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CHU-SPACES, A GROUP ALGEBRA AND INDUCED REPRESENTATIONS

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ABSTRACT. Using the Chu-construction, we define a group algebra for topological Hausdorff groups. Furthermore, for isometric, weakly continuous representations of a subgroup H of a Hausdorff group G induced representations are constructed.

1. Introduction

The group algebra kG of a finite group G is usually defined to be a vector space over the field k whose basis is the group elements. The group multiplication gives a multiplication on this basis which can be extended by linearity to kG. There is a canonical bijection between representations of the group and modules over the group algebra, more precisely an isomorphism of categories. The group algebra is used to define induced representations as the tensor product over the subgroup algebra of a right and a left module (see for example [13]).

In this paper we will define a group algebra LG for a Hausdorff topological group G. Furthermore, we will introduce a notion of weakly continuous representations and for a given isometric weakly continuous subgroup representation define an induced isometric weakly continuous representation of the group using the group algebra.

The algebra LG will be defined in the category of separated extensional Chu-spaces over the autonomous category of Banach spaces and contracting linear maps. The Chu-algebra LG is a group algebra in the sense that there is a bijection between weakly continuous isometric representations and LG-Chu-modules, more precisely a canonical equivalence of categories. This is done in the third section. Let H be a subgroup of G, there is a natural map from LH to LG that restricts an LG-Chu-module to an LH-Chu-module. In the fourth and last section, we will define an induced LG-Chu-module for any given left LH-Chu-module.

Note that the construction of the group algebra LG is done on general Hausdorff topological groups. In particular we do not suppose the existence of a Haar measure as in the classic case and there are no measure-theoretic arguments.

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In the next section we will briefly explain the Chu-construction for the readers unfamiliar with this notion. For the proofs, which are rather technical we will refer to [5].

2. Preliminaries

2.1. The Chu-construction. The construction that is now named after him was first described by P.- H. Chu in his M. Sc. thesis [5], another more recent reference is [1]. It associates to an autonomous (symmetric monoidal closed) category $\mathcal{V}$ with pullbacks and a fixed object K in $\mathcal{V}$ a $*$-autonomous category called $\mathrm{Chu}(\mathcal{V}, K)$. This construction gives us many examples of $*$-autonomous categories, which are difficult to obtain directly (see [2]). For a more detailed discussion of the advantages of using the Chu construction to define and study the $*$-autonomous categories of [2] see [3].

2.2. The category $\mathrm{Chu}(\mathcal{V}, K)$. Let $\mathcal{V}$ be an autonomous category that has pullbacks and let K be an object in $\mathcal{V}$. Then define a category $\mathrm{Chu}(\mathcal{V}, K)$ as follows. The objects are of the form $(V, V', \langle -,- \rangle)$ where V and V' are objects of $\mathcal{V}$ and $\langle -,- \rangle : V \otimes V' \longrightarrow K$ is a morphism of $\mathcal{V}$ called a *pairing*. Normally, we will not cite the pairing explicitly and often use the notation $\mathbf{V} = (V_1, V_2)$ or $\mathbf{V} = (V, V')$ for an object in $\mathrm{Chu}(\mathcal{V}, K)$. A morphism $(f, f') : (V, V') \longrightarrow (W, W')$ in $\mathrm{Chu}(\mathcal{V}, K)$ is a pair $f : V \longrightarrow W$ and $f' : W' \longrightarrow V'$ of arrows of $\mathcal{V}$, such that the following diagram commutes.

$$\begin{tikzcd}
V \otimes W' \arrow[r, "\mathrm{id} \otimes f'"] \arrow[d, "f \otimes \mathrm{id}"'] & V \otimes V' \arrow[d, "\langle -,- \rangle"] \\
W \otimes W' \arrow[r, "\langle -,- \rangle"] & K
\end{tikzcd}$$

This can be symbolized by the equation $\langle fv, w' \rangle = \langle v, f'w' \rangle$. Let $V \multimap W$ be the internal hom on $\mathcal{V}$ between V and W, define an internal hom in $\mathrm{Chu}(\mathcal{V}, K)$ as follows. The canonical morphism $\langle -,- \rangle : V \otimes V' \longrightarrow K$ determines two morphisms, the transposes of this map, in $\mathcal{V}$

$$V \longrightarrow (V' \multimap K) \qquad \text{and} \qquad V' \longrightarrow (V \multimap K).$$

Using these maps and the isomorphisms

$$V \multimap (W' \multimap K) \cong (V \otimes W') \multimap K \cong W' \multimap (V \multimap K)$$

we define $\mathcal{V}((V,V'),(W,W'))$ to be the object occurring in the following pullback diagram

$$\begin{tikzcd}
\mathcal{V}((V,V'),(W,W')) \arrow[rr] \arrow[d] & & V \multimap W \arrow[d] \\
W' \multimap V' \arrow[r] & W' \multimap (V \multimap K) \cong (V \otimes W') \multimap K & \cong V \multimap (W' \multimap K)
\end{tikzcd}$$

This diagram defines not only an object but also an arrow $\mathcal{V}((V,V'),(W,W')) \to (V \otimes W') \multimap K$, hence define $(V,V') \multimap (W,W')$ to be $(\mathcal{V}((V,V'),(W,W')), V \otimes W')$.

Let $\top$ be the unit for the tensor product in $\mathcal{V}$ and $\top \otimes K \to K$ the canonical isomorphism. The duality on $\mathrm{Chu}(\mathcal{V},K)$ is defined by the map that takes (V,V') to (V',V) and in the same way on morphisms. It is not difficult to prove that $(K,\top)$ is the dualizing object, that is $(V',V) \cong (V,V') \multimap (K,\top)$. This definition renders $\mathrm{Chu}(\mathcal{V},K)$ self-dual.

Once the internal hom and the reflexive duality are known, the tensor product on $\mathrm{Chu}(\mathcal{V},K)$ is explicitly given by

$$(V,V') \otimes (W,W') = ((V,V') \multimap (W,W')^*)^* = (V \otimes W, \mathcal{V}((V,V'),(W',W))).$$

and $\mathrm{Chu}(\mathcal{V},K)$ is closed. The unit for the tensor product is $(\top, K)$.

The category $\mathrm{Chu}(\mathcal{V},K)$ with the duality, internal hom and tensor product as explained above is a $*$-autonomous category. See [5] for details.

An object in the category $\mathrm{Chu}(\mathcal{V},K)$ will be called a *Chu-space*. This is motivated by the following example, due to Vaughan Pratt. An object of $\mathrm{Chu}(\mathrm{Set},2)$ is a set S together with a set S' equipped with a map $S' \longrightarrow 2^S$. If this map is injective (see discussion below), then S' can be identified with a set of subsets of S, that is the beginnings of a topology.

2.3. The category of separated-extensional Chu-spaces. Suppose a factorisation system $\mathcal{E}/\mathcal{M}$ on $\mathcal{V}$ is given. A Chu space (V,V') is called $\mathcal{M}$-*extensional* – or just *extensional* if it clear which factorisation system we use – if the transpose $V' \longrightarrow V \multimap K$ of the pairing belongs to $\mathcal{M}$. If the transpose $V \longrightarrow V' \multimap K$ of the pairing is in $\mathcal{M}$ then the Chu-space (V,V') is called $\mathcal{M}$-*separated* or just *separated*. The elements of $\mathcal{E}$ will often be noted by arrows $\twoheadrightarrow$ and the elements of $\mathcal{M}$ by $\rightarrowtail$.

We shall write $\mathrm{Chu_s}(\mathcal{V},K)$, $\mathrm{Chu_e}(\mathcal{V},K)$ and $\mathrm{Chu_{se}}(\mathcal{V},K) = \mathrm{chu}(\mathcal{V},K)$ for the full subcategories of $\mathcal{M}$-separated, $\mathcal{M}$-extensional, and both $\mathcal{M}$-separated and $\mathcal{M}$-extensional objects respectively. The set $\mathcal{E}$ does not necessarily consist of all epis, nor does $\mathcal{M}$ consist of all monos. We will often omit the explicit mention of the inclusion of either Chu_s or Chu_e or chu in Chu. We shall see in the next section an example of such a Chu-construction and also why the extensional and separated Chu spaces are particularly interesting. But first we cite a lemma and then some results by Barr from [4], that we shall need later.

2.4. Proposition. [4, Proposition 3.2] *The inclusion* $\mathrm{Chu_s} \longrightarrow \mathrm{Chu}$ *has a left adjoint* s *(and the inclusion* $\mathrm{Chu_e} \longrightarrow \mathrm{Chu}$ *has a right adjoint* e*).*

Proof. Let $\mathbf{A}$ be a Chu-space. The morphism $A_1 \longrightarrow A_2 \multimap K$ splits into

$$A_1 \twoheadrightarrow \widetilde{A}_1 \rightarrowtail A_2 \multimap K$$

by the properties of a factorisation system. Define $s\mathbf{A}$ to be $(\widetilde{A}_1, A_2)$. The construction of $e\mathbf{A}$ is similar. ■

In what follows, we need the following two conditions for a given factorisation system $\mathcal{E}/\mathcal{M}$ on a category $\mathcal{V}$ with internal hom

($*$) Every arrow in $\mathcal{E}$ is an epimorphism.

($**$) If $f : A \longrightarrow A'$ is in $\mathcal{M}$, then for any object B, the arrow $B \multimap f : B \multimap A \longrightarrow B \multimap A'$ is in $\mathcal{M}$.

2.5. Proposition. [4, Proposition 3.3] *The condition* ($*$) *implies that when the object* (A, A') *of* $\mathrm{Chu}(\mathcal{A}, \perp)$ *is separated, so is* $e(A, A')$ *and similarly if* (A, A') *is extensional, so is* $s(A, A')$.

2.6. Proposition. [4, Proposition 3.4] *The condition* ($**$) *implies that when* (A, A') *and* (B, B') *are extensional, so is* $(A, A') \otimes (B, B')$. *(Dually, when* (A, A') *is extensional and* (B, B') *is separated, then* $(A, A') \multimap (B, B')$ *is separated).*

In general, the tensor product of two separated spaces is not separated and the internal hom of an extensional and a separated space is not extensional. In particular, define a monoidal closed structure on $\mathrm{chu}(\mathcal{A}, K)$ by slightly modifying the tensor product and internal hom in the following way. For (A, A') and (B, B') in $\mathrm{chu}(\mathcal{A}, K)$, we define the tensor product $(A, A') \boxtimes (B, B')$ to be $s((A, A') \otimes (B, B'))$ and the internal hom $(A, A') \multimap\!\!\square\, (B, B')$ to be $e((A, A') \multimap (B, B'))$.

