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AN EMBEDDING THEOREM FOR HILBERT CATEGORIES

CHRIS HEUNEN

ABSTRACT. We axiomatically define (pre-)Hilbert categories. The axioms resemble those for monoidal Abelian categories with the addition of an involutive functor. We then prove embedding theorems: any locally small pre-Hilbert category whose monoidal unit is a simple generator embeds (weakly) monoidally into the category of pre-Hilbert spaces and adjointable maps, preserving adjoint morphisms and all finite (co)limits. An intermediate result that is important in its own right is that the scalars in such a category necessarily form an involutive field. In case of a Hilbert category, the embedding extends to the category of Hilbert spaces and continuous linear maps. The axioms for (pre-)Hilbert categories are weaker than the axioms found in other approaches to axiomatizing 2-Hilbert spaces. Neither enrichment nor a complex base field is presupposed. A comparison to other approaches will be made in the introduction.

1. Introduction

Modules over a ring are fundamental to algebra. Distilling their categorical properties results in the definition of Abelian categories, which play a prominent part in algebraic geometry, cohomology and pure category theory. The prototypical Abelian category is that of modules over a fixed ring. Indeed, Mitchell's famous embedding theorem states that any small Abelian category embeds into the category of modules over some ring [Mitchell, 1965, Freyd, 1964].

Likewise, the category **Hilb** of (complex) Hilbert spaces and continuous linear transformations is of paramount importance in quantum theory and functional analysis. So is the category **preHilb** of (complex) pre-Hilbert spaces and adjointable maps. Although they closely resemble the category of modules (over the complex field), neither **Hilb** nor **preHilb** is Abelian. At the heart of the failure of **Hilb** and **preHilb** to be Abelian is the existence of a functor providing adjoint morphisms, called a dagger, that witnesses self-duality. Hence the proof method of Mitchell's embedding theorem does not apply.

This article evens the situation, by combining ideas from Abelian categories and dagger categories. The latter have been used fruitfully in modeling aspects of quantum physics recently [Abramsky & Coecke, 2004, Selinger, 2007, Selinger, 2008]. We axiomatically define *(pre-)Hilbert categories*. The axioms closely resemble those of a monoidal Abelian category, with the addition of a dagger. Their names are justified by proving appropriate

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embedding theorems: roughly speaking, pre-Hilbert categories embed into **preHilb**, and Hilbert categories embed into **Hilb**. These embeddings are in general not full, and only weakly monoidal. But otherwise they preserve all the structure of pre-Hilbert categories, including all finite (co)limits, and adjoint morphisms (up to an isomorphism of the induced base field).

To sketch the historical context of these embedding theorems, let us start by recalling that a category is called Abelian when:

1. it has finite biproducts;
2. it has (finite) equalisers and coequalisers;
3. every monomorphism is a kernel, and every epimorphism is a cokernel.

We can point out already that Definition 2.1 below, of pre-Hilbert category, is remarkably similar, except for the occurrence of a dagger. From the above axioms, enrichment over Abelian groups follows. For the Abelian embedding theorem, there are (at least) two 'different' proofs, one by Mitchell [Mitchell, 1965], and one by Lubkin [Lubkin, 1960]. Both operate by first embedding into the category **Ab** of Abelian groups, and then adding a scalar multiplication. This approach can be extended to also take tensor products into account [Hài, 2002]. However, as **Ab** is not a self-dual category, this strategy does not extend straightforwardly to the setting of Hilbert spaces.

Several authors have used an involution on the given category in this context before. Specifically, by a *dagger* on a category $\mathbf{C}$ we mean a functor $\dagger\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{C}$ that satisfies $X^\dagger = X$ on objects and $f^{\dagger\dagger} = f$ on morphisms. For example, [Ghez, Lima & Roberts, 1985, Proposition 1.14] proves that any *C*-category* embeds into **Hilb**. Here, a C*-category is a category such that:

1. it is enriched over complex Banach spaces and linear contractions;
2. it has an antilinear dagger;
3. every $f\colon X \to Y$ satisfies $f^\dagger f = 0 \Rightarrow f = 0$,
   and there is a $g\colon X \to X$ with $f^\dagger f = g^\dagger g$;
4. $\|f\|^2 = \|f^\dagger f\|$ for every morphism f.

