*Theory and Applications of Categories*, Vol. 20, No. 6, 2008, pp. 85–115.

# BALANCED CATEGORY THEORY

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Abstract.

Some aspects of basic category theory are developed in a finitely complete category $\mathcal{C}$, endowed with two factorization systems which determine the same discrete objects and are linked by a simple reciprocal stability law. Resting on this axiomatization of final and initial functors and discrete (op)fibrations, concepts such as components, slices and coslices, colimits and limits, left and right adjunctible maps, dense maps and arrow intervals, can be naturally defined in $\mathcal{C}$, and several classical properties concerning them can be effectively proved.

For any object $X$ of $\mathcal{C}$, by restricting $\mathcal{C}/X$ to the slices or to the coslices of $X$, two dual "underlying categories" are obtained. These can be enriched over internal sets (discrete objects) of $\mathcal{C}$: internal hom-sets are given by the components of the pullback of the corresponding slice and coslice of $X$. The construction extends to give functors $\mathcal{C} \to \mathbf{Cat}$, which preserve (or reverse) slices and adjunctible maps and which can be enriched over internal sets too.

## 1. Introduction

Following [Pisani, 2007c], we further pursue the goal of an axiomatization of the category of categories based on the comprehensive factorization system. Here, the emphasis is on the simultaneous consideration of both the comprehensive factorization systems $(\mathcal{E}, \mathcal{M})$ and $(\mathcal{E}', \mathcal{M}')$ on $\mathbf{Cat}$, where $\mathcal{E}$ ($\mathcal{E}'$) is the class of final (initial) functors and $\mathcal{M}$ ($\mathcal{M}'$) is the class of discrete (op)fibrations. Rather than on the duality functor $\mathbf{Cat} \to \mathbf{Cat}$, we rest on the following "reciprocal stability" property: *the pullback of an initial (final) functor along a discrete (op)fibration is itself initial (final)* (Proposition 2.3).

A "balanced factorization category" is a finitely complete category $\mathcal{C}$ with two factorization systems on it, which are reciprocally stable in the above sense and generate the same subcategory of "internal sets": $\mathcal{S} := \mathcal{M}/1 = \mathcal{M}'/1 \hookrightarrow \mathcal{C}$. It turns out that these simple axioms are remarkably powerful, allowing one to develop within $\mathcal{C}$ some important aspects of category theory in an effective and transparent way.

In particular, the reflection of an object $X \in \mathcal{C}$ in $\mathcal{S}$ gives the internal set of components $\pi_0 X \in \mathcal{S}$, while the reflections of a "point" $x : 1 \to X$ in $\mathcal{M}/X$ and $\mathcal{M}'/X$ are the

Received by the editors 2008-02-15 and, in revised form, 2008-03-17.

Transmitted by F. W. Lawvere. Published on 2008-03-20.

2000 Mathematics Subject Classification: 18A99.

Key words and phrases: factorization systems, final maps and discrete fibrations, initial maps and discrete opfibrations, reflections, internal sets and components, slices and coslices, colimiting cones, adjunctible maps, dense maps, cylinders and homotopy, arrow intervals, enrichment, complements.

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slice $X/x$ and the coslice $x\backslash X$, with their maps to $X$. One can define two functors $\mathcal{C} \to \mathbf{Cat}$, which on $X \in \mathcal{C}$ give the "underlying" categories $\overline{X}(x,y) := \mathcal{C}/X(X/x, X/y)$ and $\overline{X}'(x,y) := \mathcal{C}/X(x\backslash X, y\backslash X)$. Furthermore, if $X[x,y] := x\backslash X \times_X X/y$, the internal set $X(x,y) := \pi_0 X[x,y]$ enriches both $\overline{X}(x,y)$ and $\overline{X}'(y,x)$. Also composition in $\overline{X}$ and $\overline{X}'$ can be enriched to a natural "symmetrical" composition map $X(x,y) \times X(y,z) \to X(x,z)$, and it follows in particular that the underlying categories are dual.

In fact, the term "balanced" is intended to reflect the fact that the objects of $\mathcal{C}$ carry two dual underlying categories, coexisting on the same level. It is the choice of one of the two factorization systems which determines one of them as "preferred": in fact, we will make only terminologically such a choice, by calling "final" ("initial") the maps in $\mathcal{E}$ ($\mathcal{E}'$) and "discrete fibrations" ("opfibrations") those in $\mathcal{M}$ ($\mathcal{M}'$). For example, we will see that the underlying functor $\mathcal{C} \to \mathbf{Cat}$ determined by $(\mathcal{E}, \mathcal{M})$ preserves discrete fibrations and opfibrations, final and initial points, slices and coslices and left and right adjunctible maps, while the other one reverses them.

The structure of balanced factorization category also supports the notion of "balanced" cylinder and "balanced" homotopy. In particular, for any $x, y : 1 \to X$ we have a cylinder (see [Lawvere, 1994])

$$X(x,y) \underset{e}{\overset{i}{\rightrightarrows}} X[x,y] \xrightarrow{c} X(x,y)$$

with $i \in \mathcal{E}'$, $e \in \mathcal{E}$ and $c \in \mathcal{E} \cap \mathcal{E}'$, and pulling back along an element $\alpha : 1 \to X(x,y)$ (corresponding to an arrow of the underlying categories) we get the "arrow interval" $[\alpha]$:

$$1 \underset{e}{\overset{i}{\rightrightarrows}} [\alpha]$$

which in **Cat** is the category of factorizations of $\alpha$ with the two factorizations through identities.

Another consequence of the axioms is the following exponential law (Proposition 5.9): if $m \in \mathcal{M}/X$, $n \in \mathcal{M}'/X$ and the exponential $n^m$ exists in $\mathcal{C}/X$, then it is in $\mathcal{M}'/X$. Thus, if $m \in \mathcal{M}/X$ has a "complement" (see [Pisani, 2007a] and [Pisani, 2007b]):

$$\neg m : \mathcal{S} \to \mathcal{C}/X \qquad S \mapsto (\pi_S)^m$$

(where $\pi_S : S \times X \to X$), it is valued in $\mathcal{M}'/X$, and conversely. In the balanced factorization category of posets (see Section 5.11), the internal sets are the truth values and the above enrichment gives the standard identification of posets with 2-valued categories, while the complement operator becomes the usual (classical) one, relating upper-sets and lower-sets.

## 2. Some categorical notions

We here stress some properties of **Cat** and of the comprehensive factorization systems $(\mathcal{E}, \mathcal{M})$ and $(\mathcal{E}', \mathcal{M}')$, on which we will model the abstraction of Sections 3 and 4.

2.1. Slices and intervals. Recall that given two functors $p : P \to X$ and $q : Q \to X$, the map (or "comma") construction $(p, q)$ yields as particular cases the slices $X/x = (\mathrm{id}, x)$ or coslices $x\backslash X = (x, \mathrm{id})$, for any object $x : 1 \to X$. More generally, $p/x = (p, x)$ and $x\backslash q = (x, q)$ can be obtained as the pullbacks

$$\begin{array}{ccc} p/x & \longrightarrow & X/x \\ \downarrow & & \downarrow \\ P & \xrightarrow{p} & X \end{array} \qquad \begin{array}{ccc} x\backslash q & \longrightarrow & x\backslash X \\ \downarrow & & \downarrow \\ Q & \xrightarrow{q} & X \end{array} \tag{1}$$

In particular, the pullback

$$\begin{array}{ccc} [x, y] & \longrightarrow & X/y \\ \downarrow & & \downarrow \\ x\backslash X & \longrightarrow & X \end{array} \tag{2}$$

is the category of factorizations between $x$ and $y$, which has as objects consecutive arrows $\beta : x \to z$ and $\gamma : z \to y$ in $X$, and as arrows the diagonals $\delta$ as below:

$$\begin{array}{ccc} x & \xrightarrow{\beta'} & z' \\ {\scriptstyle\beta}\downarrow & \overset{\delta}{\nearrow} & \downarrow{\scriptstyle\gamma'} \\ z & \xrightarrow{\gamma} & y \end{array} \tag{3}$$

Applying the components functor $\pi_0 : \mathbf{Cat} \to \mathbf{Set}$ we get $\pi_0[x, y] \cong X(x, y)$. Indeed $[x, y]$ is the sum $\sum[\alpha], \alpha : x \to y$, where $[\alpha]$ is the "arrow interval" category of factorizations of $\alpha$; $\alpha = \gamma\beta$ is an initial (terminal) object of $[\alpha]$ iff $\beta$ ($\gamma$) is an isomorphism.

For any functor $f : X \to Y$ and any $x \in X$, there is an obvious "slice functor" of $f$ at $x$:

$$e_{f,x} : X/x \to Y/fx \tag{4}$$

2.2. The comprehensive factorization systems. Recall that a functor is a discrete fibration if it is orthogonal to the codomain functor $t : 1 \to 2$, and that a functor $p : P \to X$ is final if $\pi_0(x\backslash p) = 1$ for any $x \in X$. For example, an object $e : 1 \to X$ is final iff it is terminal ($\pi_0(x\backslash e) = \pi_0 X(x, e) = \pi_0 1 = 1$), the identity (or any isomorphism) $X \to X$ is final ($\pi_0(x\backslash \mathrm{id}) = \pi_0(x\backslash X) = 1$, since $x\backslash X$ has an initial object and so it is connected) and $! : X \to 1$ is final iff $X$ is connected ($\pi_0(0\backslash !) = \pi_0(X) = 1$).

Final functors and discrete fibrations form the "left" comprehensive factorization system $(\mathcal{E}, \mathcal{M})$ on **Cat**, which is in fact the (pre)factorization system generated by $t$. Since any $(\mathcal{E}, \mathcal{M})$-factorization gives a reflection in $\mathcal{M}/X$ (see Section 3) we have a left adjoint

$\downarrow_X \dashv i_X$ to the full inclusion of $\mathcal{M}/X \simeq \mathbf{Set}^{X^{\mathrm{op}}}$ in $\mathbf{Cat}/X$. The value of $\downarrow p$ at $x$ is given by the "reflection formula" (see [Pisani, 2007a] and [Pisani, 2007b], and the references therein):

$$(\downarrow p)x \cong \pi_0(x\backslash p) \tag{5}$$

Dually, the domain functor $s : 1 \to 2$ generates the "right" comprehensive factorization system $(\mathcal{E}', \mathcal{M}')$: initial functors and discrete opfibrations (that is the one originally considered in [Street & Walters, 1973]).

In particular, the reflection of an object $x : 1 \to X$ in $\mathcal{M}/X$ is the slice projection $\downarrow x : X/x \to X$ (corresponding to the representable presheaf $X(-, x)$) and the universal property of the reflection reduces to the (discrete fibration version of the) Yoneda Lemma. So, a functor to $X$ is isomorphic in $\mathbf{Cat}/X$ to the slice projection $X/x \to X$ iff it is a discrete fibration whose domain has a terminal object over $x$. Dually, denoting by $\uparrow_X : \mathbf{Cat}/X \to \mathcal{M}'/X$ the reflection in discrete opfibrations, $\uparrow x : x\backslash X \to X$ is the coslice projection.

On the other hand, the reflection of $X \to 1$ in $\mathcal{M}/1$ gives the components of $X$, that is, $\downarrow_1 : \mathbf{Cat}/1 \cong \mathbf{Cat} \to \mathcal{M}/1 \simeq \mathbf{Set}$ can be identified with the components functor $\pi_0$. The same of course holds for $\uparrow_1 : \mathbf{Cat} \to \mathcal{M}'/1 \simeq \mathbf{Set}$.

