Theory and Applications of Categories, Vol. 22, No. 11, 2009, pp. 302–312.

ENRICHED ORTHOGONALITY AND EQUIVALENCES

M. GOLASIŃSKI AND L. STRAMACCIA

ABSTRACT. In this paper, we consider an enriched orthogonality for classes of spaces, with respect to groupoids, simplicial sets and spaces themselves. This point of view allows one to characterize homotopy equivalences, shape and strong shape equivalences. We show that there exists a class of spaces, properly containing ANR-spaces, for which shape and strong shape equivalences coincide. For such a class of spaces homotopy orthogonality implies enriched orthogonality.

Introduction

A topological strong shape equivalence [14] is a map inducing an isomorphism in the strong shape category $sSh(\mathbf{Top}, \mathbf{ANR})$. It turns out [17] that $f : X \to Y$ is such a map if it gives, by composition, an equivalence $f_Z^* : \mathbf{Gpd}(Y, Z) \to \mathbf{Gpd}(X, Z)$ between fundamental groupoids for all $Z \in \mathbf{ANR}$. In other words, one may say that strong shape equivalences form the orthogonal class of $\mathbf{ANR} \subseteq \mathbf{Top}$. Here, the notion of orthogonality is intended in an enriched sense, over the category **Gpd** of groupoids, and is denoted by $\mathbf{ANR}^{\perp_{\mathbf{Gpd}}}$ [2, 3, 4].

Since the category **Top** of compactly generated spaces, besides **Gpd**, can be enriched as well over itself and over the category **Sets** of simplicial sets [12], Section 1 starts from the study of some interesting relations between different notions of enriched orthogonality. Every class of spaces has the same orthogonal with respect to **Top** and **Sets**, which is properly contained in its orthogonal with respect to **Gpd**. In Theorem 1.4, we obtain characterizations of homotopy equivalences, strong shape equivalences and shape equivalences. In particular, we are led to a general definition of a class of strong shape equivalences for any pair of categories $(\mathbf{C}, \mathbf{K})$, enriched over a monoidal closed model category $\mathcal{V}$. Such a concept, in the topological context, appears to be a primitive one since it arises naturally and produces that of homotopy equivalence and also the class of strong shape equivalences.

The first author would like to thank INDAM for financial support and the Dipartimento di Matematica e Informatica, Universitá di Perugia, for its hospitality during his staying on September 15 - October 15, 2006.

Received by the editors 2007-04-19 and, in revised form, 2009-05-27.

Transmitted by W. Tholen. Published on 2009-06-05.

2000 Mathematics Subject Classification: 55P10, 55P15, 55P5.

Key words and phrases: ANR-space, enriched orthogonality, groupoid, homotopy structure, prospace, shape equivalence, simplicial set, $\mathcal{V}$-orthogonal object, strongly fibered space.

Enriched orthogonality over a monoidal model category always implies orthogonality in the homotopy category, in the usual sense, but the converse is false generally. In Section 2, we deal with a class of spaces containing properly ANR-spaces, called strongly fibered. This is a class of spaces which plays a key role in shape and strong shape [5, 6, 7, 16]. Another example in this direction, but of a different nature, was given in [2]. By Theorem 2.4, for the class of strongly fibred spaces, shape and strong shape equivalences coincide, so that, in such a case homotopy orthogonality implies enriched orthogonality.

1. Enriched Orthogonality

In the following, $\mathcal{V}$ denotes a monoidal, closed model category, whose class of weak equivalences is denoted by $\mathbb{W}$. The homotopy category $\mathrm{ho}\,\mathcal{V}$ is then the localization of $\mathcal{V}$ with respect to $\mathbb{W}$.

Let $\mathbf{C}$ be a category enriched over $\mathcal{V}$, called also a $\mathcal{V}$-*category*. For $X, Y \in \mathrm{Ob}\ \mathbf{C}$, let $\mathcal{V}(X, Y)$ denote the object of $\mathcal{V}$ whose underlying set is $\mathrm{Hom}_{\mathbf{C}}(\mathrm{X}, \mathrm{Y})$.

1.1. Definition. An object $Z \in \mathbf{C}$ and a morphism $f \in \mathrm{Hom}_{\mathbf{C}}(\mathrm{X}, \mathrm{Y})$ are said to be $\mathcal{V}$-*orthogonal* if the morphism

$$f_Z^* : \mathcal{V}(Y, Z) \to \mathcal{V}(X, Z)$$

determined by composition with f, is a weak equivalence in $\mathcal{V}$, that is $f_Z^* \in \mathbb{W}$. Given $\mathbf{K} \subseteq \mathrm{Ob}\mathbf{C}$, the $\mathcal{V}$-orthogonal of $\mathbf{K}$, written $\mathbf{K}^{\perp_{\mathcal{V}}}$, is the class of morphisms in $\mathbf{C}$ that are $\mathcal{V}$-orthogonal to all objects of $\mathbf{K}$. In a similar way, if $\Sigma \subseteq \mathrm{Mor}\ \mathbf{C}$, its $\mathcal{V}$-orthogonal $\Sigma^{\top_{\mathcal{V}}}$ is the class of objects in $\mathbf{C}$ that are $\mathcal{V}$-orthogonal to all morphisms of Σ. A class of objects $\mathbf{K}$, (resp. a class of morphisms Σ), of $\mathbf{C}$ is called *internally saturated* if $\mathbf{K} = \mathbf{K}^{\perp_{\mathcal{V}}\top_{\mathcal{V}}}$ (resp. $\Sigma = \Sigma^{\top_{\mathcal{V}}\perp_{\mathcal{V}}}$). The class $\Sigma \subseteq \mathrm{Mor}\ \mathbf{C}$ is called *externally saturated* if it is the class of morphisms that are sent to isomorphisms by some functor $F : \mathbf{C} \to \mathbf{D}$ [2, 4]. We point out that this notion of an enriched orthogonality differs from that given in [13].

