Theory and Applications of Categories, Vol. 5, No. 3, 1999, pp. 70–80.

A NOTE ON THE EXACT COMPLETION OF A REGULAR CATEGORY, AND ITS INFINITARY GENERALIZATIONS

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Transmitted by Walter Tholen

ABSTRACT. A new description of the exact completion $\mathcal{C}_{\mathrm{ex/reg}}$ of a regular category $\mathcal{C}$ is given, using a certain topos $\mathrm{Shv}(\mathcal{C})$ of sheaves on $\mathcal{C}$; the exact completion is then constructed as the closure of $\mathcal{C}$ in $\mathrm{Shv}(\mathcal{C})$ under finite limits and coequalizers of equivalence relations. An infinitary generalization is proved, and the classical description of the exact completion is derived.

1. Introduction

A category $\mathcal{C}$ with finite limits is said to be *regular* if every morphism factorizes as a strong epimorphism followed by a monomorphism, and moreover the strong epimorphisms are stable under pullback; it follows that the strong epimorphisms are precisely the regular epimorphisms, namely those arrows which are the coequalizer of their kernel pair. Every kernel pair is an equivalence relation; a regular category is said to be *exact* if every equivalence relation is a kernel pair. Thus a regular category is a category with finite limits and coequalizers of *kernel pairs*, satisfying certain exactness conditions; while an exact category is a category with finite limits and coequalizers of *equivalence relations*, satisfying certain exactness conditions. Regular and exact categories were introduced by Barr [1], but the definition given here is due to Joyal.

A functor between regular categories is said to be *regular* if it preserves finite limits and strong (=regular) epimorphisms. There is a 2-category **Reg** of regular categories, regular functors, and natural transformations, and it has a full sub-2-category **Ex** comprising the exact categories. The inclusion of **Ex** into **Reg** has a left biadjoint, and the value of this biadjoint at a regular category $\mathcal{C}$ is what we mean by the exact completion of the regular category $\mathcal{C}$.

Free regular and free exact categories have received a great deal of attention. A syntactic description of the free exact category on a category with finite limits was given by Carboni and Celia Magno in [2]. It was then observed that the same construction could be carried out starting with a category not with finite limits, but only with *weak* finite limits; this construction, along with its universal property, was described by Carboni and Vitale in [3]. The same paper also contained a description of the free *regular* category on

The support of the Australian Research Council is gratefully acknowledged.
Received by the editors 1998 November 6 and, in revised form, 1999 January 15.
Published on 1999 January 19.
1991 Mathematics Subject Classification: 18A35, 18A40, 18E10, 18F20.
Key words and phrases: regular category, exact category, exact completion, category of sheaves.

a category with weak finite limits. Meanwhile, a quite different account of the free exact category on a finitely complete one was given by Hu in [10], namely as the full subcategory of the functor category $[\mathrm{Lex}(\mathcal{C}, \mathbf{Set}), \mathbf{Set}]$ comprising those functors which preserve finite products and filtered colimits, where $\mathrm{Lex}(\mathcal{C}, \mathbf{Set})$ is the category of finite-limit-preserving functors from $\mathcal{C}$ to $\mathbf{Set}$. Finally, Hu and Tholen described in [11] the free exact category, and the free regular category, on a category $\mathcal{C}$ with weak finite limits, as full subcategories of the presheaf category $[\mathcal{C}^{op}, \mathbf{Set}]$: the free exact category was constructed as the closure of $\mathcal{C}$ in $[\mathcal{C}^{op}, \mathbf{Set}]$ under coequalizers of equivalence relations, while the free regular category was constructed as the closure of $\mathcal{C}$ in $[\mathcal{C}^{op}, \mathbf{Set}]$ under coequalizers of kernel pairs.

In a different line, the exact completion of a regular category had already been described by Lawvere in [14], and more fully by R. Succi Cruciani [5]; see also the account of Freyd and Scedrov in [7]. It was also used in [3] in constructing the free exact category on a category with weak finite limits. The construction is conceptually attractive: one forms the (bi)category $\mathrm{Rel}(\mathcal{C})$ of relations in the regular category $\mathcal{C}$, and then freely splits those idempotents which correspond to equivalence relations in $\mathcal{C}$; this gives the (bi)category of relations, $\mathrm{Rel}(\mathcal{C}_{\mathrm{ex/reg}})$, in the exact completion $\mathcal{C}_{\mathrm{ex/reg}}$ of the original regular category, and one can extract $\mathcal{C}_{\mathrm{ex/reg}}$ as the category of *maps* in the bicategory $\mathrm{Rel}(\mathcal{C}_{\mathrm{ex/reg}})$, that is, as the category of those arrows which have a right adjoint in the bicategory.

The purpose of this note is to provide an alternative description of $\mathcal{C}_{\mathrm{ex/reg}}$, analogous to the description in [11] of the exact and regular completions of a category with weak finite limits, in which $\mathcal{C}_{\mathrm{ex/reg}}$ is seen as a full subcategory of the presheaf category $[\mathcal{C}^{op}, \mathbf{Set}]$.