2.3. Proposition. *The pullback of an initial (final) functor along a discrete (op)fibration is itself initial (final).*

Proof. The two statements are clearly dual. Suppose that in the diagram below $q$ is final, $f$ is a discrete opfibration and both squares are pullbacks: we want to show that $x\backslash p$ is connected.

$$\begin{array}{ccccc} x\backslash p & \longrightarrow & P & \longrightarrow & Q \\ \downarrow & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ x\backslash X & \xrightarrow{\uparrow x} & X & \xrightarrow{f} & Y \end{array} \tag{6}$$

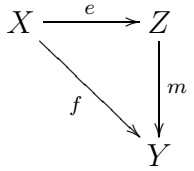
Since $f \circ \uparrow x$ is a discrete opfibration whose domain has an initial object over $fx$, it is isomorphic to the coslice $fx\backslash Y$ of $Y$; thus the vertex of the pullback rectangle is $x\backslash p \cong fx\backslash q$, which is connected by hypothesis. ∎

2.4. Remark. A functor $f : X \to Y$ is a discrete fibration iff the slice functor (4) is an isomorphism for any $x \in X$; or, equivalently, if its object mapping is bijective for any $x \in X$.

## 3. $(\mathcal{E}, \mathcal{M})$ category theory

Throughout the section, $\mathcal{C}$ will be an $(\mathcal{E}, \mathcal{M})$-category, that is a finitely complete category with a factorization system on it. For a brief survey on factorization systems and the

associated bifibrations on $\mathcal{C}$ we refer also to [Pisani, 2007c]. We just recall that an $(\mathcal{E},\mathcal{M})$-factorization of a map $f : X \to Y$ in $\mathcal{C}$

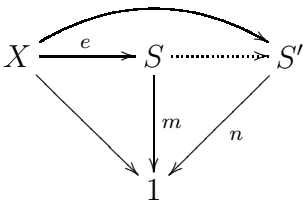



gives both a reflection of $f \in \mathcal{C}/Y$ in $\mathcal{M}/Y$ (with $e$ as reflection map) and a coreflection of $f \in X\backslash\mathcal{C}$ in $X\backslash\mathcal{E}$ (with $m$ as coreflection map). Conversely, any such (co)reflection map gives an $(\mathcal{E},\mathcal{M})$-factorization.

We assume that a "canonical" factorization of any arrow in $\mathcal{C}$ has been fixed, and denote by $\downarrow_X \dashv i_X$ the corresponding left adjoint to the full inclusion $i_X : \mathcal{M}/X \hookrightarrow \mathcal{C}/X$.

We now define and analyze several $(\mathcal{E},\mathcal{M})$-concepts, which assume their usual meaning when $\mathcal{C} = \mathbf{Cat}$ with the left comprehensive factorization system (see also [Pisani, 2007c], where the same topics are treated in a slightly different way, and other classical theorems are proved). Accordingly, we say that a map $e \in \mathcal{E}$ is **final**, while a map $m \in \mathcal{M}$ is a **discrete fibration**.

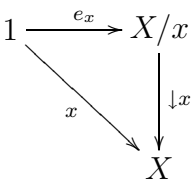
3.1. Sets and components. If $S \to 1$ is a discrete fibration, we say that $S$ is an **internal set** or also a **$\mathcal{C}$-set** (or simply a set) and we denote by $\mathcal{S} := \mathcal{M}/1 \hookrightarrow \mathcal{C}$ the reflective full subcategory of $\mathcal{C}$-sets; $\mathcal{S}$ is closed with respect to limits in $\mathcal{C}$, and so is itself finitely complete. The reflection $\pi_0 := \downarrow_1 : \mathcal{C} \to \mathcal{S}$ is the **components** functor. So a final map $e : X \to S$ to a set, that is an $(\mathcal{E},\mathcal{M})$-factorization of the terminal map $X \to 1$, gives the set $S \cong \pi_0 X$ of components of $X$, with the following universal property:



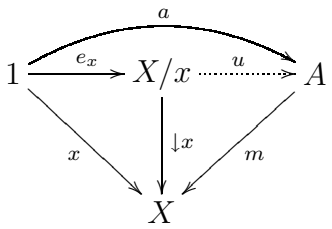

(7)

That is, $\pi_0 X$ is initial among *sets* with a map from $X$ (and it is also final among objects with a *final* map from $X$). In particular, a space $X$ is **connected** ($\pi_0 X \cong 1$) iff $X \to 1$ is final, iff any map to a set is constant.

3.2. Slices. The other important particular case is the reflection $\downarrow x$ of a "point" (global element) $x : 1 \to X$, which is obtained by the $(\mathcal{E},\mathcal{M})$-factorization of the point itself:



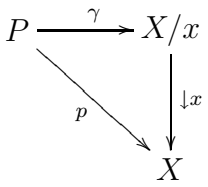
We say that $X/x$ is the **slice** of $X$ at $x$. The corresponding universal property is the following:



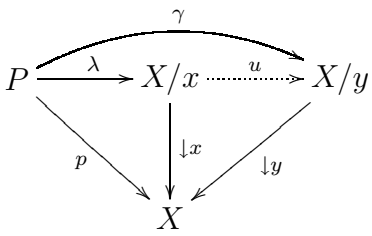

(8)

which in **Cat** becomes the Yoneda Lemma: for any discrete fibration $m : A \to X$ and any object $a$ of $A$ over $x$, there is a unique factorization $a = ue_x$ over $X$ through the "universal element" $e_x$. Note that $u$ is a discrete fibration too, and so displays $X/x$ as a slice $A/a$ of $A$.

3.3. Cones and arrows. Given a map $p : P \to X$ and a point $x$ of $X$, a **cone** $\gamma : p \to x$ is a map over $X$ from $p$ to the slice $X/x$:



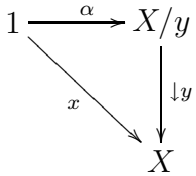

A cone $\lambda : p \to x$ is **colimiting** if it is universal among cones with domain $p$:




(9)

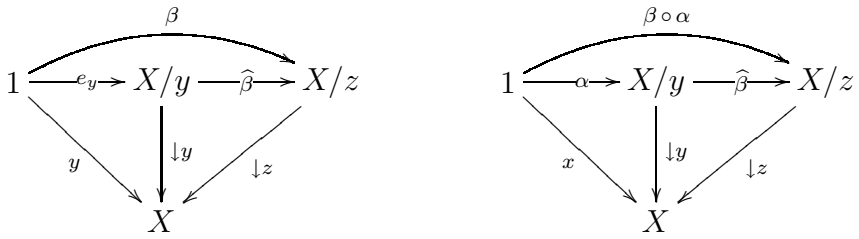
That is, a colimiting cone gives a reflection of $p \in \mathcal{C}/X$ in the full subcategory $\overline{X}$ generated by the slices of $X$ (the **underlying category** of $X$). If $\gamma : p \to x$ is final, it is an **absolute** colimiting cone. Note that $\gamma$ is indeed colimiting, since it gives a reflection of $p$ in $\mathcal{M}/X$.

If $P = 1$, a cone $\alpha : x \to y$ is an **arrow** from $x$ to $y$:




Arrows can be composed via the Kleisli construction: if $\alpha : x \to y$, $\beta : y \to z$ and $\widehat{\beta}$ is universally defined by the left hand diagram below, then $\beta \circ \alpha$ is given by the right hand

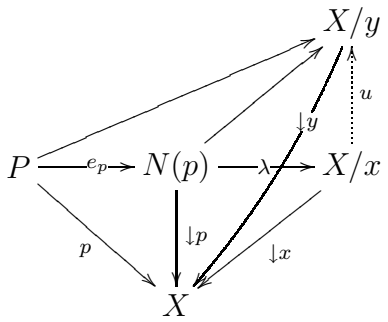
diagram:




(10)

We so get a category (with the $e_x$ as identities) which is clearly isomorphic to the underlying category $\overline{X}$, and which will be identified with it whenever opportune. An arrow $\alpha : x \to y$ is colimiting iff it is absolutely so, iff it is an isomorphism in $\overline{X}$.

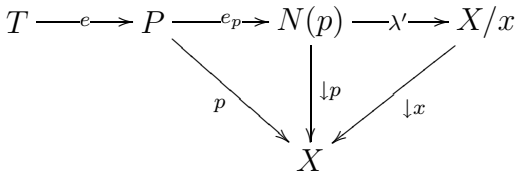
If $\downarrow p : N(p) \to X$ is the reflection of $p : P \to X$ in $\mathcal{M}/X$, the reflection map $e_p$ induces a bijection between cones $p \to x$ and cones $\downarrow p \to x$, which is easily seen to restrict to a bijection between the colimiting ones:




(11)

3.4. PROPOSITION. *If $e : T \to P$ is a final map and $p : P \to X$, then a cone $\lambda : p \to x$ is colimiting iff $\lambda \circ e : p \circ e \to x$ is colimiting.*

PROOF. Since $e_p \circ e : p \circ e \to \downarrow p$ is a reflection map of $p \circ e$ in $\mathcal{M}/X$, by the above remark, $\lambda = \lambda' \circ e_p$ is colimiting iff $\lambda'$ is colimiting, iff $\lambda' \circ e_p \circ e$ is colimiting.




(12)

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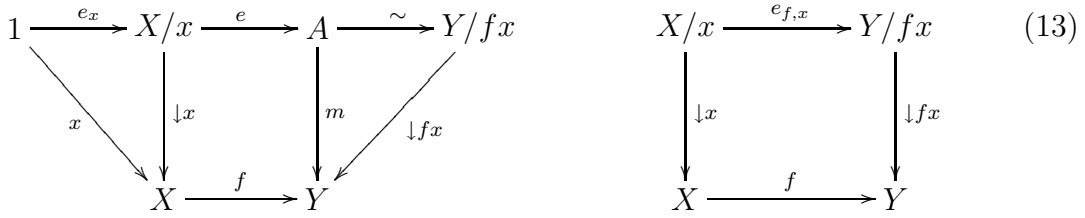
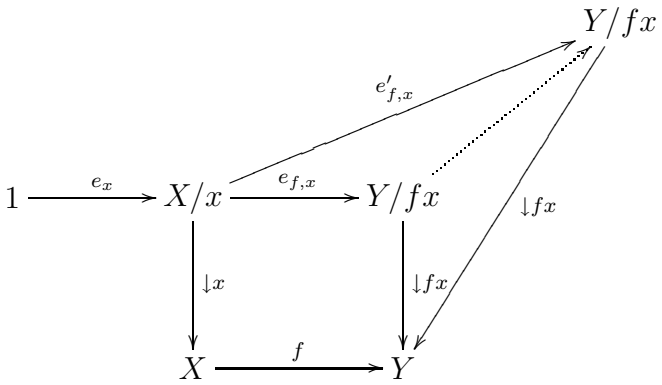
3.5. SLICE MAPS. A map $f : X \to Y$ in $\mathcal{C}$ acts on cones and arrows via an induced map between slices as follows (see (4)). By factorizing $f \circ \downarrow x$ as in the left hand diagram below, we get a discrete fibration over $Y$ with a final point over $fx$. Then there is a (unique) isomorphism $A \to Y/fx$ over $Y$ which respects the selected final points and, by composition, we get the right square below. Its upper edge $e_{f,x}$, which takes the final

point of $X/x$ to that of $Y/fx$, is the **slice map** of $f$ (at $x$):



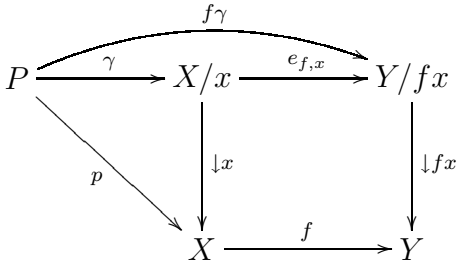

(13)

Note that $e_{f,x}$ is uniquely characterized, among (final) maps $X/x \to Y/fx$ over $Y$, by the equation $e_{f,x} \circ e_x = e_{fx}$. Indeed, given another such $e'_{f,x}$, the dotted arrow induced among the $(\mathcal{E}, \mathcal{M})$-factorizations is the identity, since it takes the "universal point" $e_{fx}$ to itself:




(14)

The slice map of $f$ acts by composition on cones: it takes $\gamma : p \to x$ to its "image cone" $f\gamma : fp \to fx$:




(15)

which has the usual meaning in **Cat**.