For a $\mathcal{V}$-category $\mathbf{C}$ and a full subcategory $\mathbf{K} \subseteq \mathbf{C}$, we call $\mathbf{K}^{\perp_{\mathcal{V}}}$ the class of *strong shape equivalences* for the pair $(\mathbf{C}, \mathbf{K})$.

In view of [4], every class of morphisms in $\mathbf{C}$ of the form $\mathbb{M} = \mathbf{K}^{\perp_{\mathcal{V}}}$ is both internally and externally saturated. The first assertion is clear since $\mathbb{M} = \mathbf{K}^{\perp_{\mathcal{V}}} = \mathbf{K}^{\perp_{\mathcal{V}}\top_{\mathcal{V}}\perp_{\mathcal{V}}} = \mathbb{M}^{\top_{\mathcal{V}}\perp_{\mathcal{V}}}$. For the second, let $\gamma_{\mathbf{K}} : \mathbf{C} \to [\![\mathbf{K}, \mathcal{V}]\!]^{op}$ be the functor which is defined on objects as $\gamma_{\mathbf{K}}(X) = \mathcal{V}(X, E(-))$, being $E : \mathbf{K} \to \mathbf{C}$ the inclusion functor. Then, $\mathbb{M}$ is the class of morphisms in $\mathbf{C}$ that are inverted by $\gamma_{\mathbf{K}}$. Moreover, a class $\mathbb{M}$ of morphisms of $\mathbf{C}$ is the class of strong shape equivalences for some pair $(\mathbf{C}, \mathbf{K})$ exactly when it is internally saturated with respect to $\mathcal{V}$; it suffices to take $\mathbf{K} = \mathbb{M}^{\top_{\mathcal{V}}}$.

In this paper, $\mathcal{V}$ is specialized to one of the following categories:

- **Gpd**, the category of groupoids and functors between them, with $\mathbb{W}$ the class of those functors that are equivalences of categories;

- **Sets**, the category of simplicial sets and simplicial maps, with $\mathbb{W}$ the class of simplicial weak homotopy equivalences;

- **Top**, the category of compactly generated Hausdorff spaces (k-spaces) and continuous maps, with $\mathbb{W}$ the class of weak homotopy equivalences.

At the same time, we consider three different kinds of enrichments of **Top**: over itself and over the categories **Gpd** and **Sets**. Given two spaces $X, Y \in \mathrm{Ob}\,\mathbf{Top}$, by $\mathrm{Map}(X, Y)$ we mean the set of continuous maps $X \to Y$. Then:

(1) $\mathbf{Gpd}(X, Y)$ is the track groupoid whose set of objects is $\mathrm{Map}(X, Y)$ and whose morphisms (tracks) $\alpha : f \Rightarrow g$ are the relative homotopy classes of homotopies connecting $f, g : X \to Y$;

(2) $\mathbf{Sets}(X, Y)$ is the simplicial set whose n-simplices are elements of $\mathbf{Sets}(X, Y)_n = \mathrm{Map}(X \times \delta[n], Y)$, for $n \geq 0$, and face and degeneracy maps are induced by the standard maps between the $\delta[n]'s$. Here, $\delta[n]$ denotes the standard topological n-simplex;

(3) $\mathbf{Top}(X, Y)$ is the topological space obtained giving $\mathrm{Map}(X, Y)$ the modified compact-open topology.

The ordinary set of homotopy classes of maps $X \to Y$ in one of the categories above, is denoted as usual by $[X, Y]$. It coincides with the set of components of $\mathbf{Top}(X, Y)$, $\mathbf{Gpd}(X, Y)$ and $\mathbf{Sets}(X, Y)$, each in the appropriate sense. It follows that there is a natural bijection between $[X, Y]$ and each of the sets $\mathrm{ho}\,\mathbf{Top}(\mathrm{X}, \mathrm{Y})$, $\mathrm{ho}\,\mathbf{Gpd}(\mathrm{X}, \mathrm{Y})$ and $\mathrm{ho}\,\mathbf{Sets}(\mathrm{X}, \mathrm{Y})$, respectively [2]. According to the classical terminology, we say that $f : X \to Y$ is *homotopy orthogonal* to a class $\mathbf{K} \subseteq \mathrm{Ob}\,\mathbf{Top}$ if the induced map

$$\tilde{f}_Z : [Y, Z] \to [X, Z]$$

is a bijection, for all $Z \in \mathbf{K}$. We adopt in this case the notation $f \in \mathbf{K}^{\perp_h}$.

The category **Top** is a cartesian closed category, that is, for all $X \in \mathbf{Top}$, the two functors $X \times (-), \mathbf{Top}(X, -) : \mathbf{Top} \to \mathbf{Top}$ are adjoints, so that, for spaces Y and Z, there are natural bijections

$$\mathrm{Map}(X \times Y, Z) \cong \mathrm{Map}(Y, \mathbf{Top}(X, Z))$$

which extend to equivalences in the enriched cases.

Consider the following result.

1.2. PROPOSITION. *For every class of spaces $\mathbf{K} \subseteq \mathrm{Ob}\,\mathbf{Top}$ the following relations hold*

$$\mathbf{K}^{\perp_{\mathbf{Top}}} = \mathbf{K}^{\perp_{\mathbf{Sets}}} \subseteq \mathbf{K}^{\perp_{\mathbf{Gpd}}}.$$

Moreover, if $f : X \to Y$ belongs to any of the classes above then also $f \in \mathbf{K}^{\perp_h}$.

