{E}$.

ζ is well-defined with respect to the interchange relations:

- for $q = 0$ or $r = 0$: trivial,
- for $q = r = 1$, $p = 0$: trivial from interchange in $\mathbb{D} \otimes \mathbb{E}$.

ζ is natural in each of its variables: immediate.

Define a functor $\vartheta : \mathbb{C}\otimes(\mathbb{D}\otimes\mathbb{E}) \to \mathbb{C}\otimes\mathbb{D}\otimes\mathbb{E}$ as follows. On generators, $\zeta(c\otimes(d\otimes e)) = c\otimes d\otimes e$, which is extended to generators $c\otimes(\{\Delta_1, \ldots, \Delta_{p'}\})$ by requiring ϑ to be functorial in the second variable. I have to check that this makes ϑ well-defined on generators, that this has the right faces, and that this is well-defined with respect to the relations. But this is exactly the reverse of the calculations for ζ above.

ϑ is natural in each of its variables: immediate.

ζ and ϑ are mutually inverse: straightforward. ■

7.2. Proposition. *The associativity of the tensor product of* **Gray**-*categories is coherent.*

Proof. As usual, this will be done via the quadruple tensor product.

The definition of $\mathbb{C} \otimes \mathbb{D} \otimes \mathbb{E} \otimes \mathbb{F}$, and of the relevant functors, is left to the reader. These relevant functors are well-defined by arguments similar to the ones above.

Finally, all these functors are completely determined by what they do to $c\otimes d\otimes e\otimes f$ and bracketings of this, and for these, commutativity of the required diagrams is immediate. ■

Let $\mathcal{I}^0$ be the one-point **Gray**-category.

7.3. Lemma. *For any* **Gray**-*category* $\mathbb{C}$*, there are canonical isomorphisms* $\mathbb{C}\otimes\mathcal{I}^0 \cong \mathbb{C} \cong \mathcal{I}^0 \otimes \mathbb{C}$. ■

7.4. Theorem. *The tensor product* $\otimes$ *and unit* $\mathcal{I}^0$ *give* **Gray-Cat** *the structure of a monoidal category.*

Proof. Associativity is proposition 7.1, coherence is proposition 7.2, and the axioms for the unit are easy. ■

7.2. **Gray-Cat** is not monoidal closed

The interchange axiom is the obstruction to this monoidal structure on **Gray-Cat** being monoidal closed. The easiest counter-example is given by

$$C \longrightarrow C' \longrightarrow C'' \otimes D \longrightarrow D' \longrightarrow D'' \quad ,$$

which is not equal to

$$(C \longrightarrow C' \longrightarrow C'' \otimes D \longrightarrow D') \cup (C \longrightarrow C' \longrightarrow C'' \otimes D' \longrightarrow D'')$$

precisely because the latter is the *non-commuting* interchange diagram.

8. Quasi-r-transfors of two and quasi-functors of three variables

Having proven associativity and coherence of the tensor product of **Gray**-categories, it is possible to state further universal properties succinctly. There are two kinds: a universal property for the (binary) tensor product with respect to *quasi-r-transfors of two variables*, which should be thought of as "bi-transforial mappings", and a universal property for multiple tensor products with respect to *quasi-functors of more variables*, which should be thought of as "multi-functorial mappings".

8.1. Quasi-r-transfors of two variables

Just as quasi-functors of two variables of **Gray**-categories extend Gray's quasi-functors of two variables of 2-categories, so will quasi-r-transfors of two variables extend Gray's quasi-natural transformations between quasi-functors of two variables and modifications between quasi-natural transformations [16, p. 58-59].

First I need to define *mixed* transfors.

8.1. DEFINITION. For $p + r \leq 3$, a *(p, r)-transfor* $\mathbb{C} \to \mathbb{D}$ is a functor $2_p \otimes \mathbb{C} \otimes 2_r \to \mathbb{D}$.

The left and right source and target of a (p, r)-transfor $\mathbb{C} \to \mathbb{D}$ are given by $2_{p-1} \otimes \mathbb{C} \otimes 2_r \rightrightarrows 2_p \otimes \mathbb{C} \otimes 2_r \to \mathbb{D}$ and $2_p \otimes \mathbb{C} \otimes 2_{r-1} \rightrightarrows 2_p \otimes \mathbb{C} \otimes 2_r \to \mathbb{D}$ respectively. $\diamond$

I could also have written this out explicitly, just as for q-transfors: for every $c \in \mathbb{C}$ of dimension q an element of dimension $p + q + r$ in $\mathbb{D}$, etcetera.