2.7. Theorem. [4, Theorem 3.1] *Let* $\mathcal{A}$ *be an autonomous category with pullbacks and* $\mathcal{E}/\mathcal{M}$ *a factorisation system satisfying the conditions* ($*$) *and* ($**$). *Then for any object* $\perp$ *of* $\mathcal{A}$ *the category* $\mathrm{chu}(\mathcal{A}, \perp)$ *is a* $*$*-autonomous category.*

Observe also that we do not assume $se = es$, this is not true in general. For example, take $\mathcal{V} = \mathrm{Ban}_\infty$ and $K = \mathbb{C}$, with the factorisation system of maps with dense image and isometric embeddings. Consider $(l^1(2), l^2(2))$. Then $es(l^1(2), l^2(2)) = s(l^1(2), l^2(2)) = (l^2(2), l^2(2))$, but $se(l^1(2), l^2(2)) = (l^1(2), l^\infty(2))$.

2.8. Example: $\mathrm{Chu}(\mathrm{Ban}_1, \mathbb{C})$. Let Ban_1 be the category of Banach spaces with linear maps of norm at most 1. These maps shall be called linear contracting.

The tensor product on Ban_1 of two Banach spaces, denoted by $\widehat{\otimes}$, is the usual one, the projective tensor product. The internal hom of two Banach spaces is the vector space of bounded linear maps between the two spaces, equipped with the operator norm and is noted $[A, B]$.

In this and the following sections, we shall often use simply Chu or chu to denote $\mathrm{Chu}(\mathrm{Ban}_1, \mathbb{C})$ and $\mathrm{chu}(\mathrm{Ban}_1, \mathbb{C})$ respectively.

It can be shown that the pullback appearing in the definition of the internal hom is the following subset of $[A_1, B_1] \times [B_2, A_2]$ with product topology

$$\begin{aligned}\mathrm{Ban}_1((A_1, A_2), (B_1, B_2)) = &\{(f, g); f : A_1 \longrightarrow B_1, g : B_2 \longrightarrow A_2,\ f, g \text{ bounded} \\ &\text{linear maps }, \langle a_1, g(b_2)\rangle_{\mathbf{A}} = \langle f(a_1), b_2\rangle_{\mathbf{B}}\}\end{aligned}$$

with the induced topology. We also use the notation $[A_1, B_1]\tilde{\times}[B_2, A_2]$ for the pullback. The tilde should remind us that this is not the cartesian product since only pairs of maps preserving the pairing are admitted. So, the internal hom in Chu of $\mathbf{A}$ and $\mathbf{B}$ is given by $([A_1, B_1]\tilde{\times}[B_2, A_2], A_1\hat{\otimes}B_2)$ with the pairing

$$\begin{array}{rlll}\langle -, - \rangle : & ([A_1, B_1]\tilde{\times}[B_2, A_2])\hat{\otimes}(A_1\hat{\otimes}B_2) & \longrightarrow & \mathbb{C} \\ & (f, g) \otimes (a_1 \otimes b_2) & \longmapsto & \langle a_1, g(b_2)\rangle_{\mathbf{A}} = \langle f(a_1), b_2\rangle_{\mathbf{B}}.\end{array}$$

The tensor product in Chu between $\mathbf{A}$ and $\mathbf{B}$ is given by

$$\mathbf{A} \otimes \mathbf{B} = (\mathbf{A} \multimap \mathbf{B}^*)^* = (A_1\hat{\otimes}B_1, [A_1, B_2]\tilde{\times}[B_1, A_2]).$$

By definition, the category $\mathrm{Chu}(\mathrm{Ban}_1, \mathbb{C})$ with the duality, the internal hom and the tensor product mentioned above is $*$-autonomous.

The example of Chu-spaces defined on the category of Banach-spaces can give a motivation for the definition of extensional and separated Chu-spaces. The factorisation system $\mathcal{E}/\mathcal{M}$ we shall use on Ban_1 is the following

$$\begin{array}{rcc}\mathcal{M} & := & \{\text{linear isometric embeddings}\} \\ \mathcal{E} & := & \{\text{linear contracting maps with dense image}\}\end{array}$$

Observe that the maps in $\mathcal{M}$ are closed.

2.9. Lemma. *A morphism $f : A \longrightarrow B$ in Ban_1 is epi if and only if its image is dense.*

Proof. We have only to show that the image of an epi is dense as the other implication is evident. Suppose f to be an epi. Let $p : B \longrightarrow B/\overline{f(A)}$ be the canonical projection and $0 : B \longrightarrow B/\overline{f(A)}$ the trivial map. But then $p \circ f = 0 = 0 \circ f$ which implies $p = 0$ and $\overline{f(A)} = B$. ■

Lemma 2.9 shows that every arrow in $\mathcal{E}$ is an epimorphism, in particular the condition $(*)$ is fulfilled. It is evident that for the mentioned factorisation system also $(**)$ is fulfilled. Then Theorem 2.7 shows that $\mathrm{chu}(\mathrm{Ban}_1, \mathbb{C})$ is a $*$-autonomous category.

In the classic theory of topological vector spaces, a *duality* is defined as a bilinear form $\langle -, - \rangle$ on the product $F \times G$ of two topological vector spaces. It is called *separated* if the following two conditions are satisfied

$$\begin{array}{llll}(S_1) & \text{if } \langle x_0, y\rangle = 0 \text{ for all } y \in G, & \text{then} & x_0 = 0 \\ (S_2) & \text{if } \langle x, y_0\rangle = 0 \text{ for all } x \in F, & \text{then} & y_0 = 0.\end{array}$$

If the duality is separated we can identify G with a subspace of the algebraic dual of F (and vice versa). A topology τ on F is called *consistent with the duality* if the topological dual $(F,\tau)'$ is isomorphic to G as a set. If the duality is separated in G (condition (S_2)) then the weak topology on F is defined to be the weakest topology consistent with duality and is given by the family of semi-norms $x \longmapsto |\langle x, y\rangle|$ for y in G. It is usually written $\sigma(F,G)$. The space $(F, \sigma(F,G))$ is locally convex and is Hausdorff if and only if the duality is separated in F (condition (S_1)). Any topology consistent with the duality is stronger than the weak topology and weaker than the *Mackey* topology. More details and all the results can be found in [12].

They motivate why the notion of separated and/or extensional Chu-spaces on Ban_1 is interesting.

3. A group algebra

Now we have all the ingredients to define an algebra in chu associated to a Hausdorff group, such that there exists a bijection between modules over this algebra and isometric (resp. unitary) weakly continuous representation of the group in a Banach (resp. Hilbert space). Note that we speak of "algebra" even though there is no additive structure on Chu-spaces. But we are principally interested in the first component of a Chu-space which is a vector space and will be seen to have a weakly continuous multiplication.

All the groups will be topological groups and Hausdorff or separated (for which T_0 is sufficient).

3.1. Preliminaries.

3.2. Lemma. *Let $\mathbf{A}, \mathbf{B}$ be in Chu_e. The first component of a given Chu-morphism $f : \mathbf{A} \longrightarrow \mathbf{B}$ determines the second one. Dually, for $\mathbf{A}, \mathbf{B}$ in Chu_s, the second component of a Chu-morphism $f : \mathbf{A} \longrightarrow \mathbf{B}$ determines the first one.*

Proof. The functor s is left adjoint to the inclusion and the functor e is right adjoint to the inclusion. ■

Note that the lemma does not say that for a given f_1 there is always an f_2 such that (f_1, f_2) is a Chu-morphism but on separated extensional Chu-spaces, it is equivalent for a linear contracting map $f_1 : A_1 \longrightarrow B_1$ to be weakly continuous with respect to the pairings and to be the first component of a Chu-morphism.

Let $\mathcal{H}$ be the category of Hausdorff spaces and continuous maps and let $\mathcal{CR}$ be the full subcategory consisting of the completely regular or Tychonoff spaces. For an object X in $\mathcal{H}$, define the following Banach spaces

$$\begin{aligned} l^1(X) &= \{\text{formal sums } \sum_{i\geqslant 0} a^i \cdot x^i;\ a^i \in \mathbb{C}, x^i \in X, \text{ and } \sum_{i\geqslant 0} |a^i| < \infty\} \\ \mathcal{C}(X) &= \{f : X \longrightarrow \mathbb{C};\ f \text{ continuous}, \ ||f|| = \sup_{x\in X} |f(x)| < \infty\}. \end{aligned}$$

The space $l^1(X)$ is a Banach space with l^1-norm and $\mathcal{C}(X)$ is a Banach space with the sup-norm. Observe that $l^1(X)$ can also be defined as the completion of the free vector space over X with the l^1-norm or as the space of measures $f : X \longrightarrow \mathbb{C}$ satisfying $||f|| = \sum_{x\in X} |f(x)| < \infty$. The pairing is defined by

$$T : \mathcal{C}(X) \longrightarrow (l^1(X))^* \qquad \varphi \longmapsto (\textstyle\sum_{i\geqslant 0} a^i x^i \mapsto \sum_{i\geqslant 0} a^i\varphi(x^i))$$

which comes from the evaluation. This map is well-defined and contracting.

Define a functor $L : \mathcal{H} \longrightarrow$ Chu on the objects by $LX = (l^1(X), \mathcal{C}(X))$ and on the arrows in a canonical way. Restrict this functor to the full subcategory $\mathcal{CR}$ of Tychonoff spaces.

3.3. Proposition. *The Chu-space LX is in* chu *if X is a Tychonoff space.*

Proof. The Chu-space LX is extensional since the map

$$F_1 : \mathcal{C}(X) \longrightarrow l^1(X)^* \qquad \psi \longmapsto (\sum_{i\geqslant 0} a^i x^i \mapsto \sum_{i\geqslant 0} a^i\psi(x^i))$$

is an isometric embedding.