PROOF. Note that, for $X, Z \in \mathbf{Top}$, the simplicial set $\mathbf{Sets}(X, Y)$ is the total singular complex of $\mathbf{Top}(X, Y)$, that is $\mathbf{Sets}(X, Y) = \mathrm{Sing}\,\mathbf{Top}(\mathrm{X}, \mathrm{Y})$ and, since a continuous map is a weak homotopy equivalence if and only if its total singular complex is a simplicial weak homotopy equivalence, it follows that $\mathbf{K}^{\perp_{\mathbf{Top}}} = \mathbf{K}^{\perp_{\mathbf{Sets}}}$. The inclusion $\mathbf{K}^{\perp_{\mathbf{Sets}}} \subseteq \mathbf{K}^{\perp_{\mathbf{Gpd}}}$ is obtained from the fact that $\mathbf{Gpd}(X, Y) = \Pi\mathbf{Sets}(X, Y)$, where the fundamental groupoid functor $\Pi : \mathbf{Sets} \to \mathbf{Gpd}$ takes simplicial weak homotopy equivalences to equivalences of $\mathbf{Gpd}$.

For the second assertion recall that every (simplicial) weak homotopy equivalence induces a bijection between the sets of path-components [18, Theorem 7.17], while a functor of groupoids is an equivalence if and only if it determines a bijection of the sets of components and isomorphisms of isotropy groups [1, 17]. ∎

1.3. REMARK. (1) The inclusion in Proposition 1.2 cannot be reversed in general, as it is shown by the following example, for which we are indebted to the referee.
Let Z be a non-contractible 2-connected space. Then the map $f : * \to \mathbb{S}^1$ sending the one-point space $*$ to the basepoint of the 1-sphere $\mathbb{S}^1$ is $\mathbf{Gpd}$-orthogonal to Z. Indeed, $\mathbf{Gpd}(\mathbb{S}^1, Z)$ has trivial set of components and trivial isotropy group at every point. However, f is not $\mathbf{Top}$-orthogonal to Z, since the loop space ΩZ is the homotopy fiber of the map $f^*_Z : \mathbf{Top}(\mathbb{S}^1, Z) \to \mathbf{Top}(*, Z)$ and ΩZ has non-trivial homotopy groups in higher dimensions.

(2) For $\mathbf{K} = \mathbf{Top}$ it is clear that

$$\mathbf{Top}^{\perp_{\mathbf{Top}}} = \mathbf{Top}^{\perp_{\mathbf{Sets}}} = \mathbf{Top}^{\perp_{\mathbf{Gpd}}} = \mathbf{Top}^{\perp_h} = \mathbb{H},$$

where $\mathbb{H}$ is the class of homotopy equivalences. In fact, a map $f : X \to Y$ that induces a bijection $\tilde{f}_Z : [Y, Z] \to [X, Z]$, for all $Z \in \mathbf{Top}$, has to be a homotopy equivalence. The converse is trivial.

(3) Write $\mathbf{Cat}$ for the category of small categories and Δ for the category of finite ordinals. Next, consider the functor $\chi : \Delta \to \mathbf{Cat}$ defined by $\chi([n]) = (\Delta \downarrow [n])$, the comma category for $[n] \in \Delta$. According to [11], such a functor induces a pair of adjoint functors:

- $\sigma : \mathbf{Sets} \to \mathbf{Cat}$, the *simplex category* or the *category of simplices*;
- $\gamma : \mathbf{Cat} \to \mathbf{Sets}$, right adjoint to σ, defined as $\gamma(\mathbf{C})_n = \mathbf{Cat}((\Delta \downarrow [n]), \mathbf{C})$ for $\mathbf{C} \in \mathrm{Ob}\,\mathbf{Cat}$.

Given spaces $X, Y \in \mathrm{Ob}\,\mathbf{Top}$ and $\mathbf{C} \in \mathrm{Ob}\,\mathbf{Cat}$, define:

- $\mathbf{Cat}(X, Y) = \sigma\mathbf{Sets}(X, Y)$,
- $X \otimes \mathbf{C} = X \times |N(\mathbf{C})|$, where $| - |$ is the geometric realization functor and N the nerve functor.

Since σ and N are homotopy inverse and N is a full embedding, so

$$\mathbf{Top}(X \otimes \mathbf{C}, Y) = \mathbf{Top}(X \times |N(\mathbf{C})|, Y) \cong \mathbf{Top}(|N(\mathbf{C})|, \mathbf{Top}(X, Y)) \cong$$
$$\mathbf{Sets}(N(\mathbf{C}), \mathbf{Sets}(X, Y)) \cong \mathbf{Sets}(N(\mathbf{C}), N(\sigma(\mathbf{Sets}(X, Y)))) \cong$$
$$\mathbf{Cat}(\mathbf{C}, \sigma(\mathbf{Sets}(X, Y))) = \mathbf{Cat}(\mathbf{C}, \mathbf{Cat}(X, Y)).$$

Hence, $\mathbf{Top}$ (and also $\mathbf{Sets}$) becomes tensored over $\mathbf{Cat}$.

Let now $\mathbf{K} \subseteq \mathrm{Ob}\ \mathbf{Top}$ and $f : X \to Y$ be a map in $\mathbf{Top}$. Assume $f \in \mathbf{K}^{\perp_{\mathbf{Cat}}}$, this means that for every $K \in \mathbf{K}$ the induced functor

$$f^*_K : \mathbf{Cat}(Y, K) \to \mathbf{Cat}(X, K)$$

is an equivalence of categories.