8.2. DEFINITION. For $0 < q \leq 3$, a *(right) quasi-r-transfor of two variables* $(\mathbb{C}, \mathbb{D}) \to \mathbb{E}$ consists of the following data:

- for every $c \in C_p$ a (p, r)-transfor $\chi(c, -) : \mathbb{D} \to \mathbb{E}$,
- for every $d \in D_q$ a right $(q + r)$-transfor $\chi(-, d) : \mathbb{C} \to \mathbb{E}$,

satisfying the following conditions:

- $\chi(c, -)(d) = \chi(-, d)(c) \stackrel{\text{def}}{=} \chi(c, d)$. $\diamond$

Note that interchange does not occur for $r > 0$ because objects of $\mathbb{D}$ cannot be composed.

8.3. PROPOSITION. *A quasi-r-transfor of two variables $\chi : (\mathbb{C}, \mathbb{D}) \to \mathbb{E}$ corresponds to an r-transfor $\mathbb{C} \otimes \mathbb{D} \to \mathbb{E}$.*

PROOF. Both correspond to a functor $\mathbb{C} \otimes \mathbb{D} \otimes 2_r \to \mathbb{E}$. ∎

Thus $\mathbb{C} \otimes \mathbb{D}$ is also the universal recipient of quasi-r-transfors of two variables from $(\mathbb{C}, \mathbb{D})$.

8.2. Quasi-functors of three variables

Quasi-functors of three (and more) variables of **Gray**-categories extend Gray's quasi-functors of n variables of 2-categories [16, p. 69-70]. But, like the ω-categorical [10, p. 143] and unlike the 2-categorical situation, quasi-functors of three variables can not be expressed only in terms of quasi-functors of two variables, but only in terms of appropriate (quasi-)*transfors*.

8.4. Definition. A *"middle" quasi-q-transfor of two variables* $(\mathbb{C},\mathbb{D}) \to \mathbb{E}$ is a functor $\mathbb{C} \otimes 2_q \otimes \mathbb{D} \to \mathbb{E}$. $\diamond$

8.5. Definition. A *quasi-functor of three variables* $(\mathbb{C},\mathbb{D},\mathbb{E}) \to \mathbb{F}$ consists of the following data:

- for every $c \in C_p$ a left p-transfor of two variables $\chi(c,-,?) : (\mathbb{D},\mathbb{E}) \to \mathbb{F}$,
- for every $d \in D_q$ a middle quasi-q-transfor of two variables $\chi(-,d,?) : (\mathbb{C},\mathbb{E}) \to \mathbb{F}$,
- for every $e \in E_r$ a right r-transfor $\chi(-,?,e) : \mathbb{C} \to \mathbb{E}$,

satisfying the following conditions:

- $\chi(c,-,?)(d,e) = \chi(-,d,?)(c,e) = \chi(-,?,e)(c,d) \stackrel{\text{def}}{=} \chi(c,d,e)$. $\diamond$

Alternatively, a quasi-functor of three variables $(\mathbb{C},\mathbb{D},\mathbb{E}) \to \mathbb{F}$ consists of:

- for every $c \in C_p$ and $d \in D_q$ a left $(p+q)$-transfor $\chi(c,d,-) : \mathbb{E} \to \mathbb{F}$,
- for every $c \in C_p$ and $e \in E_r$ a (p,r)-transfor $\chi(c,-,e) : \mathbb{D} \to \mathbb{F}$,
- for every $d \in D_q$ and $e \in E_r$ a right $(q+r)$-transfor $\chi(-,d,e) : \mathbb{D} \to \mathbb{F}$,

satisfying:

- $\chi(c,d,-)(e) = \chi(c,-,e)(d) = \chi(-,d,e)(c) \stackrel{\text{def}}{=} \chi(c,d,e)$,
- interchange axioms.

