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CONVERGENCE IN EXPONENTIABLE SPACES

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ABSTRACT. Exponentiable spaces are characterized in terms of convergence. More precisely, we prove that a relation $R : \mathcal{U}X \rightharpoonup X$ between ultrafilters and elements of a set X is the convergence relation for a quasi-locally-compact (that is, exponentiable) topology on X if and only if the following conditions are satisfied:

1. $\mathrm{id} \subseteq R \circ \eta$
2. $R \circ \mathcal{U}R = R \circ \mu$

where $\eta : X \to \mathcal{U}X$ and $\mu : \mathcal{U}(\mathcal{U}X) \to \mathcal{U}X$ are the unit and the multiplication of the ultrafilter monad, and $\mathcal{U} : \mathbf{Rel} \to \mathbf{Rel}$ extends the ultrafilter functor $\mathcal{U} : \mathbf{Set} \to \mathbf{Set}$ to the category of sets and relations. $(\mathcal{U}, \eta, \mu)$ fails to be a monad on $\mathbf{Rel}$ only because η is not a strict natural transformation. So, exponentiable spaces are the lax (with respect to the unit law) algebras for a lax monad on $\mathbf{Rel}$. Strict algebras are exponentiable and T_1 spaces.

1. Introduction

In [4] it was implicitly proved that a topological space is exponentiable if and only if its lattice of open sets is a continuous lattice [6, 8], so fixing an important topological property. This property, often called quasi-local-compactness though other names have been used, is a slight generalization of local compactness (meaning that every point has a basis of compact neighborhoods) and for sober spaces coincides with it.

In the present paper we give a further characterization of quasi-local-compactness in terms of ultrafilter convergence. The key idea is the following. Consider the typical space which is not locally compact, $\mathbf{Q} \subseteq \mathbf{R}$, and a sequence x_n in $\mathbf{R} \setminus \mathbf{Q}$ converging to $x \in \mathbf{Q}$. Next, for every n, consider a sequence x^n_m in $\mathbf{Q}$ converging to x_n. So we have a sequence of sequences in $\mathbf{Q}$ that "globally converges" to x, but such that there exists no sequence converging to x and to which it "converges" (this sequence ought to be x_n, which is not in $\mathbf{Q}$). Then we may try to capture local compactness by requiring, for a topological space X, that such a sequence always exists. Using ultrafilters in place of sequences, this condition may be restated as:

$$\forall\, \Xi\, x \quad \mu\Xi \,\mathrm{Lim}\, x \;\Rightarrow\; \exists\, \xi\, (\, \Xi\, \mathcal{U}\mathrm{Lim}\, \xi\, ,\; \xi \,\mathrm{Lim}\, x\,) \tag{1}$$

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where $\xi \in \mathcal{U}X$ is an ultrafilter on X; $\Xi \in \mathcal{U}(\mathcal{U}X)$ is an ultrafilter on the set of ultrafilters; $\mathrm{Lim} : \mathcal{U}X \rightharpoonup X$ is the convergence relation in X; $\mu : \mathcal{U}(\mathcal{U}X) \to \mathcal{U}X$, the multiplication of the ultrafilter monad, is used to formalize "global convergence" (in terms of nets, μx_m^n is a net which converges to x if, for any neighborhood A of x, every net x_m^n with n sufficiently big is eventually in A [9]); and $\mathcal{U}\mathrm{Lim} : \mathcal{U}(\mathcal{U}X) \rightharpoonup \mathcal{U}X$, defined in section 4, formalizes "convergence" of a sequence of sequences to a sequence (in terms of nets, x_m^n "converges" to y_k if, for any k_0, every net x_m^n with n sufficiently big converges to a point y_k with $k \geq k_0$).

In section 8 we prove that condition (1) is exactly quasi-local-compactness (theorem 8.2).

Now we give an outline of the contents of the other sections. In section 2 we fix notations and recall the definition of the filter and the ultrafilter monads [3, 10, 13]. In section 3 we recall the Extension-Exclusion Lemma (EEL), one form of the Prime Ideal Theorem [12] (we give directly the version for prime filters in $\mathcal{P}X$, that is ultrafilters). This lemma is used in a systematic way throughout the paper. In section 4 another important tool is presented (see the discussion above): we define a functor $\mathcal{U} : \mathbf{Rel} \to \mathbf{Rel}$ that extends the usual ultrafilter functor $\mathcal{U} : \mathbf{Set} \to \mathbf{Set}$.

Next, after briefly recalling some equivalent definitions of pretopological space in section 5, we devote section 6 to find a characterization of convergence in a topological space (theorems 6.9 and 6.10). Namely, we show that a relation $\mathrm{Lim} : \mathcal{U}X \rightharpoonup X$ is the ultrafilter convergence relation for a topology on X if and only if the reverse of implication (1)

$$\forall\, \Xi\, \xi\, x \quad \Xi\, \mathcal{U}\mathrm{Lim}\, \xi\,,\ \xi\, \mathrm{Lim}\, x \ \Rightarrow\ \mu\Xi\, \mathrm{Lim}\, x \tag{2}$$

holds, together with convergence of "constant sequences" (principal ultrafilters). Intuitively, condition (2) states that if a sequence of sequences "converges" to a sequence that converges to a point, then it "globally converges" itself to that point (to see why this holds in a topological space, just consider an open neighborhood of the point). This theorem (that has a predecessor in the characterization of net convergence in a topological space [9]) was first proved in [1], although our proof is entirely independent from that one. Furthermore, there is an important difference in the definition of $\mathcal{U} : \mathbf{Rel} \to \mathbf{Rel}$. Even if the two definitions can be proved to be equivalent, ours seems to be more handy. In particular, it allows us to prove strict functoriality (proposition 4.3) and its direct generalization to filters gives a parallel characterization of filter convergence in a topological space (theorem 6.10), so avoiding the Axiom of Choice.