The embedding of a C*-category into **Hilb** uses powerful analytical methods, as it is basically an extension of the Gelfand-Naimark theorem [Gelfand & Naimark, 1943] showing that every C*-algebra (*i.e.* one-object C*-category) can be realized concretely as an algebra of operators on a Hilbert space. Compare the previous definition to Definition 2.1 below: the axioms of (pre-)Hilbert categories are much weaker. For example, nothing about the base field is built into the definition. In fact, one of our main results derives the fact that the base semiring is a field. For the same reason, our situation also differs from *Tannakian categories* [Deligne, 1990], that are otherwise somewhat similar to our (pre-)Hilbert categories. Moreover, (pre-)Hilbert categories do not presuppose any enrichment, but derive it from prior principles.

A related embedding theorem is [Doplicher & Roberts, 1989] (see also [Halvorson & Müger, 2007] for a categorical account). It characterizes categories that are equivalent to the category of finite-dimensional unitary representations of a uniquely determined compact supergroup. Without explaining the postulates, let us mention that the categories **C** considered:

1. are enriched over complex vector spaces;
2. have an antilinear dagger;
3. have finite biproducts;
4. have tensor products $(I, \otimes)$;
5. satisfy $\mathbf{C}(I, I) \cong \mathbb{C}$;
6. every projection dagger splits;
7. every object is compact.

Our definition of (pre-)Hilbert category also requires 2,3, and 4 above. Furthermore, we will also use an analogue of 5, namely that I is a *simple generator*. But notice, again, that 1 above presupposes a base field $\mathbb{C}$, and enrichment over complex vector spaces, whereas (pre-)Hilbert categories do not. As will become clear, our definition and theorems function regardless of dimension; we will come back to dimensionality and the compact objects in 7 above in Subsection 7.1.

This is taken a step further by [Baez, 1997], which follows the "categorification" programme originating in homotopy theory [Kapranov & Voevodsky, 1994]. A *2-Hilbert space* is a category that:

1. is enriched over **Hilb**;
2. has an antilinear dagger;
3. is Abelian;

The category **2Hilb** of 2-Hilbert spaces turns out to be monoidal. Hence it makes sense to define a *symmetric 2-H*-algebra* as a commutative monoid in **2Hilb**, in which furthermore every object is compact. Then, [Baez, 1997] proves that every symmetric 2-H*-algebra is equivalent to a category of continuous unitary finite-dimensional representations of some compact supergroupoid. Again, the proof is basically a categorification of the Gelfand-Naimark theorem. Although the motivation for 2-Hilbert spaces is a categorification of a single Hilbert space, they resemble our (pre-)Hilbert categories, that could be seen as a characterisation of the category of all Hilbert spaces. However, there are important differences. First of all, axiom 1 above again presupposes both the complex numbers as a base field, and a nontrivial enrichment. For example, as (pre-)Hilbert categories assume no enrichment, we do not have to consider coherence with conjugation. Moreover, [Baez, 1997] considers only finite dimensions, whereas the category of all Hilbert spaces, regardless of dimension, is a prime example of a (pre-)Hilbert category (see also Subsection 7.1). Finally, a 2-Hilbert space is an Abelian category, whereas a (pre-)Hilbert category need not be (see Appendix A).

Having sketched how the present work differs from existing work, let us end this introduction by making our approach a bit more precise while describing the structure of this paper. Section 2 introduces our axiomatisation. We then embark on proving embedding theorems for such categories $\mathbf{H}$, under the assumption that the monoidal unit I is a generator. First, we establish a functor $\mathbf{H} \to \mathbf{sHMod}_S$, embedding $\mathbf{H}$ into the category of strict *Hilbert semimodules* over the involutive semiring $S = \mathbf{H}(I, I)$. Section 3 deals with this rigorously. This extends previous work, that shows that a category $\mathbf{H}$ with just biproducts and tensor products is enriched over S-semimodules [Heunen, 2008]. If moreover I is simple, Section 4 proves that S is an involutive field of characteristic zero. This is an improvement over [Vicary, 2008], on which Section 4 draws for inspiration. Hence $\mathbf{sHMod}_S = \mathbf{preHilb}_S$, and S embeds into a field isomorphic to the complex numbers. *Extension of scalars* gives an embedding $\mathbf{preHilb}_S \to \mathbf{preHilb}_{\mathbb{C}}$, discussed in Section 5. Finally, when $\mathbf{H}$ is a Hilbert category, Section 6 shows that Cauchy completion induces an embedding into $\mathbf{Hilb}$ of the image of $\mathbf{H}$ in $\mathbf{preHilb}$. Composing these functors then gives an embedding $\mathbf{H} \to \mathbf{Hilb}$. Along the way, we also discuss how a great deal of the structure of $\mathbf{H}$ is preserved under this embedding: in addition to being (weakly) monoidal, the embedding preserves all finite limits and colimits, and preserves adjoint morphisms up to an isomorphism of the complex field. Section 7 concludes the main body of the paper, and Appendix A considers relevant aspects of the category $\mathbf{Hilb}$ itself.