8.6. Proposition. *A quasi-functor of three variables $\chi : (\mathbb{C},\mathbb{D},\mathbb{E}) \to \mathbb{F}$ corresponds to a functor $\mathbb{C} \otimes \mathbb{D} \otimes \mathbb{E} \to \mathbb{F}$.* ■

Thus $\mathbb{C} \otimes \mathbb{D} \otimes \mathbb{E}$ is the universal recipient of quasi-functors of three variables from $(\mathbb{C},\mathbb{D},\mathbb{E})$.

It is, of course, completely analogous to define quasi-functors of more variables, and even quasi-r-transfors of more variables, and to give the corresponding universal properties.

8.3. Associativity revisited

It would have been possible to define quasi-functors of three variables without any reference to the triple tensor product (here it occurs in the definition of a (p,r)-transfor and of a middle q-transfor). Then comparing the universal property of quasi-functors of three variables $(\mathbb{C},\mathbb{D},\mathbb{E}) \to \mathbb{F}$ with the universal property of quasi-functors of two variables $(\mathbb{C},\mathbb{D} \otimes \mathbb{E}) \to \mathbb{F}$ would give an alternative proof of associativity. But this proof would only be different in appearance, and not in actual mathematical content.

9. Composition of transfors

Because $\boldsymbol{\omega}$**-Cat** is monoidal biclosed and hence enriched over itself, with lax-q-transformations as elements of the internal homs, lax-q-transformations of ω-categories can be composed [10, Section 3-12]. Although **Gray-Cat** is not monoidal biclosed, it is still possible to talk about composition of transfors of **Gray**-categories. As for lax-q-transformations, there are two kinds of composition: one which "morally" takes place in the would-be internal hom **Gray-Cat**$(\mathbb{C},\mathbb{D})$, and which comes from composition in $\mathbb{D}$, and one which "morally" takes place in the would-be enrichment of **Gray-Cat** over itself, and which comes from *substitution*. There is one caveat: the composition of 1-transfors need not always result in a 1-transfor again.

9.1. Composition from pasting

A q-transfor $\mathbb{C} \to \mathbb{D}$ assigns to a p-dimensional element of $\mathbb{C}$ a $(p+q)$-dimensional element of $\mathbb{D}$. The functoriality conditions about n-composition in $\mathbb{C}$ on a q-transfor are statements about certain $(n+q)$-composites in $\mathbb{D}$. The remaining directions of composition in $\mathbb{D}$ can be used to define compositions of transfors. But note that because transfors are dimension raising maps, their composition does also involve the other compositions in $\mathbb{D}$.

For $\alpha : F \to G$ and $\beta : G \to H$ 1-transfors $\mathbb{C} \to \mathbb{D}$, define (a degree 1 map) $\beta \#_0 \alpha : \mathbb{C} \to \mathbb{D}$ by:

- $(\beta \#_0 \alpha)_C = \beta_C \#_0 \alpha_C$,
- $(\beta \#_0 \alpha)_f = (\beta_{C'} \#_0 \alpha_f) \#_1 (\beta_f \#_0 \alpha_C)$,
- $(\beta \#_0 \alpha)_\gamma = \begin{array}{c} ((\beta_{C'} \#_0 \alpha_{f'}) \#_1 (\beta_\gamma \#_0 \alpha_C)) \\ \#_2 \\ ((\beta_{C'} \#_0 \alpha_\gamma) \#_1 (\beta_f \#_0 \alpha_C)) \end{array}$.

9.1. Lemma. *Let $\alpha : F \to G$ and $\beta : G \to H$ be 1-transfors $\mathbb{C} \to \mathbb{D}$. Then $\beta \#_0 \alpha$ satisfies all the axioms for a 1-transfor $F \to H$ except for functoriality with respect to composition of arrows.* ■

The problem with functoriality is as follows: $(\beta \#_0 \alpha)_{f' \#_0 f}$ equals, by definition,

$$\begin{tikzcd}
 & & \bullet \arrow[dr] & & \\
 & \bullet \arrow[ur] \arrow[dr] & & \bullet \arrow[dr] & \\
\bullet \arrow[ur] \arrow[dr] & & \bullet \arrow[dr] \arrow[ur, phantom, "\Downarrow\beta_{f'\#_0 f}"] & & \bullet \\
 & \bullet \arrow[dr] \arrow[ur, phantom, "\Downarrow\alpha_{f'\#_0 f}"] & & \bullet \arrow[ur] & \\
 & & \bullet \arrow[ur] & &
\end{tikzcd},$$