Then, the induced simplicial map

$$f_K^* : \mathbf{Sets}(Y, K) \to \mathbf{Sets}(X, K)$$

is a simplicial weak homotopy equivalence, i.e., $f \in K^{\perp_{\mathbf{Sets}}}$. The converse goes along the same lines. Finally,

$$K^{\perp_{\mathbf{Top}}} = K^{\perp_{\mathbf{Sets}}} = K^{\perp_{\mathbf{Cat}}} \subseteq K^{\perp_{\mathbf{Gpd}}}.$$

Observe that each of the three categories **Top**, **Sets** and **Cat** is tensored over the others, but not over **Gpd**. Let $\mathrm{loc} : \mathbf{Cat} \to \mathbf{Gpd}$ be the reflection functor consider e.g., in [9], i.e., $\mathrm{loc}(\mathbf{C}) = \mathbf{C}[(\mathrm{Mor}\,\mathbf{C})^{-1}]$ for $\mathbf{C} \in \mathrm{Ob}\mathbf{Cat}$. Then, $\mathrm{loc} \circ \sigma = \Pi$ is the fundamental groupoid functor. Hence, the functor loc, as it should be expected, is responsible for the destruction of the tensoring and the symmetry in the relation above.

There is another interesting case in which the inclusion of Proposition 1.2 can be reversed. Let **ANR** be the category of spaces having the homotopy type of ANR-spaces. Recall that a map $f : X \to Y$ is a (topological) *shape equivalence* [14] if it fulfils the following two properties:

(se.1) for each map $g : X \to Z$ with $Z \in \mathrm{Ob}\,\mathbf{ANR}$, there is a map $h : Y \to Z$ such that $h \circ f \simeq g$;

(se.2) if $h_1, h_2 : Y \to Z$ with $Z \in \mathbf{ANR}$ are maps such that $h_1 \circ f \simeq h_2 \circ f$ then $h_1 \simeq h_2$.

While f is a (topological) *strong shape equivalence* if it fulfils (se.1) and the following strengthened form of (se.2) holds:

(sse.2) given maps $h_1, h_2 : Y \to Z$ with $Z \in \mathrm{Ob}\,\mathbf{ANR}$ and a homotopy $G : X \times I \to Z$, $G : h_1 \circ f \simeq h_2 \circ f$, there is a homotopy $H : Y \times I \to Z$ with $H : h_1 \simeq h_2$ and such that $H \circ (f \times \mathrm{id}_I) \simeq G$, rel $X \times I \times \partial I$.

Note that, in light of [6], the homotopy H in (sse.2) is uniquely determined up to homotopies, rel $X \times I \times \partial I$.

Denote by Σ the class of topological strong shape equivalences, defined by the properties (se.1) and (sse.2). Then, it is clear that $\Sigma = \mathbf{ANR}^{\perp_{\mathbf{Gpd}}}$ is the class of strong shape equivalences for the pair $(\mathbf{Top}, \mathbf{ANR})$, while $\mathbf{ANR}^{\perp_{\mathbf{h}}}$ is the class of shape equivalences for that pair.

The following theorem is essentially contained in [6] (see also [8]) for sake of completeness we give here a direct proof.

1.4. Theorem. $\mathbf{ANR}^{\perp_{\mathbf{Top}}} = \mathbf{ANR}^{\perp_{\mathbf{Sets}}} = \mathbf{ANR}^{\perp_{\mathbf{Gpd}}} = \Sigma$.

Proof. Only the inclusion $\Sigma \subseteq \mathbf{ANR}^{\perp_{\mathbf{Top}}}$ has to be proved. Let $f : X \to Y$ be **Gpd**-orthogonal to **ANR**. By the Whitehead's Lemma [17] f induces bijections on the sets of components $f_Z : [Y, Z] \to [X, Z]$, for all $Z \in \mathrm{Ob}\,\mathbf{ANR}$. Let $\mathbb{S}^n_+$, $n \geq 0$, be the n-sphere with a disjoint base point. Then, by ([15, Theorem 4, p. 38]) it holds $\mathbf{Top}(\mathbb{S}^n_+, Z) \in \operatorname{Ob}\mathbf{ANR}$, for all $Z \in \operatorname{Ob}\mathbf{ANR}$. The bijection

$$f^*_{\mathbf{Top}(\mathbb{S}^n_+,Z)} : [Y, \mathbf{Top}(\mathbb{S}^n_+, Z)] \to [X, \mathbf{Top}(\mathbb{S}^n_+, Z)]$$

gives by adjunction a bijection

$$\phi : [\mathbb{S}^n_+, \mathbf{Top}(Y, Z)] \to [\mathbb{S}^n_+, \mathbf{Top}(X, Z)]$$

for all $n \geq 0$, induced by composition with f^*_Z and $Z \in \operatorname{Ob}\mathbf{ANR}$. By [3, Theorem 1.4], it follows that f^*_Z is a weak homotopy equivalence, for all $Z \in \operatorname{Ob}\mathbf{ANR}$, hence f is a strong shape equivalence. ∎

1.5. Corollary. *A map $f : X \to Y$ is a strong shape equivalence for the pair* $(\mathbf{Top}, \mathbf{ANR})$ *if and only if it induces a bijection* $[\mathbb{S}^n_+, \mathbf{Top}(X, Z)] \cong [\mathbb{S}^n_+, \mathbf{Top}(Y, Z)]$, *for all* $Z \in \operatorname{Ob}\mathbf{ANR}$ *and* $n \geq 0$.

