Theory and Applications of Categories, Vol. 22, No. 12, 2009, pp. 313–320.

WEAK DISTRIBUTIVE LAWS

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ABSTRACT. Distributive laws between monads (triples) were defined by Jon Beck in the 1960s; see [1]. They were generalized to monads in 2-categories and noticed to be monads in a 2-category of monads; see [2]. Mixed distributive laws are comonads in the 2-category of monads [3]; if the comonad has a right adjoint monad, the mate of a mixed distributive law is an ordinary distributive law. Particular cases are the entwining operators between algebras and coalgebras; for example, see [4]. Motivated by work on weak entwining operators (see [5] and [6]), we define and study a weak notion of distributive law for monads. In particular, each weak distributive law determines a wreath product monad (in the terminology of [7]); this gives an advantage over the mixed case.

1. Introduction

Distributive laws between monads (triples) were defined by Jon Beck [1] in the 1960s. In [2] they were generalized to monads in 2-categories and were noticed to be monads in a 2-category of monads. The 2-categories can easily be replaced by bicategories. Mixed distributive laws are comonads in the bicategory of monads. Entwining structures between a coalgebra and an algebra were introduced in [8] and [9]. At the level of entwining structures $\psi : C \otimes A \to A \otimes C$ between a comonoid C and a monoid A in a monoidal category $\mathcal{C}$ (as in [4] for example), the concept is the same as a mixed distributive law. On the one hand, the monoidal category $\mathcal{C}$ can be regarded as the endohom category of a one-object bicategory so that C is a comonad and A is a monad in that bicategory, while ψ is a mixed distributive law. On the other hand, we obtain an ordinary comonad $G = C \otimes -$ and a monad $T = A \otimes -$ on the category $\mathcal{C}$, and $\psi \otimes -$ is a mixed distributive law $GT \to TG$.

Any mixed distributive law $\psi : GT \to TG$ for which the comonad G has a right adjoint monad S (which it always does *qua* profunctor) is the mate [10] of a distributive law $\lambda : TS \to ST$ between two monads. The advantage of this is that we obtain a composite monad ST.

With the introduction of weak entwining operators (see [5] and [6]), the subject of the present paper is naturally to look at the counterpart in terms of comonads and monads. The weakening here has to do with the compatibility of ψ with the comonad's counit and monad's unit. The main result is to obtain a new monad $S \circ_\lambda T$ from a weak distributive

Received by the editors 2009-03-25 and, in revised form, 2009-06-09.
Transmitted by Jiri Rosicky. Published on 2009-06-15.
2000 Mathematics Subject Classification: 18C15; 18D10; 18D05.
Key words and phrases: monad; triple; distributive law; weak bialgebra.

law $\lambda : TS \to ST$ by splitting a certain idempotent κ on the composite ST.

In my talk on weak distributive laws in the Australian Category Seminar on 21 January 2009, I mentioned that there seemed to be two popular uses for the adjective "weak". One comes from the literature on higher categories where a "weak 2-category" means a "bicategory". I must take some blame for this use because, in [11], it was linked to the use by Freyd of the term "weak limit" (existence without uniqueness). The other use, which is the sense intended in this paper, comes from the quantum groups literature: see [12], [13]. The weakening here is to do with units (identity cells). In the talk, I speculated as to whether there was a connection between the two uses. Before the end of January, I had the basic form of this paper typed. A month later Steve Lack, expecting there to be some connection to my work, drew my attention to the posting [14] by Gabriella Böhm in which a "weak" version of the EM construction in [7] was developed. Gabriella, Steve and the author followed this with a sequence of interesting emails. The weak distributive laws here are a special kind of weak wreath in the sense of [14]; and Theorem 4.1 can be extracted from Proposition 3.7 of [14]. Finally, however, also in our communications, the sense in which the two uses of "weak" are related is emerging; publication of this will undoubtedly follow soon.

2. Weak distributive laws

For any monad T on a category $\mathcal{A}$, we write $\mu : TT \to T$ and $\eta : 1 \to T$ for the multiplication and unit.

Let S and T be monads on a category $\mathcal{A}$ (however, we could take them to be monads on an object $\mathcal{A}$ of any bicategory).