which by functoriality of α and β is equal to

$$
\begin{tikzcd}
 & & \bullet \arrow[dr] & & \\
 & \bullet \arrow[ur] \arrow[dr] & \Downarrow\beta_f & \bullet \arrow[dr] & \\
\bullet \arrow[ur] \arrow[dr] & \alpha_f\Downarrow \quad (3) & \bullet \arrow[ur] \arrow[dr] & (2) \quad \Downarrow\beta_{f'} & \bullet \\
 & \bullet \arrow[ur] \arrow[dr] & \Downarrow\alpha_{f'} & \bullet \arrow[ur] & \\
 & & \bullet \arrow[ur] & &
\end{tikzcd}
\quad .
$$

On the other hand,

$$
\begin{tikzcd}
 & & \bullet \arrow[dr] & & \\
 & \bullet \arrow[ur] & \Downarrow(\beta\#_0\alpha)_f & \bullet \arrow[dr] & \\
\bullet \arrow[ur] \arrow[dr] & & \bullet \arrow[ur] & \Downarrow(\beta\#_0\alpha)_{f'} & \bullet \\
 & \bullet \arrow[ur] \arrow[dr] & & \bullet \arrow[ur] & \\
 & & \bullet \arrow[ur] & &
\end{tikzcd}
$$

equals, again by definition,

$$
\begin{tikzcd}
 & & \bullet \arrow[dr] & & \\
 & \bullet \arrow[ur] \arrow[dr] & \Downarrow\beta_f & \bullet \arrow[dr] & \\
\bullet \arrow[ur] \arrow[dr] & \alpha_f\Downarrow \quad (2) & \bullet \arrow[ur] \arrow[dr] & (3) \quad \Downarrow\beta_{f'} & \bullet \\
 & \bullet \arrow[ur] \arrow[dr] & \Downarrow\alpha_{f'} & \bullet \arrow[ur] & \\
 & & \bullet \arrow[ur] & &
\end{tikzcd}
\quad ,
$$

which is not equal to the previous composite, as required for functoriality, there only being a 3-arrow between these two composites.

There are instances where the composite of 1-transfors *is* a 1-transfor again because the 3-arrow between the two sides of functoriality actually is an identity.

9.2. DEFINITION. Let $\alpha : F \to G$ and $\beta : G \to H$ be 1-transfors $\mathbb{C} \to \mathbb{D}$. α and β are said to be *truly composable* if for every $C \xrightarrow{f} C' \xrightarrow{f'} C''$ in $\mathbb{C}$

$$\begin{array}{c} (\beta_{C''} \#_0 \alpha_{f'} \#_0 F(f)) \\ \#_1 \\ (\beta_{C''} \#_0 G(f') \#_0 \alpha_f) \\ \#_1 \\ (\beta_{f'} \#_0 G(f') \#_0 \alpha_C) \\ \#_1 \\ (H(f') \#_0 \beta_f \#_0 \alpha_C) \end{array} = \begin{array}{c} (\beta_{C''} \#_0 \alpha_{f'} \#_0 F(f)) \\ \#_1 \\ (\beta_{f'} \#_0 \alpha_{C'} \#_0 F(f)) \\ \#_1 \\ (H(f') \#_0 \beta_{C'} \#_0 \alpha_f) \\ \#_1 \\ (H(f') \#_0 \beta_f \#_0 \alpha_C) \end{array},$$

and

$$\begin{array}{c} (\beta_{C''} \#_0 \alpha_{f'} \#_0 F(f)) \\ \#_1 \\ (\beta_{f'} \#_0 \alpha_f) \\ \#_1 \\ (H(f') \#_0 \beta_f \#_0 \alpha_C) \end{array}$$

is equal to this identity. $\diamond$

When α and β are truly composable, α is said to be *(right) social with respect to* β. One can also speak of α being social with respect to a class B of 1-transfors. In particular, α is said to be *social* if α is social with respect to every β. And one can say that a class B of 1-transfors is social if every pair of elements of B is truly composable.

As an example of social transfors, every 1-transfor into a 3-category is social. Another example of "sociality" is that the collection of 1-transfors "tensoring with an arrow" is social: every pair of composable 1-transfors $- \otimes g$ and $- \otimes g' : \mathbb{C} \to \mathbb{C} \otimes \mathbb{D}$ is truly composable precisely because of the interchange axiom for the tensor product.