1.6. Remark. (1) The condition $f \in \mathbf{ANR}^{\perp_{\mathbf{Top}}}$ was used in [6] to define topological strong shape equivalences. Hence, the results above also give a quick proof of the equivalence of this definition with the classical topological one of [14] given above.

(2) By Remark 1.3.(2), the class $\mathbb{H}$ of homotopy equivalences is the class of strong shape equivalences for the pair $(\mathbf{Top}, \mathbf{Top})$. On the other hand, denoting by $\mathtt{T}$ the terminal object of $\mathbf{Top}$, one gets $\mathtt{T}^{\perp_{\mathbf{Gpd}}} = \{\text{all continuous maps}\}$. Moreover, for classes of objects $\mathbf{H} \subseteq \mathbf{K} \subseteq \operatorname{Ob}\mathbf{Top}$, it holds

$$\mathbb{H} = \mathbf{Top}^{\perp_{\mathbf{Gpd}}} \subseteq \mathbf{K}^{\perp_{\mathbf{Gpd}}} \subseteq \mathbf{H}^{\perp_{\mathbf{Gpd}}} \subseteq \{\text{all continuous maps}\}.$$

It follows that the (relative) concept of strong shape equivalence is so general to be considered as a primitive one.

The following proposition can be used to characterize strong shape equivalences in terms of shape equivalences. We shall say more on that in the next section.

1.7. Proposition. *Let $f : X \to Y$ be a map and $\mathbf{K} \subseteq \operatorname{Ob}\mathbf{Top}$. Then, $f \in \mathbf{K}^{\perp_{\mathbf{Top}}}$ if and only if $f \times \mathrm{id}_C \in \mathbf{K}^{\perp_h}$ for all CW-complexes C.*

Proof. Let $f \in \mathbf{K}^{\perp_{\mathbf{Top}}}$, then $f^*_Z : \mathbf{Top}(Y, Z) \to \mathbf{Top}(X, Z)$ is a weak homotopy equivalence for all $Z \in \mathbf{K}$. This amounts [18] to the fact that the map $[C, \mathbf{Top}(Y, Z)] \to [C, \mathbf{Top}(X, Z)]$, induced by f^*_Z is a bijection, for all CW-complexes C. Its adjoint map $[Y \times C, Z] \to [X \times C, Z]$, induced by $f \times \mathrm{id}_C : X \times C \to Y \times C$, is also bijective. This fact says that $f \times \mathrm{id}_C$ has to be a shape equivalence for the pair $(\mathbf{Top}, \mathbf{K})$, for all CW-complexes C. ∎

2. Strong Shape Equivalences

In the previous section, we have seen that enriched orthogonality (whatever) implies homotopy orthogonality. The converse is false, in general, apart from the cases $\mathbf{K} = \mathbf{Top}$ and $\mathbf{K} = \{\mathtt{T}\}$, see [2] for an algebraic example in this direction.
In this section, we exhibit a class of spaces, called *strongly fibered*, containing properly ANR-spaces, for which the converse holds, in particular its homotopy orthogonal class coincides with the class Σ of topological strong shape equivalences.

Recall from [16], that a map $f : X \to Y$ is said to have the *strong homotopy extension property* (SHEP) with respect to a class $\mathbf{K}$ of spaces if the following diagram

$$\begin{tikzcd} X \arrow[r, shift left, "e_0^X"] \arrow[r, shift right, "e_1^X"'] \arrow[d, "f"'] & X \times I \arrow[d, "f \times \mathrm{id}_I"] \\ Y \arrow[r, shift left, "e_0^Y"] \arrow[r, shift right, "e_1^Y"'] & Y \times I \end{tikzcd}$$

is a weak colimit with respect to $\mathbf{K}$, where $e_0^X, e_1^X : X \to X \times I$ are the canonical imbeddings. That is, for given maps $u, v : Y \to P$ in $\mathbf{K}$ and a homotopy $H : X \times I \to P$ with $H : u \circ f \simeq v \circ f$, there exists a homotopy $G : Y \times I \to P$ with $G : u \simeq v$ and such that $G \circ (f \times \mathrm{id}_I) = H$.
It was proved in [16, Theorem 3.6] that for a map $f : X \to Y$ which is a (Hurewicz) cofibration and has property (se.1), to be a strong shape equivalence amounts to have the SHEP with respect to **ANR**.

The class of strong shape equivalences for $(\mathbf{Top}, \mathbf{ANR})$ contains all homotopy equivalences and is closed under composition. Consequently, if $f = f_1 \circ f_0$ is the mapping cylinder decomposition of f, with f_0 a cofibration and f_1 a homotopy equivalence then f is a strong shape equivalence if and only if f_0 is so. It follows that (when speaking of (strong) shape equivalences) there is no loss of generality in assuming that the maps involved are cofibrations.
In the following, we use also the facts that, if f is a strong shape equivalence (resp. a cofibration) then also $f \times \mathrm{id}_I : X \times I \to Y \times I$ is a strong shape equivalence (resp. a cofibration), see [7] and [18], respectively.