The proof of the main theorem in section 8 is preceded by a brief account of quasi-locally-compact spaces in section 7.

In the conclusions we briefly discuss some possible developments of this work, especially regarding connections with the theory of monads and continuous lattices.

Finally, a word on the proofs. They have often the form of a chain of equivalent formulae, written one above the other, the formulae themselves being manipulated either by application of the (EEL) or using pure logic (or, if you prefer, **2**-enriched category theory).

2. Basic definitions

If X is a set, we write $\mathcal{F}X$ for the set of filters on X, and $\mathcal{U}X$ for the subset of ultrafilters. We use Greek letters, φ, ψ and χ, to denote filters; ξ, υ and ζ will usually denote ultrafilters.

Given a function $f : X \to Y$, one defines

$$\mathcal{F}f : \mathcal{F}X \to \mathcal{F}Y \qquad \text{and} \qquad \mathcal{U}f : \mathcal{U}X \to \mathcal{U}Y$$

by

$$\varphi \mapsto \{B \subseteq Y : f^{-1}B \in \varphi\}$$

so obtaining two functors $\mathcal{F} : \mathbf{Set} \to \mathbf{Set}$ and $\mathcal{U} : \mathbf{Set} \to \mathbf{Set}$.

One can also define

$$\mathrm{e} : \mathcal{P}X \to \mathcal{P}(\mathcal{F}X) \qquad \text{and} \qquad \mathrm{e} : \mathcal{P}X \to \mathcal{P}(\mathcal{U}X)$$

by

$$\varphi \in \mathrm{e}A \quad \Leftrightarrow \quad A \in \varphi$$

No confusion should arise from using the same symbol for two different maps. The choice of the name is due to the fact that it may be seen as an evaluation map. Note that the second map is a Boolean algebra homomorphism.

Finally, one defines

$$\eta : X \to \mathcal{F}X \qquad \text{and} \qquad \eta : X \to \mathcal{U}X$$

by

$$x \mapsto \{A \subseteq X : x \in A\}$$

and

$$\mu : \mathcal{F}(\mathcal{F}X) \to \mathcal{F}X \qquad \text{and} \qquad \mu : \mathcal{U}(\mathcal{U}X) \to \mathcal{U}X$$

by

$$\Phi \mapsto \{A \subseteq X : \mathrm{e}A \in \Phi\}$$

or, equivalently, by

$$\Phi \mapsto \bigcup_{U \in \Phi} \bigcap_{\varphi \in U} \varphi$$

Recall that η and μ are (the components at X of) the unit and the multiplication for the filter monad and the ultrafilter monad [3, 10, 13]. Again, the context should clear up the ambiguity due to conflicting terminology.

3. The Extension-Exclusion Lemma

The main tool to be used in the present work for proving theorems is the well-known Extension-Exclusion Lemma:

3.1. LEMMA. [EEL] *Let $\mathcal{A}$ and $\mathcal{B}$ be two families of subsets of X such that $\mathcal{A}$ is filtered and $\mathcal{B}$ is directed. Then there exists an ultrafilter ξ that extends $\mathcal{A}$ (that is, $\mathcal{A} \subseteq \xi$) and excludes $\mathcal{B}$ (that is, $\xi \cap \mathcal{B} = \emptyset$) if and only if*

$$\forall\, A\, B \quad A \in \mathcal{A}\,,\; B \in \mathcal{B} \;\Rightarrow\; A \not\subseteq B$$

The (EEL) actually holds in an arbitrary distributive lattice and in that form it is equivalent to the Prime Ideal Theorem [12]. While the proof of the (EEL) depends on the Axiom of Choice, the following formally similar proposition has a trivial proof:

3.2. LEMMA. [eel] *Let $\mathcal{A}$ and $\mathcal{B}$ be two families of subsets of X and let $\mathcal{A}$ be filtered. Then there exists a filter that extends $\mathcal{A}$ and excludes $\mathcal{B}$ if and only if*

$$\forall\, A\, B \quad A \in \mathcal{A}\,,\; B \in \mathcal{B} \;\Rightarrow\; A \not\subseteq B$$

This lemma will allow us to save space and time in section 6, where analogous results will be proved both for filters and ultrafilters. Indeed, many proofs worked out for ultrafilters are also valid for filters, provided that the (eel) is used in place of the (EEL).

We shall also use the following form of the (EEL), which states that ultrafilters are dense in filters:

3.3. PROPOSITION. *For any two filters φ, $\psi \in \mathcal{F}X$ the following are equivalent:*

1. $\varphi \subseteq \psi$
2. $\forall\, \xi \in \mathcal{U}X \quad \psi \subseteq \xi \;\Rightarrow\; \varphi \subseteq \xi$