2. (Pre-)Hilbert categories

This section introduces the object of study. Let $\mathbf{H}$ be a category. A functor $\dagger \colon \mathbf{H}^{\mathrm{op}} \to \mathbf{H}$ with $X^\dagger = X$ on objects and $f^{\dagger\dagger} = f$ on morphisms is called a *dagger*; the pair $(\mathbf{H}, \dagger)$ is then called a *dagger category*. Such categories are automatically isomorphic to their opposite. We can consider coherence of the dagger with respect to all sorts of structures. For example, a morphism m in such a category that satisfies $m^\dagger m = \mathrm{id}$ is called a *dagger mono(morphism)* and is denoted $\rightarrowtail$. Likewise, e is a *dagger epi(morphism)*, denoted $\twoheadrightarrow$, when $ee^\dagger = \mathrm{id}$. A morphism is called a *dagger isomorphism* when it is both dagger epic and dagger monic. Similarly, a biproduct on such a category is called a *dagger biproduct* when $\pi^\dagger = \kappa$, where π is a projection and κ an injection. This is equivalent to demanding $(f \oplus g)^\dagger = f^\dagger \oplus g^\dagger$. Also, an equaliser is called a *dagger equaliser* when it can be represented by a dagger mono, and a kernel is called a *dagger kernel* when it can be represented by a dagger mono. Finally, a dagger category $\mathbf{H}$ is called *dagger monoidal* when it is equipped with monoidal structure $(\otimes, I)$ that is compatible with the dagger, in the sense that $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$, and the coherence isomorphisms are dagger isomorphisms.

2.1. Definition. *A category is called a* pre-Hilbert category *when:*

- *it has a dagger;*
- *it has finite dagger biproducts;*
- *it has (finite) dagger equalisers;*
- *every dagger mono is a dagger kernel;*
- *it is symmetric dagger monoidal.*

Notice that no enrichment of any kind is assumed. Instead, it will follow. Also, no mention is made of the complex numbers or any other base field. This is a notable difference with other approaches mentioned in the Introduction.

Our main theorem will assume that the monoidal unit I is a *generator*, *i.e.* that $f = g \colon X \to Y$ when $fx = gx$ for all $x \colon I \to X$. A final condition we will use is the following: the monoidal unit I is called *simple* when $\mathrm{Sub}(I) = \{0, I\}$ and $\mathbf{H}(I, I)$ is at most of continuum cardinality. Intuitively, a simple object I can be thought of as being "one-dimensional". The definition of a simple object in abstract algebra is usually given without the size requirement, which we require to ensure that the induced base field is not too large. With an eye toward future generalisation, this paper postpones assuming I simple as long as possible.

The category **Hilb** itself is a locally small pre-Hilbert category whose monoidal unit is a simple generator, and so is its subcategory **fdHilb** of finite-dimensional Hilbert spaces (see Appendix A).

Finally, a pre-Hilbert category whose morphisms are bounded is called a *Hilbert category*. It is easier to define this last axiom rigorously after a discussion of scalars, and so we defer this to Section 6.

3. Hilbert semimodules

In this section, we study Hilbert semimodules, to be defined in Definition 3.1 below. It turns out that the structure of a pre-Hilbert category $\mathbf{H}$ gives rise to an embedding of $\mathbf{H}$ into a category of Hilbert semimodules. Let us first recall the notions of semiring and semimodule in some detail, as they might be unfamiliar to the reader.