The Chu-space LX is separated if the map

$$F_2 : l^1(X) \longrightarrow (\mathcal{C}(X))^* \qquad \sum_{i\geqslant 0} a^i x^i \longmapsto (\psi \mapsto \sum_{i\geqslant 0} a^i\psi(x^i))$$

is isometric. In other words, we have to prove that

$$\sup\{\sum_{i\geqslant 0} a^i\psi(x^i);\ ||\psi|| \leqslant 1\} = \sum_{i\geqslant 0} |a^i| \quad \text{ for all } \sum_{i\geqslant 0} a^i x^i \in l^1(X).$$

Fix $\sum_{i\geqslant 0} a^i x^i$ in $l^1(X)$, without restriction $a_i \neq 0$ and the x_i distinct. For all $\varepsilon > 0$ there exists n_ε in $\mathbb{N}$ such that $\sum_{i\geqslant n_0} |a^i| < \frac{\varepsilon}{2}$ for all $n_0 \geqslant n_\varepsilon$. Since a Tychonoff space is in particular regular we can construct disjoint open neighbourhoods $(U_i)_{0\leqslant i\leqslant n_\varepsilon}$ of the first n_ε elements of the sequence $(x^i)_{i\in\mathbb{N}}$. For each neighbourhood U_i there exists a function $\psi_i : X \longrightarrow [-1, 1]$ which has the property: $\psi_i(x_i) = 1$ and $\psi_i(x) = 0$ for all $x \in X \setminus U_i$. In particular $\psi_i(x_j) = 0$ for all $j \neq i$, $0 \leqslant j \leqslant n_\varepsilon$. Then define ψ to be $\sum_{1\leqslant i\leqslant n_\varepsilon} \frac{\bar{a}_i}{|a_i|}\psi_i$. Then ψ is continuous with image in $O\mathbb{C}$ by construction and

$$\begin{aligned}
|\sum_{i\geqslant 0} |a^i| - |\sum_{i\geqslant 0} a^i\psi(x^i)|| &\leqslant |\sum_{i\geqslant 0} |a^i| - \sum_{i\geqslant 0} a^i\psi(x^i)| \\
&= |\sum_{i\geqslant 0} |a^i| - \sum_{0\leqslant i\leqslant n_\varepsilon} |a_i| - \sum_{i>n_\varepsilon} a^i\psi(x^i)| \\
&\leqslant \sum_{i>n_\varepsilon} |a^i| + |\sum_{i>n_\varepsilon} a^i\psi(x^i)| \\
&\leqslant \sum_{i>n_\varepsilon} |a^i| + \sum_{i>n_\varepsilon} |a^i| \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

∎

From now on, we consider L as a functor from $\mathcal{CR}$ to chu.

We now define a functor $V : \text{chu} \longrightarrow \mathcal{CR}$. For each Chu-space (A_1, A_2) define $V(A_1, A_2)$ to be $O(A_1, \sigma(A_1, A_2))$, that is furnish A_1 with the weak topology and take the unit ball as a topological space. Then $V(A_1, A_2)$ is locally convex and a completely regular space, since the family of semi-norms on a locally convex vector space defines a uniformity which induces the topology, and the topology on a space X can be induced by a uniformity on the set X if and only if X is completely regular, see [7, Theorem 8.1.20].

3.4. PROPOSITION. *The functor L is left adjoint to the functor V.*

Proof. Let X be a Tychonoff space. Remember that by definition $LX = (l^1(X), \mathcal{C}(X))$ and then $VLX = O(l^1(X), \sigma(l^1(X), \mathcal{C}(X)))$. Define the map $\eta : X \longrightarrow VLX$ by $x \longmapsto 1 \cdot x$.

We will first show that η is continuous. Let p be any continuous semi-norm on $(l^1(X), \sigma)$, then by definition $p(\sum_{i\geqslant 0} a^i x^i) = |\sum_{i\geqslant 0} a^i \varphi(x^i)|$ for a φ in $\mathcal{C}(X)$. Then, for x in X, we have $p \circ \eta(x) = |\varphi(\eta(x))| = |\varphi(x)|$ so $p \circ \eta$ is continuous on X for every semi-norm p.

The map η is a natural transformation and the unit of the adjunction as will be seen below. For each separated extensional Chu-space (B_1, B_2) and each continuous map $\Phi : X \longrightarrow V(B_1, B_2)$ there exists a Chu-morphism $f : LX \longrightarrow (B_1, B_2)$ satisfying $\Phi = Vf \circ \eta$. Define f to be (f_1, f_2) where f_1 and f_2 are given as follows

$$\begin{array}{ll} f_1 : l^1(X) \longrightarrow B_1 & \sum_{i\geqslant 0} a^i x^i \longmapsto \sum_{i\geqslant 0} a^i \Phi(x^i), \\ f_2 : B_2 \longrightarrow \mathcal{C}(X) & b_2 \longmapsto (x \mapsto \langle \Phi(x), b_2 \rangle). \end{array}$$

The map f_1 is well defined because by assumption the image of Φ is in the unit ball of B_1. This shows also that f_1 is contracting. The map f_2 is well defined since the pairing $\langle -, - \rangle$ is in Ban_1 and the image of Φ is in the unit ball of B_1, so $f_2(b_2)$ is bounded. The map $f_2(b_2)$ is continuous on X since the map $\Phi : X \longrightarrow O(B_1, \sigma(B_1, B_2))$ is continuous in the weak topology. The map f_2 is also contracting. Then the pair (f_1, f_2) is a Chu-morphism since $\langle f_1(\sum_{i\geqslant 0} a^i x^i), b_2 \rangle = \langle \sum_{i\geqslant 0} a^i x^i, f_2(b_2) \rangle$.

We have also

$$Vf(\eta(x)) = Vf(1 \cdot x) = f_1(1 \cdot x) = \Phi(x).$$

For a given Φ, the map f is unique. Suppose that $f = (f_1, f_2)$ and $g = (g_1, g_2)$ are two chu-morphisms that satisfy

$$Vf(\eta(x)) = f_1(x) = \Phi(x) = g_1(x) = Vg(\eta(x)).$$

Then f_1 and g_1 coincide on every formal linear combination of elements in X by linearity and by boundedness coincide on every element of $l^1(X)$. Then also f_2 and g_2 are equal using lemma 3.2. ∎

3.5. Definition of a group algebra. First, we shall define a natural map $i' : LX \otimes LY \longrightarrow L(X \times Y)$ in the following way. Recall that

$$\begin{aligned} LX \otimes LY &= (l^1(X)\hat{\otimes} l^1(Y), [l^1(X), \mathcal{C}(Y)] \tilde{\times} [l^1(Y), \mathcal{C}(X)]) \\ L(X \times Y) &= (l^1(X \times Y), \mathcal{C}(X \times Y)). \end{aligned}$$

The first component of i' is defined by the bilinear map b given by

$$b : l^1(X) \times l^1(Y) \longrightarrow l^1(X \times Y) \qquad (\sum_{i\geqslant 0} a^i x^i, \sum_{j\geqslant 0} b^j y^j) \longmapsto (\sum_{i,j\geqslant 0} a^i b^j (x^i, y^j)).$$

This map b is well defined and bilinearity is evident. Furthermore, the norm of b is at most 1, then b defines a unique linear continuous map i'_1 on the projective tensor product with the same norm (see [8]). The second component of i' is defined by

$$i'_2 : \mathcal{C}(X \times Y) \longrightarrow [l^1(X), \mathcal{C}(Y)] \tilde{\times} [l^1(Y), \mathcal{C}(X)] \qquad \psi \longmapsto (\alpha_1, \alpha_2).$$

where

$$\alpha_1(\sum_{i\geqslant 0} a^i x^i) = \sum_{i\geqslant 0} a^i \psi(x^i, -) \qquad \alpha_2(\sum_{j\geqslant 0} b^j y^j) = \sum_{j\geqslant 0} b^j \psi(-, y^j)$$

The pair (α_1, α_2) preserves the canonical map. Each $a^i\psi(x^i, -)$ is continuous on Y and the series of the functions $a^i\psi(x^i, -)$ converges uniformly. Take $\varepsilon > 0$, if ψ is not equal to zero, then there exists an n_ε such that $\sum_{i>n_0} |a^i| < \frac{\varepsilon}{||\psi||}$ for all $n_0 \geqslant n_\varepsilon$. Then we have

$$\begin{aligned} ||\sum_{i>n_0} a^i \psi(x^i, -)|| &= \sup_{y\in Y} |\sum_{i>n_0} a^i \psi(x^i, y)| \\ &\leqslant \sup_{y\in Y} \sum_{i>n_0} |a^i \psi(x^i, y)| \leqslant ||\psi|| \sum_{i>n_0} |a^i| < \varepsilon. \end{aligned}$$

Both maps α_1 and α_2 are continuous and bounded by $||\psi||$. This implies that i'_2 is well defined and is also contracting.

Notice that linear combinations of tensors in the image of the universal map, called basic tensors, are dense in the projective tensor product. To check that the pair (i'_1, i'_2) preserves the pairing, consider linear combinations of basic tensors, use linearity and continuity of the involved functions and conclude by density.

To see that the map $i'_{X,Y}$ is natural in both components, calculate explicitly the commutativity of the diagram. Then apply 2.4 to get a map $i_{X,Y}$ on $LX \boxtimes LY$ that is also natural.

Furthermore, for each fixed e in the Tychonoff space X there exists a chu-morphism $f^e : L\{*\} \longrightarrow LX$, where $\{*\}$ is the topological space with one point. The chu-morphism f^e is defined by its components

$$\begin{aligned} f^e_1 &: \mathbb{C} \longrightarrow l^1(X) \qquad z \longmapsto z \cdot e \\ f^e_2 &: \mathcal{C}(X) \longrightarrow \mathbb{C} \qquad \psi \longmapsto \psi(e). \end{aligned}$$

Then (f_1, f_2) is a chu-morphism.

3.6. Theorem. *The functor L is monoidal.*

We will first prove some lemmas used in the proof of the theorem.

3.7. Lemma. *Let X, Y and Z be Tychonoff spaces. The diagram*

$$\begin{tikzcd}
LX \otimes (LY \otimes LZ) \arrow[rr] \arrow[d, "id \otimes i'_{Y,Z}"'] & & (LX \otimes LY) \otimes LZ \arrow[d, "i'_{X,Y} \otimes id"] \\
LX \otimes L(Y \times Z) \arrow[d, "i'_{X,Y\times Z}"'] & & L(X \times Y) \otimes LZ \arrow[d, "i'_{X\times Y,Z}"] \\
L(X \times (Y \times Z)) \arrow[rr] & & L((X \times Y) \times Z)
\end{tikzcd}$$

commutes.