The proof can be given as a chain of equivalent formulae:

$$\forall\, \xi \quad \psi \subseteq \xi \;\Rightarrow\; \varphi \subseteq \xi$$

$$\forall\, \xi \quad \psi \subseteq \xi \;\Rightarrow\; \forall\, A\, (\, A \in \varphi \;\Rightarrow\; A \in \xi\,)$$

$$\forall\, A\, \xi \quad \psi \subseteq \xi\,,\; A \in \varphi \;\Rightarrow\; A \in \xi$$

$$\forall\, A\, \xi \quad \psi \subseteq \xi\,,\; A \notin \xi \;\Rightarrow\; A \notin \varphi$$

$$\forall\, A \quad \exists\, \xi\, (\, \psi \subseteq \xi\,,\; A \notin \xi\,) \;\Rightarrow\; A \notin \varphi$$

$$\forall\, A \quad \forall\, B\, (\, B \in \psi \;\Rightarrow\; B \not\subseteq A\,) \;\Rightarrow\; A \notin \varphi \qquad \text{(EEL)}$$

$$\forall\, A \quad A \notin \psi \;\Rightarrow\; A \notin \varphi$$

$$\varphi \subseteq \psi$$

3.4. COROLLARY. *Every filter is the intersection of the ultrafilters that extend it.*

4. Filters and relations

Let **Rel** be the category of sets and relations. In the following we consider the set $\mathcal{F}X$ of filters on X ordered "geometrically" by

$$\varphi \leq \psi \quad \Leftrightarrow \quad \psi \subseteq \varphi$$

4.1. DEFINITION. A relation $R : X \rightharpoonup Y$ between two ordered sets is *compatible* if it is a **2**-bimodule, that is if $\forall\, x,\ x' \in X \quad \forall\, y,\ y' \in Y$

$$x' \leq x\ ,\ x R\, y\ ,\ y \leq y' \ \Rightarrow\ x' R\, y'$$

The composition of two compatible relations is also compatible, while the order relation on X is a compatible relation which acts as the identity. We denote by $\mathbf{Rel}^*$ the category of ordered sets and compatible relations.

4.2. DEFINITION. If $R : X \rightharpoonup Y$ is a relation, $\mathcal{U}R : \mathcal{U}X \rightharpoonup \mathcal{U}Y$ is the relation defined by

$$\xi\, \mathcal{U}R\, v \quad \Leftrightarrow \quad \forall\, B\ (\ B \in v \ \Rightarrow\ R^{op}B \in \xi\ )$$

where $R^{op}B = \{\, x \in X : \exists\, y\ (\ y \in B\ ,\ xR\,y\ )\,\}$.
In the same way we define a compatible relation $\mathcal{F}R : \mathcal{F}X \rightharpoonup \mathcal{F}Y$.

4.3. PROPOSITION. *The correspondence $R \mapsto \mathcal{U}R$ defines a functor $\mathcal{U} : \mathbf{Rel} \to \mathbf{Rel}$ which extends the ultrafilter functor $\mathcal{U} : \mathbf{Set} \to \mathbf{Set}$.*

Since we are restricting ourselves to ultrafilters, it is easy to see that $\mathcal{U}$ extends the ultrafilter functor of section 2 and preserves identities. As for composition:

$\xi\ (\mathcal{U}S \circ \mathcal{U}R)\ \zeta$

$\exists\, v\ (\ \xi\, \mathcal{U}R\, v\ ,\ v\, \mathcal{U}S\ \zeta\ )$

$\exists\, v\ (\ \forall\, B\ (\ B \in v \ \Rightarrow\ R^{op}B \in \xi\ )\ ,\ \forall\, C\ (\ C \in \zeta \ \Rightarrow\ S^{op}C \in v\ )\ )$

$\exists\, v\ (\ \forall\, B\ (\ R^{op}B \notin \xi \ \Rightarrow\ B \notin v\ )\ ,\ \forall\, C\ (\ C \in \zeta \ \Rightarrow\ S^{op}C \in v\ )\ )$

$\forall\, B\ C \quad R^{op}B \notin \xi\ ,\ C \in \zeta \ \Rightarrow\ S^{op}C \not\subseteq B \qquad \text{(EEL)}$

$\forall\, C \quad C \in \zeta \ \Rightarrow\ \forall\, B\ (\ R^{op}B \notin \xi \ \Rightarrow\ S^{op}C \not\subseteq B\ )$

$$\forall\, C \quad C \in \zeta \;\Rightarrow\; \forall\, B\ (\ S^{op}C \subseteq B \;\Rightarrow\; R^{op}B \in \xi\ )$$

$$\forall\, C \quad C \in \zeta \;\Rightarrow\; R^{op}(S^{op}C) \in \xi$$

$$\forall\, C \quad C \in \zeta \;\Rightarrow\; (S \circ R)^{op}C \in \xi$$

$$\xi\, \mathcal{U}(S \circ R)\, \zeta$$

The application of the (EEL) in the above proof is justified by the following facts, which will be used also in subsequent applications:

1. If $F : \mathcal{P}X \to \mathcal{P}Y$ is a monotone map and φ is a filter on X (respectively, an ideal) then $\{FA : A \in \varphi\}$ is filtered (respectively, directed).
2. If ξ is an ultrafilter on X, then $\{A \in \mathcal{P}X : A \notin \xi\}$ is an ideal.
3. If $F : \mathcal{P}X \to \mathcal{P}Y$ preserves finite intersections (respectively, finite unions) and φ is a filter on Y (respectively, an ideal) then $\{A : FA \in \varphi\}$ is a filter on X (respectively, an ideal).

Using the (eel) in place of the (EEL) in the above proof, we easily obtain

4.4. Proposition. *The correspondence $R \mapsto \mathcal{F}R$ defines a functor $\mathcal{F} : \mathbf{Rel} \to \mathbf{Rel}^*$.*

5. Pretopological spaces

Pretopological spaces are obtained by weakening the axioms for topology. They can be presented in several ways.

5.1. Definition. [pretopology via interior operator] A pretopological space is a set X with a map $\mathrm{Int} : \mathcal{P}X \to \mathcal{P}X$ such that $\forall\, A,\ B \subseteq X$

1. $A \subseteq B \;\Rightarrow\; \mathrm{Int}\, A \subseteq \mathrm{Int}\, B$
2. $\mathrm{Int}\, X = X$
3. $\mathrm{Int}(A \cap B) = \mathrm{Int}\, A \cap \mathrm{Int}\, B$
4. $\mathrm{Int}\, A \subseteq A$

5.2. Definition. [pretopology via neighborhoods] A pretopological space is a set X with a map $\mathrm{N} : X \to \mathcal{F}X$ such that $\forall\, x \quad \mathrm{N}x \subseteq \eta x$.