A *semiring* is roughly a "ring that does not necessarily have subtraction". All the semirings we use will be commutative. Explicitly, a commutative semiring consists of a set S, two elements $0, 1 \in S$, and two binary operations $+$ and $\cdot$ on S, such that the following equations hold for all $r, s, t \in S$:

$$
\begin{aligned}
0 + s &= s, & 1 \cdot s &= s,\\
r + s &= s + r, & r \cdot s &= s \cdot r,\\
r + (s + t) &= (r + s) + t, & r \cdot (s \cdot t) &= (r \cdot s) \cdot t,\\
s \cdot 0 &= 0, & r \cdot (s + t) &= r \cdot s + r \cdot t.
\end{aligned}
$$

Semirings are also known as *rigs*. For more information we refer to [Golan, 1999].

A *semimodule* over a commutative semiring is a generalisation of a module over a commutative ring, which in turn is a generalisation of a vector space over a field. Explicitly, a semimodule over a commutative semiring S is a set M with a specified element $0 \in M$, equipped with functions $+\colon M \times M \to M$ and $\cdot\colon S \times M \to M$ satisfying the following equations for all $r, s \in S$ and $l, m, n \in M$:

$$
\begin{aligned}
s \cdot (m + n) &= s \cdot m + s \cdot n, & 0 + m &= m,\\
(r + s) \cdot m &= r \cdot m + s \cdot m, & m + n &= n + m,\\
(r \cdot s) \cdot m &= r \cdot (s \cdot m), & l + (m + n) &= (l + m) + n,\\
0 \cdot m &= 0, & 1 \cdot m &= m,\\
s \cdot 0 &= 0.
\end{aligned}
$$

A function between S-semimodules is called S-semilinear when it preserves $+$ and $\cdot$. Semimodules over a commutative semiring S and S-semilinear transformations form a category $\mathbf{SMod}_S$ that largely behaves like that of modules over a commutative ring. For example, it is symmetric monoidal closed. The tensor product of S-semimodules M and N is generated by elements of the form $m \otimes n$ for $m \in M$ and $n \in N$, subject to the following relations:

$$
\begin{aligned}
(m + m') \otimes n &= m \otimes n + m' \otimes n,\\
m \otimes (n + n') &= m \otimes n + m \otimes n',\\
(s \cdot m) \otimes n &= m \otimes (s \cdot n),\\
k \cdot (m \otimes n) &= (k \cdot m) \otimes n = m \otimes (k \cdot n),\\
0 \otimes n &= 0 = m \otimes 0,
\end{aligned}
$$

for $m, m' \in M$, $n, n' \in N$, $s \in S$ and $k \in \mathbb{N}$. It satisfies a universal property that differs slightly from that of modules over a ring: every function from $M \times N$ to a commutative monoid T that is semilinear in both variables separately factors uniquely through a semilinear function from $M \otimes N$ to $T/\sim$, where $t \sim t'$ iff there is a $t'' \in T$ with $t + t'' = t' + t''$. For more information about semimodules, we refer to [Golan, 1999], or [Heunen, 2008] for a categorical perspective.

A commutative *involutive semiring* is a commutative semiring S equipped with a semilinear involution $\ddagger\colon S \to S$. An element s of an involutive semiring is called *positive*, denoted $s \geq 0$, when it is of the form $s = t^{\ddagger}t$. The set of all positive elements of an involutive semiring S is denoted S^+. For every semimodule M over a commutative involutive semiring, there is also a semimodule $M^\ddagger$, whose carrier set and addition are the same as before, but whose scalar multiplication sm is defined in terms of the scalar multiplication of M by $s^\ddagger m$. An S-semilinear map $f\colon M \to N$ also induces a map $f^\ddagger\colon M^\ddagger \to N^\ddagger$ by $f^\ddagger(m) = f(m)$. Thus, an involution $\ddagger$ on a commutative semiring S induces an involutive functor $\ddagger\colon \mathbf{SMod}_S \to \mathbf{SMod}_S$.

Now, just as pre-Hilbert spaces are vector spaces equipped with an inner product, we can consider semimodules with an inner product.