Being truly composable is preserved by composition, in the following sense.

9.3. Proposition. *Let $\alpha : F \to G$, $\beta : G \to H$ and $\gamma : H \to K$ be 1-transfors $\mathbb{C} \to \mathbb{D}$. If α is truly composable with β and β truly composable with γ then $\beta \#_0 \alpha$ is truly composable with γ and α is truly composable with $\gamma \#_0 \beta$.* ∎

In particular, composition of social 1-transfors gives a social 1-transfor again. However, sociality is not preserved by composition from substitution, to be considered shortly.

From now on, whenever I mention composition of 1-transfors, it will be assumed that these transfors are truly composable. If I want to emphasize this assumption, such a composite will be denoted by $\#_0^{\top}$.

For $\alpha : F \to G$ a 1-transfor, define (a degree 2 map) $\mathrm{id}_\alpha : \mathbb{C} \to \mathbb{D}$ by:

- $(\mathrm{id}_\alpha)_C = id_{\alpha_C}$,
- $(\mathrm{id}_\alpha)_f = id_{\alpha_f}$.

9.4. Lemma. *Let $\alpha : F \to G$ be a 1-transfor. Then id_α is a 2-transfor $\alpha \to \alpha$.* ∎

For $\mu : \alpha \to \alpha'$ and $\nu : \alpha' \to \alpha''$ 2-transfors, define (a degree 2 map) $\nu \#_1 \mu : \mathbb{C} \to \mathbb{D}$ by:

- $(\nu \#_1 \mu)_C = \nu_C \#_1 \mu_C$,
- $(\nu \#_1 \mu)_f = \begin{array}{c} ((\nu_{C'} \#_0 F(f)) \#_1 \mu_f) \\ \#_2 \\ (\nu_f \#_1 (G(f) \#_0 \mu_C)) \end{array}$.

9.5. Lemma. *Let $\mu : \alpha \to \alpha'$ and $\nu : \alpha' \to \alpha''$ be 2-transfors. Then $\nu \#_1 \mu$ is a 2-transfor $\alpha \to \alpha''$.* ∎

For $\mu : \alpha \to \alpha'$ a 2-transfor and $\beta : G \to H$ a 1-transfor, define (a degree 2 map) $\beta \#_0 \mu : \mathbb{C} \to \mathbb{D}$ by:

- $(\beta \#_0 \mu)_C = \beta_C \#_0 \mu_C$,
- $(\beta \#_0 \mu)_f = \begin{array}{c} ((\beta_{C'} \#_0 \mu_f) \#_1 (\beta_f \#_0 \alpha_C)) \\ \#_2 \\ ((\beta_{C'} \#_0 \alpha'_f) \#_1 (\beta_f \#_0 \mu_C)^{-1}) \end{array}$.

For $\alpha : F \to G$ a 1-transfor and $\nu : \beta \to \beta'$ a 2-transfor, define (a degree 2 map) $\nu \#_0 \alpha : \mathbb{C} \to \mathbb{D}$ by:

- $(\nu \#_0 \alpha)_C = \nu_C \#_0 \alpha_C$,
- $(\nu \#_0 \alpha)_f = \begin{array}{c} ((\nu_{C'} \#_0 \alpha_f) \#_1 (\beta_f \#_0 \alpha_C)) \\ \#_2 \\ ((\beta'_{C'} \#_0 \alpha_f) \#_1 (\nu_f \#_0 \alpha_C)) \end{array}$.

9.6. Lemma. *Let $\mu : \alpha \to \alpha'$ and $\nu : \beta \to \beta'$ be 2-transfors. Then $\beta \#_0 \mu$ is a 2-transfor $\beta \#_0^\top \alpha \to \beta \#_0^\top \alpha'$ and $\nu \#_0 \alpha$ is a 2-transfor $\beta \#_0^\top \alpha \to \beta' \#_0^\top \alpha$.* ∎

I will sometimes write $\beta \#_0^\top \mu$ and $\nu \#_0^\top \alpha$ if I want to emphasize that the 1-transfors inhere must be truly composable.

For $\mu : \alpha \to \beta$ a 2-transfor, define (a degree 3 map) $\mathrm{id}_\mu : \mathbb{C} \to \mathbb{D}$ by:

- $(\mathrm{id}_\mu)_C = id_{\mu_C}$.