5.3. Definition. [pretopology via filter convergence] A pretopological space is a set X with a relation $\lim : \mathcal{F}X \rightharpoonup X$ such that

1. it is a compatible relation
2. $\forall\, x \quad \bigcap_{\psi \lim x} \psi \ \lim\ x$
3. $\forall\, x \quad \eta x\ \lim\ x$

The first two conditions condense to:

$$\forall\, \varphi\, x \qquad \bigcap_{\psi \lim x} \psi\ \subseteq \varphi\ \Rightarrow\ \varphi \lim x$$

5.4. Definition. [pretopology via ultrafilter convergence] A pretopological space is a set X with a relation $\mathrm{Lim} : \mathcal{U}X \rightharpoonup X$ such that

1. $\forall\, \xi\, x \quad \bigcap_{\upsilon \,\mathrm{Lim}\, x} \upsilon\ \subseteq \xi\ \Rightarrow\ \xi \,\mathrm{Lim}\, x$
2. $\forall\, x \quad \eta x\ \mathrm{Lim}\, x$

It is easy to see that definitions 5.1 and 5.2 are equivalent, a bijective correspondence being given by transposition:

$$A \in \mathrm{N}x \quad \Leftrightarrow \quad x \in \mathrm{Int}\, A$$

while

$$\varphi \lim x \quad \Leftrightarrow \quad \mathrm{N}x \subseteq \varphi \qquad \text{and} \qquad \mathrm{N}x = \bigcap_{\psi \lim x} \psi$$

give an equivalence between definitions 5.2 and 5.3, and similarly

$$\xi \,\mathrm{Lim}\, x \quad \Leftrightarrow \quad \mathrm{N}x \subseteq \xi \qquad \text{and} \qquad \mathrm{N}x = \bigcap_{\upsilon \,\mathrm{Lim}\, x} \upsilon$$

give an equivalence between definitions 5.2 and 5.4, as a consequence of corollary 3.3.

The following propositions will be used in the next section (we omit the simple proofs):

5.5. Proposition. *The first condition of definition 5.4 is equivalent to*

$$\forall\, \xi\, x \quad \forall\, A\, (\, \forall\, \upsilon\, (\, \upsilon \,\mathrm{Lim}\, x\ \Rightarrow\ A \in \upsilon\ )\ \Rightarrow\ A \in \xi\ )\ \Rightarrow\ \xi \,\mathrm{Lim}\, x$$

5.6. Proposition. *The first two conditions of definition 5.3 are equivalent to*

$$\forall\, \varphi\, x \quad \forall\, A\, (\, \forall\, \psi\, (\, \psi \lim x\ \Rightarrow\ A \in \psi\ )\ \Rightarrow\ A \in \varphi\ )\ \Rightarrow\ \varphi \lim x$$

Before going on, it is useful to extend from points to filters the notion of neighborhood filter in a pretopological space:

5.7. Definition. If φ is a filter in a pretopological space, the filter $\mathrm{N}\varphi$ is defined by

$$A \in \mathrm{N}\varphi \quad \Leftrightarrow \quad \mathrm{Int}\, A \in \varphi$$

Coherently, we write $\varphi \lim \psi$ for $\mathrm{N}\psi \subseteq \varphi$, and similarly, if ξ is an ultrafilter we write $\xi \,\mathrm{Lim}\, \psi$ for $\mathrm{N}\psi \subseteq \xi$.

6. Convergence and topological spaces

Now we want to find conditions on a relation $R : \mathcal{U}X \rightharpoonup X$ (or $R : \mathcal{F}X \rightharpoonup X$) that force it to be the convergence relation for a topology on X. Recall that a topological space may be seen as a pretopological space X (defined via interior) such that

$$\forall\, A \subseteq X \quad \mathrm{Int}\, A \subseteq \mathrm{Int}(\mathrm{Int}\, A)$$

(that is, as a left exact comonad on $\mathcal{P}X$).

6.1. Proposition. *A pretopological space* $\mathrm{Lim} : \mathcal{U}X \rightharpoonup X$ *is a topological space if and only if*

$$\forall\, \xi\ \upsilon\ x \quad \xi \,\mathrm{Lim}\, \upsilon\, ,\ \upsilon \,\mathrm{Lim}\, x \ \Rightarrow\ \xi \,\mathrm{Lim}\, x$$

Proof:

$$\forall\, A \quad \mathrm{Int}\, A \subseteq \mathrm{Int}(\mathrm{Int}\, A)$$

$$\forall\, A\ x \quad x \in \mathrm{Int}\, A \ \Rightarrow\ x \in \mathrm{Int}(\mathrm{Int}\, A)$$

$$\forall\, A\ x \quad A \in \mathrm{N}x \ \Rightarrow\ \mathrm{Int}\, A \in \mathrm{N}x$$

$$\forall\, A\ x \quad A \in \mathrm{N}x \ \Rightarrow\ \forall\, \upsilon\ (\ \mathrm{N}x \subseteq \upsilon \ \Rightarrow\ \mathrm{Int}\, A \in \upsilon\ ) \qquad \text{(EEL)}$$

$$\forall\, A\ x \quad A \in \mathrm{N}x \ \Rightarrow\ \forall\, \upsilon\ (\ \upsilon \,\mathrm{Lim}\, x \ \Rightarrow\ A \in \mathrm{N}\upsilon\ )$$