3.1. Definition. *Let S be a commutative involutive semiring. An S-semimodule M is called a* Hilbert semimodule *when it is equipped with a morphism $\langle - \,|\, - \rangle\colon M^\ddagger \otimes M \to S$ of $\mathbf{SMod}_S$, satisfying*

- $\langle m \,|\, n \rangle = \langle n \,|\, m \rangle^\ddagger$,
- $\langle m \,|\, m \rangle \geq 0$, *and*
- $\langle m \,|\, - \rangle = \langle n \,|\, - \rangle \Rightarrow m = n$.

The Hilbert semimodule is called strict *if moreover*

- $\langle m \,|\, m \rangle = 0 \Rightarrow m = 0$.

For example, S itself is a Hilbert S-semimodule by $\langle s \,|\, t \rangle_S = s^\ddagger t$. Recall that a semiring S is *multiplicatively cancellative* when $sr = st$ and $s \neq 0$ imply $r = t$ [Golan, 1999]. Thus S is a strict Hilbert S-semimodule iff S is multiplicatively cancellative.

The following choice of morphisms is also the standard choice of morphisms between Hilbert C*-modules [Lance, 1995].[1]

3.2. Definition. *A semimodule homomorphism $f\colon M \to N$ between Hilbert S-semimodules is called* adjointable *when there is a semimodule homomorphism $f^\dagger\colon N \to M$ such that $\langle f^\ddagger(m) \,|\, n \rangle_N = \langle m \,|\, f^\dagger(n) \rangle_M$ for all $m \in M^\ddagger$ and $n \in N$.*

The *adjoint* $f^\dagger$ is unique since the power transpose of the inner product is a monomorphism. However, it does not necessarily exist, except in special situations like (complete) Hilbert spaces ($S = \mathbb{C}$ or $S = \mathbb{R}$) and bounded semilattices (S is the Boolean semiring $\mathbb{B} = (\{0,1\}, \max, \min)$, see [Paseka, 1999]). Hilbert S-semimodules and adjointable maps organise themselves in a category $\mathbf{HMod}_S$. We denote by $\mathbf{sHMod}_S$ the full subcategory of strict Hilbert S-semimodules. The choice of morphisms ensures that $\mathbf{HMod}_S$ and $\mathbf{sHMod}_S$ are dagger categories. Let us study some of their properties. The following lemma could be regarded as an analogue of the Riesz-Fischer theorem [Reed & Simon, 1972, Theorem III.1].

[1] There is another analogy for this choice of morphisms. Writing $M^* = M \multimap S$ for the dual S-semimodule of M, Definition 3.1 resembles that of a 'diagonal' object of the Chu construction on $\mathbf{SMod}_S$. The Chu construction provides a 'generalised topology', like an inner product provides a vector space provides with a metric and hence a topology [Barr, 1999].

3.3. Lemma. *$\mathbf{HMod}_S$ is enriched over $\mathbf{SMod}_S$, and*

$$\mathbf{HMod}_S(S, X) = \mathbf{SMod}_S(S, X) \cong X,$$

where we suppressed the forgetful functor $\mathbf{HMod}_S \to \mathbf{SMod}_S$.

Proof. For $X, Y \in \mathbf{HMod}_S$, the zero map $X \to Y$ in $\mathbf{SMod}_S$ is self-adjoint, and hence a morphism in $\mathbf{HMod}_S$. If $f, g \colon X \to Y$ are adjointable, then so is $f + g$, as its adjoint is $f^\dagger + g^\dagger$. If $s \in S$ and $f \colon X \to Y$ is adjointable, then so is sf, as its adjoint is $s^\ddagger f^\dagger$:

$$\langle sf(x) \,|\, y \rangle_Y = s^\ddagger \langle f(x) \,|\, y \rangle_Y = s^\ddagger \langle x \,|\, f^\dagger(y) \rangle_X = \langle x \,|\, s^\ddagger f^\dagger(y) \rangle_X.$$

Since composition is bilinear, $\mathbf{HMod}_S$ is thus enriched over $\mathbf{SMod}_S$.