A disadvantage of this approach is that it applies only to *small* regular categories, but in fact this is less serious than it might seem. When we speak of *small* sets, we mean sets whose cardinality is less than some inaccessible cardinal ∞. A small category is then a category with a small set of objects and small hom-sets. Given a regular category $\mathcal{C}$ which is not small, it will suffice to choose an inaccessible cardinal ∞' greater than the cardinality of each of the hom-sets of $\mathcal{C}$ and that of the set of objects of $\mathcal{C}$; for we may then write $\mathbf{SET}$ for the category of sets whose cardinality is less than ∞', and redefine small to mean " of cardinality less than ∞'". The exact completion of $\mathcal{C}$ can now be constructed as a full subcategory of $[\mathcal{C}^{op}, \mathbf{SET}]$, just as in Section 3 below.

A joint paper with Kelly, still in preparation, will describe the precise sense in which all of the constructions described above are free cocompletions with respect to certain classes of colimits "in the lex world".

2. Sheaves on a regular category

Say that a diagram

$$K \overset{k}{\underset{l}{\rightrightarrows}} A \xrightarrow{p} B$$

in a regular category is an *exact fork* if p is the coequalizer of k and l, and (k, l) is the kernel pair of p.

We shall, as promised, construct the free exact category on the regular category $\mathcal{C}$ as a full subcategory of the presheaf category $[\mathcal{C}^{op}, \mathbf{Set}]$, but it is not in fact the presheaf category with which we mostly work, for in passing from $\mathcal{C}$ to $\mathcal{C}_{\mathrm{ex/reg}}$ we wish to preserve coequalizers of kernel pairs in $\mathcal{C}$, and the Yoneda embedding does not preserve such coequalizers. We therefore consider only those presheaves F for which, if

$$K \underset{l}{\overset{k}{\rightrightarrows}} A \xrightarrow{p} B$$

is an exact fork in $\mathcal{C}$, then any arrow $u : yA \to F$ which coequalizes k and l must factor uniquely through p. By the Yoneda lemma, these are precisely the presheaves $F : \mathcal{C}^{op} \to \mathbf{Set}$ which preserve the equalizers of those pairs which are in fact cokernel pairs. We write $\mathrm{Shv}(\mathcal{C})$ for the full subcategory of $[\mathcal{C}^{op}, \mathbf{Set}]$ given by these presheaves; clearly it contains the representables, and moreover is reflective by [6, Theorem 5.2.1].

The reason for the name $\mathrm{Shv}(\mathcal{C})$ is of course that the objects of $\mathrm{Shv}(\mathcal{C})$ are the sheaves for a (Grothendieck) topology on $\mathcal{C}$. Since $\mathcal{C}$ is regular, there is a *pretopology* [12] (or *basis for a topology* [15]) on $\mathcal{C}$ for which the covering families of B are the singleton families $(p : A \to B)$ with p a strong epimorphism. This generates a topology on $\mathcal{C}$, called the *regular epimorphism topology* [1, Section I.4], for which the sheaves are precisely the objects of $\mathrm{Shv}(\mathcal{C})$. We conclude that $\mathrm{Shv}(\mathcal{C})$ is a Grothendieck topos, and in particular an exact category. The inclusion $Y : \mathcal{C} \to \mathrm{Shv}(\mathcal{C})$ preserves whatever limits exist in $\mathcal{C}$, since the Yoneda embedding preserves any limits which exist in $\mathcal{C}$, and the inclusion of $\mathrm{Shv}(\mathcal{C})$ in the presheaf category $[\mathcal{C}^{op}, \mathbf{Set}]$ preserves and reflects all limits. Thus $Y : \mathcal{C} \to \mathrm{Shv}(\mathcal{C})$ preserves finite limits; by construction it preserves coequalizers of kernel pairs, and so is a regular functor. This full regular embedding of $\mathcal{C}$ in the topos $\mathrm{Shv}(\mathcal{C})$, was described already in [1], as was the reflectivity of $\mathrm{Shv}(\mathcal{C})$ in $[\mathcal{C}^{op}, \mathbf{Set}]$.

We shall need the following characterization of strong epimorphisms in $\mathrm{Shv}(\mathcal{C})$ as "local surjections"; since it is a (well-known) special case of [15, Corollary III.7.5], we only sketch the proof.

2.1. Lemma. *An arrow $\alpha : F \to G$ in $\mathrm{Shv}(\mathcal{C})$ is a strong epimorphism if and only if for every $v : YB \to G$, there exists a strong epimorphism $p : A \twoheadrightarrow B$ in $\mathcal{C}$ and a factorization $v.Yp = \alpha.u$.*

Proof. Suppose that α is strong epi; for each object B of $\mathcal{C}$, define HB to be the set $\{x \in GB \mid (Gp)x = (\alpha A)w$ for some $w \in FA$ and some strong epi $p : A \twoheadrightarrow B$ in $\mathcal{C}\}$, and write $mB : HB \to GB$ for the inclusion. This is easily seen to be functorial in B, giving a functor $H : \mathcal{C}^{op} \to \mathbf{Set}$, and a natural transformation $m : H \to G$ all of whose components are monomorphisms; clearly α factorizes through m. One now shows that H is in fact a sheaf, so that the strong epimorphism α in $\mathrm{Shv}(\mathcal{C})$ factors through the subobject m, which must therefore be invertible. It follows that α satisfies the conditions of the lemma.