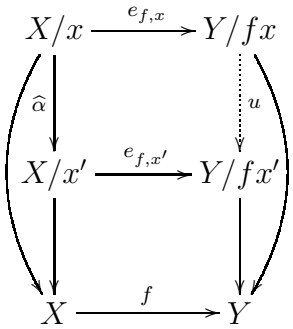
3.6. Proposition. *An absolute colimiting cone is preserved by any map.*

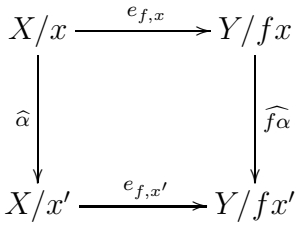
Proof. If in the diagram above $\gamma$ is final, so is also $f\gamma$. ∎

3.7. The underlying functor. In particular, $f$ induces a mapping between arrows $x \to x'$ of $X$ and arrows $fx \to fx'$ of $Y$, which preserves identities. In fact, it is the arrow mapping of a functor $\overline{f} : \overline{X} \to \overline{Y}$: since in the left hand diagram below $e_{f,x}$ is final, there is a uniquely induced dotted arrow $u$ (over $Y$) which makes the upper square commute. So,

$$u \circ e_{fx} = u \circ e_{f,x} \circ e_x = e_{f,x'} \circ \widehat{\alpha} \circ e_x = e_{f,x'} \circ \alpha = f\alpha$$

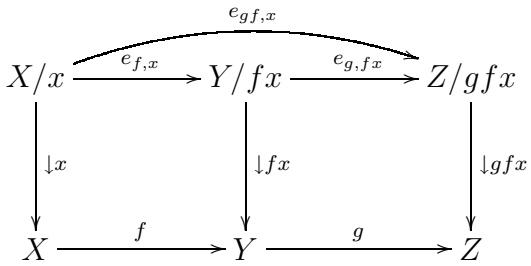
Thus, $u = \widehat{f\alpha}$ and the right hand square commutes, showing that composition is preserved too.




(16)

Furthermore, $e_{g,fx} \circ e_{f,x} = e_{gf,x}$ (since $e_{g,fx} \circ e_{f,x} \circ e_x = e_{g,fx} \circ e_{fx} = e_{gfx}$),

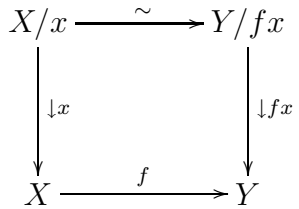



(17)

so that $\overline{g \circ f} = \overline{g} \circ \overline{f}$, and we get the **underlying functor** $\overline{(-)} : \mathcal{C} \to \mathbf{Cat}$.

3.8. Proposition. *The underlying functor preserves the terminal object, sets, slices and discrete fibrations.*

Proof. The fact that $\overline{1}$ is the terminal category is immediate. For any final point $t : 1 \to X$ in $\mathcal{C}$, the slice projection $X/t \to X$ is an isomorphism, that is a terminal object in $\mathcal{C}/X$ and so also in $\overline{X}$. If $f : X \to Y$ is a discrete fibration in $\mathcal{C}$, its slice map at any $x$ is an isomorphism:

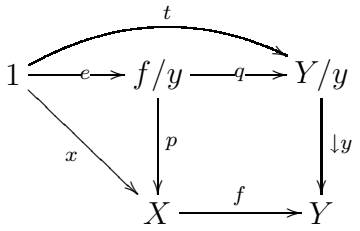



(18)

Since $e_{f,x}$ induces on elements the object mapping of $e_{\overline{f},x}$, $\overline{f}$ is a discrete fibration in **Cat** by Remark 2.4. The remaining statements now follow at once. ∎

An example in Section 5.11 shows that connected objects (and so also final maps) are not preserved in general.

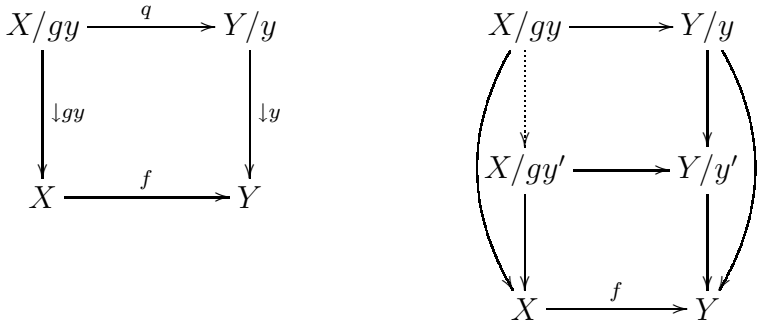
3.9. Adjunctible maps. If the vertex of the pullback square below has a final point, we say that $f$ is **adjunctible** at $y$:




(19)

and that the pair $\langle x, t : fx \to y\rangle$ is a **universal arrow** from $f$ to $y$. In that case, $f/y$ is (isomorphic to) a slice $X/x$ of $X$.

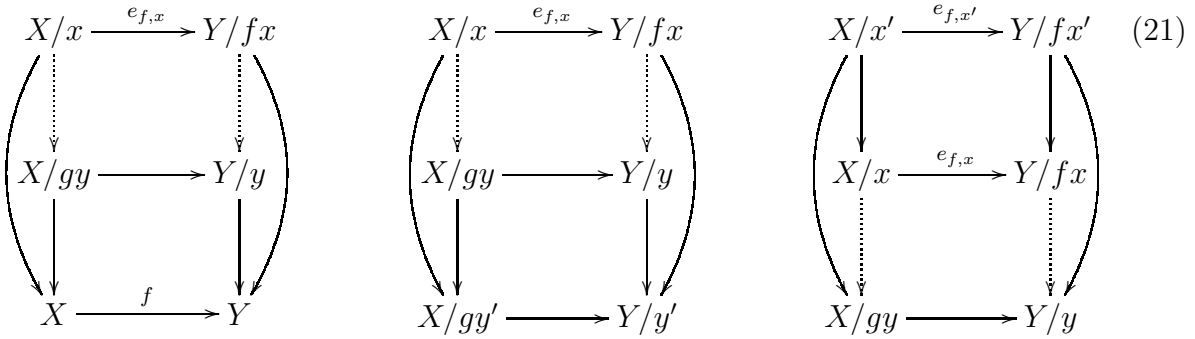
Suppose that $f : X \to Y$ is adjunctible at any point of $Y$ and denote by $gy := pe$ the point of $X$ corresponding to (a choice of) the final point $e$; then we have pullback squares as below. The right hand diagram (whose squares are pullbacks) shows that $g : \mathcal{C}(1, Y) \to \mathcal{C}(1, X)$ can be extended to a functor $\overline{g} : \overline{Y} \to \overline{X}$. (Note that we are not saying that $\overline{g}$ underlies a map $g : Y \to X$ in $\mathcal{C}$.)




(20)

3.10. Proposition. *If $f : X \to Y$ is an adjunctible map, then there is an adjunction $\overline{f} \dashv \overline{g} : \overline{Y} \to \overline{X}$.*

Proof. In the left hand diagram below, the slice map $e_{f,x}$ is final, so that any dotted arrow on the left induces a unique dotted arrow on the right, making the upper square (over $Y$) commute. Since the lower square is a pullback, the converse also holds true. The diagrams on the right display the naturality of the bijection as composition of commutative squares (see (20) and (16)).




(21)

■

3.11. Corollary. *The underlying functor preserves adjunctible maps.*

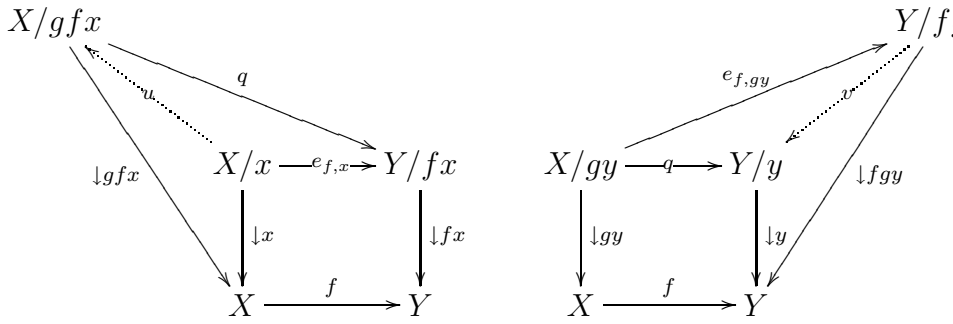
■

3.12. Remarks.

1. Diagram (19) shows that Corollary 3.11 would be obvious if, along with slices, the underlying functor would preserve also pullbacks. Up to now, the author does not know if this is true in general.

2. The adjunction of Proposition 3.10 is nothing but the adjunction $\exists_f \dashv f^*$ (where $f^* : \mathcal{M}/Y \to \mathcal{M}/X$ is the pullback functor and $\exists_f$ gives, when $\mathcal{C} = \mathbf{Cat}$, the left Kan extension along $f$), which for an adjunctible $f$ restricts to $\overline{f} \dashv \overline{g}$ (see [Pisani, 2007c]).

3. The diagrams below display the unit and the counit of the adjunction $\overline{f} \dashv \overline{g}$:

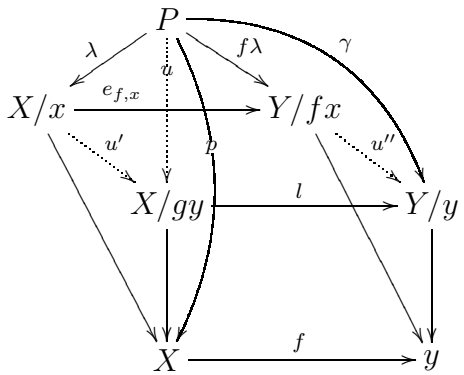



(22)

In **Cat**, $u$ is the functor $\alpha : x' \to x \quad \mapsto \quad \eta_x \circ \alpha : x' \to gfx$, and the triangle $e_{f,x} = q \circ u$ reduces to the classical expression of the arrow mapping of a left adjoint functor via transposition and the unit $\eta$. Similarly, $v$ is the functor $\beta \mapsto \varepsilon_y \circ \beta$, and the triangle $q = v \circ e_{f,gy}$ becomes the classical expression of the transposition bijection of an adjunction via the arrow mapping of the left adjoint functor and the counit $\varepsilon$.

3.13. Proposition. *An adjunctible map $f : X \to Y$ preserves colimits: if $\lambda : p \to x$ is colimiting, then so is $f\lambda : fp \to fx$.*

Proof. Suppose that $\lambda : p \to x$ is colimiting. We want to show that $f\lambda$ is colimiting as well, that is any cone $\gamma : fp \to y$ factorizes uniquely through $e_{f,x} \circ \lambda : fp \to fx$.