Now, let $\mathtt{Z} = (Z_\mu, z_{\mu\mu'}, M)$ be an inverse system of ANR's. Following the notation of [15], we assume that the index set $(M, \leq)$ is directed and cofinite, that is every $\mu \in M$ has only finitely many predecessors. Recall that a *promap* $\mathtt{g} : X \to \mathtt{Z}$ is a natural cone of maps $\{g_\mu : X \to Z_\mu \mid \mu \in M\}$, that is, for $\mu \leq \mu'$ in M, it holds $z_{\mu\mu'} \circ g_{\mu'} = g_\mu$.
A homotopy connecting promaps $\mathtt{g}, \mathtt{h} : X \to \mathtt{Z}$, is a promap $\mathtt{H} : X \times I \to \mathtt{Z}$ such that $\mathtt{H} \circ e_0 = \mathtt{h}$, $\mathtt{H} \circ e_1 = \mathtt{g}$. Note that, for every $\mu \in M$, the map $H_\mu : X \times I \to Z_\mu$ is a homotopy connecting h_μ and g_μ. This allows one to define, in a natural way, the fundamental or track groupoid $\mathbf{Gpd}(X, \mathtt{Z})$ of a space X and an inverse system $\mathtt{Z}$ of spaces.

2.1. LEMMA. *Let* $\mathtt{Z}$ *be an inverse system of spaces and* $\mathtt{p} : Z \to \mathtt{Z}$ *the limiting cone. For every space* X*, the functor* $\mathbf{Gpd}(X, Z) \to \mathbf{Gpd}(X, \mathtt{Z})$ *induced by composition with* $\mathtt{p}$*, is an equivalence.*

PROOF. Clearly, the functor $\mathbf{Gpd}(X, Z) \to \mathbf{Gpd}(X, \mathtt{Z})$ is essentially surjective. Let $\alpha, \beta : X \to Z$ be maps and $\mathtt{H} : X \times I \to \mathtt{Z}$ a homotopy $\mathtt{H} : \mathtt{p} \circ \alpha \simeq \mathtt{p} \circ \beta$. For every $\mu \in M$, we get a homotopy $H_\mu : X \times I \to Z_\mu$ with $H_\mu : p_\mu \circ \alpha \simeq p_\mu \circ \beta$. By the universal property of the limit, there is a unique homotopy $G : X \times I \to Z$ with $p_\mu \circ G = H_\mu$, for all $\mu \in M$. Then, $p_\mu \circ G \circ\ e_0 = H_\mu \circ e_0 = p_{\mu \circ \alpha}$, for all $\mu \in M$. By the universal property of the limit, it follows that $G \circ\ e_0 = \alpha$. Analogously, $G \circ\ e_1 = \beta$.
Now, let $H,\ H' : X \times I \to Z$ be two homotopies connecting α and β and suppose that there is a homotopy $\chi : X \times I \times I \to \mathtt{Z}$ with $\chi : \mathtt{p} \circ H \simeq \mathtt{p} \circ H'$ (rel $X \times I \times \partial I$). Then, for every $\mu \in M$, it holds $\chi_\mu : p_\mu \circ H \simeq p_\mu \circ H'$. By arguments similar to the above, we get a homotopy $\Gamma : X \times I \times I \to Z$ with $\Gamma : H \simeq H'$ (rel $X \times I \times \partial I$), and $\mathtt{p} \circ \Gamma = \chi$. ∎

An inverse system $\mathtt{Z}$ of ANR-spaces is called *strongly fibrant* [5, 7, 16] if the unique promap $\mathtt{Z} \to *$ is a fibration and, moreover, for every $\mu^* \in M$, the unique map $z_{\mu^*} : Z_{\mu^*} \to \lim_{\mu<\mu^*} Z_\mu$, induced by the bonding morphisms of the system, is a (Hurewicz) fibration. A *strongly fibered* space Z is a space which can be represented as the inverse limit of a strongly fibrant inverse system of ANR's. Denote by **Fib** the class of strongly fibered spaces. It is clear that $\mathbf{ANR} \subseteq \mathbf{Fib}$.

2.2. LEMMA. (cf. [6, 16]) *Let* $f : X \to Y$ *be a strong shape equivalence and* $\mathtt{Z}$ *be a strongly fibrant inverse system. Moreover, let* $\{k_\mu : Y \to Z_\mu |\ \mu \in M\}$ *be a family of maps such that* $\mathtt{g} = \{k_\mu \circ f : Z \to Z_\mu |\ \mu \in M\}$ *is a promap. Then, there is a promap* $\mathtt{h} : Y \to \mathtt{Z}$ *such that* $\mathtt{h} \circ f = \mathtt{g}$.

PROOF. We proceed by induction on the number $\#(\mu)$ of predecessors of $\mu \in M$:

- if μ has no predecessors, then put $h_\mu = k_\mu$;
- assume we have defined $h_\mu : Y \to Z_\mu$, for all $\mu \in M$ with $1 \leq \#(\mu) < n$, in such a way that $\mu' < \mu$ implies $z_{\mu'\mu} \circ h_\mu = h_{\mu'}$;
- let $\mu^* \in M$ with $\#(\mu^*) = n$. Note that the maps h_μ with $\mu < \mu^*$ induce a unique map $h : Y \to \lim_{\mu<\mu^*} Z_\mu$ such that $p_\mu \circ h = h_\mu$, for all $\mu < \mu^*$, where the p_μ's are the projections of the limit. Then, $h \circ f = z_{\mu^*} \circ k_{\mu^*} \circ f$ and, since f is a sse, there is a homotopy $H : Y \times I \to \lim_{\mu<\mu^*} Z_\mu$ with $H : h \simeq z_{\mu^*} \circ k_{\mu^*}$ and such that $H \circ (f \times \mathrm{id}_I)$ is stationary.