Suppose, conversely, that $\alpha : F \to G$ satisfies the conditions of the lemma. If α factorizes (in $\mathrm{Shv}(\mathcal{C})$) as $\alpha = m.\beta$ with $m : H \to G$ monic, then for any $v : YB \to G$ we have a commutative diagram

$$\begin{tikzcd} YA \arrow[rr, two heads, "Yp"] \arrow[d, "u"'] & & YB \arrow[d, "v"] \\ F \arrow[r, two heads, "\beta"'] & H \arrow[r, tail, "m"'] & G \end{tikzcd}$$

in $\mathrm{Shv}(\mathcal{C})$ with Yp strong epi and m mono, and so a factorization of v through m. Since the representables form a strong generator for $\mathrm{Shv}(\mathcal{C})$, it follows that m is invertible, and so that α is a strong epimorphism. ∎

We also need the following description of geometric morphisms into $\mathrm{Shv}(\mathcal{C})$; once again it is a special case of a well-known result.

2.2. LEMMA. *Let $\mathcal{E}$ be a Grothendieck topos. Composition with $Y : \mathcal{C} \to \mathrm{Shv}(\mathcal{C})$ induces an equivalence between* **Reg** $(\mathcal{C}, \mathcal{E})$ *and the category* **Geom** $(\mathcal{E}, \mathrm{Shv}(\mathcal{C}))$ *of geometric morphisms from $\mathcal{E}$ to* $\mathrm{Shv}(\mathcal{C})$*; the inverse image S^* of the geometric morphism corresponding to $S : \mathcal{C} \to \mathcal{E}$ is given by the (pointwise) left Kan extension* $\mathrm{Lan}_Y S$ *of S along $Y : \mathcal{C} \to \mathrm{Shv}(\mathcal{C})$.*

Proof. There is a well-known equivalence between $\mathrm{Geom}(\mathcal{E},[\mathcal{C}^{op}, \mathbf{Set}])$ and the category $\mathrm{Lex}(\mathcal{C},\mathcal{E})$ of finite-limit-preserving functors from $\mathcal{C}$ to $\mathcal{E}$; it is a special case, for example, of [15, Theorem VII.7.2]. By a special case of [15, Lemma VII.7.3], a geometric morphism from $\mathcal{E}$ to $[\mathcal{C}^{op}, \mathbf{Set}]$ factors through $\mathrm{Shv}(\mathcal{C})$ if and only if the corresponding finite-limit-preserving functor $S : \mathcal{C} \to \mathcal{E}$ maps strong epimorphisms in $\mathcal{C}$ to epimorphisms in $\mathcal{E}$; but, since $\mathcal{E}$ is a topos, all epimorphisms are strong, and so this just says that S is a regular functor.

Finally the inverse image S^* of the geometric morphism corresponding to a regular functor $S : \mathcal{C} \to \mathcal{E}$ satisfies $S^*L \cong \mathrm{Lan}_{JY} S$, where $L : [\mathcal{C}^{op}, \mathbf{Set}] \to \mathrm{Shv}(\mathcal{C})$ is the reflection, and so $S^* \cong S^*LJ \cong (\mathrm{Lan}_{JY} S)J \cong (\mathrm{Lan}_J \mathrm{Lan}_Y S)J \cong \mathrm{Lan}_Y S$. ∎

3. The exact completion of a regular category

We now write $\mathcal{C}_{\mathrm{ex/reg}}$ for the full subcategory of $\mathrm{Shv}(\mathcal{C})$ comprising those objects which are coequalizers in $\mathrm{Shv}(\mathcal{C})$ of equivalence relations in $\mathcal{C}$. We write $Z : \mathcal{C} \hookrightarrow \mathcal{C}_{\mathrm{ex/reg}}$ and $W : \mathcal{C}_{\mathrm{ex/reg}} \hookrightarrow \mathrm{Shv}(\mathcal{C})$ for the inclusions.

3.1. LEMMA. *If $p : YA \twoheadrightarrow F$ is a strong epimorphism and $m_i : F \to G_i$ is a finite jointly-monic family in* $\mathrm{Shv}(\mathcal{C})$*, then F lies in $\mathcal{C}$ if each G_i does; thus in particular $\mathcal{C}$ is closed in $\mathcal{C}_{\mathrm{ex/reg}}$ under subobjects.*

Proof. Since p is strong epi, it is the coequalizer of its kernel pair. Since the m_i are jointly monic, the induced arrow $m : F \to \Pi_i G_i$ is monic, and so the kernel pair of p is also the kernel pair of mp. If each G_i is in $\mathcal{C}$, then since $\mathcal{C}$ has finite products, $\Pi_i G_i$ is in $\mathcal{C}$; and since $\mathcal{C}$ has pullbacks, the kernel pair of mf lies in $\mathcal{C}$; and so F lies in $\mathcal{C}$ since $\mathcal{C}$

has coequalizers of kernel pairs. The second statement follows from the first: if F lies in $\mathcal{C}_{\mathrm{ex/reg}}$ then there is a strong epimorphism $YA \twoheadrightarrow F$; if also F is a subobject of some YB then we conclude that F lies in $\mathcal{C}$. ∎