(23)

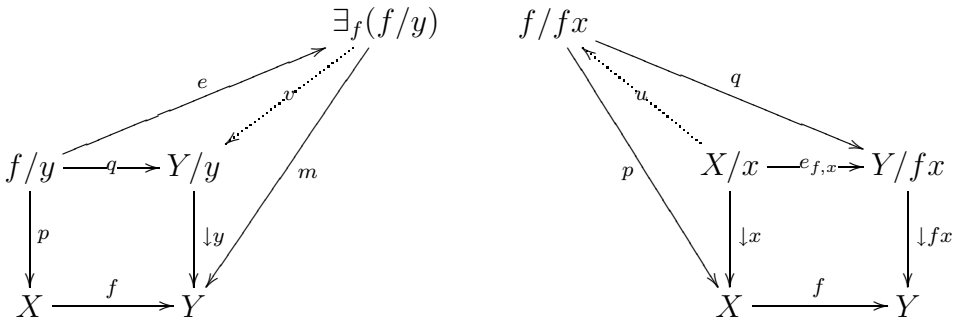
Since the rectangle is a pullback, we have a universally induced cone $u : p \to gy$ and, since $\lambda$ is colimiting, also a map (arrow) $u' : X/x \to X/gy$; the latter corresponds by adjunction ($e_{f,x}$ is final) to a unique $u'' : Y/fx \to Y/y$, which is the desired unique factorization of $\gamma$ through $f\lambda$. Note that we have followed the same steps of the most obvious classical proof, but at this abstract level we need not to verify or use any naturality condition. ∎

3.14. Dense and fully faithful maps. If in the pullback below $e$ is colimiting, we say that $f$ is **adequate** at $y$:

$$\begin{array}{ccc} f/y & \xrightarrow{e} & Y/y \\ \downarrow & & \downarrow{\scriptstyle y} \\ X & \xrightarrow{f} & Y \end{array} \tag{24}$$

while if $e$ is final we say that $f$ is **dense**. Note that in that case $e$ is an absolute colimiting cone and the diagram above displays at once dense maps as "absolutely adequate" maps, as well as "locally final" maps (see [Adámek et al., 2001] for the case $\mathcal{C} = \mathbf{Cat}$, where a different terminology is used; see also Section 5.6).

The density condition can be restated as the fact that the counit of the adjunction $f^* \dashv \exists_f : \mathcal{M}/X \to \mathcal{M}/Y$, that is the uniquely induced arrow $v$ on the left, is an isomorphism on the slices of $Y$:




(25)

On the other hand, if the unit $u$ of the same adjunction, that is the uniquely induced arrow $u$ on the right, is an isomorphism on the slice $X/x$, we say that $f$ is **fully faithful** at $x$. Note that if $f$ is fully faithful at $x$, then $f$ is adjunctible at $fx$.

## 4. Balanced factorization categories

As shown in the previous section and in [Pisani, 2007c], category theory can in part be developed in any $(\mathcal{E}, \mathcal{M})$-category. However, in order to treat "left-sided" and "right-sided" categorical concepts simultaneously and to analyze their interplay, we need to consider *two* factorization systems, related by appropriate axioms:

4.1. Definition. *A* **balanced factorization category** (bfc) *is a finitely complete category $\mathcal{C}$ with two factorization systems $(\mathcal{E}, \mathcal{M})$ and $(\mathcal{E}', \mathcal{M}')$ on it, such that*

1. $\mathcal{M}/1 = \mathcal{M}'/1 \hookrightarrow \mathcal{C}$;

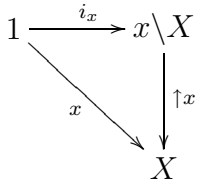
2. *the* **reciprocal stability law** *holds:* the pullback of a map $e \in \mathcal{E}$ ($e' \in \mathcal{E}'$) along a map $m' \in \mathcal{M}'$ ($m \in \mathcal{M}$) is itself in $\mathcal{E}$ ($\mathcal{E}'$).

While this definition aims at capturing some relevant features of **Cat**, other interesting instances of bfc are presented in Setion 5.11.

Note that any slice $\mathcal{C}/X$ of a bfc is itself a "weak" bfc: the condition $\mathcal{M}/1 = \mathcal{M}'/1$ may not hold.

4.2. Notations and terminology. As in Section 3, we assume that "canonical" $(\mathcal{E}, \mathcal{M})$ and $(\mathcal{E}', \mathcal{M}')$-factorizations have been fixed for the arrows of $\mathcal{C}$, and denote by $\downarrow_X : \mathcal{C}/X \to \mathcal{M}/X$ and $\uparrow_X : \mathcal{C}/X \to \mathcal{M}'/X$ the corresponding reflections. For the concepts defined in Section 3, but referred to $(\mathcal{E}', \mathcal{M}')$, we use the standard "dual" terminology when it is available. Otherwise, we distinguish the $(\mathcal{E}, \mathcal{M})$-concepts from the $(\mathcal{E}', \mathcal{M}')$-ones, by qualifying them with the attributes "left" and "right" respectively.

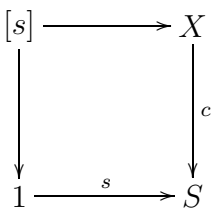
So, we say that a map $i \in \mathcal{E}'$ is **initial**, a map $n \in \mathcal{M}'$ is a **discrete opfibration** and the **coslice** of $X \in \mathcal{C}$ at $x : 1 \to X$ is given by the $(\mathcal{E}', \mathcal{M}')$-factorization of $x$:




We denote by $\mathcal{B} := \mathcal{M} \cap \mathcal{M}'$ the class of "discrete bifibrations". Of course, we also have a "right" underlying functor $\overline{(-)}' : \mathcal{C} \to \mathbf{Cat}$, with $\overline{X}'(x, y) := \mathcal{C}/X(x\backslash X, y\backslash X)$.

4.3. First properties. We begin by noting that, since $\mathcal{S} := \mathcal{M}/1 = \mathcal{M}'/1 \hookrightarrow \mathcal{C}$ (that is, "left" and "right" $\mathcal{C}$-sets coincide, and are discrete bifibrations over 1), then also any (op)fibration over a set is in fact a bifibration: $\mathcal{M}/S = \mathcal{M}'/S = \mathcal{B}/S = \mathcal{S}/S$. Furthermore, a map $X \to S$ to a set is initial iff it is final: $\mathcal{E}/S = \mathcal{E}'/S$. In that case, $S \cong \pi_0 X$, where $\pi_0 := \uparrow_1 = \downarrow_1 : \mathcal{C} \to \mathcal{S}$ is the components functor. In particular, $\mathcal{E}/1 = \mathcal{E}'/1 \hookrightarrow \mathcal{C}$ is the full subcategory of **connected** objects. Note also that a map $S \to S'$ between sets is final (or initial) iff it is an isomorphism; so, if $e : S \to X$ is a final (or initial) map from a set, then $S \cong \pi_0 X$.

4.4. Cylinders and intervals. If $c : X \to S$ is a reflection in $\mathcal{S}$, that is if $c$ is initial (or equivalently final), for any element $s : 1 \to S$ (which is in $\mathcal{B}$), the pullback




(26)

displays $[s]$ as the "component" $[s] \hookrightarrow X$ of $X$ at $s$; $[s]$ is connected by the reciprocal stability law and is included in $X$ as a discrete bifibration. (Note that we are not saying that $X$ is the sum of its components, although this is the case under appropriate hypothesis.) If furthermore $i : S' \to X$ is initial (or final), then $S' \cong S \cong \pi_0 X$. Thus, $i$ is monomorphic and meets any component $[s]$ in a point, which is an initial point $i_s$ of $[s]$:

$$\begin{array}{ccccc} 1 & \xrightarrow{i_s} & [s] & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle s} \\ S' & \xrightarrow{i} & X & \longrightarrow & S \end{array} \tag{27}$$

By a **balanced cylinder**, we mean a cylinder

$$B \underset{e}{\overset{i}{\rightrightarrows}} C \xrightarrow{c} B$$

(that is, a map with two sections; see [Lawvere, 1994]) where the base inclusions $i$ and $e$ are initial and final respectively. As a consequence, $c$ is both initial and final and, if the base is a $\mathcal{C}$-set $S$, then $S \cong \pi_0 C$ and $c$ is a reflection in $\mathcal{S}$. Among these "discrete cylinders", there are "intervals", that is balanced cylinders with a terminal base. Thus, intervals in $\mathcal{C}$ are (connected) objects $i, e : 1 \to I$ with an initial and a final selected point.

In particular, any component of a cylinder with discrete base is an interval:

$$\begin{array}{ccccc} 1 & \underset{e_s}{\overset{i_s}{\rightrightarrows}} & [s] & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle s} \\ S & \underset{e}{\overset{i}{\rightrightarrows}} & X & \longrightarrow & S \end{array} \tag{28}$$

(Conversely, if $\mathcal{E}$ and $\mathcal{E}'$ are closed with respect to products in $\mathcal{C}^{\to}$, by multiplying any object $B \in \mathcal{C}$ with an interval, we get a cylinder with base $B$.)

4.5. Homotopic maps. Two parallel maps $f, f' : X \to Y$ are **homotopic** if there is a balanced cylinder

$$X \underset{e}{\overset{i}{\rightrightarrows}} C \xrightarrow{c} X$$

and a map ("homotopy") $h : C \to Y$ such that $f = hi$ and $f' = he$ (see [Lawvere, 1994]).

4.6. Proposition. *The two base inclusions $i, e : X \to C$ of a balanced cylinder are coequalized by any map $C \to S$ toward a set.*

PROOF. Since the retraction $c$ of the cylinder on its base is final, the proposition follows immediately from the fact that, by orthogonality, any map $C \to S$ to a set factors uniquely through any final map $c : C \to X$:

$$\begin{array}{ccc} C & \longrightarrow & S \\ {\scriptstyle c}\downarrow & \nearrow & \downarrow \\ X & \longrightarrow & 1 \end{array} \tag{29}$$

■

4.7. COROLLARY. *Two homotopic maps $f, f' : X \to Y$ are coequalized by any map $Y \to S$ toward a set.*

■

4.8. INTERNAL HOM-SETS. As in Section 2, we denote by $X[x, y]$ (or simply $[x, y]$) the vertex of a (fixed) pullback

$$\begin{array}{ccc} [x,y] & \xrightarrow{q_{x,y}} & X/y \\ {\scriptstyle p_{x,y}}\downarrow & & \downarrow{\scriptstyle \downarrow y} \\ x\backslash X & \xrightarrow{\uparrow x} & X \end{array} \tag{30}$$

We also denote by $c_{x,y} : X[x, y] \to X(x, y)$ the "canonical" reflection map in $\mathcal{S}$. Thus $X(x, y)$ (or simply $(x, y)$) is the $\mathcal{C}$-set $\pi_0 X[x, y]$.

By further pulling back along the final (initial) point of the (co)slice, we get:

$$\begin{array}{ccccc} & & S' & \longrightarrow & 1 \\ & & {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle e_y} \\ S & \xrightarrow{i} & [x,y] & \longrightarrow & X/y \\ \downarrow & & \downarrow & & \downarrow \\ 1 & \xrightarrow{i_x} & x\backslash X & \longrightarrow & X \end{array} \tag{31}$$

where $S$ and $S'$ are sets (since the pullback of a discrete (op)fibration is still a discrete (op)fibration) and $i$ ($e$) is initial (final) by the reciprocal stability law. Then $b_i = c_{x,y} \circ i : S \to (x, y)$ and $b_e = c_{x,y} \circ e : S' \to (x, y)$ are both isomorphisms and in the above diagram

we can replace $S$ and $S'$ by $(x,y)$, $i$ by $i \circ b_i^{-1}$ and $e$ by $e \circ b_e^{-1}$. So we have the following diagram of pullback squares:

$$\begin{array}{ccccc} & & (x,y) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle e_{x,y}} & & \downarrow{\scriptstyle e_y} \\ (x,y) & \xrightarrow{i_{x,y}} & [x,y] & \xrightarrow{q_{x,y}} & X/y \\ \downarrow & & \downarrow{\scriptstyle p_{x,y}} & & \downarrow{\scriptstyle \downarrow y} \\ 1 & \xrightarrow{i_x} & x\backslash X & \xrightarrow{\uparrow x} & X \end{array} \tag{32}$$

and the balanced cylinder:

$$(x,y) \underset{e_{x,y}}{\overset{i_{x,y}}{\rightrightarrows}} [x,y] \xrightarrow{c_{x,y}} (x,y) \tag{33}$$

Considering in particular the pullbacks

$$\begin{array}{ccc} (x,y) & \xrightarrow{q_{x,y} \circ i_{x,y}} & X/y \\ \downarrow & & \downarrow \\ 1 & \xrightarrow{x} & X \end{array} \qquad \begin{array}{ccc} (x,y) & \xrightarrow{p_{x,y} \circ e_{x,y}} & x\backslash X \\ \downarrow & & \downarrow \\ 1 & \xrightarrow{y} & X \end{array} \tag{34}$$

we see that the elements of $\alpha : 1 \to X(x,y)$ correspond bijectively to "left" arrows $x \to y$ and to "right" arrows $y \to x$:

$$\mathcal{C}(1, X(x,y)) \cong \overline{X}(x,y) \cong \overline{X}'(y,x) \tag{35}$$

Then $X(x,y)$ deserves to be called the "internal hom-set of arrows from $x$ to $y$". (Note that by using $[x,y]$ and $(x,y)$ we have notationally made an arbitrary choice among one of two possible orders, causing an apparent asymmetry in the subsequent theory.) From now on, also an element $\alpha : 1 \to X(x,y)$ will be called an "arrow from $x$ to $y$". Composing it with $q_{x,y} \circ i_{x,y}$ ($p_{x,y} \circ e_{x,y}$) we get the corresponding left (right) arrow (which will be sometimes denoted with the same name).