Proof. First recall that for $\mathbf{A}$ and $\mathbf{B}$ extensional the Chu-space $\mathbf{A} \otimes \mathbf{B}$ is also extensional. Thus all the spaces in the above diagram are extensional since LX is extensional and separated for every Tychonoff space. Then lemma 3.2 implies that it is sufficient consider the first component of the Chu-morphisms. By associativity of the product of topological spaces the diagram commutes in the first component. ■

3.8. Lemma. *Let $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ be separated extensional Chu-spaces. For a given Chu-morphism $(f_1, f_2) : \mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C}) \longrightarrow \mathbf{D}$ there exists a unique Chu-morphism $(g_1, f_2) : \mathbf{A} \otimes s(\mathbf{B} \otimes \mathbf{C}) \longrightarrow \mathbf{D}$ such that the following diagram commutes.*

$$\begin{tikzcd}
\mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C}) \arrow[r] \arrow[dr, "(f_1, f_2)"'] & \mathbf{A} \otimes s(\mathbf{B} \otimes \mathbf{C}) \arrow[d, "(g_1, f_2)"] \\
& \mathbf{D}
\end{tikzcd}$$

Proof. Since $\mathbf{A} \multimap \mathbf{D}$ is separated

$$\begin{aligned}
\mathrm{Hom}(\mathbf{A} \otimes s(\mathbf{B} \otimes \mathbf{C}), \mathbf{D}) &\cong \mathrm{Hom}(s(\mathbf{B} \otimes \mathbf{C}), \mathbf{A} \multimap \mathbf{D}) \\
&\cong \mathrm{Hom}(\mathbf{B} \otimes \mathbf{C}, \mathbf{A} \multimap \mathbf{D}) \\
&\cong \mathrm{Hom}(\mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C}), \mathbf{D}).
\end{aligned}$$

■

A consequence of the lemma is that $s(\mathbf{A} \otimes s(\mathbf{B} \otimes \mathbf{C})) \cong s(\mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C}))$ this implies of course that $(\mathbf{A} \otimes s(\mathbf{B} \otimes \mathbf{C}))_2 \cong (\mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C}))_2$.

In the following propositions, we need the notion of the tensor product of two maps. We shall now describe this construction explicitly.

Let $\mathbf{A}, \mathbf{A}', \mathbf{B}, \mathbf{B}'$ be Chu-spaces and $f : \mathbf{A} \longrightarrow \mathbf{A}'$, $g : \mathbf{B} \longrightarrow \mathbf{B}'$ be two Chu-morphisms. Then $f \otimes g : \mathbf{A} \otimes \mathbf{B} \longrightarrow \mathbf{A}' \otimes \mathbf{B}'$ may be given as

$$\begin{array}{rccll} (f \otimes g)_1 : & A_1 \widehat{\otimes} B_1 & \longrightarrow & A_1' \widehat{\otimes} B_1' & \qquad (f \otimes g)_1 = f_1 \otimes g_1 \\ (f \otimes g)_2 : & [A_1', B_2'] \widetilde{\times} [B_1', A_2'] & \longrightarrow & [A_1, B_2] \widetilde{\times} [B_1, A_2] & \\ & (\alpha_1, \alpha_2) & \longmapsto & (g_2 \circ \alpha_1 \circ f_1, f_2 \circ \alpha_2 \circ g_1). & \end{array}$$

It is easy to see that the pair $f \otimes g$ preserves the canonical map (show it on linear combinations of basic tensors and use continuity and a density argument). The map $f \boxtimes g$ is then by definition $s(f \otimes g)$.

Proof. of 3.6.

Let i' denote the map defined above on Chu. To prove that L is monoidal, we have to prove the commutativity of the following three diagrams for all Tychonoff spaces X, Y and Z and $\{*\}$ the one point topological space, e a point in X.

$$\begin{tikzcd} LX \boxtimes (LY \boxtimes LZ) \arrow[r, "i \boxtimes \mathrm{id}"] \arrow[d, "\mathrm{id} \boxtimes i"'] & (LX \boxtimes LY) \boxtimes LZ \arrow[d, "i \boxtimes \mathrm{id}"] \\ LX \boxtimes L(Y \times Z) \arrow[d, "i"'] & L(X \times Y) \boxtimes LZ \arrow[d, "i"] \\ L(X \times (Y \times Z)) \arrow[r] & L((X \times Y) \times Z) \end{tikzcd}$$

$$\begin{tikzcd} L\{*\} \boxtimes LX \arrow[r, "i"] \arrow[d, "f^e \boxtimes \mathrm{id}"'] & L(\{*\} \times X) \arrow[d] \\ (\mathbb{C}, \mathbb{C}) \boxtimes LX \arrow[r] & LX \end{tikzcd} \qquad \begin{tikzcd} LX \boxtimes L\{*\} \arrow[r, "i"] \arrow[d, "\mathrm{id} \boxtimes f^e"'] & L(X \times \{*\}) \arrow[d] \\ LX \boxtimes (\mathbb{C}, \mathbb{C}) \arrow[r] & LX \end{tikzcd}$$

To prove the commutativity of the first diagram, remember that we have proved for separated extensional Chu-spaces that the second components of $\mathbf{A} \otimes s(\mathbf{B} \otimes \mathbf{C})$ and $\mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C})$ are isometrically isomorphic (see 3.8). Note also that we have to show the commutativity of a diagram involving only separated (and extensional) spaces, but then by lemma 3.2 it is sufficient to test commutativity in the second component. Consider the following diagram

$$
\begin{tikzcd}
 & (LX \otimes (LY \otimes LZ))_2 & & & ((LX \otimes LY) \otimes LZ))_2 \arrow[lll] & \\
 & & (LX \boxtimes s(LY \otimes LZ))_2 \arrow[ul, "F"'] & (s(LX \otimes LY) \boxtimes LZ)_2 \arrow[l] \arrow[ur, "\widetilde{F}"] & & \\
 & & (LX \boxtimes L(Y \times Z))_2 \arrow[u, "(\mathrm{id} \boxtimes i_{Y,Z})_2"] & (L(X \times Y) \boxtimes LZ)_2 \arrow[u, "(i_{X,Y} \boxtimes \mathrm{id})_2"'] & & \\
(LX \otimes L(Y \times Z))_2 \arrow[uuur, "(\mathrm{id} \otimes i'_{Y,Z})_2"] \arrow[urr, "="'] & & & & & (L(X \times Y) \otimes LZ)_2 \arrow[uuul, "(i'_{X,Y} \otimes \mathrm{id})_2"'] \arrow[ull, "="] \\
 & & (L(X \times (Y \times Z))_2 \arrow[ull] \arrow[uu, "(i_{X,Y \times Z})_2"] & (L(X \times Y) \times Z)_2 \arrow[l] \arrow[urr] \arrow[uu, "(i_{X \times Y,Z})_2"'] & &
\end{tikzcd}
$$

We shall show that it is commutative. The two lower triangles right and left commute because when we "separate" a space using the functor s, the second component does not change. Then by definition $i_2 = i'_2$. The other parts of the diagram all commute by the definition of the maps. Let us show it for the square on the left.

From lemma 3.2 we get that F is the Ban_1-isomorphism (!)

$$F : (LX \boxtimes (LY \boxtimes LZ))_2 \longrightarrow (LX \otimes (LY \otimes LZ))_2 \quad (\alpha_1, \alpha_2) \longmapsto (\alpha_1, \alpha_2 \circ k)$$

where k is the first component of $LY \otimes LZ \longrightarrow s(LY \otimes LZ)$.

Then we have by definition of $\mathrm{id} \boxtimes i$, $\mathrm{id} \otimes i'$ and i that for (β_1, β_2) in $(LX \otimes L(Y \times Z))_2$

$$
\begin{aligned}
F \circ (\mathrm{id} \boxtimes i)_2(\beta_1, \beta_2) &= F(i_2 \circ \beta_1, \beta_2 \circ i_1) = (i_2 \circ \beta_1, \beta_2 \circ i_1 \circ k) \\
&= (i_2 \circ \beta_1, \beta_2 \circ i'_1) = (\mathrm{id} \otimes i')_2(\beta_1, \beta_2).
\end{aligned}
$$

This shows the commutativity of the left quadrilateral.

To show the associativity of the upper quadrilateral, we have just to write explicitly the associativity maps involved.

These considerations show that the inner hexagon commutes because the outer hexagon does. Then the corresponding diagram of the first components commutes also since the maps can be identified with the dual maps of the maps of this diagram.

We will only show the commutativity one of the other two diagrams. The following diagram in Chu commutes

$$\begin{array}{ccc} L\{*\}\otimes LX & \xrightarrow{i} & L(\{*\}\times X) \\ {\scriptstyle f^e\otimes\mathrm{id}}\downarrow & & \downarrow \\ (\mathbb{C},\mathbb{C})\otimes LX & \xrightarrow{u} & LX \end{array}$$

where u is the map that identifies $LX\otimes(\mathbb{C},\mathbb{C})$ with LX and f^e is the map given above. Since all spaces in this diagram are extensional, it is sufficient to test commutativity in the first component. Applying s to the above diagram shows that the required diagram commutes.

■

Let G now be a Hausdorff topological group. Since the group multiplication on G and the passage to the inverse are continuous, the group is completely regular as a topological space (see [7, Example 8.1.17]). Using the fact that L is a monoidal functor, we can define now a multiplication on LG, a chu-morphism

$$m : LG \boxtimes LG \xrightarrow{i} L(G\times G) \xrightarrow{L\widetilde{m}} LG$$

where $\widetilde{m}$ denotes the group multiplication.

If we consider the corresponding map on Chu, that is

$$m' : LG \otimes LG \xrightarrow{i'} L(G\times G) \xrightarrow{L\widetilde{m}} LG$$

we can write the two components explicitly

$$\begin{array}{rccc} m'_1: & l^1(G)\widehat{\otimes} l^1(G) & \longrightarrow & l^1(G) \\ & \sum_{i\geqslant 0} a^i g^i \otimes \sum_{j\geqslant 0} b^j h^j & \longmapsto & \sum_{i,j\geqslant 0} a^i b^j (g^i\cdot h^j) \\ m'_2: & \mathcal{C}(G) & \longrightarrow & [l^1(G),\mathcal{C}(G)]\widetilde{\times}[l^1(G),\mathcal{C}(G)] \\ & \psi & \longmapsto & (\alpha_1,\alpha_2) \end{array}$$

where

$$\alpha_1(\sum_{i\geqslant 0} a^i g^i)(k) = \sum_{i\geqslant 0} a^i\psi(g^i\cdot k) \quad \text{and} \quad \alpha_2(\sum_{j\geqslant 0} b^j h^j)(k) = \sum_{i\geqslant 0} b^j\psi(k\cdot h^j).$$

An element of $l^1(G)$ can also be seen as a particular measure, that is a function $f : G \longrightarrow \mathbb{C}$ where $\sum_{g\in G}|f(g)| < \infty$. Then the first map is the convolution of such measures (see for example [9]). The first (or the second) component of the second map is a particular version of the convolution on preduals. A description of such a convolution can be found, for example, in [11].