3.2. Proposition. *$\mathcal{C}_{\mathrm{ex/reg}}$ is closed in $\mathrm{Shv}(\mathcal{C})$ under finite limits and coequalizers of equivalence relations; thus $\mathcal{C}_{\mathrm{ex/reg}}$ is an exact category, the inclusions $Z : \mathcal{C} \hookrightarrow \mathcal{C}_{\mathrm{ex/reg}}$ and $W : \mathcal{C}_{\mathrm{ex/reg}} \hookrightarrow \mathrm{Shv}(\mathcal{C})$ are fully faithful regular functors, and $\mathcal{C}_{\mathrm{ex/reg}}$ is the closure of $\mathcal{C}$ in $\mathrm{Shv}(\mathcal{C})$ under finite limits and coequalizers of equivalence relations.*

Proof. $\mathrm{Shv}(\mathcal{C})$ is exact, the inclusion of $\mathcal{C}$ into $\mathrm{Shv}(\mathcal{C})$ is regular, and every object of $\mathcal{C}_{\mathrm{ex/reg}}$ is the coequalizer in $\mathrm{Shv}(\mathcal{C})$ of an equivalence relation in $\mathcal{C}$; thus it will suffice to prove the first statement: that $\mathcal{C}_{\mathrm{ex/reg}}$ is closed under finite limits and coequalizers of equivalence relations.

Step 1: $\mathcal{C}_{\mathrm{ex/reg}}$ is closed in $\mathrm{Shv}(\mathcal{C})$ under finite products. Of course the terminal object of $\mathrm{Shv}(\mathcal{C})$ lies not just in $\mathcal{C}_{\mathrm{ex/reg}}$ but in $\mathcal{C}$. Given a finite non-empty family $(F_i)_{i\in I}$ of objects of $\mathcal{C}_{\mathrm{ex/reg}}$, we have exact forks

$$\begin{tikzcd} YR_i \arrow[r, shift left] \arrow[r, shift right] & YA_i \arrow[r, "p"] & F_i \end{tikzcd}$$

and so, since $\mathrm{Shv}(\mathcal{C})$ is exact, an exact fork

$$\begin{tikzcd} \Pi_i YR_i \arrow[r, shift left] \arrow[r, shift right] & \Pi_i YA_i \arrow[r] & \Pi_i F_i \end{tikzcd} ;$$

but $\mathcal{C}$ has finite products, and so $\Pi_i YR_i$ and $\Pi_i YA_i$ are both in $\mathcal{C}$, whence $\Pi_i F_i$ is in $\mathcal{C}_{\mathrm{ex/reg}}$.

Step 2: If

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] & YB \arrow[d, "v"] \\ YA \arrow[r, "u"'] & G \end{tikzcd}$$

is a pullback in $\mathrm{Shv}(\mathcal{C})$, and G is in $\mathcal{C}_{\mathrm{ex/reg}}$, then P is in $\mathcal{C}$. For since G is in $\mathcal{C}_{\mathrm{ex/reg}}$, we have an exact fork

$$\begin{tikzcd} YR \arrow[r, shift left, "r"] \arrow[r, shift right, "s"'] & YC \arrow[r, "p"] & G \end{tikzcd}$$

in $\mathrm{Shv}(\mathcal{C})$, and so, by Lemma 2.1, a diagram

$$\begin{tikzcd} YA' \arrow[r, "u'"] \arrow[d, "a"'] & YC \arrow[d, "p"] & YB' \arrow[l, "v'"'] \arrow[d, "b"] \\ YA \arrow[r, "u"'] & G & YB \arrow[l, "v"] \end{tikzcd}$$

with a and b strong epis.

We can therefore form diagrams of pullbacks

$$\begin{tikzcd}
YD \arrow[r] \arrow[d] & YR_B \arrow[r] \arrow[d] & YB' \arrow[d, "v'"] \\
YR_A \arrow[r] \arrow[d] & YR \arrow[r, "s"] \arrow[d, "r"'] & YC \arrow[d, "p"] \\
YA' \arrow[r, "u'"'] & YC \arrow[r, "p"'] & G
\end{tikzcd}
\qquad
\begin{tikzcd}
YD \arrow[r, two heads, "a''"] \arrow[d, "b''"'] & P_B \arrow[r] \arrow[d, "b'"] & YB' \arrow[d, "b"] \\
P_A \arrow[r, two heads, "a'"'] \arrow[d] & P \arrow[r, "p_2"] \arrow[d, "p_1"'] & YB \arrow[d, "v"] \\
YA' \arrow[r, two heads, "a"'] & YA \arrow[r, "u"'] & G
\end{tikzcd}$$

in $\mathrm{Shv}(\mathcal{C})$, giving

$$YD \xrightarrow{b'a''} P \xrightarrow{\binom{p_1}{p_2}} YA \times YB\ ;$$

and now P lies in $\mathcal{C}$ by Lemma 3.1.