As in (28), to any arrow $\alpha : 1 \to X(x,y)$ there also corresponds an "arrow interval" $[\alpha]$, the component at $\alpha$ of the balanced cylinder (33):

$$\begin{array}{ccccc} 1 & \underset{e}{\overset{i}{\rightrightarrows}} & [\alpha] & \longrightarrow & 1 \\ \downarrow{\scriptstyle \alpha} & & \downarrow & & \downarrow \\ (x,y) & \underset{e_{x,y}}{\overset{i_{x,y}}{\rightrightarrows}} & [x,y] & \longrightarrow & (x,y) \end{array} \tag{36}$$

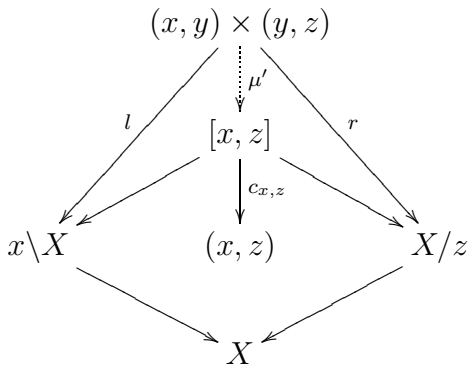
4.9. PROPOSITION. *Two points $x, y : 1 \to X$ are homotopic iff there is an arrow from $x$ to $y$.*

PROOF. If $\alpha : x \to y$ is an arrow of $X$, then $[\alpha] \to X$ gives a homotopy from $x$ to $y$. Conversely, given a homotopy $1 \underset{e}{\overset{i}{\rightrightarrows}} C \xrightarrow{h} X$ there is a (unique) arrow $i \to e$ of $C$, and so its image under $h$ is the desired arrow of $X$. ∎

4.10. ENRICHED COMPOSITION. The "enriched composition" map

$$\mu_{x,y,z} : X(x,y) \times X(y,z) \to X(x,z)$$

is defined as $\mu_{x,y,z} := c_{x,z} \circ \mu'_{x,y,z}$, where $\mu'$ is universally induced by the pullback:

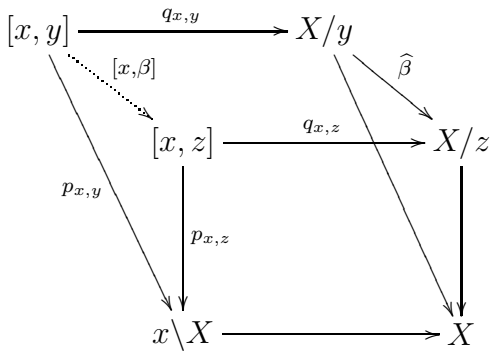



(37)

in which the maps $l$ and $r$ are the product projections followed by $p_{x,y} \circ e_{x,y}$ and $q_{y,z} \circ i_{y,z}$, respectively.

4.11. PROPOSITION. *The map $\mu_{x,y,z}$ induces on the elements the composition mapping $\overline{X}(x,y) \times \overline{X}(y,z) \to \overline{X}(x,z)$.*

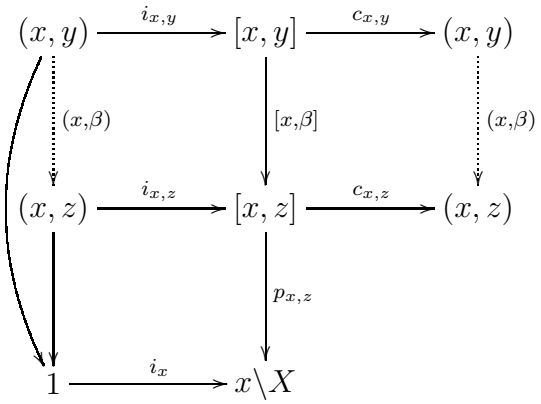
PROOF. For any arrow $\beta : 1 \to X(y,z)$ we have a map $[x,\beta] : [x,y] \to [x,z]$:




(38)

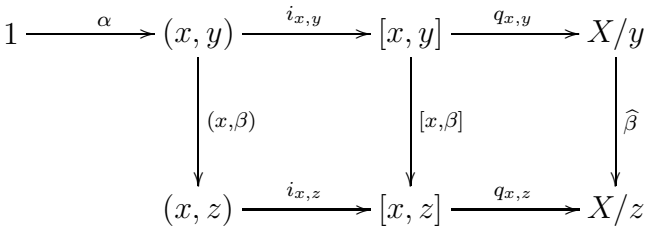
In the diagram below, the left dotted arrow (induced by the universality of the lower pullback square) and the right one ($(x,\beta) := \pi_0[x,\beta] : (x,y) \to (x,z)$, induced by the

universality of the reflection $c_{x,y} : [x, y] \to (x, y)$) are the same, since the horizontal edges of the rectangle are identities:




(39)

So we also have the commutative diagram




(40)

showing that $(x, \beta)$ acts on an arrow $\alpha : 1 \to X(x, y)$ in the same way as $\widehat{\beta}$ acts on the corresponding left arrow $q_{x,y} \circ i_{x,y} \circ \alpha : 1 \to X/y$, that is as $- \circ \beta : \overline{X}(x, y) \to \overline{X}(x, z)$.

Now observe that since $(x, \beta) = c_{x,z} \circ [x, \beta] \circ i_{x,y}$, and since $i_{x,y}$ and $e_{x,y}$ are trivially homotopic, also $(x, \beta) = c_{x,z} \circ [x, \beta] \circ e_{x,y}$. But composing $[x, \beta] \circ e_{x,y} : (x, y) \to [x, z]$ with the pullback projections $p_{x,z}$ and $q_{x,z}$ we get $p_{x,y} \circ e_{x,y}$ and the constant map through $\beta : 1 \to X/y$, so that the map $[x, \beta] \circ e_{x,y}$ coincides with the composite

$$(x, y) \xrightarrow{\sim} (x, y) \times 1 \xrightarrow{\mathrm{id} \times \beta} (x, y) \times (y, z) \xrightarrow{\mu'} [x, z]$$

Summing up, we have seen that:

$$(x, \beta) = \pi_0[x, \beta] = c_{x,z} \circ [x, \beta] \circ i_{x,y} = c_{x,z} \circ [x, \beta] \circ e_{x,y} = \mu \circ (\mathrm{id} \times \beta) \circ \langle \mathrm{id}, ! \rangle$$

So, $\mu(\alpha, \beta) = (x, \beta) \circ \alpha$ corresponds to $\overline{\mu}(\alpha, \beta)$, where $\overline{\mu}$ is the composition mapping of $\overline{X}$, proving the thesis. ∎

"Dually", for any $\alpha : x \to y$ there are maps $[\alpha, z] : [y, z] \to [x, z]$ and $(\alpha, z) : (y, z) \to (x, z)$ such that

$$(\alpha, z) = \pi_0[\alpha, z] = c_{x,z} \circ [\alpha, z] \circ i_{y,z} = c_{x,z} \circ [\alpha, z] \circ e_{y,z} = \mu \circ (\alpha \times \mathrm{id}) \circ \langle !, \mathrm{id} \rangle$$

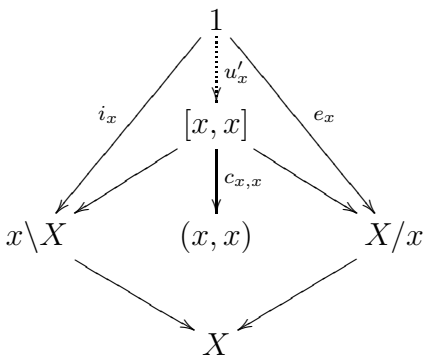
So, $\mu(\alpha, \beta) = (\alpha, z) \circ \beta$ corresponds to $\overline{\mu}'(\beta, \alpha)$, where $\overline{\mu}'$ is the composition in $\overline{X}'$.

4.12. Corollary. *The two underlying categories are related by duality:* $\overline{X}' \cong \overline{X}^{\mathrm{op}}$.

∎

4.13. Internally enriched categories?. Up to now, the author has not been able to prove the associativity of the enriched composition, nor to find a counter-example. On the other hand, that the identity arrow $u_x := c_{x,x} \circ u'_x : 1 \to (x, x)$

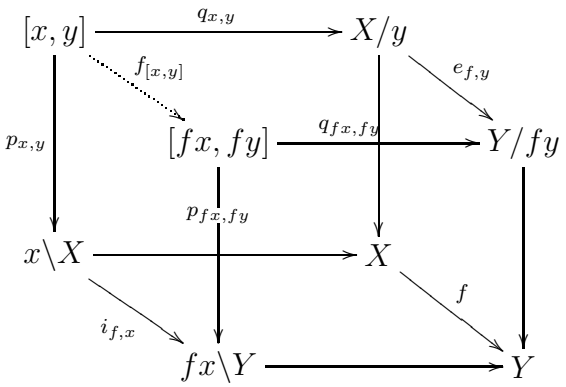
(41)




satisfies the unity laws for the enriched composition, follows from the previous section or can be easily checked directly.

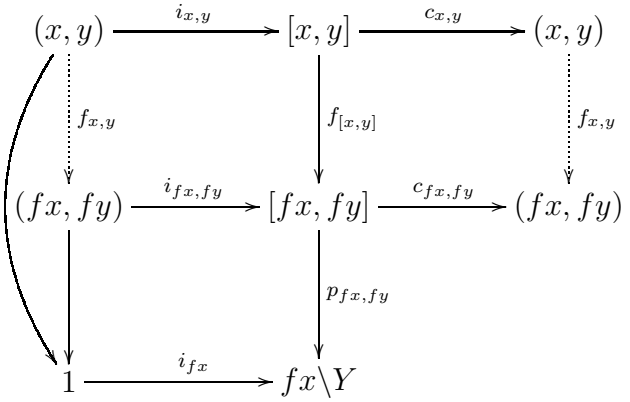
4.14. The action of a map on internal hom-sets. Given a map $f : X \to Y$ in $\mathcal{C}$, we have an induced map $f_{[x,y]} : X[x, y] \to Y[fx, fy]$:

(42)



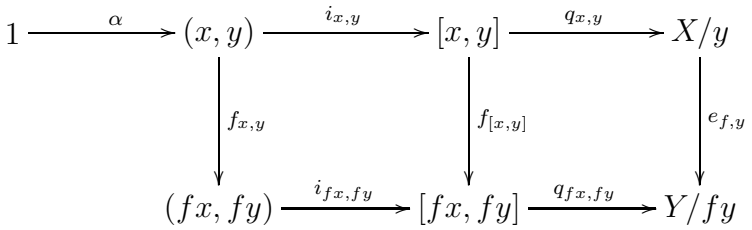

In the diagram below the left dotted arrow (induced by the universality of the lower pullback square) and the right one ($f_{x,y} := \pi_0 f_{[x,y]} : X(x, y) \to Y(fx, fy)$, induced by the universality of the reflection $c_{x,y} : [x, y] \to (x, y)$) are the same, since the horizontal edges

of the rectangle are identities.