2.1. Definition. *A* weak distributive law of T over S *is a natural transformation (2-cell) $\lambda : TS \to ST$ satisfying the following three conditions.*

$$TTS \xrightarrow{\mu S} TS \xrightarrow{\lambda} ST \quad = \quad TTS \xrightarrow{T\lambda} TST \xrightarrow{\lambda T} STT \xrightarrow{S\mu} ST \tag{1}$$

$$TSS \xrightarrow{S\mu} TS \xrightarrow{\lambda} ST \quad = \quad TSS \xrightarrow{\lambda S} STS \xrightarrow{S\lambda} SST \xrightarrow{\mu T} ST \tag{2}$$

$$ST \xrightarrow{\eta ST} TST \xrightarrow{\lambda T} STT \xrightarrow{S\mu} ST \quad = \quad ST \xrightarrow{ST\eta} STS \xrightarrow{S\lambda} SST \xrightarrow{\mu T} ST \tag{3}$$

2.2. Proposition. *Equation 3 is equivalent to the following two conditions:*

$$S \xrightarrow{\eta S} TS \xrightarrow{\lambda} ST \quad = \quad S \xrightarrow{S\eta\eta} STS \xrightarrow{S\lambda} SST \xrightarrow{\mu T} ST \tag{4}$$

$$T \xrightarrow{T\eta} TS \xrightarrow{\lambda} ST \quad = \quad T \xrightarrow{\eta\eta T} TST \xrightarrow{\lambda T} STT \xrightarrow{S\mu} ST. \tag{5}$$

Proof. Given equation 3, we have

$$\mu T.S\lambda.S\eta\eta = S\mu.\lambda T.\eta ST.\eta S = S\mu.\lambda T.TS\eta.\eta S = S\mu.ST\eta.\lambda.\eta S = \lambda.\eta S.$$

This proves equation 4, while equation 5 is dual. Conversely, given the equations of the Proposition,

$$S\mu.\lambda T.\eta ST = S\mu. \quad \mu TT.S\lambda T.S\eta\eta T = \mu T.SS\mu.S\lambda T.S\eta\eta T = \mu T.S\lambda.ST\eta.$$

∎

Recall from [1] that a distributive law of T over S is a 2-cell $\lambda : TS \to ST$ satisfying equations 1 and 2 and the following two unit conditions:

$$S \xrightarrow{\eta S} TS \xrightarrow{\lambda} ST \quad = \quad S \xrightarrow{S\eta} ST \tag{6}$$

$$T \xrightarrow{T\eta} TS \xrightarrow{\lambda} ST \quad = \quad T \xrightarrow{\eta T} ST. \tag{7}$$

These clearly imply equations 4 and 5

We define the endomorphism $\kappa : ST \to ST$ to be either side of equation 3. This is an identity in the non-weak case.

2.3. Proposition. *The endomorphism κ is idempotent and satisfies the following two conditions:*

$$\kappa.\lambda = \lambda \tag{8}$$

$$\mu\mu.S\lambda T.\kappa\kappa \quad = \kappa.\mu\mu.S\lambda T. \tag{9}$$

Proof. While string diagrams are a better way to prove this, here are some equations (using only monad properties and equations 1 and 2):

$$\begin{gathered}
\kappa.\kappa = S\mu.\lambda T.\eta ST.S\mu.\lambda T.\eta ST = S\mu.ST\mu.\lambda TT.\eta STT.\lambda T.\eta ST = \\
S\mu.S\mu T.\lambda TT.T\lambda T.\eta TST.\eta ST = S\mu.\lambda T. \quad \eta ST = \kappa \quad , \\
\kappa.\lambda = S\mu.\lambda T.\eta ST.\lambda = S\mu.\lambda T.T\lambda.\eta TS = \lambda.\mu S.\eta TS = \lambda, \quad \text{and} \\
\mu\mu.S\lambda T.\kappa\kappa \quad = S\mu.\mu\mu T.S\lambda TT.S\mu STT.\lambda T\lambda T.\eta ST\eta ST = \\
= S\mu.\mu\mu T.SS\mu TT.S\lambda TTT.ST\lambda TT.\lambda T\lambda T.\eta ST\eta ST = \\
S\mu.S\mu T.\lambda TT.T\mu T\mu.TS\lambda TT.TST\lambda T.\eta ST\eta ST = \\
S\mu.\lambda T.\eta ST.\mu\mu.S\lambda T.S\mu ST. \quad ST\eta ST = \kappa.\mu\mu.S\lambda T.
\end{gathered}$$

∎

Define a multiplication on ST to be the composite

$$\mu = \left(STST \xrightarrow{S\lambda T} SSTT \xrightarrow{\mu\mu} ST \right).$$

The usual calculation as with a distributive law using equations 1 and 2 shows that this multiplication is associative. However, in the weak case we do not have a monad ST since $1 \xrightarrow{\eta\eta} ST$ is not generally a unit.