Step 3: $\mathcal{C}_{\mathrm{ex/reg}}$ is closed in $\mathrm{Shv}(\mathcal{C})$ under pullbacks. If

$$\begin{tikzcd}
P \arrow[r, "p_2"] \arrow[d, "p_1"'] & F' \arrow[d, "v"] \\
F \arrow[r, "u"'] & G
\end{tikzcd}$$

is a pullback in $\mathrm{Shv}(\mathcal{C})$ with F, F', and G in $\mathcal{C}_{\mathrm{ex/reg}}$, we have strong epis $q : YA \twoheadrightarrow F$ and $q' : YA' \twoheadrightarrow F'$, and so a diagram

$$\begin{tikzcd}
Q \arrow[r, two heads, "q_2"] \arrow[d, "q_2'"'] & P_1' \arrow[r] \arrow[d, "q_1'"] & YA' \arrow[d, "q'"] \\
P_1 \arrow[r, two heads, "q_1"'] \arrow[d] & P \arrow[r, "p_2"] \arrow[d, "p_1"'] & F' \arrow[d, "v"] \\
YA \arrow[r, two heads, "q"'] & F \arrow[r, "u"'] & G
\end{tikzcd}$$

of pullbacks in $\mathrm{Shv}(\mathcal{C})$. Thus we have

$$Q \xrightarrow{q_1'q_2} P \xrightarrow{\binom{p_1}{p_2}} F \times F'$$

with Q in $\mathcal{C}$ by Step 2, and $F \times F'$ in $\mathcal{C}_{\mathrm{ex/reg}}$ by Step 1. Now $q_1'q_2$ is strong epi, and so must be the coequalizer of its kernel pair. But $\binom{p_1}{p_2}$ is monic, and so the kernel pair of $q_1'q_2$ is the kernel pair of $\binom{p_1}{p_2}q_1'q_2$, which, by Step 2, must lie in $\mathcal{C}$. We conclude that P is the coequalizer of an equivalence relation in $\mathcal{C}$, and so lies in $\mathcal{C}_{\mathrm{ex/reg}}$.

Step 4: $\mathcal{C}_{\mathrm{ex/reg}}$ is closed in $\mathrm{Shv}(\mathcal{C})$ under coequalizers of equivalence relations. Let

$$\begin{tikzcd}
F \arrow[r, shift left, "r"] \arrow[r, shift right, "s"'] & G \arrow[r, "q"] & H
\end{tikzcd}$$

be an exact fork in $\mathrm{Shv}(\mathcal{C})$, with both F and G in $\mathcal{C}_{\mathrm{ex/reg}}$. Then we have a diagram

$$\begin{tikzcd}
P \arrow[r, two heads, "p_1'"] \arrow[d, "p_2'"'] & F_B \arrow[r, two heads, "s'"] \arrow[d, "p_2"'] & YB \arrow[d, "p"] \\
F_A \arrow[r, two heads, "p_1"] \arrow[d, "r'"'] & F \arrow[r, two heads, "s"] \arrow[d, "r"'] & G \arrow[d, "q"] \\
YB \arrow[r, two heads, "p"'] & G \arrow[r, two heads, "q"'] & H
\end{tikzcd}$$

of pullbacks in $\mathrm{Shv}(\mathcal{C})$, in which YB, F, and G are in $\mathcal{C}_{\mathrm{ex/reg}}$, and so F_A and F_B are in $\mathcal{C}_{\mathrm{ex/reg}}$, and so P is in $\mathcal{C}_{\mathrm{ex/reg}}$. Thus there is a strong epi $q' : YD \twoheadrightarrow P$; but also there is a mono $\left(\begin{smallmatrix} s'p_1' \\ r'p_2' \end{smallmatrix}\right) : P \to YB \times YB$, and so, by Lemma 3.1, we conclude that P is in $\mathcal{C}$. Finally

$$\begin{tikzcd}
P \arrow[r, shift left, "s'p_1'"] \arrow[r, shift right, "r'p_2'"'] & YB \arrow[r, two heads, "qp"] & H
\end{tikzcd}$$

is an exact fork in $\mathrm{Shv}(\mathcal{C})$ with P and YB in $\mathcal{C}$, and so we conclude that H is in $\mathcal{C}_{\mathrm{ex/reg}}$. ∎

We now prove the universal property of $\mathcal{C}_{\mathrm{ex/reg}}$; the proof is reminiscent of [13]:

3.3. Theorem. *If $\mathcal{D}$ is an exact category, then composition with $Z : \mathcal{C} \to \mathcal{C}_{\mathrm{ex/reg}}$ induces an equivalence of categories $\mathbf{Reg}(Z, \mathcal{D}) : \mathbf{Reg}(\mathcal{C}_{\mathrm{ex/reg}}, \mathcal{D}) \to \mathbf{Reg}(\mathcal{C}, \mathcal{D})$ to which the inverse equivalence is given by (pointwise) left Kan extension along Z; thus $\mathcal{C}_{\mathrm{ex/reg}}$ is the exact completion of $\mathcal{C}$.*