9.7. Lemma. *Let $\mu : \alpha \to \beta$ be a 2-transfor. Then id_μ is a 3-transfor $\mu \to \mu$.* ∎

For $\mu : \alpha \to \alpha'$ and $\nu : \beta \to \beta'$ 2-transfors, define (a degree 3 map) $\nu \#_0 \mu : \mathbb{C} \to \mathbb{D}$ by:

- $(\nu \#_0 \mu)_C = \nu_C \#_0 \mu_C$.

9.8. Lemma. *Let $\mu : \alpha \to \alpha'$ and $\nu : \beta \to \beta'$ be 2-transfors. Then $\nu \#_0 \mu$ is a 3-transfor $(\beta' \#_0^\top \mu) \#_1 (\nu \#_0^\top \alpha) \to (\nu \#_0^\top \alpha') \#_1 (\beta \#_0^\top \mu)$.* ∎

Again, I will sometimes write $\nu \#_0^\top \mu$.

For $u : \mu \to \mu'$ and $v : \mu' \to \mu''$ 3-transfors, define (a degree 3 map) $v \#_2 u : \mathbb{C} \to \mathbb{D}$ by:

- $(v \#_2 u)_C = v_C \#_2 u_C$.

9.9. Lemma. *Let $u : \mu \to \mu'$ and $v : \mu' \to \mu''$ be 3-transfors. Then $v \#_2 u$ is a 3-transfor $\mu \to \mu''$.* ■

For **Gray**-categories $\mathbb{C}$ and $\mathbb{D}$, denote by **Gray-Cat**$(\mathbb{C},\mathbb{D})$ the 3-truncated globular set having as i-dimensional elements i-transfors $\mathbb{C} \to \mathbb{D}$, with faces as defined in section 5.

9.10. Proposition. *For* **Gray**-*categories $\mathbb{C}$ and $\mathbb{D}$, the globular set* **Gray-Cat**$(\mathbb{C},\mathbb{D})$, *with compositions and identity given as above, and with identity given by* id_{-} *above, satisfies all the axioms for a* **Gray**-*category, except that 0-composition of 1-transfors is only defined for truly composable pairs.* ■

Denote by **Gray-Cat**$(\mathbb{C},\mathbb{D})_{\mathrm{soc}}$ the 3-truncated globular set having as i-dimensional elements *social* i-transfors $\mathbb{C} \to \mathbb{D}$, with faces as defined in section 5.

9.11. Proposition. *For* **Gray**-*categories $\mathbb{C}$ and $\mathbb{D}$, the globular set* **Gray-Cat**$(\mathbb{C},\mathbb{D})_{\mathrm{soc}}$, *with compositions and identity given as above, and with identity given by* id_{-} *above, is a* **Gray**-*category.* ■

9.2. Composition from substitution

Composition of functors $F : \mathbb{C} \to \mathbb{D}$ and $G : \mathbb{D} \to \mathbb{E}$ is defined in the obvious way: $(G \circ F)(c) = G(F(c))$. Composition of 1-transfors with functors (on either side) is equally obvious.

Composition like this does result in a 1-transfor, but composition of a social 1-transfor with a functor need not give a social 1-transfor again: e.g., compose the 1-transfor with an embedding in a bigger **Gray**-category; this latter can be chosen to be one which universally makes the composite fail to be social (add squares which will be the images of elements of $\mathbb{C}$ under a new transfor, and then the composite of the original 1-transfor with the embedding is not social with respect to this new 1-transfor because there are no interchange conditions on the newly added squares).

For 1-transfors $\alpha : \mathbb{C} \to \mathbb{D}$ and $\beta : \mathbb{D} \to \mathbb{E}$, define (a degree 2 map) $m(\alpha, \beta) : \mathbb{C} \to \mathbb{E}$, also denoted $\beta \circ \alpha$, by

- $m(\alpha, \beta)_C = \beta_{\alpha_C}$.
- $m(\alpha, \beta)_f = \beta_{\alpha_f}$.