(43)

So we also have the commutative diagram




(44)

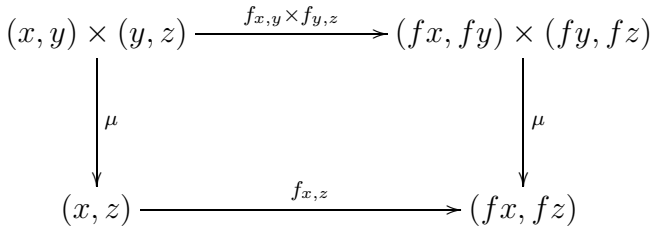
showing that $f_{x,y}$ acts on an arrow $\alpha : 1 \to X(x,y)$ in the same way as the slice map of $f$ acts on the corresponding left arrow $q_{x,y} \circ i_{x,y} \circ \alpha : 1 \to X/y$. So, $f_{x,y} : X(x,y) \to Y(fx,fy)$ enriches $\overline{f}_{x,y} : \overline{X}(x,y) \to \overline{Y}(\overline{f}x, \overline{f}y)$. We also have the "dual" commuting diagrams, with $e_{x,y}$ in place of $i_{x,y}$ and so on, showing that $f_{x,y}$ also enriches $\overline{f}'_{y,x} : \overline{X}'(y,x) \to \overline{Y}'(\overline{f}y, \overline{f}x)$.

4.15. Corollary. *The two underlying functors are related by duality:* $\overline{(-)}' \cong \overline{(-)}^{\mathrm{op}}$.

■

4.16. Proposition. *The maps $f_{x,y}$ are "functorial" with respect to enriched composition.*

Proof. We want to show that the following diagram commutes:




(45)

Since the lower rectangle below commutes

$$\begin{array}{ccc}
(x,y)\times(y,z) & \xrightarrow{f_{x,y}\times f_{y,z}} & (fx,fy)\times(fy,fz) \\
\downarrow{\scriptstyle \mu'} & & \downarrow{\scriptstyle \mu'} \\
[x,z] & \xrightarrow{f_{[x,z]}} & [fx,fz] \\
\downarrow{\scriptstyle c} & & \downarrow{\scriptstyle c} \\
(x,z) & \xrightarrow{f_{x,z}} & (fx,fz)
\end{array} \tag{46}$$

we have to show that the upper one also commutes, that is that composing with the pullback projections (say, the second one $q_{fx,fz} : [fx, fz] \to Y/fz$) we get the same maps:

$$\begin{array}{ccc}
(x,y)\times(y,z) & \xrightarrow{\pi} & (y,z) \\
\downarrow{\scriptstyle \mu'} & & \downarrow{\scriptstyle q_{y,z}\circ i_{y,z}} \\
[x,z] & \xrightarrow{q_{x,z}} & X/z \\
\downarrow{\scriptstyle f_{[x,z]}} & & \downarrow{\scriptstyle e_{f,z}} \\
[fx,fz] & \xrightarrow{q_{fx,fz}} & Y/fz
\end{array}
\qquad
\begin{array}{ccc}
(x,y)\times(y,z) & \xrightarrow{\pi} & (y,z) \\
\downarrow{\scriptstyle f_{x,y}\times f_{y,z}} & & \downarrow{\scriptstyle f_{y,z}} \\
(fx,fy)\times(fy,fz) & \xrightarrow{\pi} & (fy,fz) \\
\downarrow{\scriptstyle \mu'} & & \downarrow{\scriptstyle q_{fy,fz}\circ i_{fy,fz}} \\
[fx,fz] & \xrightarrow{q_{fx,fz}} & Y/fz
\end{array} \tag{47}$$

Indeed, this is the case because all the squares commute and the composites on the right are the two paths in the rectangle (44). ∎

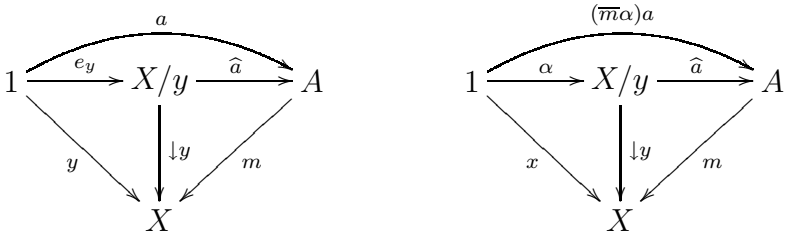
4.17. Enriching discrete fibrations. To any discrete fibration $m : A \to X$ in $\mathcal{C}$ there corresponds a presheaf $\overline{m}$ on $\overline{X}$, as can be seen in several ways:

1. As we have seen in Proposition 3.8, by applying $\overline{(-)} : \mathcal{C} \to \mathbf{Cat}$ to $m$, we obtain a discrete fibration $\overline{m} : \overline{A} \to \overline{X}$ in $\mathbf{Cat}$, corresponding to a presheaf $\overline{m} : \overline{X}^{\mathrm{op}} \to \mathbf{Set}$.

2. Via the inclusion $i : \overline{X} \hookrightarrow \mathcal{M}/X$, any object $A \in \mathcal{M}/X$ is "interpreted" as a presheaf $\mathcal{M}/X(i-, A)$ on $\overline{X}$.

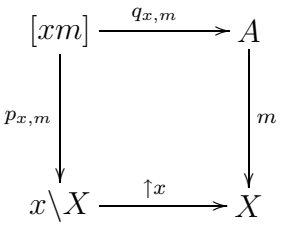
3. To any object $x : 1 \to X$ of $\overline{X}$ there corresponds the set $\mathcal{C}/X(x, m)$ of points of $A$ over $x$, and to any arrow $\alpha : x \to y$ in $\overline{X}$ there corresponds a mapping $\overline{m}\alpha : \mathcal{C}/X(y, m) \to \mathcal{C}/X(x, m)$, whose value at $a \in A$ over $y$ is given by the right

hand diagram below (where $\widehat{a}$ is defined by the left hand one):



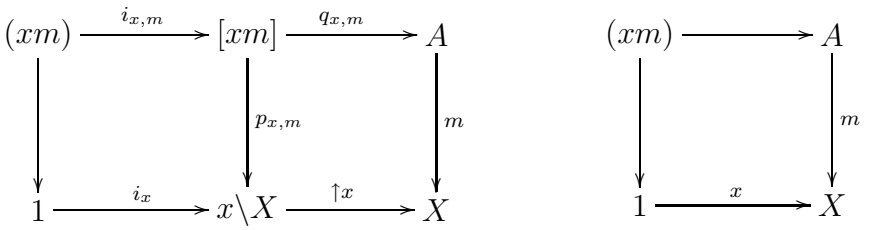

(48)

Thus, we have mappings $\overline{X}(x,y)\times y\overline{m} \to x\overline{m}$, and we now show how they can be enriched as maps in $\mathcal{S}$. (This enriching should not be confused with the $m_{a,b} : A(a,b) \to X(ma,mb)$ of Proposition 4.11, associated to any map in $\mathcal{C}$.) Following and generalizing the results of Section 4.10, we define $[xm]$ as the pullback




(49)

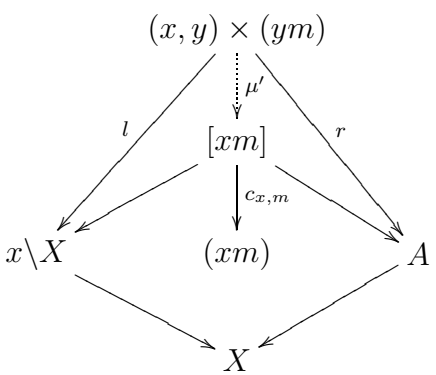
and $(xm) := \pi_0[xm]$, with reflection map $c_{x,m} : [xm] \to (xm)$. Then we also have the pullbacks:




(50)

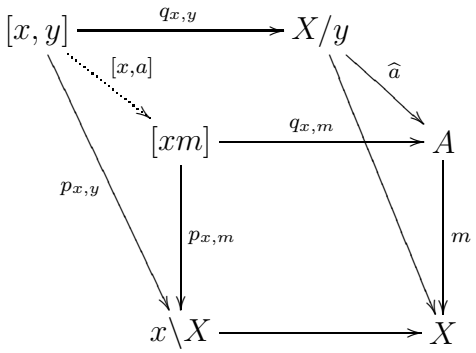
where we can assume that $c_{x,m} \circ i_{x,m}$ is the identity of the set $(xm)$, and which show that the elements of $(xm)$ correspond bijectively to those of $x\overline{m}$.

Now we define (see (37)) $\mu_{x,y,m} : X(x,y) \times (ym) \to (xm)$ as $c_{x,m} \circ \mu'$:



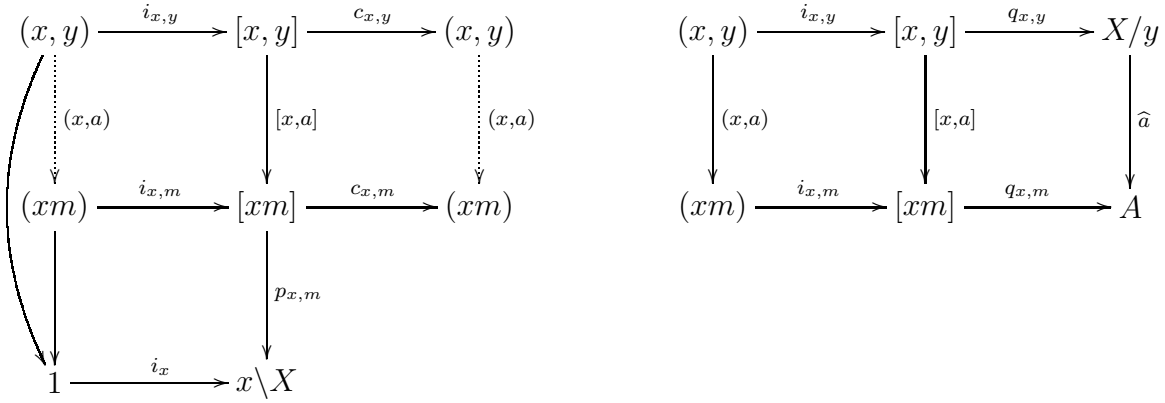

(51)

where $\mu'$ is universally induced by the maps $l$ and $r$, the product projections followed by $p_{x,y} \circ e_{x,y}$ and $q_{y,m} \circ i_{y,m}$, respectively. To prove that this is the desired enrichment, for a fixed point $a \in A$ over $y$ we consider the map $[x, a] : [x, y] \to [xm]$:




(52)

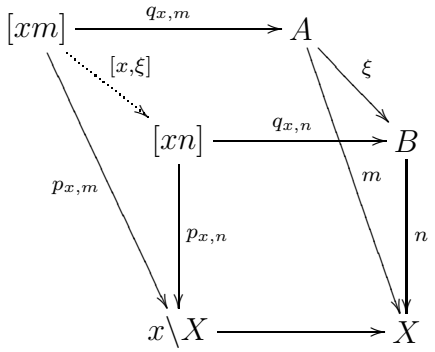
and define $(x, a) := \pi_0[x, a] : (x, y) \to (xm)$, getting:




(53)

The right hand diagram above shows that $(x, a)$ acts on an arrow $\alpha : 1 \to X(x, y)$ in the same way as $\widehat{a}$ acts on the corresponding left arrow $q_{x,y} \circ i_{x,y} \circ \alpha : 1 \to X/y$. Furthermore we obtain that $(x, a) = \pi_0[x, a] = c_{x,m} \circ [x, a] \circ i_{x,y} = c_{x,m} \circ [x, a] \circ e_{x,y} = \mu \circ (\mathrm{id} \times a) \circ \langle \mathrm{id}, ! \rangle$ Then, $\mu(\alpha, a) = (x, a) \circ \alpha$ corresponds to $(\overline{m}\alpha)a$, proving the thesis.