Proof. Let $S : \mathcal{C} \to \mathcal{D}$ be a regular functor; then the composite $YS : \mathcal{C} \to \mathrm{Shv}(\mathcal{D})$ of S with $Y : \mathcal{D} \to \mathrm{Shv}(\mathcal{D})$ is regular, and so by Lemma 2.2 induces a geometric morphism from $\mathrm{Shv}(\mathcal{D})$ to $\mathrm{Shv}(\mathcal{C})$, whose inverse image we call $S^* : \mathrm{Shv}(\mathcal{C}) \to \mathrm{Shv}(\mathcal{D})$.

Consider now the full subcategory $\mathcal{C}_1$ of $\mathrm{Shv}(\mathcal{C})$ containing those objects F for which S^*F lies in $\mathcal{D}$; it is the pseudopullback of $Y : \mathcal{D} \to \mathrm{Shv}(\mathcal{D})$ along S^*. Since S^* preserves finite limits and coequalizers of equivalence relations, and since $\mathcal{D}$ is closed in $\mathrm{Shv}(\mathcal{D})$ under finite limits and coequalizers of equivalence relations, it follows that $\mathcal{C}_1$ is closed in $\mathrm{Shv}(\mathcal{C})$ under finite limits and coequalizers of equivalence relations. Clearly $\mathcal{C}_1$ contains $\mathcal{C}$, and so, by Proposition 3.2, $\mathcal{C}_1$ must contain $\mathcal{C}_{\mathrm{ex/reg}}$; thus S^* restricts to a functor $\bar{S} : \mathcal{C}_{\mathrm{ex/reg}} \to \mathcal{D}$. Since S^* and W preserve finite limits and coequalizers of equivalence relations, and since $Y : \mathcal{D} \to \mathrm{Shv}(\mathcal{D})$ reflects them, we deduce that $\bar{S}$ is a regular functor. (It follows that $\mathbf{Reg}(Z, \mathcal{D}) : \mathbf{Reg}(\mathcal{C}_{\mathrm{ex/reg}}, \mathcal{D}) \to \mathbf{Reg}(\mathcal{C}, \mathcal{D})$ is essentially surjective on objects.)

By Lemma 2.2 we have $S^* \cong \mathrm{Lan}_Y(YS)$, and so $Y\bar{S} \cong S^*W \cong \mathrm{Lan}_Y(YS)W \cong (\mathrm{Lan}_W \mathrm{Lan}_Z(YS))W \cong \mathrm{Lan}_Z(YS)$. This says precisely that $\mathcal{C}_{\mathrm{ex/reg}}(Z, F) * YS \cong Y\bar{S}F$ for each object F of $\mathcal{C}_{\mathrm{ex/reg}}$, where $\mathcal{C}_{\mathrm{ex/reg}}(Z, F) * YS$ denotes the colimit of $YS : \mathcal{C} \to \mathrm{Shv}(\mathcal{D})$, weighted by the functor $\mathcal{C}_{\mathrm{ex/reg}}(Z, F) : \mathcal{C}^{op} \to \mathbf{Set}$ taking an object C of $\mathcal{C}$ to $\mathcal{C}_{\mathrm{ex/reg}}(ZC, F)$. But Y is fully faithful and so reflects colimits, giving $\mathcal{C}(Z, F) * S \cong \bar{S}F$; which is to say that $\mathrm{Lan}_Z(S) \cong \bar{S}$.

Thus we obtain a functor $\mathrm{Lan}_Z : \mathbf{Reg}(\mathcal{C}, \mathcal{D}) \to \mathbf{Reg}(\mathcal{C}_{\mathrm{ex/reg}}, \mathcal{D})$, with the composite $\mathbf{Reg}(Z, \mathcal{D}).\mathrm{Lan}_Z$ naturally isomorphic to the identity, since Z is fully faithful. It remains to show that the canonical map $\eta : \mathrm{Lan}_Z(TZ) \to T$ is invertible, for any regular functor $T : \mathcal{C}_{\mathrm{ex/reg}} \to \mathcal{D}$. Now $\mathrm{Lan}_Z(TZ)$ and T both preserve coequalizers of equivalence relations, and ηC is invertible for objects of $\mathcal{C}$, hence by Proposition 3.2 must be invertible for all objects of $\mathcal{C}_{\mathrm{ex/reg}}$. ■

4. Infinitary generalizations

In [11], the more general case of κ-regular and κ-exact categories was considered, for a regular cardinal κ. We recall that a κ-regular category is a regular category with κ-products, such that the strong epimorphisms are closed under κ-products; and that a κ-exact category is a κ-regular category that is exact. These classes of categories were introduced by Makkai [16] and further studied by Hu [9]; in the additive context, κ-regular categories were already considered by Grothendieck in [8].