3.9. THEOREM. *Let G be a Hausdorff topological group and e in the definition of f^e the unit of this group. Then LG equipped with the multiplication m is a monoid.*

Proof. The theorem is a corollary of 3.6. ■

There is no addition defined on LG. It is nevertheless reasonable to call LG an algebra associated to G since it is the first component of the chu-space LG — which is a vector-space — that is interesting.

Furthermore, there exists an involution on LG, an anti-Chu-morphism in a sense explained below. It is given by the inversion of the group G. Define a map $* = (*_1, *_2) : LG \longrightarrow LG$ by

$$\begin{array}{ll} *_1 : l^1(G) \longrightarrow l^1(G) & \qquad \sum_{i\geqslant 0} a^i g^i \longmapsto \sum_{i\geqslant 0} \bar{a}^i (g^i)^{-1} \\ *_2 : \mathcal{C}(G) \longrightarrow \mathcal{C}(G) & \qquad \psi \longmapsto (g \mapsto \overline{\psi(g^{-1})}). \end{array}$$

The maps $*_1$ and $*_2$ are conjugate linear by definition, and they are anti-Chu-morphisms in the following sense

$$\begin{aligned} \langle *_1(\sum_{i\geqslant 0} a^i g^i), \psi\rangle &= \sum_{i\geqslant 0} \bar{a}^i \langle (g^i)^{-1}, \psi\rangle = \sum_{i\geqslant 0} \bar{a}^i \psi((g^i)^{-1}) \\ &= \sum_{i\geqslant 0} \bar{a}^i \langle g^i, \overline{*_2(\psi)}\rangle = \langle \sum_{i\geqslant 0} \bar{a}^i g^i, \overline{*_2(\psi)}\rangle \\ &= \overline{\langle \sum_{i\geqslant 0} a^i g^i, *_2(\psi)\rangle}. \end{aligned}$$

This involution has the property that $*$ composed with $*$ gives the identity, this is clear on the first component, but then also on the second by extensionality.

As appears in the propositions and theorems above, it is not easy to understand explicitly the elements of $LG \otimes LG$ and even more difficult for $LG \boxtimes LG$. So let us try to give an interpretation to "elements" of $LG \otimes LG$. We have proved that there is a map from $(\mathbb{C}, \mathbb{C}) \otimes (\mathbb{C}, \mathbb{C})$ to $LG \otimes LG$ (use 3.6), but $(\mathbb{C}, \mathbb{C}) \otimes (\mathbb{C}, \mathbb{C}) \cong (\mathbb{C}, \mathbb{C})$, so this gives us a map from $(\mathbb{C}, \mathbb{C})$ to $LG \otimes LG$. Then we can see an element of $LG \otimes LG$ as a pair (μ_1, μ_2) with μ_1 in $[\mathbb{C}\hat{\otimes}\mathbb{C}, l^1(G) \otimes l^1(G)]$ and μ_2 in $[[l^1(G), \mathcal{C}(G)] \widetilde{\times} [l^1(G), \mathcal{C}(G)], [\mathbb{C}, \mathbb{C}] \widetilde{\times} [\mathbb{C}, \mathbb{C}]]$.

Then we can write for example the multiplication on elements using the diagram below.

$$\begin{tikzcd} LG \otimes LG \arrow[d, "m"'] & (\mathbb{C}, \mathbb{C}) \otimes (\mathbb{C}, \mathbb{C}) \arrow[l] & (\mathbb{C}, \mathbb{C}) \arrow[l, "\cong"'] \arrow[dll] \\ LG & & \end{tikzcd}$$

The multiplication applies then (μ_1, μ_2) to (ν_1, ν_2) with

$$\begin{aligned} \nu_1(z) &= z \cdot (\sum_{i,j\geqslant 0} a^i b^j (g^i h^j)) \quad \text{where } \mu_1(1 \otimes 1) = \sum_{i\geqslant 0} a^i g^i \otimes \sum_{j\geqslant 0} b^j h^j \\ \nu_2(\psi) &= \sum_{i,j\geqslant 0} a^i b^j \psi(g^i h^j). \end{aligned}$$

3.10. LG AND REPRESENTATIONS. Before we describe how LG and a representation of the group G are related, we introduce some useful notions. Recall that for Banach spaces A, B the Banach space of the bounded linear maps with the operator norm from A to B is written as $[A, B]$.

Let G be a topological group, $\mathbf{B} = (B_1, B_2)$ a separated extensional Chu-space and $\rho : G \longrightarrow [B_1, B_1]$ a linear isometric representation of G. Then ρ is called *weakly continuous* if the map

$$G \longrightarrow \mathbb{C} \qquad g \longmapsto \langle \rho(g)b_1, b_2\rangle$$

is continuous for each b_1 in B_1 and b_2 in B_2 and the map

$$(B_1, \sigma(B_1, B_2)) \longrightarrow \mathbb{C} \qquad b_1 \longmapsto \langle \rho(g)b_1, b_2\rangle$$

is continuous on B_1 with respect to weak topology defined by the pairing $\langle -, -\rangle : B_1 \widehat{\otimes} B_2 \longrightarrow \mathbb{C}$ for each g in G and for each b_2 in B_2. Note that the second condition is essential for the existence of an associated LG-module structure on $\mathbf{B}$ (see proof of the following theorem) but not too restrictive. If for a given Banach space B_1 the space $B_2 = B_1^*$, that is the pairing is the canonical one, then the second condition is satisfied. For a linear endomorphism of a Banach space, it is equivalent to be bounded and to be weakly-(B, B^*)-continuous with (see for example [6]).

3.11. LEMMA. *There exists a natural multiplication on* $\mathbf{B} \mathbin{-\!\Box} \mathbf{B}$.

Proof. The multiplication on $\mathbf{B} \mathbin{-\!\Box} \mathbf{B}$ is defined in a canonical way as the transpose of the following map

$$(\mathbf{B} \mathbin{-\!\Box} \mathbf{B}) \boxtimes (\mathbf{B} \mathbin{-\!\Box} \mathbf{B} \boxtimes \mathbf{B}) \xrightarrow{(\mathbf{B} \mathbin{-\!\Box} \mathbf{B}) \boxtimes ev} (\mathbf{B} \mathbin{-\!\Box} \mathbf{B}) \boxtimes \mathbf{B} \xrightarrow{ev} \mathbf{B}$$

where $ev : (\mathbf{B} \mathbin{-\!\Box} \mathbf{B}) \boxtimes \mathbf{B} \longrightarrow \mathbf{B}$ is the transpose of the identity on $\mathbf{B} \mathbin{-\!\Box} \mathbf{B}$ (see for example [10]). The multiplication is associative by construction. ∎

3.12. LEMMA. *Let* $\mathbf{A}$ *be an extensional and* $\mathbf{B}$ *a separated Chu-space,* $(f_1, f_2) : \mathbf{A} \longrightarrow \mathbf{B}$ *a given Chu-morphism. Then* $es(f_1, f_2) = se(f_1, f_2)$.

Proof. The map $\varepsilon_A : A \longrightarrow sA$ is in $\mathcal{E}$ and $\eta_B : eB \longrightarrow B$ is in $\mathcal{M}$. Consider then the following diagrams

$$\begin{tikzcd} A \arrow[r, "\varepsilon_A"] \arrow[d, "f"'] \arrow[dr, "ef"] & sA = seA \arrow[d, "sef"] \\ B & seB = eB \arrow[l, "\eta_B"] \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\varepsilon_A"] \arrow[d, "f"'] & sA = seA \arrow[d, "esf"] \arrow[dl, "sf"'] \\ B & seB = eB \arrow[l, "\eta_B"] \end{tikzcd}$$

where ef (resp. sf) is the unique morphism that makes commute the lower triangle of the first (resp. second) diagram and $s(ef) = sef$ (resp. esf) is the unique map that makes commute the upper triangle. Then we have

$$\begin{aligned} \eta_B \circ sef \circ \varepsilon_A &= \eta_B \circ ef = f \\ \eta_B \circ esf \circ \varepsilon_A &= sf \circ \varepsilon_A = f \end{aligned}$$

Since ε_A is epic and η_B is monic, we have $esf = sef$. ∎

Using the previous lemma, an explicit description of this multiplication is possible. Let

$$M' : (\mathbf{B} \multimap \mathbf{B}) \otimes (\mathbf{B} \multimap \mathbf{B}) \longrightarrow \mathbf{B} \multimap \mathbf{B}$$

be define by

$$\begin{array}{rccc} M'_1 : & ([B_1, B_1]\tilde{\times}[B_2, B_2])\hat{\otimes}([B_1, B_1]\tilde{\times}[B_2, B_2]) & \longrightarrow & [B_1, B_1]\tilde{\times}[B_2, B_2] \\ & (f_1, f_2) \otimes (g_1, g_2) & \longmapsto & (f_1 \circ g_1, g_2 \circ f_2) \end{array}$$

$$\begin{array}{rccc} M'_2 : & B_1\hat{\otimes}B_2 & \longrightarrow & \Big[[B_1, B_1]\tilde{\times}[B_2, B_2], B_1\hat{\otimes}B_2\Big] \,\tilde{\times}\, \Big[[B_1, B_1]\tilde{\times}[B_2, B_2], B_1\hat{\otimes}B_2\Big] \\ & b_1\hat{\otimes}b_2 & \longmapsto & (\alpha_1, \alpha_2) \end{array}$$

with $\alpha_1(f_1, f_2) = b_1 \otimes f_2(b_2)$ and $\alpha_2(g_1, g_2) = g_1(b_1) \otimes b_2$

Then a Chu-morphism

$$M'' : (\mathbf{B} \mathbin{-\!\Box} \mathbf{B}) \otimes (\mathbf{B} \mathbin{-\!\Box} \mathbf{B}) \longrightarrow \mathbf{B} \multimap \mathbf{B}$$

exists. Explicitly, M''_1 is M'_1 (since $(e(\mathbf{B} \multimap \mathbf{B}))_1 = (\mathbf{B} \multimap \mathbf{B})_1$) and M'_2 is defined by

$$\begin{array}{rccc} M''_2 : & B_1\hat{\otimes}B_2 & \longrightarrow & \Big[[B_1, B_1]\tilde{\times}[B_2, B_2], \widetilde{B_1\hat{\otimes}B_2}\Big] \,\tilde{\times}\, \Big[[B_1, B_1]\tilde{\times}[B_2, B_2], \widetilde{B_1\hat{\otimes}B_2}\Big] \\ & b_1 \otimes b_2 & \longmapsto & (k \circ \alpha_1, k \circ \alpha_2) \end{array}$$

where k is the map

$$k : B_1\hat{\otimes}B_2 \longrightarrow \widetilde{B_1\hat{\otimes}B_2} \subset ([B_1, B_1]\tilde{\times}[B_2, B_2])^*.$$

Then the multiplication on $\mathbf{B} \mathbin{-\!\Box} \mathbf{B}$ is $(M_1, M_2) = se(M''_1, M''_2) = es(M''_1, M''_2)$.