If $n : B \to X$ is another discrete fibration over $X$ and $\xi : m \to n$ a map over $X$, we have a corresponding morphism of discrete fibrations $\overline{\xi} : \overline{m} \to \overline{n}$ in $\mathbf{Cat}/X$, or equivalently a natural transformation. On the other hand, $\xi$ also induces maps $[x, \xi] : [xm] \to [xn]$:




(54)

and one can check that the maps $(x,\xi) := \pi_0[x,\xi] : (xm) \to (xn)$ define a morphism of left $\mathcal{S}$-modules on $X$, which enriches the natural transformation $\overline{\xi}$.

## 5. Balanced category theory

Balanced category theory is category theory developed in a balanced factorization category $\mathcal{C}$, playing the role of **Cat** with the comprehensive factorization systems. We here present just a few of its aspects, others still needing to be analyzed.

5.1. The underlying functors. Summarizing the results of Section 4, we have two "underlying" functors: the "left" one $\overline{(-)} : \mathcal{C} \to \mathbf{Cat}$ and the "right" one $\overline{(-)}' : \mathcal{C} \to \mathbf{Cat}$, where $\overline{X}(x,y) = \mathcal{C}/X(X/x, X/y)$ and $\overline{X}'(x,y) = \mathcal{C}/X(x\backslash X, y\backslash X)$. They are isomorphic up to the duality functor $(-)^{\mathrm{op}} : \mathbf{Cat} \to \mathbf{Cat}$:

$$\begin{array}{ccc} & \overset{\overline{(-)}}{\nearrow} & \mathbf{Cat} \\ \mathcal{C} & & \downarrow{\scriptstyle (-)^{\mathrm{op}}} \\ & \underset{\overline{(-)}'}{\searrow} & \mathbf{Cat} \end{array} \tag{55}$$

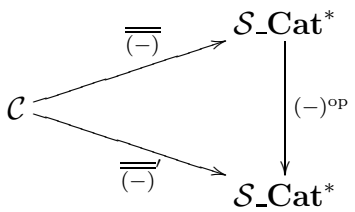
By propositions 3.8 and 3.11, both the underlying functors preserve the terminal object and sets; the left one preserves final and initial points, discrete fibrations and opfibrations, slices and coslices and left and right adjunctible maps, while the right one reverses them. This apparent asymmetry is only the effect of our naming of the arrows in $\mathcal{E}$, $\mathcal{E}'$, $\mathcal{M}$ and $\mathcal{M}'$.

The duality of the underlying functors (corollaries 4.12 and 4.15) is a consequence of the fact that for any object $X$ of $\mathcal{C}$ we have an $\mathcal{S}$-enriched weak category (that is, the associativity of the composition $\mu$ may not hold; see Section 4.13), and for any map $f : X \to Y$ an $\mathcal{S}$-enriched functor $f_{x,y} : X(x,y) \to Y(fx,fy)$ such that the following diagrams in **Set** commute:

$$\begin{array}{ccccc} \overline{X}'(z,y) \times \overline{X}'(y,x) & \xleftarrow{\sim} & \mathcal{C}(1, X(x,y) \times X(y,z)) & \xrightarrow{\sim} & \overline{X}(x,y) \times \overline{X}(y,z) \\ \downarrow{\scriptstyle \overline{\mu}'} & & \downarrow{\scriptstyle \mathcal{C}(1,\mu)} & & \downarrow{\scriptstyle \overline{\mu}} \\ \overline{X}'(z,x) & \xleftarrow{\sim} & \mathcal{C}(1, X(x,z)) & \xrightarrow{\sim} & \overline{X}(x,z) \end{array} \tag{56}$$

$$\begin{array}{ccccc} \overline{X}'(y,x) & \xleftarrow{\sim} & \mathcal{C}(1, X(x,y)) & \xrightarrow{\sim} & \overline{X}(x,y) \\ \downarrow{\scriptstyle \overline{f}'_{y,x}} & & \downarrow{\scriptstyle \mathcal{C}(1,f_{x,y})} & & \downarrow{\scriptstyle \overline{f}_{x,y}} \\ \overline{Y}'(fy,fx) & \xleftarrow{\sim} & \mathcal{C}(1, Y(fx,fy)) & \xrightarrow{\sim} & \overline{Y}(x,y) \end{array} \tag{57}$$

Then, denoting by $\mathcal{S}\text{-}\mathbf{Cat}^*$ the category of $\mathcal{S}$-enriched weak categories and $\mathcal{S}$-enriched functors, diagram (55) can be enriched as



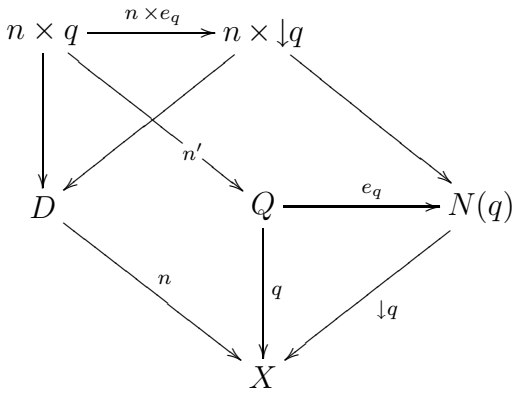

(58)

where $\overline{\overline{X}}(x,y) = \overline{\overline{X}}'(y,x) = X(x,y)$, while $\overline{\overline{f}}_{x,y} = \overline{\overline{f}}'_{y,x} = f_{x,y} : X(x,y) \to X(fx,fy)$.

Furthermore, for any discrete fibration $m : A \to X$, the corresponding presheaf on $\overline{X}$ can also be enriched as a left $\mathcal{S}$-module on the weak $\mathcal{S}$-category $X$, and the functor $\mathcal{M}/X \to \mathbf{Set}^{\overline{X}^{\mathrm{op}}}$ can be enriched to a functor $\mathcal{M}/X \to X\text{-}\mathbf{mod}(\mathcal{S})$ (and dually for $\mathcal{M}'/X \to \mathbf{Set}^{\overline{X}}$).

5.2. The tensor functor. Given $p, q \in \mathcal{C}/X$, their **tensor product** is the components set $\mathrm{ten}_X(p,q) := \pi_0(p \times_X q)$ of their product over $X$. We so get a functor $\mathrm{ten}_X : \mathcal{C}/X \times \mathcal{C}/X \to \mathcal{S}$. In particular, $\mathrm{ten}(x\backslash X, X/y) = X(x,y)$ and, if $m \in \mathcal{M}/X$, $\mathrm{ten}(x\backslash X, m) = (xm)$. If $m \in \mathcal{M}/X$, $n \in \mathcal{M}'/X$ and $\mathcal{C} = \mathbf{Cat}$, $\mathrm{ten}(m,n)$ gives the classical tensor product (coend) $n \otimes m$ of the corresponding set functors $m : X^{\mathrm{op}} \to \mathbf{Set}$ and $n : X \to \mathbf{Set}$.

5.3. The inversion law. If $q : Q \to X$ is any map, and $n : D \to X$ is a discrete opfibration over $X$, pulling back the $(\mathcal{E}, \mathcal{M})$-factorization $q = {\downarrow}q \circ e_q$ along $n$ we get:




(59)

where $\times$ denotes the product in $\mathcal{C}/X$. Since $n'$ is a discrete opfibration, by the reciprocal stability law

$$n \times e_q : n \times q \to n \times {\downarrow}q$$

is a final map over $X$; thus its domain and codomain have the same reflection in $\mathcal{M}/X$:

5.4. Proposition. *If $n \in \mathcal{M}'/X$ and $q \in \mathcal{C}/X$, ${\downarrow_X}(n \times q) \cong {\downarrow_X}(n \times {\downarrow_X}q)$; dually, if $m \in \mathcal{M}/X$, ${\uparrow_X}(m \times q) \cong {\uparrow_X}(m \times {\uparrow_X}q)$*

■

Similarly, given two maps $p : P \to X$ and $q : Q \to X$, we have the following symmetrical diagram:

$$\begin{array}{ccccc} & & \uparrow p \times q & \longrightarrow & Q \\ & & \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle e_q} \\ p \times\downarrow q & \xrightarrow{i} & \uparrow p \times \downarrow q & \longrightarrow & N(q) \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \downarrow q} \\ P & \xrightarrow{i_p} & N(p) & \xrightarrow{\uparrow p} & X \end{array} \tag{60}$$

Since objects linked by initial or final maps have the same reflection in $\mathcal{M}/1 = \mathcal{S}$, we get the following "inversion law":

$$\mathrm{ten}(\uparrow p, q) \;\cong\; \mathrm{ten}(\uparrow p, \downarrow q) \;\cong\; \mathrm{ten}(p, \downarrow q) \tag{61}$$

5.5. The reflection formula. If $p$ is a point $x : 1 \to X$, diagram (60) becomes:

$$\begin{array}{ccccc} & & x\backslash q & \longrightarrow & Q \\ & & \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle e_q} \\ (x \downarrow q) & \xrightarrow{i} & [x \downarrow q] & \longrightarrow & N(q) \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \downarrow q} \\ 1 & \xrightarrow{i_x} & x\backslash X & \xrightarrow{\uparrow x} & X \end{array} \tag{62}$$

and the inversion law becomes the "reflection formula":

$$(x \downarrow q) = \mathrm{ten}(x\backslash X, \downarrow q) \;\cong\; \pi_0(x\backslash q) = \mathrm{ten}(x\backslash X, q) \in \mathcal{S} \tag{63}$$

giving the enriched value of the reflection $\downarrow q \in \mathcal{M}/X$ at $x$. If the functor

$$\mathcal{M}/X \to \mathcal{S}^{\overline{X}^{\mathrm{op}}} \qquad m \mapsto \mathrm{ten}(-\backslash X, m) \tag{64}$$

reflects isomorphisms, then the formula $\mathrm{ten}(-\backslash X, q)$ determines $\downarrow q$ up to isomorphisms. In particular, in this case, $q$ is final iff the inverse image $x\backslash q = q^*(x\backslash X)$ of any slice of $X$ is connected: $\pi_0(x\backslash q) = 1$, $x \in X$.

If $\mathcal{C} = \mathbf{Cat}$, we find again the "classical" formula (5):

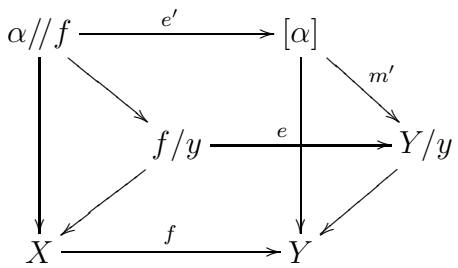
$$\downarrow q \;\cong\; \pi_0(-\backslash q)$$

which gives the presheaf corresponding to the discrete fibration $\downarrow q$.

5.6. Dense maps. In Section 3, we have already discussed dense maps, from a "left-sided" perspective. But density is clearly a "balanced" concept. Indeed, the following is among the characterizations of dense functors in **Cat** given in [Adámek et al., 2001]:

5.7. Proposition. *If $f : X \to Y$ is dense, then for any arrow $\alpha$ of $Y$ the object $\alpha//f$, obtained by pulling back the arrow interval $[\alpha]$ along $f$, is connected. If the functor* (64) *reflects isomorphisms, the converse also holds.*

Proof. The pullback square factors through the two pullback rectangles below:




(65)

If $f$ is dense, the map $e$ is final by definition; $m'$ is a discrete opfibration, because it is the composite of the component inclusion $[\alpha] \to [x, y]$ and the pullback projection $q_{x,y} : [x, y] \to Y/y$, which are both discrete opfibrations (see diagrams (30) and (36)). Then, by the reciprocal stability law also $e'$ is final, and the connectedness of $[\alpha]$ implies that of $\alpha//f$.