Suppose $X \in \mathbf{HMod}_S$, and $f \colon S \to X$ is a morphism of $\mathbf{SMod}_S$. Define a morphism $f^\dagger \colon X \to S$ of $\mathbf{SMod}_S$ by $f^\dagger = \langle f(1) \,|\, - \rangle_X$. Then

$$\langle f(s) \,|\, x \rangle_X = \langle sf(1) \,|\, x \rangle_X = s^\ddagger \langle f(1) \,|\, x \rangle_X = s^\ddagger f^\dagger(x) = \langle s \,|\, f^\dagger(x) \rangle_S.$$

Hence $f \in \mathbf{HMod}_S(S, X)$. Obviously $\mathbf{HMod}_S(S, X) \subseteq \mathbf{SMod}_S(S, X)$. The fact that S is a generator for $\mathbf{HMod}_S$ proves the last claim $\mathbf{SMod}_S(S, X) \cong X$. ■

Notice from the proof of the above lemma that the inner product of X can be reconstructed from $\mathbf{HMod}_S(S, X)$. Indeed, if we temporarily define $\underline{x} \colon S \to X$ by $1 \mapsto x$ for $x \in X$, then we can use the adjoint by

$$\langle x \,|\, y \rangle_X = \langle \underline{x}(1) \,|\, y \rangle_X = \langle 1 \,|\, \underline{x}^\dagger(y) \rangle_S = \underline{x}^\dagger(y) = \underline{x}^\dagger \circ \underline{y}(1).$$

We can go further by providing $\mathbf{HMod}_S(S, X)$ itself with the structure of a Hilbert S-semimodule: for $f, g \in \mathbf{HMod}_S(S, X)$, put $\langle f \,|\, g \rangle_{\mathbf{HMod}_S(S,X)} = f^\dagger \circ g(1)$. Then the above lemma can be strengthened as follows.

3.4. Lemma. *There is a dagger isomorphism $X \cong \mathbf{HMod}_S(S, X)$ in $\mathbf{HMod}_S$.*

Proof. Define $f \colon X \to \mathbf{HMod}_S(S, X)$ by $f(x) = x \cdot (-)$, and $g \colon \mathbf{HMod}_S(S, X) \to X$ by $g(\varphi) = \varphi(1)$. Then $f \circ g = \mathrm{id}$ and $g \circ f = \mathrm{id}$, and moreover $f^\dagger = g$:

$$\begin{aligned}\langle x \,|\, g(\varphi) \rangle_X &= \langle x \,|\, \varphi(1) \rangle_X = (x \cdot (-))^\dagger \circ \varphi(1) \\ &= \langle x \cdot (-) \,|\, \varphi \rangle_{\mathbf{HMod}_S(S,X)} = \langle f(x) \,|\, \varphi \rangle_{\mathbf{HMod}_S(S,X)}\end{aligned}$$

■

Recall that (a subset of) a semiring is called *zerosumfree* when $s + t = 0$ implies $s = t = 0$ for all elements s and t in it [Golan, 1999].

3.5. Proposition. *$\mathbf{HMod}_S$ has finite dagger biproducts. When S^+ is zerosumfree, $\mathbf{sHMod}_S$ has finite dagger biproducts.*

PROOF. Let $H_1, H_2 \in \mathbf{HMod}_S$ be given. Consider the S-semimodule $H = H_1 \oplus H_2$. Equip it with the inner product

$$\langle h \,|\, h' \rangle_H = \langle \pi_1(h) \,|\, \pi_1(h') \rangle_{H_1} + \langle \pi_2(h) \,|\, \pi_2(h') \rangle_{H_2}. \tag{1}$$

Suppose that $\langle h \,|\, - \rangle_H = \langle h' \,|\, - \rangle_H$. For every $i \in \{1,2\}$ and $h'' \in H_i$ then

$$\langle \pi_i(h) \,|\, h'' \rangle_{H_i} = \langle h \,|\, \kappa_i(h'') \rangle_H = \langle h' \,|\, \kappa_i(h'') \rangle_H = \langle \pi_i(h') \,|\, h'' \rangle_{H_i},$$

whence $\pi_i(h) = \pi_i(h')$, and so $h = h'$. Thus H is a Hilbert semimodule. The maps κ_i are morphisms of $\mathbf{HMod}_S$, as their adjoints are given by $\pi_i \colon H \to H_i$:

$$\langle h \,|\, \kappa_i(h') \rangle_H = \langle \pi_1(h) \,|\, \pi_1 \kappa_i(h') \rangle_{H_1} + \langle \pi_2(h) \,|\, \pi_2 \kappa_i(h') \rangle_{H_2} = \langle \pi_i(h) \,|\, h' \rangle_{H_i}.$$