Since z_{μ^*} is a fibration, the following commutative diagram

$$\begin{tikzcd} Y \arrow[r, "k_{\mu^*}"] \arrow[d, "e_0"'] & Z_{\mu^*} \arrow[d, "z_{\mu^*}"] \\ Y \times I \arrow[r, "H"] \arrow[ur, "H^*"] & \lim_{\mu<\mu^*} Z_\mu \end{tikzcd}$$

has a lifting $H^* : Y \times I \to Z_{\mu^*}$ such that $H^* \circ e_0 = k_{\mu^*},\ z_{\mu^*} \circ H^* = H$ and $H^* \circ (f \times \mathrm{id}_I)$ is stationary. At this point, putting h_{μ^*} to be $H^* \circ e_1 : Y \to Z_{\mu^*}$, the definition of a promap $\mathtt{h} : Y \to \mathtt{Z}$ with $\mathtt{h} = \{h_\mu\}$ and such that $\mathtt{h} \circ f = \mathtt{g}$ is completed. ∎

2.3. PROPOSITION. *Let $f : X \to Y$ be a strong shape equivalence for* $(\mathbf{Top}, \mathbf{ANR})$. *Then, f induces an equivalence of groupoids*

$$f^*_{\mathtt{Z}} : \mathbf{Gpd}(Y, \mathtt{Z}) \to \mathbf{Gpd}(X, \mathtt{Z})$$

for every strongly fibrant prospace $\mathtt{Z}$.

PROOF. We may assume, without restrictions, that f is a cofibration. Let $\mathtt{g} : X \to \mathtt{Z}$ be a promap. Then, from (se.1) and ([12, p. 11]), for every $\mu \in M$, there is a map $k_\mu : Y \to Z_\mu$ such that $k_\mu \circ f = g_\mu$. From Lemma 2.2, it follows that $f^*_{\mathtt{Z}}$ is essentially surjective.
Let $\mathtt{h}$, $\mathtt{h}' : Y \to \mathtt{Z}$ and $\mathtt{G} : X \times I \to \mathtt{Z}$ be a homotopy $G : \mathtt{h} \circ f \simeq \mathtt{h}' \circ f$. Then, for every $\mu \in M$ the map $G_\mu : X \times I \to Z_\mu$ is a homotopy connecting $h_\mu \circ f$ and $h'_\mu \circ f$. By the SHEP, the diagram

$$\begin{tikzcd}
X \arrow[r, shift left, "e_0^X"] \arrow[r, shift right, "e_1^X"'] \arrow[d, "f"'] & X \times I \arrow[d, "f \times \mathrm{id}_I"] \arrow[ddr, "G_\mu"] & \\
Y \arrow[r, shift left, "e_0^Y"] \arrow[r, shift right, "e_1^Y"'] \arrow[drr, shift left, "h_\mu"] \arrow[drr, shift right, "h'_\mu"'] & Y \times I \arrow[dr, "H_\mu"] & \\
 & & Z_\mu
\end{tikzcd}$$

yields, for every $\mu \in M$, a homotopy $H_\mu : h_\mu \simeq h'_\mu$ such that $H_\mu \circ (f \times \mathrm{id}_I) = G_\mu$. Again from Lemma 2.2, the H_μ's give a homotopy $\mathtt{H} : Y \times I \to \mathtt{Z}$ with $\mathtt{H} : \mathtt{h} \simeq \mathtt{h}'$ and such that $\mathtt{H} \circ (f \times 1) = \mathtt{G}$. Hence, $f^*_{\mathtt{Z}}$ is a full functor.
Finally, let $\mathtt{H}$, $\mathtt{H}' : Y \times I \to \mathtt{Z}$ be homotopies connecting $\mathtt{h}$ and $\mathtt{h}'$ and assume that there is a homotopy $\Gamma : \mathtt{H} \simeq \mathtt{H}'$ (rel $X \times I \times \partial I$). It follows that $\Gamma_\mu : H_\mu \simeq H'_\mu$ (rel $X \times I \times \partial I$), for every $\mu \in M$. Since $f \times \mathrm{id}_I$ is a strong shape equivalence and a cofibration, by the SHEP for $f \times 1$, there is, for every $\mu \in M$, a homotopy $\chi_\mu : H_\mu \simeq H'_\mu$ such that $\chi_\mu \circ (f \times \mathrm{id}_I) = \Gamma_\mu$. Using again Lemma 2.2, it follows the existence of a homotopy $\chi : \mathtt{H} \simeq \mathtt{H}'$ (rel $Y \times I \times \partial I$). Consequently, f^*_μ is also faithful. ∎

Finally, we are in a position to state

2.4. THEOREM. *For the pair* $(\mathbf{Top}, \mathbf{Fib})$ *the notions of shape equivalence and strong shape equivalence coincide.*

PROOF. Let $f : X \to Y$ be a shape equivalence for the pair $(\mathbf{Top}, \mathbf{Fib})$. Since every ANR-space is strongly fibered, it follows that f is a shape equivalence for $(\mathbf{Top}, \mathbf{ANR})$. Then, for all $Z \in \mathrm{Ob}\,\mathbf{Fib}$, the induced map $f^*_Z : [Y, Z] \to [X, Z]$ is a bijection. Take

$\mathtt{Z} = \Phi(X)$ the strongly fibrant prospace associated with X and constructed in [16, §2] and let $\eta : X \to \Phi(X)$ the related expansion which is a strong shape equivalence. Then, there is a promap $\mathtt{g} : Y \to \Phi(X)$ such that $\mathtt{g} \circ f = \eta$. From [6, Theorem 1.8], it follows that f is a strong shape equivalence for $(\mathbf{Top}, \mathbf{ANR})$. By Proposition 2.3, f induces equivalences $\mathbf{Gpd}(Y, \mathtt{Z}) \simeq \mathbf{Gpd}(X, \mathtt{Z})$, for all strongly fibrant prospaces $\mathtt{Z}$. In view of Lemma 2.1, the map f induces equivalences $\mathbf{Gpd}(Y, Z,) \simeq \mathbf{Gpd}(X, Z)$ for $Z = \lim \mathtt{Z}$, hence it is a strong shape equivalence for the pair $(\mathbf{Top}, \mathbf{Fib})$. ∎