We write κ-**Reg** for the 2-category of κ-regular categories, functors preserving κ-limits and strong epimorphisms, and natural transformations; and we write κ-**Ex** for the full sub-2-category of κ-**Reg** comprising the κ-exact categories. The value at a κ-regular category $\mathcal{C}$ of a left biadjoint to the inclusion of κ-**Ex** in κ-**Reg** is called the κ-exact completion of the κ-regular category $\mathcal{C}$.

If $\mathcal{C}$ is a κ-regular category, and $(F_i)_{i\in I}$ is a κ-small family of objects of $\mathcal{C}_{\mathrm{ex/reg}}$, then for each i there is an exact fork

$$YR_i \rightrightarrows YA_i \longrightarrow F_i$$

in $\mathrm{Shv}(\mathcal{C})$, and so, since $\mathrm{Shv}(\mathcal{C})$ is κ-exact, an exact fork

$$\Pi_i YR_i \rightrightarrows \Pi_i YA_i \longrightarrow \Pi_i F_i$$

in $\mathrm{Shv}(\mathcal{C})$; but since $\mathcal{C}$ has κ-products, it follows that $\Pi_i YR_i$ and $\Pi_i YA_i$ are in $\mathcal{C}$, and so that $\Pi_i F_i$ is in $\mathcal{C}_{\mathrm{ex/reg}}$. Thus $\mathcal{C}_{\mathrm{ex/reg}}$ is closed in $\mathrm{Shv}(\mathcal{C})$ under κ-products, and so is κ-exact, since $\mathrm{Shv}(\mathcal{C})$ is so. Thus $\mathcal{C}_{\mathrm{ex/reg}}$ is a κ-exact category, and $Z : \mathcal{C} \to \mathcal{C}_{\mathrm{ex/reg}}$ is a κ-regular functor.

Moreover, if $\mathcal{D}$ is a κ-exact category, and $S : \mathcal{C} \to \mathcal{D}$ a κ-exact functor, then it is clear by the construction of κ-products in $\mathcal{C}_{\mathrm{ex/reg}}$ that $\mathrm{Lan}_Z S : \mathcal{C}_{\mathrm{ex/reg}} \to \mathcal{D}$ preserves κ-products, and so we have:

4.1. Theorem. *If $\mathcal{C}$ is κ-regular, and $\mathcal{D}$ is κ-exact, then composition with $Z : \mathcal{C} \to \mathcal{C}_{\mathrm{ex/reg}}$ induces an equivalence of categories κ-$\mathbf{Reg}(Z, \mathcal{D}) : \kappa$-$\mathbf{Reg}(\mathcal{C}_{\mathrm{ex/reg}}, \mathcal{D}) \to \kappa$-$\mathbf{Reg}(\mathcal{C}, \mathcal{D})$ to which the inverse equivalence is given by (pointwise) left Kan extension along Z; thus $\mathcal{C}_{\mathrm{ex/reg}}$ is the κ-exact completion of $\mathcal{C}$.* ■

5. A concrete description of the exact completion

We can now apply Theorem 3.3 to derive a concrete description of the exact completion, using the language of relations in a regular category. For further details on the calculus of relations see [7, 4]; in fact all that we need is summarized in [3]. We write $R : A \not\rightarrow B$ for a relation from A to B, by which we mean an equivalence class of subobjects of $A \times B$. We further write $S \circ R : A \not\rightarrow C$ for the composite of relations $R : A \not\rightarrow B$ and $S : B \not\rightarrow C$, and $R^\circ : B \not\rightarrow A$ for the opposite relation of R. Recall that an equivalence relation is a relation R satsifying $R \circ R = R$, $R = R^\circ$, and $1 \leq R$.

We shall construct a category $\mathcal{E}$ equipped with a functor $\Phi : \mathcal{E} \to \mathcal{C}_{\mathrm{ex/reg}}$ which is fully faithful and essentially surjective on objects, and so an equivalence. We take the objects of $\mathcal{E}$ to be the equivalence relations in $\mathcal{C}$, and define Φ on objects to take such an equivalence relation to its coequalizer in $\mathcal{C}_{\mathrm{ex/reg}}$. Since every object of $\mathcal{C}_{\mathrm{ex/reg}}$ is the coequalizer of an equivalence relation in $\mathcal{C}$, the functor Φ, once defined, will be essentially surjective on objects.

Given an equivalence relation $R : A \not\rightarrow A$, recall that the coequalizer $r : A \to A/R$ of R satisfies $r^\circ \circ r = R$ and $r \circ r^\circ = 1$, and so $r \dashv r^\circ$. If $S : B \not\rightarrow B$ is another equivalence relation, and $s : B \to B/S$ is its coequalizer, an arrow in $\mathcal{C}_{\mathrm{ex/reg}}$ from A/R to B/S determines a relation from A to B as follows. We form the pullback

$$\begin{tikzcd}
D \arrow[rr, "f"] \arrow[d, "g"'] & & B \arrow[d, "s"] \\
A \arrow[r, two heads, "r"'] & A/R \arrow[r, "\alpha"'] & B/S
\end{tikzcd}$$

in $\mathcal{C}_{\mathrm{ex/reg}}$; of course this is a relation in $\mathcal{C}_{\mathrm{ex/reg}}$, but since D is a subobject of $A \times B$, it follows by Lemma 3.1 that D is in fact in $\mathcal{C}$, and so f and g are the legs of a relation from A to B in $\mathcal{C}$. Since g and r are strong epimorphisms, there is clearly at most one $\alpha : A/R \to B/S$ giving rise in this way to a particular relation $U : A \not\rightarrow B$, but not every U does so arise.