For the reverse implication, note that $[\alpha]$, having an initial point, is a coslice of $Y/y$:

$$[\alpha] \cong \alpha\backslash(Y/y) \tag{66}$$

Thus, $\alpha//f$ is the pullback $\alpha\backslash e$, and the fact that it is connected for any $\alpha$ implies, by the hypothesis, the finality of $e$. ∎

5.8. Exponentials and complements. Till now, we have not considered any of the exponentiability properties of **Cat**. In this subsection, we assume that for any $X \in \mathcal{C}$, the discrete fibrations and opfibrations over $X$ are exponentiable in $\mathcal{C}/X$.

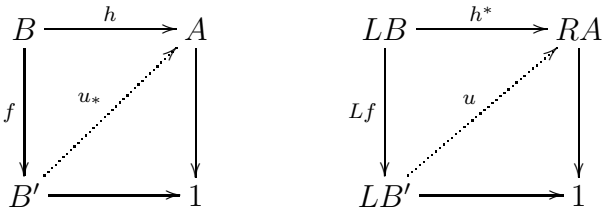
5.9. Proposition. *If $m \in \mathcal{M}/X$ and $n \in \mathcal{M}'/X$ then $m^n \in \mathcal{M}/X$ and, dually, $n^m \in \mathcal{M}'/X$.*

Proof. If $n$ is a discrete opfibration over $X$ and $q : Q \to X$ is any object in $\mathcal{C}/X$, by Proposition 5.4,

$$n \times \eta_q : n \times q \to n \times \downarrow q$$

is a final map over $X$ (recall that $e_q : q \to \downarrow q$ is the reflection map of $q \in \mathcal{C}/X$ in $\mathcal{M}/X$, that is the unit $\eta_q$ of the adjunction $\downarrow_X \dashv i_X : \mathcal{M}/X \hookrightarrow \mathcal{C}/X$). Then it is orthogonal to any discrete fibration over $X$, in particular to $m$. In turn, this implies that any unit

$\eta_q : q \to \downarrow q$ is orthogonal to $m^n$, since in general if $L \dashv R : \mathcal{A} \to \mathcal{B}$, an arrow $f : B \to B'$ in $\mathcal{B}$ is orthogonal to the object $RA$ iff $Lf : LB \to LB'$ is orthogonal to $A$ in $\mathcal{A}$:




(67)

Indeed, in one direction the right hand diagram above gives a unique $u$, whose transpose $u_*$ makes the left hand one commute: $u_* \circ f = (u \circ Lf)_* = (h^*)_* = h$. The converse implication is equally straightforward.

Then, taking $q = m^n$ we see that $\eta_{m^n}$ is a reflection map with a retraction, and so $m^n \in \mathcal{M}/X$ (see [Pisani, 2007c] or also [Borceux, 1994]). ∎

Any $m \in \mathcal{M}/X$ has a "complement"

$$\mathrm{ten}(m, -) \dashv \neg m : \mathcal{S} \to \mathcal{C}/X \qquad S \mapsto (\pi_S)^m \tag{68}$$

where $\pi_S : S \times X \to X$ is the constant bifibration over $X$ with value $S$. By Proposition 5.9, $\neg m$ takes values in $\mathcal{M}'/X$, giving a broad generalization of the fact that the (classical) complement of a lower-set of a poset $X$ is an upper-set. (These items are treated at length in [Pisani, 2007a] and [Pisani, 2007b]. There, we conversely used the exponential law to prove the inversion law in **Cat**; however, the present one seems by far the most natural path to follow, also in the original case $\mathcal{C} = \mathbf{Cat}$.)

5.10. Codiscrete and groupoidal objects. An object $X \in \mathcal{C}$ is **codiscrete** if all its points are both initial and final (that is, all the (co)slice projections are isomorphisms). An object $X \in \mathcal{C}$ is **groupoidal** if all its (co)slices are codiscrete. (Note that codiscrete objects and sets are groupoidal.) Since these concepts are self dual and the underlying functors preserve (or dualize) slices, it is immediate to see that they preserve both codiscrete and groupoidal objects (which in **Cat** have the usual meaning).

5.11. Further examples of balanced factorization categories.

- If $\mathcal{C}$ is an $(\mathcal{E}, \mathcal{M})$-category such that $\mathcal{E}$ is $\mathcal{M}$-stable (or, equivalently, satisfying the Frobenius law; see [Clementino et al., 1996]), then $\mathcal{C}$ is a bfc with $(\mathcal{E}, \mathcal{M}) = (\mathcal{E}', \mathcal{M}')$.

- In particular, for any lex category $\mathcal{C}$ we have the "discrete" bfc $\mathcal{C}^{\mathrm{d}}$ and the "codiscrete" bfc $\mathcal{C}^{\mathrm{c}}$ on $\mathcal{C}$. In $\mathcal{C}^{\mathrm{d}}$, $\mathcal{E} = \mathcal{E}' = \mathrm{iso}\mathcal{C}$ and $\mathcal{M} = \mathcal{M}' = \mathrm{ar}\mathcal{C}$. Conversely in $\mathcal{C}^{\mathrm{c}}$, $\mathcal{E} = \mathcal{E}' = \mathrm{ar}\mathcal{C}$ and $\mathcal{M} = \mathcal{M}' = \mathrm{iso}\mathcal{C}$. In $\mathcal{C}^{\mathrm{d}}$ all object are discrete: $\mathcal{S} = \mathcal{C}$, while 1 is the only connected object. Conversely, in $\mathcal{C}^{\mathrm{c}}$ all objects are connected, while 1 is the only set. So, for any $X \in \mathcal{C}^{\mathrm{d}}$ ($X \in \mathcal{C}^{\mathrm{c}}$) $\mathrm{id} : X \to X$ ($! : X \to 1$) is the reflection map of $X$ in sets and, for any $x : 1 \to X$, $X/x = x\backslash X = x$ ($X/x = x\backslash X = \mathrm{id}_X$). Then, for any $X \in \mathcal{C}^{\mathrm{d}}$ ($X \in \mathcal{C}^{\mathrm{c}}$), $\overline{X}$ is the (co)discrete category on $\mathcal{C}(1, X)$.

Furthermore, if $X \in \mathcal{C}^{\mathrm{d}}$ and $x, y : 1 \to X$, then $X[x, y]$ is pointless if $x \neq y$, while $X[x, x] = 1$. So the same is true for $X(x, y)$. On the other hand, if $X \in \mathcal{C}^{\mathrm{c}}$, then $X[x, y] = X$ and $X(x, y) = 1$, for any $x, y : 1 \to X$. Indeed, for any arrow $u_{x,y} : x \to y$, the arrow interval $[u_{x,y}]$ is simply $x, y : 1 \to X$.

- On the category **Pos** of posets, we can consider the bfc in which $m \in \mathcal{M}$ ($\mathcal{M}'$) iff it is, up to isomorphisms, a lower-set (upper-set) inclusion, and $e : P \to X$ is in $\mathcal{E}$ ($\mathcal{E}'$) iff it is a cofinal (coinitial) mapping in the classical sense: for any $x \in X$, there is $a \in P$ such that $x \leq ea$ ($ea \leq x$).

  The category of internal sets is $\mathcal{S} = 2$, and the components functor $\mathbf{Pos} \to 2$ reduces to the "non-empty" predicate. For any poset $X$ and any point $x \in X$, the slice $X/x$ ($x\backslash X$) is the principal lower-set (upper-set) generated by $x$. As in **Cat**, $\overline{X}$ is isomorphic to $X$ itself. This example also shows that the underlying functor may not preserve final maps: a poset $X$ is "internally" connected iff it is not empty, while $\overline{X} \cong X$ may well be not connected in **Cat**.

  Given a map $p : P \to X$, a colimiting cone $\lambda : p \to x$ simply indicates that $x$ is the sup of the set of points $pa$, $a \in P$. The colimit is absolute iff such a sup is in fact a maximum. A map $f : X \to Y$ is adequate iff any $y \in Y$ is the sup of the $fx$ which are less than or equal to $y$.

  Furthermore, given $x, y : 1 \to X$, $[x, y] \subseteq X$ has its usual meaning: it is the interval of the points $z \in X$ such that $x \leq z \leq y$. So, $X(x, y)$ is the truth value of the predicate $x \leq y$, and the enriching of $\overline{X}$ in internal sets gives the usual identification of posets with categories enriched over 2.

  By restricting to discrete posets, we get the bfc associated to the epi-mono factorization system on **Set**.

- Let **Gph** be the category of reflexive graphs, with the factorization systems $(\mathcal{E}, \mathcal{M})$ and $(\mathcal{E}', \mathcal{M}')$ generated by $t : 1 \to 2$ and $s : 1 \to 2$ respectively, where 2 is the arrow graph (see [Pisani, 2007a] and [Pisani, 2007b]). In this case $\overline{X}$ is the free category on $X$, and the underlying functor is the free category functor $\mathbf{Gph} \to \mathbf{Cat}$.

  Furthermore, if $X \in \mathbf{Gph}$ and $x, y : 1 \to X$, then $X[x, y]$ is the graph which has as objects the pairs of consecutive paths $\langle \alpha, \beta \rangle$, with $\alpha : x \to z$ and $\beta : z \to y$, while there is an arrow $\langle \alpha, \beta \rangle \to \langle \alpha', \beta' \rangle$ over $a : z \to z'$ iff $\alpha' = a\alpha$ and $\beta = \beta' a$. In **Gph**, $\mathcal{S} = \mathbf{Set}$ and the components of $X[x, y]$ are the paths $x \to y$. Thus, the enriching $X(x, y)$ of $\overline{X}(x, y)$ is in fact an isomorphism.

## 6. Conclusions

There are at least three well-estabilished abstractions (or generalizations) of category theory: enriched categories, internal categories and 2-category theory. Each of them is best suited to enlighten certain of its aspects and to capture new instances; for example,

monads and adjunctions (via the triangular identities) are surely 2-categorical concepts, while enriched categories subsume many important structures and support quantifications.

Here, we have based our abstraction on final and initial functors and discrete fibrations and opfibrations, whose decisive relations are encoded in the concept of balanced factorization category. In this context, natural transformations and/or exponentials are no more basic notions; rather, we have seen how the universal exactness properties of $\mathcal{C}$ and those depending on the two factorization systems, complement each other in an harmonious way (in particular, pullbacks and the reflection in discrete (op)fibrations) to give an effective and natural tool for proving categorical facts.

Thus we hope to have shown that balanced category theory offers a good perspective on several basic categorical concepts and properties, helping to distinguish the "trivial" ones (that is, those which depend only on the bfc structure of **Cat**) from those regarding peculiar aspects of **Cat** (such as colimits, lextensivity, power objects and the arrow object). The latter have been partly considered in [Pisani, 2007c] and deserve further study. Summarizing, balanced category theory is

- *simple*: a bfc is a lex category with two reciprocally stable factorization systems generating the same (internal) sets;
- *expressive*: many categorical concepts can be naturally defined in any bfc;
- *effective*: simple universal properties guide and almost "force" the proving of categorical properties relative to these concepts;
- *symmetrical* (or "balanced"): the category **Cat** in itself does not allow one to distinguish an object from its dual; this is fully reflected in balanced category theory;
- *self-founded*: it is largely enriched over its own internal sets, providing in a sense its own foundation (see for example [Lawvere, 2003] and [Lawvere, 1966]).

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