Assume the idempotent κ splits in $\mathcal{A}$ (or in the category of endomorphisms of $\mathcal{A}$ when we are in a bicategory). We have

$$\kappa = \left(ST \xrightarrow{\upsilon} K \xrightarrow{\iota} ST\right) \quad \text{and} \quad \left(K \xrightarrow{\iota} ST \xrightarrow{\upsilon} K\right) = 1_K.$$

Now we obtain candidates for a multiplication and unit on $K : \mathcal{A} \to \mathcal{A}$ defined as follows:

$$\mu = \left(KK \xrightarrow{\iota\iota} STST \xrightarrow{\mu} ST \xrightarrow{\upsilon} K\right) \quad \text{and} \quad \eta = \left(1 \xrightarrow{\eta\eta} ST \xrightarrow{\upsilon} K\right).$$

2.4. THEOREM. *With this multiplication and unit, K is a monad.*

PROOF. By equation 9, the idempotent κ preserves the associative multiplication on ST so the splitting K has an induced associative multiplication as defined above. It remains to show that η is the unit.

$$\begin{gathered}\mu.K\eta = \upsilon.\mu\mu.S\lambda T.\iota\iota.K\iota.K\eta\eta \;= \upsilon.\mu\mu.S\lambda T.\iota \;\; ST.KS\mu.K\lambda T.K\eta\eta\eta \;= \\ \upsilon.\mu\mu.S\lambda T.\iota \;\; ST.K\lambda T.K\eta\eta \;= \upsilon.\mu T.S\lambda \;\; .S\mu T.\iota \;\; ST.K\eta\eta \;= \\ \upsilon.\mu T.S\lambda \;\; .ST\eta.\iota = \upsilon.\kappa.\iota = 1_K.\end{gathered}$$

Similarly, $\mu.\eta K = 1_K$. ∎

2.5. DEFINITION. *Following [7], we call K the* wreath product of T over S with respect *to λ; the notation is $K = S \circ_\lambda T$.*

2.6. LEMMA. *The following three equations hold:*

$$STST \xrightarrow{\upsilon\upsilon} KK \xrightarrow{\mu} K \;=\; STST \xrightarrow{S\lambda T} SSTT \xrightarrow{\mu\mu} ST \xrightarrow{\upsilon} K; \tag{10}$$

$$KK \xrightarrow{\iota\iota} STST \xrightarrow{S\lambda T} SSTT \xrightarrow{\mu\mu} ST \xrightarrow{\kappa} ST \;=\; KK \xrightarrow{\iota\iota} STST \xrightarrow{S\lambda T} SSTT \xrightarrow{\mu\mu} ST; \tag{11}$$

$$K \xrightarrow{\iota} ST \xrightarrow{S\eta\eta T} SSTT \xrightarrow{\mu\mu} ST \xrightarrow{\upsilon} K \;=\; K \xrightarrow{1_K} K. \tag{12}$$

PROOF. This is fairly easy in light of Proposition 2.3. ∎

3. Weak mixed distributive laws

For any comonad G on $\mathcal{A}$, we write $\delta : G \to GG$ and $\varepsilon : G \to 1$ for the comultiplication and counit. Let T be a monad on $\mathcal{A}$.