To say that a relation $U : A \not\rightarrow B$ arises, as in the previous paragraph, from $\alpha : A/R \to B/S$, is precisely to say that $U = s^\circ \circ \alpha \circ r$. Since $s \circ s^\circ = 1$ and $r \circ r^\circ = 1$, it then follows that $\alpha = s \circ s^\circ \circ \alpha \circ r \circ r^\circ = s \circ U \circ r^\circ$. Thus U arises from a (necessarily unique) α if and only if $U = s^\circ \circ s \circ U \circ r^\circ \circ r$ and $s \circ U \circ r^\circ$ is a map (rather than a general relation).

The first condition says that $U = S \circ U \circ R$, which, since S and R are idempotents, is equivalent to the two conditions:

$$U = S \circ U$$
$$U = U \circ R.$$

It remains to express the condition that $\alpha = s \circ U \circ r^\circ$ be a map; that is, that $\alpha \circ \alpha^\circ \leq 1$ and $1 \leq \alpha^\circ \circ \alpha$. Now $\alpha \circ \alpha^\circ = (s \circ U \circ r^\circ) \circ (s \circ U \circ r^\circ)^\circ = s \circ U \circ r^\circ \circ r \circ U^\circ \circ s^\circ =$

$s \circ U \circ R \circ U^\circ \circ s^\circ = s \circ U \circ U^\circ \circ s^\circ$; and, since $s \dashv s^\circ$, the inequality $s \circ U \circ U^\circ \circ s^\circ \leq 1$ is equivalent to $U \circ U^\circ \circ s^\circ \leq s^\circ$ and so to $U \circ U^\circ \leq s^\circ \circ s = S$; whence $\alpha \circ \alpha^\circ \leq 1$ is equivalent to

$$U \circ U^\circ \leq S.$$

On the other hand $\alpha^\circ \circ \alpha = (s \circ U \circ r^\circ)^\circ \circ (s \circ U \circ r^\circ) = r \circ U^\circ \circ s^\circ \circ s \circ U \circ r^\circ = r \circ U^\circ \circ S \circ U \circ r^\circ = r \circ U^\circ \circ U \circ r^\circ$; and $1 \leq r \circ U^\circ \circ U \circ r^\circ$ implies that $R = r^\circ \circ r \leq r^\circ \circ r \circ U^\circ \circ U \circ r^\circ \circ r = R^\circ \circ U^\circ \circ U \circ R = (U \circ R)^\circ \circ (U \circ R) = U^\circ \circ U$, while $R \leq U^\circ \circ U$ implies that $1 = r \circ r^\circ \circ r \circ r^\circ = r \circ R \circ r^\circ \leq r \circ U^\circ \circ U \circ r^\circ$; and so $1 \leq \alpha^\circ \alpha$ is equivalent to

$$R \leq U^\circ \circ U.$$

We now define an arrow in $\mathcal{E}$ from R to S to be a relation $U : A \not\rightarrow B$ in $\mathcal{C}$ satisfying the three conditions $S \circ U \circ R = U$, $U \circ U^\circ \leq S$, and $R \leq U^\circ \circ U$; and define $\Phi : \mathcal{E}(R, S) \to \mathcal{C}_{\mathrm{ex/reg}}(A/R, B/S)$ be the bijection taking U to $s \circ U \circ r^\circ$.

Composition in $\mathcal{C}_{\mathrm{ex/reg}}$ now induces a composition in $\mathcal{E}$ making $\mathcal{E}$ into a category and Φ an equivalence of categories from $\mathcal{E}$ to $\mathcal{C}_{\mathrm{ex/reg}}$; to complete the description of $\mathcal{E}$ it remains only to calculate the induced composition and identities.

Suppose then that $T : C \not\rightarrow C$ is another equivalence relation in $\mathcal{C}$, and $t : C \to C/T$ is its coequalizer. Let $\beta : B/S \to C/T$ be an arrow in $\mathcal{C}_{\mathrm{ex/reg}}$, and $V = t^\circ \circ \beta \circ s$ the corresponding relation. Then $V \circ U = t^\circ \circ \beta \circ s \circ s^\circ \circ \alpha \circ r = t^\circ \circ \beta \circ \alpha \circ r$, which is the relation corresponding to $\beta\alpha$; thus composition in $\mathcal{E}$ is just composition of relations. On the other hand, the identity arrow $1 : A/R \to A/R$ corresponds to the relation $R : A \not\rightarrow A$, which is therefore the identity arrow in $\mathcal{E}$ at the object R. Thus $\mathcal{E}$ is now seen to be precisely the exact completion as described in [3].

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