2.5. Remark. The class **Fib** of strongly fibered spaces properly contains the class **ANR** of absolute neighborhood retracts. In fact, it follows from Proposition 2.3 that a continuous map f is a strong shape equivalence for the pair $(\mathbf{Top}, \mathbf{ANR})$ if and only if it is an ordinary shape equivalence for the pair $(\mathbf{Top}, \mathbf{Fib})$. Since it is known that there are shape equivalences for $(\mathbf{Top}, \mathbf{ANR})$ that are not strong [7], the claim follows. Furthermore, Theorem 2.4 asserts that $\mathbf{Fib}^{\perp_{\mathbf{Gpd}}} = \mathbf{Fib}^{\perp_h}$.

References

[1] R. Brown, *Topology*, Ellis Horwood (1988).

[2] C. Casacuberta, B.Chorny, *The Orthogonal Subcategory Problem in Homotopy Theory*, Contemp. Math., vol. 399 (2006), 4-53.

[3] C. Casacuberta, J.L. Rodriguez, *On Weak Homotopy Equivalences Between Mapping Spaces*, Topology, vol. 37, no. 4 (1998), 709-717.

[4] C. Casacuberta, A. Frei, *Localizations as Idempotent Approximations to Completions*, J. Pure Appl. Alg., vol. 142 (1999), 25-33.

[5] F.W. Cathey, J. Segal, *Strong Shape Theory and Resolutions*, Top. Appl. 15 (1983), 119–130.

[6] J. Dydak, S. Nowak, *Strong Shape for Topological Spaces*, Trans. Amer. Math. Soc. 323, (2) (1991), 765–796.

[7] J. Dydak, S. Nowak, *Function Spaces and Shape Theories*, Fund. Math., vol. 171 (2002), 117-154.

[8] M. Golasiński, L. Stramaccia, *Weak Homotopy Equivalences of Mapping Spaces and Vogt's Lemma*, Cah. Topol. Géom. Différ. Catèg. 49 (2008), 69-80.

[9] P. Gabriel, M. Zisman, *Calculus of Fractions and Homotopy Theory*, Springer Verlag, Berlin-Heidelberg-New York (1968).

[10] P.S. Hirschhorn, *Model Categories and Their Localizations*, Mathematical Surveys and Monographs, 99. American Mathematical Society, Providence, RI (2003).

[11] D.M. Kan, *On c.s.s. Complexes*, Trans. Amer. Math. Soc. 87 (1958), 330-346.

[12] H. Kamps, T. Porter, *Abstract Homotopy and Simple Homotopy Theory*, World Scientific (1997).

[13] G.M. Kelly, *Basic Concepts of Enriched Category Theory*, Repr. Theory Appl. Categ. no. 10 (2005).

[14] S. Mardešić, *Strong Shape and Homology*, Springer Monographs in Mathematics, Springer Verlag, Berlin-Heidelberg-New York (2000).

[15] S. Mardešić, J. Segal, *Shape Theory*, North Holland (1982).

[16] L. Stramaccia, *Metrizable Shape and Strong Shape Equivalences*, Homology Homotopy Appl., vol. 4 (2002), 71-85.

[17] L. Stramaccia, 2-*Categorical Aspects of Strong Shape*, Topology Appl. 153 (2006), 3007-3018.

[18] G.W. Whitehead, *Elements of Homotopy Theory*, Graduate Texts in Mathematics, Springer Verlag, Berlin-Heidelberg-New York (1978).

Faculty of Mathematics and Computer Science
Nicholaus Copernicus University
87-100 Toruń, Chopina 12/18 - Poland

Dipartimento di Matematica e Informatica
Universitá di Perugia
via Vanvitelli, 06123 Perugia - Italia
Email: marek@mat.uni.torun.pl
stra@dipmat.unipg.it

This article may be accessed at http://www.tac.mta.ca/tac/ or by anonymous ftp at ftp://ftp.tac.mta.ca/pub/tac/html/volumes/22/11/22-11.{dvi,ps,pdf}

THEORY AND APPLICATIONS OF CATEGORIES (ISSN 1201-561X) will disseminate articles that significantly advance the study of categorical algebra or methods, or that make significant new contributions to mathematical science using categorical methods. The scope of the journal includes: all areas of pure category theory, including higher dimensional categories; applications of category theory to algebra, geometry and topology and other areas of mathematics; applications of category theory to computer science, physics and other mathematical sciences; contributions to scientific knowledge that make use of categorical methods.

Articles appearing in the journal have been carefully and critically refereed under the responsibility of members of the Editorial Board. Only papers judged to be both significant and excellent are accepted for publication.

Full text of the journal is freely available in .dvi, Postscript and PDF from the journal's server at `http://www.tac.mta.ca/tac/` and by ftp. It is archived electronically and in printed paper format.

Subscription information. Individual subscribers receive abstracts of articles by e-mail as they are published. To subscribe, send e-mail to `tac@mta.ca` including a full name and postal address. For institutional subscription, send enquiries to the Managing Editor, Robert Rosebrugh, `rrosebrugh@mta.ca`.

Information for authors. The typesetting language of the journal is TeX, and LaTeX2e strongly encouraged. Articles should be submitted by e-mail directly to a Transmitting Editor. Please obtain detailed information on submission format and style files at `http://www.tac.mta.ca/tac/`.