3.13. Lemma. *Let $\mathbf{A}$ be a separated Chu-space and note $e\mathbf{A} = (A_1, \tilde{A}_2)$. Then $V(e\mathbf{A}) = O(A_1, \sigma(A_1, \tilde{A}_2)) = O(A_1, \sigma(A_1, A_2)) = V(\mathbf{A})$.*

Proof. Remember that the functor V associates to a Chu space $\mathbf{A}$ the unit ball of the first component with the weak topology.

The space $e\mathbf{A}$ is defined by the decomposition of the following transpose of the pairing

$$\begin{tikzcd} A_2 \arrow[rr, "F"] \arrow[dr] & & A_1^* \\ & \tilde{A}_2 = \overline{\operatorname{im} F} \arrow[ur] & \end{tikzcd}$$

We have only to show that the topology defined by the image of F is stronger than the topology defined by the closed image of F, the other inclusion being evident.

Let φ be in $\overline{\operatorname{im} F}$, then there is a sequence $(\varphi_i)_i$ with φ_i in $\operatorname{im} F$ such that $\varphi = \lim_{i\to\infty} \varphi_i$, that is $\lim_{i\to\infty}(\sup_{||a_1||\leqslant 1} |\varphi(a_1) - \varphi_i(a_1)|) = 0$.

We want to show that for each $\varepsilon > 0$ there is an $\varepsilon_0 > 0$ and an i_0 in $\mathbb{N}$ such that

$$\{a_1;\ ||a_1|| \leqslant 1 \text{ and } |\varphi(a_1)| < \varepsilon\} \supset \{a_1;\ ||a_1|| \leqslant 1 \text{ and } |\varphi_{i_0}(a_1)| < \varepsilon_0\}.$$

Take $\varepsilon_0 = \frac{\varepsilon}{2}$. Then there exists i_0 in $\mathbb{N}$ such that $\sup_{||a_1||\leqslant 1} |\varphi(a_1) - \varphi_i(a_1)| < \varepsilon_0$ for all $i \geqslant i_0$. Then in particular $|\varphi(a_1) - \varphi_{i_0}(a_1)| < \varepsilon_0$ for all a_1 in A_1 with $||a_1|| \leqslant 1$. This is equivalent to $|\varphi(a_1)| < |\varphi_{i_0}(a_1)| + \varepsilon_0$ if $||a_1|| \leqslant 1$. Then, for such an a_1,

$$|\varphi_{i_0}(a_1)| < \varepsilon_0 \quad \text{implies} \quad |\varphi(a_1)| < |\varphi_{i_0}(a_1)| + \varepsilon_0 < \varepsilon_0 + \varepsilon_0 = \varepsilon. \qquad \blacksquare$$

3.14. Proposition. *Let ρ be a weakly continuous isometric representation. Then there is a unique morphism of Chu-algebras $R : LG \longrightarrow \mathbf{B} \mathbin{-\!\Box} \mathbf{B}$ such that $(R_1(1 \cdot g))_1 = \rho(g)$.*

Proof. The second part of the definition of a weakly continuous representation is that $\rho(g) : B_1 \longrightarrow B_1$ is weakly continuous with respect to the duality $\langle -, - \rangle : B_1 \widehat{\otimes} B_2 \longrightarrow \mathbb{C}$. In this case $\rho(g)' : (B_1, \sigma(B_1, B_2))' \cong B_2 \longrightarrow B_2 \cong (B_1, \sigma)'$ exists and the previous lemma and first part of the definition of a weakly continuous representation show that

$$G \longrightarrow O([B_1, B_1] \widetilde{\times} [B_2, B_2], \widetilde{B_1 \widehat{\otimes} B_2}) \quad g \longmapsto (\rho(g), \rho(g)')$$

is continuous.

Then, by theorem 3.4, ρ defines

$$R_1 : l^1(G) \longrightarrow [B_1, B_1] \widetilde{\times} [B_2, B_2] \qquad 1 \cdot g \longmapsto (\rho(g), \rho(g)').$$

and $R_2 : B_1 \widehat{\otimes} B_2 \longrightarrow \mathcal{C}(G)$ by

$$R_2 : b_1 \otimes b_2 \longmapsto (g \longmapsto \langle (\rho(g), \rho(g)'), b_1 \otimes b_2 \rangle = \langle \rho(g) b_1, b_2 \rangle).$$

The morphism (R_1, R_2) is also an LG-morphism, that is the following diagram commutes.

$$\begin{tikzcd} LG \boxtimes LG \arrow[r, "m"] \arrow[d, "R \boxtimes R"'] & LG \arrow[d, "R"] \\ (\mathbf{B} \mathbin{-\!\Box} \mathbf{B}) \boxtimes (\mathbf{B} \mathbin{-\!\Box} \mathbf{B}) \arrow[r, "M"] & \mathbf{B} \mathbin{-\!\Box} \mathbf{B} \end{tikzcd}$$

To show this, it is sufficient to prove the commutativity of the following diagram

$$\begin{tikzcd} LG \otimes LG \arrow[r, "m"] \arrow[d, "R \otimes R"'] & LG \arrow[d, "R"] \\ (\mathbf{B} \mathbin{-\!\Box} \mathbf{B}) \otimes (\mathbf{B} \mathbin{-\!\Box} \mathbf{B}) \arrow[r, "M"] & \mathbf{B} \mathbin{-\!\Box} \mathbf{B} \end{tikzcd}$$

which commutes since the following diagram does as direct calculation shows and all the spaces are extensional.

$$\begin{array}{ccc} l^1(G)\widehat{\otimes} l^1(G) & \xrightarrow{\quad m_1 \quad} & l^1(G) \\ {\scriptstyle R_1\widehat{\otimes} R_1}\Big\downarrow & & \Big\downarrow{\scriptstyle R_1} \\ ([B_1,B_1]\widetilde{\times}[B_2,B_2])\widehat{\otimes}([B_1,B_1]\widetilde{\times}[B_2,B_2]) & \xrightarrow{M_1'} & [B_1,B_1]\widetilde{\times}[B_2,B_2] \end{array}$$

■

Let G be a Hausdorff topological group and $\mathbf{B}$ a separated-extensional Chu-space. Write $\mathcal{R}ep_G$ the category of isometric weakly continuous representations of G with intertwining (weakly continuous) operators and $\mathcal{CB}_G$ the category of LG-module-structures with Chu-module-morphisms.

The previous proposition defines a functor $F : \mathcal{R}ep_G \longrightarrow \mathcal{CB}_G$ on objects, on arrows F is defined in a natural way.

A functor $H : \mathcal{CB}_G \longrightarrow \mathcal{R}ep_G$ is defined as follows. Write the Chu-module-structure on $\mathbf{B}$ as $f : LG \boxtimes \mathbf{B} \longrightarrow \mathbf{B}$, then the associated map $\widetilde{f} : LG \longrightarrow \mathbf{B} \multimap \mathbf{B}$ is a Chu-algebra-morphism and the image by the functor H is given by

$$\rho = H(LG \xrightarrow{\widetilde{f}} \mathbf{B} \multimap \mathbf{B}) : G \longrightarrow [B_1, B_1] \qquad\qquad g \longmapsto (\widetilde{f}_1(1\cdot g))_1$$

On arrows, H is defined in a natural way.

3.15. Theorem. *The functors F and H define an equivalence of the categories $\mathcal{R}ep_G$ and $\mathcal{CB}_G$.*

Proof. Obviously, we have $H \circ F = \mathbf{1}$ and the fact that the second component of a separated extensional Chu-space is isomorphic to weak dual of the first component shows that $F \circ H \cong \mathbf{1}$. ■

4. Induced representations

In the following text, let G be a topological Hausdorff group and H a subgroup of G. Let $\mathbf{B}$ be a separated extensional Chu-space and an LH-Chu-module. In an obvious way there is a right (and also a left) multiplication with elements of H defined on the space $l^1(G)$.

Define the following subspace $\mathcal{S}'$ of the vector space $l^1(G)\widehat{\otimes} B_1$ to be

$$\mathcal{S}' = \mathrm{span}\{(z\cdot h)\otimes b - z\otimes(h\cdot b),\ \text{ for } \ z\in l^1(G),\, b\in B_1,\, h\in H\}.$$

Then let $\mathcal{S}$ be the closure $\overline{\mathcal{S}'}$ of $\mathcal{S}'$ in the Banach space $l^1(G)\widehat{\otimes} B_1$. We may consider the quotient space $(l^1(G)\widehat{\otimes} B_1)/\mathcal{S}$ and equip it with the quotient norm

$$||[z]|| = \inf_{w\in\mathcal{S}} ||z+w||_{l^1(G)\widehat{\otimes} B_1}.$$

Define for two $l^1(H)$-modules C_1 and C_2

$$_{l^1(H)}\operatorname{Ban}_1(C_1, C_2) = \{f : C_1 \longrightarrow C_2 \in \operatorname{Ban}_\infty;\ h \cdot f(c_1) = f(h \cdot c_1),\ \text{for all } h \in H\}$$

Then we can define the induced module $LG \otimes_{LH} \mathbf{B}$ to be $s\mathbf{N}$ where

$$\begin{aligned} N_1 &= (l^1(G)\widehat{\otimes} B_1)/\mathcal{S} \\ N_2 &= \{\lambda \in {}_{l^1(H)}\operatorname{Ban}(B_1, \mathcal{C}(G)); \lambda(-)(g) \in (B_1, \sigma)' \cong B_2, \text{for all } g \in G\}. \end{aligned}$$

The $l^1(H)$-module structure of $\mathcal{C}(G)$ is defined as $(h \cdot \lambda)(g) = \lambda(g \cdot h)$.