$$\forall\, A\ \upsilon\ x \quad A \in \mathrm{N}x\, ,\ \upsilon \,\mathrm{Lim}\, x \ \Rightarrow\ A \in \mathrm{N}\upsilon$$

$$\forall\, \upsilon\ x \quad \upsilon \,\mathrm{Lim}\, x \ \Rightarrow\ \forall\, A\ (\ A \in \mathrm{N}x \ \Rightarrow\ A \in \mathrm{N}\upsilon\ )$$

$$\forall\, \upsilon\ x \quad \upsilon \,\mathrm{Lim}\, x \ \Rightarrow\ \mathrm{N}x \subseteq \mathrm{N}\upsilon$$

$$\forall\, \upsilon\ x \quad \upsilon \,\mathrm{Lim}\, x \ \Rightarrow\ \forall\, \xi\ (\ \mathrm{N}\upsilon \subseteq \xi \ \Rightarrow\ \mathrm{N}x \subseteq \xi\ ) \qquad \text{(EEL)}$$

$$\forall\, \upsilon\ x \quad \upsilon \,\mathrm{Lim}\, x \ \Rightarrow\ \forall\, \xi\ (\ \xi \,\mathrm{Lim}\, \upsilon \ \Rightarrow\ \xi \,\mathrm{Lim}\, x\ )$$

$$\forall\, \xi\ \upsilon\ x \quad \xi \,\mathrm{Lim}\, \upsilon\, ,\ \upsilon \,\mathrm{Lim}\, x \ \Rightarrow\ \xi \,\mathrm{Lim}\, x$$

The same proof clearly works for filters in place of ultrafilters:

6.2. PROPOSITION. *A pretopological space* $\lim : \mathcal{F}X \rightharpoonup X$ *is a topological space if and only if*

$$\forall\, \varphi\, \psi\, x \quad \varphi \lim \psi\ ,\ \psi \lim x \ \Rightarrow\ \varphi \lim x$$

The next step is to reduce convergence between filters to convergence between filters and points:

6.3. LEMMA. *In a pretopological space* $\xi\,\mathrm{Lim}\,\upsilon$ *if and only if*

$$\forall\, A\, B \quad \mathrm{Lim}^{op} B \subseteq \mathrm{e}A\ ,\ B \in \upsilon \ \Rightarrow\ A \in \xi$$

Proof:

$\xi\,\mathrm{Lim}\,\upsilon$

$\mathrm{N}\upsilon \subseteq \xi$

$\forall\, A \quad A \in \mathrm{N}\upsilon \ \Rightarrow\ A \in \xi$

$\forall\, A \quad \mathrm{Int}\, A \in \upsilon \ \Rightarrow\ A \in \xi$

$\forall\, A \quad \exists\, B\ (\ B \subseteq \mathrm{Int}\, A\ ,\ B \in \upsilon\ ) \ \Rightarrow\ A \in \xi$

$\forall\, A\, B \quad B \subseteq \mathrm{Int}\, A\ ,\ B \in \upsilon \ \Rightarrow\ A \in \xi$

$\forall\, A\, B \quad \forall\, z\ (\ z \in B \ \Rightarrow\ z \in \mathrm{Int}\, A\ )\ ,\ B \in \upsilon \ \Rightarrow\ A \in \xi$

$\forall\, A\, B \quad \forall\, z\ (\ z \in B \ \Rightarrow\ A \in \mathrm{N}z\ )\ ,\ B \in \upsilon \ \Rightarrow\ A \in \xi$

$\forall\, A\, B \quad \forall\, z\ (\ z \in B \ \Rightarrow\ \forall\, \zeta\ (\ \mathrm{N}z \subseteq \zeta \ \Rightarrow\ A \in \zeta\ )\ )\ ,\ B \in \upsilon \ \Rightarrow\ A \in \xi$
(EEL)

$\forall\, A\, B \quad \forall\, \zeta\, z\ (\ z \in B\ ,\ \zeta\,\mathrm{Lim}\, z \ \Rightarrow\ A \in \zeta\ )\ ,\ B \in \upsilon \ \Rightarrow\ A \in \xi$

$\forall\, A\, B \quad \forall\, \zeta\ (\ \exists\, z\ (\ z \in B\ ,\ \zeta\,\mathrm{Lim}\, z\ ) \ \Rightarrow\ A \in \zeta\ )\ ,\ B \in \upsilon \ \Rightarrow\ A \in \xi$

$\forall\, A\, B \quad \forall\, \zeta\ (\ \zeta \in \mathrm{Lim}^{op} B \ \Rightarrow\ \zeta \in \mathrm{e}A\ )\ ,\ B \in \upsilon \ \Rightarrow\ A \in \xi$

$\forall\, A\, B \quad \mathrm{Lim}^{op} B \subseteq \mathrm{e}A\ ,\ B \in \upsilon \ \Rightarrow\ A \in \xi$

Once again, we have also proved

6.4. LEMMA. *In a pretopological space* $\varphi \lim \psi$ *if and only if*

$$\forall\, A\, B \quad \lim^{op} B \subseteq \mathrm{e}A\ ,\ B \in \psi \ \Rightarrow\ A \in \varphi$$

Combining lemma 6.3 and proposition 6.1 we obtain

6.5. Corollary. *A pretopological space* $\mathrm{Lim} : \mathcal{U}X \rightharpoonup X$ *is a topological space if and only if* $\forall\, \xi\, \upsilon\, x$

$$\forall\, A\, B\,(\,\mathrm{Lim}^{op}\, B \subseteq \mathrm{e}A\,,\ B \in \upsilon\ \Rightarrow\ A \in \xi\,)\,,\ \upsilon\,\mathrm{Lim}\,x\ \Rightarrow\ \xi\,\mathrm{Lim}\,x$$