9.12. Lemma. *Let $\alpha : F \to F'$ and $\beta : G \to G'$ be 1-transfors $\mathbb{C} \to \mathbb{D}$ and $\mathbb{D} \to \mathbb{E}$ respectively. Then $m(\alpha, \beta)$ is a 2-transfor $(G' \circ \alpha) \#_0^{\top} (\beta \circ F) \to (\beta \circ F') \#_0^{\top} (G \circ \alpha)$.* ■

Remember the assumption about 1-transfors being truly composable! I could, but will not, write $m^{\top}(\alpha, \beta)$, or $\beta \circ^{\top} \alpha$.

For a 2-transfor $\mu : \mathbb{C} \to \mathbb{D}$ and a 1-transfor $\beta : \mathbb{D} \to \mathbb{E}$, define (a degree-3-map) $m(\mu, \beta) : \mathbb{C} \to \mathbb{E}$, also denoted $\beta \circ \mu$, by:

- $m(\mu, \beta)_C = \beta_{\mu_C}$.

For a 1-transfor $\alpha : \mathbb{C} \to \mathbb{D}$ and a 2-transfor $\nu : \mathbb{D} \to \mathbb{E}$, define (a degree-3-map) $m(\alpha, \nu) : \mathbb{C} \to \mathbb{E}$, also denoted $\nu \circ \alpha$, by:

- $m(\alpha, \nu)_C = \nu_{\alpha_C}$.

9.13. Lemma. *Let $\mu : \alpha \to \alpha'$ and $\nu : \beta \to \beta'$ be 2-transfors $\mathbb{C} \to \mathbb{D}$ and $\mathbb{D} \to \mathbb{E}$ respectively. Then $m(\mu, \beta)$ is a 3-transfor*

$$\begin{array}{ccc} ((\beta \circ F') \#_0^\top (G \circ \mu)) & & m(\alpha', \beta) \\ \#_1 & \to & \#_1 \\ m(\alpha, \beta) & & ((G' \circ \mu) \#_0^\top (\beta \circ F)) \end{array}$$

and $m(\alpha, \nu)$ is a 3-transfor

$$\begin{array}{ccc} m(\alpha, \beta') & & ((\nu \circ F') \#_0^\top (G \circ \alpha)) \\ \#_1 & \to & \#_1 \\ ((G' \circ \alpha) \#_0^\top (\nu \circ F)) & & m(\alpha, \beta) \end{array}.$$

■

9.14. Proposition. *m satisfies all the axioms for a functor* $\mathbf{Gray\text{-}Cat}(\mathbb{C}, \mathbb{D}) \otimes \mathbf{Gray\text{-}Cat}(\mathbb{D}, \mathbb{E}) \to \mathbf{Gray\text{-}Cat}(\mathbb{C}, \mathbb{E})$*, except that operations are restricted to situations where all 0-composites of 1-transfors are "true".*

Proof. This follows immediately because m is substitution in transfors which themselves satisfy all these properties, except interchange which holds because composition of 1-transfors is restricted to truly composable pairs. ■

9.3. Gray-Cat as a partial $(\mathbf{Gray\text{-}Cat})_\otimes$-CATegory

9.15. Theorem. **Gray-Cat** *satisfies all the axioms for an $(\mathbf{Gray\text{-}Cat})_\otimes$-CATegory, except that operations are restricted to situations where all 0-composites of 1-transfors are "true".*

Proof. Everything has been done before, except associativity of m, which is immediate because substitution is associative. ■

9.4. Gray-Cat is not closed

Even though it has been established before that **Gray-Cat** is not monoidal closed, this still leaves room for $\mathbf{Gray\text{-}Cat}(\mathbb{C}, \mathbb{D})_{\mathrm{soc}}$ to be the internal hom of a (separate) *closed* structure. However, the only candidate for the functor $\mathbf{Gray\text{-}Cat}(\mathbb{D}, \mathbb{E})_{\mathrm{soc}} \to \mathbf{Gray\text{-}Cat}(\mathbf{Gray\text{-}Cat}(\mathbb{C}, \mathbb{D})_{\mathrm{soc}}, \mathbf{Gray\text{-}Cat}(\mathbb{C}, \mathbb{E})_{\mathrm{soc}})_{\mathrm{soc}}$ is to send a transfor $\varrho : \mathbb{D} \to \mathbb{E}$ to right composition by ϱ. But as seen before, even right composition by a functor does not need to preserve sociality. And also, on another level, there is no reason for right composition by a (social) transfor to be a *social* transfor itself.

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