The space N_2 is a normed space and a Banach space because

$$\begin{aligned} {}_{l^1(H)}\operatorname{Ban}(B_1, \mathcal{C}(G)) = \{f : B_1 \longrightarrow \mathcal{C}(G); f \in \operatorname{Ban}_\infty;\ f(-)(g) \in B_1^*, \\ \text{for all } g \in G \text{ and } f(hb_1) = hf(b_1)\} \end{aligned}$$

and $B_2 \cong (B_1, \sigma(B_1, B_2))' \subset B_1^*$. Then B_1' is closed in B_1^* as a Banach space. The fact that the representation of H is isometric finishes the proof of the fact that N_2 is complete.

The duality is defined in the following way. Let λ in N_2 be fixed, define the bilinear map ψ by

$$\psi : l^1(G) \times B_1 \longrightarrow \mathbb{C} \qquad (\sum_{i \geqslant 0} a_i g_i, b_1) \longmapsto \sum_{i \geqslant 0} a_i \lambda(b_1)(g_i)$$

It is bounded by $||\lambda||$.

Then there exists a linear bounded map with the same norm on the projective tensor product. The following sequence of equalities imply that then there is also a linear map on the quotient space with the same norm

$$\begin{aligned} \psi(\sum_{i \geqslant 0} a_i g_i h, b_1) &= \sum_{i \geqslant 0} a_i \lambda(b_1)(g_i h) = \sum_{i \geqslant 0} a_i (h \cdot \lambda(b_1))(g_i) \\ &= \sum_{i \geqslant 0} a_i \lambda(hb_1)(g_i) = \psi(\sum_{i \geqslant 0} a_i g_i, hb_1). \end{aligned}$$

Varying λ gives a bilinear map on $N_1 \times N_2$ of norm at most 1 which can be extended to the tensor product and this defines the pairing on (N_1, N_2).

The two previous remarks construct (N_1, N_2) as a Chu-space that is even extensional. To show this, we have to prove that the map $N_2 \longrightarrow N_1^*$ is isometric (it is contracting by definition), that is that for f in N_2

$$\sup\{|\sum_{i \geqslant 0} a_i f(b_i)(g_i)|;\ ||\sum_{i \geqslant 0} a_i [g_i \otimes b_i]|| \leqslant 1\} = ||f||$$

There exists a b_0 in B_1 with $||b_0|| \leqslant 1$ and a g_0 in G such that $||f|| - |f(b_0)(g_0)| < \varepsilon$.

We do not know if (N_1, N_2) is separated in general, but $s(N_1, N_2)$ is a separated-extensional Chu-space.

There exists an LG-Chu-module structure on this space. Define first a map f' : $LG \otimes (N_1, N_2) \longrightarrow (N_1, N_2)$ in the following way

$$\begin{array}{rrcl} f_1': & l^1(G) \otimes ((l^1(G)\widehat{\otimes} B_1)/\mathcal{S}) & \longrightarrow & (l^1(G)\widehat{\otimes} B_1)/\mathcal{S} \\ & 1 \cdot g \otimes [1 \cdot g_1 \otimes b] & \longmapsto & [1 \cdot g \cdot g_1 \otimes b] \\ f_2': & N_2 & \longrightarrow & [l^1(G), N_2]\widetilde{\times}[N_1, \mathcal{C}(G)] \\ & \lambda & \longmapsto & (\alpha_1, \alpha_2) \end{array}$$

where α_1 and α_2 are defined as follows.

Let $1 \cdot g$ be in $l^1(G)$, b_1 in B_1 and g_1 in G. Then we have $\alpha_1(1 \cdot g)(b_1)(g_1) = \lambda(b_1)(gg_1)$. Note that $\alpha_1(1 \cdot g)$ is in N_2 because we require that by definition $\lambda(-)(g)$ is in $(B_1, \sigma)'$ for all g in G. The map α_2 is defined by $\alpha_2([1 \cdot g \otimes b_1])(g_1) = \lambda(b_1)(g_1 g)$. It is well defined because multiplication in a topological group is continuous by definition and $\lambda(b_1)$ is in $\mathcal{C}(G)$ for all b_1 in B_1. Then (α_1, α_2) preserves the duality. Both maps are linear by definition and bounded with norm less or equal than $||\lambda||$. Furthermore, the map (f_1', f_2') preserves the duality and both maps are contracting.

The associated map $(\widetilde{f_1'}, \widetilde{f_2'}) : LG \longrightarrow (N_1, N_2) \multimap (N_1, N_2)$ is equal to

$$\widetilde{f_1'} : l^1(G) \longrightarrow [N_1, N_1]\widetilde{\times}[N_2, N_2] \qquad \sum_{i \geqslant 0} a_i g_i \longmapsto (\beta_1, \beta_2)$$

where

$$\begin{array}{l} \beta_1([g \otimes b]) = f_1'(\sum_{i\geqslant 0} a_i g_i \otimes [g \otimes b]) = \sum_{i \geqslant 0} a_i [g_i g \otimes b] \\ \beta_2(\lambda) = (f_2'(\lambda))_1(\sum_{i\geqslant 0} a_i g_i) = \sum_{i\geqslant 0} a_i \lambda(-)(g_i -) \in N_2 \end{array}$$

and

$$\widetilde{f_2'} : N_1 \widehat{\otimes} N_2 \longrightarrow \mathcal{C}(G) \qquad [g \otimes b] \otimes \lambda \longmapsto (f_2'(\lambda))_2([g \otimes b]) = \lambda(b)(-g)$$

The map $f' : LG \otimes (N_1, N_2) \longrightarrow (N_1, N_2)$ defines a map

$$LG \otimes (N_1, N_2) \longrightarrow (N_1, N_2) \longrightarrow s(N_1, N_2)$$

which is a Chu-morphism by definition and this map defines a

$$F' : LG \otimes s(N_1, N_2) \longrightarrow s(N_1, N_2)$$

using the following chain of adjunctions.

$$\begin{array}{c} \mathrm{Hom}(LG \otimes \mathbf{N}, s\mathbf{N}) \cong \mathrm{Hom}(\mathbf{N}, LG \multimap s\mathbf{N}) \cong \mathrm{Hom}(s\mathbf{N}, LG \multimap s\mathbf{N}) \\ \cong \mathrm{Hom}(LG \otimes s\mathbf{N}, s\mathbf{N}). \end{array}$$

Note that the space $LG \multimap s\mathbf{N}$ is separated since LG is extensional and $s\mathbf{N}$ is separated.

The first component of the space $s(N_1, N_2)$ can be seen as the completion of a quotient space. Take z in N_1 and let $[\![z]\!] = \langle z, - \rangle$ be the notation for the equivalence class with respect to this quotient. Note that the fact that the functor s is adjoint to the inclusion means, in this special case, that if a Chu-morphism f from $\mathbf{A}$ to a separated space is given then there is a Chu-morphism f' on $s\mathbf{A}$ with $f'_1([\![z]\!]) = f_1(z)$ for any element z of the equivalence class.

The first component is explicitly given by the following formula. Let $[\![[g \otimes b]]\!]$ be in sN_1, λ in N_2.

$$[\![F'_1(\sum_{i\geqslant 0} a_i g_i \otimes [\![[g\otimes b]]\!]]\!](\lambda) = \sum_{i\geqslant 0} a_i\lambda(b)(g_i g) = [\![\sum_{i\geqslant 0} a_i[g_i \cdot g \otimes b]]\!](\lambda)$$

and the second component is given by

$$F'_2(\lambda) = (\beta_1, \beta_2) \quad \text{with} \quad \begin{array}{l} \beta_1(\sum_{i\geqslant 0} a_i g_i)(b)(g) = \sum_{i\geqslant 0} a_i \lambda(b)(g_i g) \\ \beta_2([\![[g_1 \otimes b_1]]\!])(g) = \lambda(b)(gg_1) \end{array}$$

and the associated map $\widetilde{F}' : LG \longrightarrow s\mathbf{N} \multimap s\mathbf{N}$ by

$$\begin{array}{lll} \widetilde{F}'_1(\sum_{i\geqslant 0} a_i g_i) = (\gamma_1, \gamma_2) & \text{with} & \gamma_1([\![[g\otimes b]]\!]) = [\![\sum_{i\geqslant 0} a_i[g_i g \otimes b]]\!] \\ & & \gamma_2(\lambda)(b)(g) = \sum_{i\geqslant 0} a_i\lambda(b)(g_i g) \\ \widetilde{F}'_2([\![[g\otimes b]]\!] \otimes \lambda) = \sum_{i\geqslant 0} a_i \lambda(b)(g_i g). & & \end{array}$$

To prove that $(\widetilde{F}'_1, \widetilde{F}'_2)$ is an algebra morphism, it is sufficient to prove the commutativity of the following diagram in the first component since all the spaces are extensional.

$$\begin{tikzcd} LG \otimes LG \arrow[r] \arrow[d] & (s\mathbf{N} \multimap s\mathbf{N}) \otimes (s\mathbf{N} \multimap s\mathbf{N}) \arrow[d] \\ LG \arrow[r] & s\mathbf{N} \multimap s\mathbf{N} \end{tikzcd}$$

The diagram in the first component is

$$\begin{tikzcd} l^1(G)\widehat{\otimes} l^1(G) \arrow[r, "\widetilde{F}'_1 \widehat{\otimes} \widetilde{F}'_1"] \arrow[d, "m_1"'] & ([sN_1, sN_1]\widetilde{\times}[N_2, N_2])\widehat{\otimes}([sN_1, sN_1]\widetilde{\times}[N_2, N_2]) \arrow[d, "M'_1"] \\ l^1(G) \arrow[r, "\widetilde{F}'_1"] & [sN_1, sN_1]\widetilde{\times}[N_2, N_2] \end{tikzcd}$$