Similarly, from lemma 6.4 and proposition 6.2 we get

6.6. Corollary. *A pretopological space* $\lim : \mathcal{F}X \rightharpoonup X$ *is a topological space if and only if* $\forall\, \varphi\, \psi\, x$

$$\forall\, A\, B\,(\,\lim^{op}\, B \subseteq \mathrm{e}A\,,\ B \in \psi\ \Rightarrow\ A \in \varphi\,)\,,\ \psi\lim x\ \Rightarrow\ \varphi\lim x$$

We are now in a position to prove

6.7. Proposition. *A relation* $R : \mathcal{U}X \rightharpoonup X$ *is the ultrafilter convergence relation for a topology on* X *if and only if*

1. $\forall\, x \quad \eta x\, R\, x$
2. $\forall\, \xi\, \upsilon\, x \quad \forall\, A\, B\,(\,R^{op}B \subseteq \mathrm{e}A\,,\ B \in \upsilon\ \Rightarrow\ A \in \xi\,)\,,\ \upsilon R\, x\ \Rightarrow\ \xi R\, x$

In one direction, this is an immediate consequence of corollary 6.5. In the other, substituting υ with ηx in the second condition and using the first one, we get

$$\forall\, \xi\, x \quad \forall\, A\, B\,(\,R^{op}B \subseteq \mathrm{e}A\,,\ x \in B\ \Rightarrow\ A \in \xi\,)\ \Rightarrow\ \xi R\, x$$

$$\forall\, \xi\, x \quad \forall\, A\,(\,\exists\, B\,(\,R^{op}B \subseteq \mathrm{e}A\,,\ x \in B\,)\ \Rightarrow\ A \in \xi\,)\ \Rightarrow\ \xi R\, x$$

$$\forall\, \xi\, x \quad \forall\, A\,(\,R^{op}\{x\} \subseteq \mathrm{e}A\ \Rightarrow\ A \in \xi\,)\ \Rightarrow\ \xi R\, x$$

$$\forall\, \xi\, x \quad \forall\, A\,(\,\forall\, \upsilon\,(\,\upsilon R\, x\ \Rightarrow\ A \in \upsilon\,)\ \Rightarrow\ A \in \xi\,)\ \Rightarrow\ \xi R\, x$$

which is the condition of proposition 5.5. Thus R is the convergence relation for a pretopological space and the assertion follows again from corollary 6.5.

Similarly, from proposition 5.6 and corollary 6.6 we obtain

6.8. Proposition. *A relation* $R : \mathcal{F}X \rightharpoonup X$ *is the filter convergence relation for a topology on* X *if and only if*

1. $\forall\, x \quad \eta x\, R\, x$
2. $\forall\, \varphi\, \psi\, x \quad \forall\, A\, B\,(\,R^{op}B \subseteq \mathrm{e}A\,,\ B \in \psi\ \Rightarrow\ A \in \varphi\,)\,,\ \psi R\, x\ \Rightarrow\ \varphi R\, x$

So we have succeeded in characterizing those relations that are the convergence relations for a topology, but the second condition we have obtained appears cumbersome. We now show how it can be formulated in a more compact way.

6.9. THEOREM. [Barr] *A relation* $R : \mathcal{U}X \rightharpoonup X$ *is the ultrafilter convergence relation for a topology on* X *if and only if*

1. $\mathrm{id} \subseteq R \circ \eta$
2. $R \circ \mathcal{U}R \subseteq R \circ \mu$

It is immediate to see that the first condition is equivalent to the first condition of proposition 6.7. So, let us analyze the second condition (we denote elements of $\mathcal{U}(\mathcal{U}X)$ by uppercase Greek letters):

$$R \circ \mathcal{U}R \subseteq R \circ \mu$$

$$\forall\, \Xi\, x \quad \Xi(R \circ \mathcal{U}R)\,x \;\Rightarrow\; \Xi(R \circ \mu)\,x$$

$$\forall\, \Xi\, \upsilon\, x \quad \Xi\,\mathcal{U}R\,\upsilon\,,\; \upsilon R x \;\Rightarrow\; \mu\Xi\; R\, x$$

$$\forall\, \Xi\, \upsilon\, x \quad \Xi\,\mathcal{U}R\,\upsilon\,,\; \upsilon R x \;\Rightarrow\; \forall\, \xi\; (\; \mu\Xi \subseteq \xi \;\Rightarrow\; \xi R x\;)$$

(because $\mu\Xi$ and ξ are ultrafilters)

$$\forall\, \Xi\, \xi\, \upsilon\, x \quad \Xi\,\mathcal{U}R\,\upsilon\,,\; \upsilon R x\,,\; \mu\Xi \subseteq \xi \;\Rightarrow\; \xi R x$$

$$\forall\, \xi\, \upsilon\, x \quad \exists\, \Xi\; (\; \Xi\,\mathcal{U}R\,\upsilon\,,\; \mu\Xi \subseteq \xi\;)\,,\; \upsilon R x \;\Rightarrow\; \xi R x$$

$$\forall\, \xi\, \upsilon\, x \quad \exists\, \Xi\; (\; \forall\, B\; (\; B \in \upsilon \;\Rightarrow\; R^{op}B \in \Xi\;)\,,\; \forall\, A\; (\; A \in \mu\Xi \;\Rightarrow\; A \in \xi\;)\;)\,,\; \upsilon R x \;\Rightarrow\; \xi R x$$