3.1. DEFINITION. *A* weak (mixed) distributive law of a monad T over a comonad G *is a 2-cell $\psi : GT \to TG$ satisfying the following four conditions in which*

$\xi = (G \xrightarrow{G\eta} GT \xrightarrow{\psi} TG \xrightarrow{T\varepsilon} T)$.

$$GTT \xrightarrow{G\mu} GT \xrightarrow{\psi} TG \quad = \quad GTT \xrightarrow{\psi T} TGT \xrightarrow{T\psi} TTG \xrightarrow{\mu G} TG \tag{13}$$

$$GT \xrightarrow{\psi} TG \xrightarrow{T\delta} TGG \quad = \quad GT \xrightarrow{\delta T} GGT \xrightarrow{G\psi} GTG \xrightarrow{\psi G} TGG \tag{14}$$

$$G \xrightarrow{G\eta} GT \xrightarrow{\psi} TG \quad = G \xrightarrow{\delta} GG \xrightarrow{\xi G} TG \tag{15}$$

$$GT \xrightarrow{\psi} TG \xrightarrow{T\varepsilon} T \quad = \quad GT \xrightarrow{\xi T} TT \xrightarrow{\mu} T \tag{16}$$

3.2. Proposition. *For a weak mixed distributive law $\psi : GT \to TG$, the following two composites are idempotents.*

$$\rho = \left(TG \xrightarrow{TG\eta} TGT \xrightarrow{T\psi} TTG \xrightarrow{\mu G} TG\right) \tag{17}$$

$$\sigma = \left(GT \xrightarrow{\delta T} GGT \xrightarrow{G\psi} GTG \xrightarrow{GT\varepsilon} GT\right) \tag{18}$$

Moreover, ψ is a morphism of idempotents; that is,

$$\psi\sigma = \psi = \rho\psi. \tag{19}$$

Proof. Apart from monad properties, proving the composite 17 idempotent only requires equation 13. Similarly, apart from comonad properties, proving the composite 18 idempotent only requires equation 14. ∎

Recall [15] that if we have a right adjoint $G \dashv S$ to G with counit $\alpha : GS \to 1$ and unit $\beta : 1 \to SG$ then S becomes a monad, the *right adjoint monad* of G, with multiplication and unit

$$\mu = \left(SS \xrightarrow{\beta SS} SGSS \xrightarrow{S\delta SS} SGGSS \xrightarrow{SG\alpha S} SGS \xrightarrow{S\alpha} S\right) \quad \text{and} \quad \eta = \left(1 \xrightarrow{\beta} SG \xrightarrow{S\varepsilon} S\right).$$

Moreover, each 2-cell $\psi : GT \to TG$ has a mate $\lambda : TS \to ST$ (in the sense of [10]) defined as the composite

$$TS \xrightarrow{\beta TS} SGTS \xrightarrow{S\psi S} STGS \xrightarrow{ST\alpha} ST.$$

3.3. Proposition. *Suppose G is a comonad with a right adjoint monad S and suppose T is any monad. A 2-cell $\psi : GT \to TG$ is a weak mixed distributive law if and only if its mate $\lambda : TS \to ST$ is a weak distributive law.*

Proof. This is an exercise in the calculus of mates. One sees (easily using string diagrams!) that equation 1 is equivalent to equation 13, equation 2 is equivalent to equation 14, equation 4 is equivalent to equation 15, and equation 5 is equivalent to equation 16. ∎

4. Modules

For a weak distributive law $\lambda : TS \to ST$ of monad T over monad S, a (T, S, λ)- *module* is a triple (A, a_T, a_S) where $a_T : TA \to A$ is a T-algebra and $a_S : SA \to A$ is an S-algebra in the sense of Eilenberg-Moore [15] such that

$$TSA \xrightarrow{Ta_S} TA \xrightarrow{a_T} A \quad = \quad TSA \xrightarrow{\lambda_A} STA \xrightarrow{Sa_T} SA \xrightarrow{a_S} A. \tag{20}$$

A *module morphism* $f : (A, a_T, a_S) \to (B, b_T, b_S)$ is a morphism $f : A \to B$ in $\mathcal{A}$ which is both a morphism of T-algebras and S-algebras. We write $\mathcal{A}^{(T,S,\lambda)}$ for the category of (T, S, λ)-modules.

4.1. THEOREM. *There is an isomorphism of categories*

$$\mathcal{A}^{S \circ_\lambda T} \cong \mathcal{A}^{(T,S,\lambda)}$$

over $\mathcal{A}$ where the left-hand side is the category of Eilenberg-Moore algebras for the monad of Theorem 2.4.