We have

$$\widetilde{F}'_1 \circ m_1(\sum_{i\geqslant 0} a_i g_i \otimes \sum_{j\geqslant 0} c_j k_j) = \widetilde{F}'_1(\sum_{i,j\geqslant 0} a_i c_j (g_i \cdot k_j)) = (\gamma_1, \gamma_2)$$

with

$$\begin{array}{l} \gamma_1([\![[g\otimes b]]\!]) = \sum_{i,j\geqslant 0} a_i c_j [\![[(g_i k_j)\cdot g \otimes b]]\!] \\ \gamma_2(\lambda)(b)(g) = \sum_{i,j\geqslant 0} a_i c_j \lambda(b)(g_i k_j g) \end{array}$$

and

$$M_1' \circ \widetilde{F}_1' \otimes \widetilde{F}_1' = M_1'((\alpha_1, \alpha_2) \otimes (\beta_1, \beta_2)) = (\alpha_1 \circ \beta_1, \beta_2 \circ \alpha_2)$$

with

$$\begin{aligned}
\alpha_1 \circ \beta_1([\![[g \otimes b]]\!]) &= \alpha_1(\sum_{j \geqslant 0} c_j [\![[k_j \cdot g \otimes b]]\!]) = \sum_{j \geqslant 0} c_j \alpha_1([\![[k_j g \otimes b]]\!]) \\
&= \sum_{j \geqslant 0} c_j \sum_{i \geqslant 0} a_i [\![[g_i k_j g \otimes b]]\!] = \sum_{ij \geqslant 0} c_j a_i [\![[g_i k_j g \otimes b]]\!]
\end{aligned}$$

and

$$\begin{aligned}
(\beta_2 \circ \alpha_2(\lambda))(b)(g) &= \beta_2(\underbrace{\sum_{i \geqslant 0} a_i \lambda(-)(g_i -)}_{\lambda'})(b)(g) \\
&= \sum_{j \geqslant 0} c_j \lambda'(b)(k_j g) = \sum_{j \geqslant 0} c_j (\sum_{i \geqslant 0} a_i \lambda(b)(g_i k_j g)).
\end{aligned}$$

4.1. Lemma. *The following diagram commutes*

$$\begin{tikzcd}
LG \otimes (N_1, N_2) \arrow[r, "\mathbf{1} \otimes p"] \arrow[d, "n"] & LG \otimes s(N_1, N_2) \arrow[d, "l"] \\
(N_1, N_2) \arrow[r, "p"] & s(N_1, N_2)
\end{tikzcd}$$

where the vertical arrows are the LG-module-structures on (N_1, N_2) and $s(N_1, N_2)$ respectively.

Proof. By direct calculation prove that the diagram commutes in the first component which is sufficient since all the spaces are extensional. ■

Let now the following map $i : (B_1, B_2) \longrightarrow (N_1, N_2)$ be defined

$$\begin{array}{ll}
i_1 : B_1 \longrightarrow N_1 & b_1 \longmapsto [1 \cdot e \otimes b_1] \\
i_2 : N_2 \longrightarrow B_2 \cong (B_1, \sigma)' & \lambda \longmapsto (b_1 \mapsto \lambda(b_1)(e))
\end{array}$$

where e is the unit of the group and we know that by definition $\lambda(-)(e)$ is in $(B_1, \sigma)'$.

4.2. Proposition. *Let* $\mathbf{B}$ *be an LH-Chu-module,* $\mathbf{D}$ *an LG-module,* $f : \mathbf{B} \longrightarrow \mathbf{D}$ *an LH-morphism. Then there exists a unique LG-morphism* $h : s(N_1, N_2) \longrightarrow \mathbf{D}$ *such that the following diagram commutes.*

$$\begin{tikzcd}
\mathbf{B} \arrow[r, "s(i)"] \arrow[dr, "f"'] & s(N_1, N_2) \arrow[d, "h"] \\
 & \mathbf{D}
\end{tikzcd}$$

Proof. Note that $s(i) = p \circ i$ where $p : (N_1, N_2) \longrightarrow s(N_1, N_2)$. Let us show first that i has the following universal property Let $\mathbf{B}$, $\mathbf{D}$ and f be as in the proposition. Then there exists a unique LG-morphism $h^0 : \mathbf{N} \longrightarrow \mathbf{D}$ such that the following diagram commutes.

$$\begin{tikzcd}
\mathbf{B} \arrow[r, "s(i)"] \arrow[dr, "f"'] & (N_1, N_2) \arrow[d, "h^0"] \\
& \mathbf{D}
\end{tikzcd}$$

Define

$$\begin{array}{lccl}
h_1^0 : N_1 \longrightarrow D_1 & [g \otimes b] & \longmapsto & g \cdot f_1(b) \\
h_2^0 : D_2 \longrightarrow N_2 & d_2 & \longmapsto & (b_1 \mapsto (k \mapsto \langle k f_1(b_1), d_2 \rangle))\,.
\end{array}$$

The map h_1^0 is well defined and contracting. Furthermore, we have to show that $h_2^0(d_2)(b_1)$ is in $\mathcal{C}(G)$ for each b_1 in B_1 and d_2 in D_2. Let the first component of the LG-module structure on $\mathbf{D}$ be written as $n_1 : l^1(G) \longrightarrow [D_1, D_1] \widetilde{\times} [D_2, D_2]$. Then

$$\begin{aligned}
h_2^0(d_2)(b_1)(g) &= \langle (n_1(g))_1(f_1(b_1)), d_2 \rangle = \langle n_1(g), f_1(b_1) \otimes d_2 \rangle \\
&= \langle 1 \cdot g, n_2(f_1(b_1) \otimes d_2) \rangle = n_2(f_1(b_1) \otimes d_2)(g)
\end{aligned}$$

but by definition $n_2(f_1(b_1) \otimes d_2)$ is in $\mathcal{C}(G)$. The map $h_2^0(d_2)$ is bounded of norm less or equal to $||d_2||$ and $h_2^0(d_2)(.)(g)$ is weakly continuous on B_1. It is also an H-morphism because f_1 is and h_2^0 is contracting.

The map (h_1^0, h_2^0) preserves the duality and is also an LG-morphism since the following diagram commutes in the first component

$$\begin{tikzcd}
LG \otimes (N_1, N_2) \arrow[r, "\mathbf{1} \otimes (h_1^0, h_2^0)"] \arrow[d] & LG \otimes \mathbf{D} \arrow[d] \\
(N_1, N_2) \arrow[r, "(h_1^0, h_2^0)"] & \mathbf{D}
\end{tikzcd}$$

$$\begin{aligned}
h_1^0(g \cdot [g_1 \otimes b_1]) &= h_1^0([g g_1 \otimes b_1]) = g g_1 \cdot f_1(b_1) \\
g \cdot h_1^0([g_1 \otimes b_1]) &= g \cdot (g_1 \cdot f_1(b)).
\end{aligned}$$

This shows the existence of h^0.

The uniqueness comes from the uniqueness of the first component. Two G-morphisms h_1^0 and h_1^1 that make commute the above diagram commute are equal since

$$\begin{aligned}
h_1^0([g_1 \otimes b_1]) &= h_1^0(g_1 \cdot [e \otimes b_1]) = g_1 \cdot h_1^0([e \otimes b_1]) = g_1 \cdot f_1([e \otimes b_1]) \\
&= g_1 \cdot h_1^1([e \otimes b_1]) = h_1^1(g_1 \cdot [e \otimes b_1]) = h_1^1([g_1 \otimes b_1]).
\end{aligned}$$

Consider now the following diagram.

$$
\begin{tikzcd}
\mathsf{B} \arrow[r, "i"] \arrow[dr, "\forall f"'] & (N_1, N_2) \arrow[r, "p"] \arrow[d, dotted, "\exists ! h^0"] & s(N_1, N_2) \arrow[dl, dotted, "\exists ! h"] \\
& \mathsf{D} &
\end{tikzcd}
$$

Using the fact that s is left adjoint to the inclusion, we know that h exists and that $h \circ p = h^0$. To show that h is an LG-morphism we have to show the commutativity of the following diagram.

$$
\begin{tikzcd}
LG \otimes s\mathsf{N} \arrow[r, "\mathbf{1} \otimes h"] \arrow[d, "l"'] & LG \otimes \mathsf{D} \arrow[d, "k"] \\
s\mathsf{N} \arrow[r, "h"] & \mathsf{D}
\end{tikzcd}
$$

All the spaces being extensional it is sufficient to show the commutativity in the first component

$$
\begin{tikzcd}
& l^1(G)\widehat{\otimes} N_1 \arrow[dl, "n_1"'] \arrow[d, "\mathbf{1} \otimes p_1"'] \arrow[dr, "\mathbf{1} \otimes h_1^0"] & \\
N_1 \arrow[dr, "p_1"'] & l^1(G)\widehat{\otimes} sN_1 \arrow[r, "\mathbf{1} \otimes h_1"'] \arrow[d, "l_1"'] & l^1(G)\widehat{\otimes} D_1 \arrow[d, "k_1"] \\
& sN_1 \arrow[r, "h_1"] & D_1
\end{tikzcd}
$$

The left square commutes because of the previous lemma. The upper triangle commutes by definition of h. Furthermore $p_1(N_1)$ is dense in sN_1. Then also $\mathbf{1}\widehat{\otimes}p_1(l^1(G)\widehat{\otimes}N_1)$ is dense in $l^1(G)\widehat{\otimes}sN_1$ which shows that $\mathbf{1}\widehat{\otimes}p_1$ is an epi (2.9). We also have

$$k_1 \circ \mathbf{1} \otimes h_1^0 = h_1^0 \circ n_1 = h_1 \circ p_1 \circ n_1$$

since h^0 is an LG-morphism. Then

$$h_1 \circ l_1 \circ \mathbf{1} \otimes p_1 = h_1 \circ p_1 \circ n_1 = k_1 \circ \mathbf{1} \otimes h_1^0 = k_1 \circ \mathbf{1} \otimes h_1 \circ \mathbf{1} \otimes p_1$$

which implies

$$h_1 \circ l_1 = k_1 \circ \mathbf{1} \otimes h_1$$

since $\mathbf{1}\widehat{\otimes}p_1$ is an epi. ■

The construction of the induced module (N_1, N_2) can be formulated in the following way. The tensor product over LH of the left LH-module LG and the right LH-module $\mathbf{B}$ in the category of Chu-spaces shall be defined in the same way as for the classic tensor product of modules over an algebra. Consider the following two maps

$$LG \otimes LH \otimes \mathbf{B} \cong (LG \otimes LH) \otimes \mathbf{B} \longrightarrow LG \otimes \mathbf{B}$$
$$LG \otimes LH \otimes \mathbf{B} \cong LG \otimes (LH \otimes \mathbf{B}) \longrightarrow LG \otimes \mathbf{B}$$

The first map is defined by the left LH-module structure of LG that is the restriction of the multiplication on LG to LH. The second map is defined by the right LH-module structure of $\mathbf{B}$. It is not difficult to check that (N_1, N_2) is then the co-equalizer of the two maps.

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