$$\forall\, \xi\, \upsilon\, x \quad \exists\, \Xi\; (\; \forall\, B\; (\; B \in \upsilon \;\Rightarrow\; R^{op}B \in \Xi\;)\,,\; \forall\, A\; (\; A \notin \xi \;\Rightarrow\; \mathrm{e}A \notin \Xi\;)\;)\,,\; \upsilon R x \;\Rightarrow\; \xi R x$$

$$\forall\, \xi\, \upsilon\, x \quad \forall\, A\, B\; (\; B \in \upsilon\,,\; A \notin \xi \;\Rightarrow\; R^{op}B \not\subseteq \mathrm{e}A\;)\,,\; \upsilon R x \;\Rightarrow\; \xi R x \qquad \text{(EEL)}$$

$$\forall\, \xi\, \upsilon\, x \quad \forall\, A\, B\; (\; R^{op}B \subseteq \mathrm{e}A\,,\; B \in \upsilon \;\Rightarrow\; A \in \xi\;)\,,\; \upsilon R x \;\Rightarrow\; \xi R x$$

and this is exactly the second condition of proposition 6.7, so the theorem is proved.

Now, this proof is also valid for filters in place of ultrafilters, except for the passage explicitly marked. We easily overcome the inconvenient by assuming R to be a compatible relation:

6.10. THEOREM. *A compatible relation* $R : \mathcal{F}X \rightharpoonup X$ *is the filter convergence relation for a topology on* X *if and only if*

1. $\mathrm{id} \subseteq R \circ \eta$
2. $R \circ \mathcal{F}R \subseteq R \circ \mu$

Note that Manes' theorem [10, 8], which identifies compact Hausdorff topological spaces with the algebras for the ultrafilter monad, follows as a corollary from theorem 6.9. Indeed, by a well-known application of the (EEL), compact Hausdorff spaces are characterized by the fact that the ultrafilter convergence relation is a function, and inclusion between functions reduces to equality. But then the conditions of theorem 6.9 become the defining conditions for monad algebras.

7. Quasi-locally-compact spaces

As mentioned in the introduction, quasi-locally-compact spaces can be characterized in several ways.

7.1. Definition. A topological space X is *quasi-locally-compact* if for every point $x \in X$ and for every neighborhood U of x, there exists a neighborhood V of x which is relatively compact in U (that is, every open cover of U has finitely many members which cover V).

Using the (EEL), it is not difficult to prove that V is relatively compact in U if and only if every ultrafilter ξ "in" V (that is, $V \in \xi$) converges to a point in U.

It is easy to see that

7.2. Proposition. *A topological space is quasi-locally-compact if and only if its lattice of open sets is a continuous lattice.*

The following theorem, that was implicitly proved in [4], shows that this property of local compactness is what is needed for the existence of well-behaved function spaces (see [7] for an overview on these arguments):

7.3. Theorem. *A topological space X is quasi-locally-compact if and only if it is an exponentiable object in the category* **Top** *of topological spaces and continuous maps.*

Furthermore, one can prove [6, 8]:

7.4. Theorem. *A quasi-locally-compact and sober space is locally compact, that is, every point has a basis of compact neighborhoods.*

Now we are going to give a further characterization of quasi-locally-compact spaces.

8. The reverse inclusions

Let us analyze what it means for a topological space to satisfy also the reverse inclusions of those of theorem 6.9. The proof of the following proposition is an easy exercise:

8.1. Proposition. *A topological space is T_1 if and only if*

$$\mathrm{Lim} \circ \eta \subseteq \mathrm{id}$$

Now we turn to the main result of this paper:

8.2. THEOREM. *A topological space is quasi-locally-compact if and only if*

$$\mathrm{Lim} \circ \mu \subseteq \mathrm{Lim} \circ \mathcal{U}\mathrm{Lim}$$

As usual, the proof is given by a chain of equivalent formulae:

$\mathrm{Lim} \circ \mu \subseteq \mathrm{Lim} \circ \mathcal{U}\mathrm{Lim}$

$\forall\, \Xi\, x \quad \Xi\,(\mathrm{Lim} \circ \mu)\, x \;\Rightarrow\; \Xi\,(\mathrm{Lim} \circ \mathcal{U}\mathrm{Lim})\, x$

$\forall\, \Xi\, x \quad \mu\Xi\, \mathrm{Lim}\, x \;\Rightarrow\; \exists\, \upsilon\; (\; \Xi\, \mathcal{U}\mathrm{Lim}\, \upsilon \;,\; \upsilon\, \mathrm{Lim}\, x \;)$

$\forall\, \Xi\, x \quad \mu\Xi\, \mathrm{Lim}\, x \;\Rightarrow\; \exists\, \upsilon\; (\; \forall\, A\; (\; A \in \upsilon \;\Rightarrow\; \mathrm{Lim}^{op}\, A \in \Xi \;) \;,\; \mathrm{N}x \subseteq \upsilon \;)$

$\forall\, \Xi\, x \quad \mu\Xi\, \mathrm{Lim}\, x \;\Rightarrow\; \exists\, \upsilon\; (\; \forall\, A\; (\; \mathrm{Lim}^{op}\, A \notin \Xi \;\Rightarrow\; A \notin \upsilon \;) \;,$
$\forall\, C\; (\; C \in \mathrm{N}x \;\Rightarrow\; C \in \upsilon \;)\;)$

$\forall\, \Xi\, x \quad \mu\Xi\, \mathrm{Lim}\, x \;\Rightarrow\; \forall\, A\, C\; (\; \mathrm{Lim}^{op}\, A \notin \Xi \;,\; C \in \mathrm{N}x \;\Rightarrow\; C \not\subseteq A \;) \qquad \text{(EEL)}$