PROOF. Put $K = S \circ_\lambda T$. Each (T, S, λ)-module (A, a_T, a_S) defines a K-algebra (A, a) where

$$a = \left(KA \xrightarrow{\iota_A} STA \xrightarrow{Sa_T} SA \xrightarrow{a_S} A\right). \tag{21}$$

On the other hand, each K-algebra (A, a) defines a (T, S, λ)-module (A, a_T, a_S) defined by

$$a_T = \left(TA \xrightarrow{\eta_{TA}} STA \xrightarrow{\upsilon_A} KA \xrightarrow{a} A\right), \tag{22}$$

$$a_S = \left(SA \xrightarrow{S\eta_A} STA \xrightarrow{\upsilon_A} KA \xrightarrow{a} A\right). \tag{23}$$

The details of the proof are fairly straightforward given Lemma 2.6; to be truthful, I wrote them using string diagrams. ∎

5. Entwining operators

Recall that a monoidal category $\mathcal{C}$ can be regarded as the endohom category of a single object bicategory. Monads and comonads in the bicategory amount to monoids and comonoids (sometimes called algebras and coalgebras) in $\mathcal{C}$. Therefore Definitions 2.1 and 3.1 become definitions of *weak entwining operators* between monoids and between a comonoid and a monoid. However, a weak entwining operator $\lambda : A \otimes B \longrightarrow B \otimes A$ between monoids A and B or $\psi : C \otimes A \longrightarrow A \otimes C$ between a comonoid C and a monoid A deliver weak distributive laws $\lambda \otimes -$ or $\psi \otimes -$ between the monads and comonads $A \otimes -$, $B \otimes -$, and $C \otimes -$.

6. Examples

Consider a braided right-closed monoidal category $\mathcal{C}$. We write X^A for the right internal hom; so $\mathcal{C}(A \otimes X, Y) \cong \mathcal{C}(X, Y^A)$. Let A be a monoid and let C be a comonoid in $\mathcal{C}$. Let $T = A \otimes - : \mathcal{C} \to \mathcal{C}$ be the monad on $\mathcal{C}$ induced by the monoid structure on A. Let $S = (-)^A : \mathcal{C} \to \mathcal{C}$ be the monad on $\mathcal{C}$ induced by the comonoid structure on A. By Proposition 3.3, a weak distributive law $\lambda : TS \to ST$ is equivalent to a weak mixed distributive law $\psi : GT \to TG$ where G is the comonad $G = C \otimes - : \mathcal{C} \to \mathcal{C}$ induced by the comonoid structure on C. Put $\psi_X : C \otimes A \otimes X \to A \otimes C \otimes X$ equal to $c_{C,A} \otimes X : C \otimes A \otimes X \to A \otimes C \otimes X$ where $c_{X,Y} : X \otimes Y \to Y \otimes X$ is the braiding on $\mathcal{C}$. It is easy to see that we obtain a distributive law.

Let A be a weak bimonoid (in the sense of [16]) in the braided right-closed monoidal category $\mathcal{C}$. Let $T = A \otimes - : \mathcal{C} \to \mathcal{C}$ be the monad on $\mathcal{C}$ induced by the monoid structure on A. Let $S = (-)^A : \mathcal{C} \to \mathcal{C}$ be the monad on $\mathcal{C}$ induced by the comonoid structure on A. Let G be the comonad $G = A \otimes - : \mathcal{C} \to \mathcal{C}$ induced by the comonoid structure on A.

6.1. Proposition. *For a weak bimonoid A, a weak mixed distributive law of the monad $T = A \otimes -$ over the comonad $G = A \otimes -$ is defined by tensoring on the left with the composite*

$$A \otimes A \xrightarrow{1 \otimes \delta} A \otimes A \otimes A \xrightarrow{c_{A,A} \otimes 1} A \otimes A \otimes A \xrightarrow{1 \otimes \mu} A \otimes A.$$

Proof. We freely use the defining and derived equations of [16]. Notice that Equations 13 and 14 for a weak mixed distributive law follow easily from property (b) of a weak bimonoid as in Definition 1.1 of [16]. Notice that the morphism ξ of Definition 3.1 is nothing other than the "target morphism" t of [16]. Then we can use the properties (4) and (2) of t in Figure 2 of [16] to prove our Equations 15 and 16. ■

In light of Theorem 4.7 of [5] and Proposition 5.8 of [14], by also looking at tensoring with A on the right, it is presumably possible to characterize weak bimonoids in terms of weak distributive laws.

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