$\forall\, \Xi\, x \quad \mu\Xi\, \mathrm{Lim}\, x \;\Rightarrow\; \forall\, A\; (\; \mathrm{Lim}^{op}\, A \notin \Xi \;\Rightarrow\; \forall\, C\; (\; C \in \mathrm{N}x \;\Rightarrow\; C \not\subseteq A \;)\;)$

$\forall\, \Xi\, x \quad \mu\Xi\, \mathrm{Lim}\, x \;\Rightarrow\; \forall\, A\; (\; \mathrm{Lim}^{op}\, A \notin \Xi \;\Rightarrow\; A \notin \mathrm{N}x \;)$

$\forall\, \Xi\, x\, A \quad \mu\Xi\, \mathrm{Lim}\, x \;,\; \mathrm{Lim}^{op}\, A \notin \Xi \;\Rightarrow\; A \notin \mathrm{N}x$

$\forall\, x\, A \quad \exists\, \Xi\; (\; \mu\Xi\, \mathrm{Lim}\, x \;,\; \mathrm{Lim}^{op}\, A \notin \Xi \;) \;\Rightarrow\; A \notin \mathrm{N}x$

$\forall\, x\, A \quad \exists\, \Xi\; (\; \mathrm{N}x \subseteq \mu\Xi \;,\; \mathrm{Lim}^{op}\, A \notin \Xi \;) \;\Rightarrow\; A \notin \mathrm{N}x$

$\forall\, x\, A \quad \exists\, \Xi\; (\; \forall\, B\; (\; B \in \mathrm{N}x \;\Rightarrow\; B \in \mu\Xi \;) \;,\; \mathrm{Lim}^{op}\, A \notin \Xi \;) \;\Rightarrow\; A \notin \mathrm{N}x$

$\forall\, x\, A \quad \exists\, \Xi\; (\; \forall\, B\; (\; B \in \mathrm{N}x \;\Rightarrow\; \mathrm{e}B \in \Xi \;) \;,\; \mathrm{Lim}^{op}\, A \notin \Xi \;) \;\Rightarrow\; A \notin \mathrm{N}x$

$\forall\, x\, A \quad \forall\, B\; (\; B \in \mathrm{N}x \;\Rightarrow\; \mathrm{e}B \not\subseteq \mathrm{Lim}^{op}\, A \;) \;\Rightarrow\; A \notin \mathrm{N}x \qquad \text{(EEL)}$

$\forall\, x\, A \quad A \in \mathrm{N}x \;\Rightarrow\; \exists\, B\; (\; B \in \mathrm{N}x \;,\; \mathrm{e}B \subseteq \mathrm{Lim}^{op}\, A \;)$

$\forall\, x\, A \quad A \in \mathrm{N}x \;\Rightarrow\; \exists\, B\; (\; B \in \mathrm{N}x \;,\; \forall\, \xi\; (\; \xi \in \mathrm{e}B \;\Rightarrow\; \xi \in \mathrm{Lim}^{op}\, A \;)\;)$

$\forall\, x\, A \quad A \in \mathrm{N}x \;\Rightarrow\; \exists\, B\; (\; B \in \mathrm{N}x \;,\; \forall\, \xi\; (\; B \in \xi \;\Rightarrow$
$\Rightarrow\; \exists\, y\; (\; y \in A \;,\; \xi\, \mathrm{Lim}\, y \;)\;)\;)$

and this is exactly quasi-local-compactness, as observed after definition 7.1.

Combining theorems 6.9 and 8.2 and proposition 8.1 we obtain

8.3. COROLLARY. *A relation $R : \mathcal{U}X \rightharpoonup X$ is the ultrafilter convergence relation for a quasi-locally-compact topology on X if and only if*

1. $\mathrm{id} \subseteq R \circ \eta$
2. $R \circ \mathcal{U}R = R \circ \mu$

The first condition holds with equality if and only if the topology is T_1.

9. Conclusions

In this work a new relationship has been proved to exist between local compactness and ultrafilter convergence. This result can lend itself to interesting developments. In particular, the evident connections with the theory of monads, which have not been explicitly considered in the present paper, will be analyzed in a work in preparation. We anticipate some results obtained in this direction.

Using the functor defined in section 4, the ultrafilter monad on **Set** can be extended to a lax monad on **Rel** (in fact, strictness fails only because the unit is no more a strict natural transformation). Then corollary 8.3 may be rephrased by saying that lax (with respect to the unit law) algebras for this monad are exactly quasi-locally-compact topological spaces. It is interesting to analyze strict morphisms for these algebras: using the same techniques adopted in this paper, one can prove that such morphisms are relations that are continuous and proper [2].

On the other hand, it was proved in [3] (see also [13]) that continuous lattices are the algebras for the filter monad (on **Set**), and that morphisms of algebras are the maps preserving arbitrary meets and directed joins. The category of continuous locales [8] and that of distributive continuous lattices then have the same objects but different morphisms. The interesting fact is that while the category of quasi-locally-compact spaces and continuous maps is “very similar” to the first one, the category of quasi-locally-compact spaces and algebra morphisms (continuous and proper relations) turns out to be “very similar” to the second one.

Moreover, by restricting to the category of sets and partial functions, we rediscover the equivalence between the category of locally compact Hausdorff spaces with continuous and proper partial maps and the category of pointed compact Hausdorff spaces [5]. Indeed, both are equivalent to the algebras for the ultrafilter monad on the category of pointed sets.

In conclusion, it seems that the techniques used in this paper can be an effective tool for studying topological properties. Furthermore, many proofs are likely to have a meaning not only for preorders (**2**-enriched categories) but also for ordinary categories.

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