For $\mathbf{sHMod}_S$ we need to verify that H is strict when H_1 and H_2 are. Suppose $\langle h \,|\, h \rangle_H = 0$. Then $\langle \pi_1(h) \,|\, \pi_1(h) \rangle_{H_1} + \langle \pi_2(h) \,|\, \pi_2(h) \rangle_{H_2} = 0$. Since S^+ is zerosumfree, we have $\langle \pi_i(h) \,|\, \pi_i(h) \rangle_{H_i} = 0$ for $i = 1, 2$. Hence $\pi_i(h) = 0$, because H_i is strict. Thus $h = 0$, and H is indeed strict. ■

3.6. PROPOSITION. *$\mathbf{HMod}_S$ is symmetric dagger monoidal. When S is multiplicatively cancellative, $\mathbf{sHMod}_S$ is symmetric dagger monoidal.*

PROOF. Let H, K be Hilbert S-semimodules; then $H \otimes K$ is again an S-semimodule. Define an equivalence relation $\sim$ on $H \otimes K$ by setting

$$h \otimes k \sim h' \otimes k' \quad \text{iff} \quad \langle h \,|\, - \rangle_H \cdot \langle k \,|\, - \rangle_K = \langle h' \,|\, - \rangle_H \cdot \langle k' \,|\, - \rangle_K \colon H \oplus K \to S.$$

This is a congruence relation (see [Golan, 1999]), so $H \otimes_H K = H \otimes K / \sim$ is again an S-semimodule. Defining an inner product on it by

$$\langle [h \otimes k]_\sim \,|\, [h' \otimes k']_\sim \rangle_{H \otimes_H K} = \langle h \,|\, h' \rangle_H \cdot \langle k \,|\, k' \rangle_K.$$

makes $H \otimes_H K$ into a Hilbert semimodule.

Now let $f \colon H \to H'$ and $g \colon K \to K'$ be morphisms of $\mathbf{HMod}_S$. Define $f \otimes_H g \colon H \otimes_H K \to H' \otimes_H K'$ by $(f \otimes_H g)([h \otimes k]_\sim) = [f(h) \otimes g(k)]_\sim$. This is a well-defined function, for if $h \otimes k \sim h' \otimes k'$, then

$$\begin{aligned}
\langle f(h) \,|\, - \rangle_{H'} \cdot \langle g(k) \,|\, - \rangle_{K'} &= \langle h \,|\, f^\dagger(-) \rangle_H \cdot \langle k \,|\, g^\dagger(-) \rangle_K \\
&= \langle h' \,|\, f^\dagger(-) \rangle_H \cdot \langle k' \,|\, g^\dagger(-) \rangle_K \\
&= \langle f(h') \,|\, - \rangle_{H'} \cdot \langle g(k') \,|\, - \rangle_{K'},
\end{aligned}$$

and hence $(f \otimes_H g)(h \otimes k) \sim (f \otimes_H g)(h' \otimes k')$. Moreover, it is adjointable, and hence a morphism of $\mathbf{HMod}_S$:

$$\begin{aligned}
\langle (f \otimes_H g)(h \otimes k) \,|\, (h' \otimes k') \rangle_{H' \otimes_H K'} &= \langle f(h) \otimes g(k) \,|\, h' \otimes k' \rangle_{H' \otimes_H K'} \\
&= \langle f(h) \,|\, h' \rangle_{H'} \cdot \langle g(k) \,|\, k' \rangle_{K'} \\
&= \langle h \,|\, f^\dagger(h') \rangle_H \cdot \langle k \,|\, g(k') \rangle_K \\
&= \langle h \otimes k \,|\, f^\dagger(h') \otimes g^\dagger(k') \rangle_{H \otimes_H K} \\
&= \langle h \otimes k \,|\, (f^\dagger \otimes g^\dagger)(h' \otimes k') \rangle_{H \otimes_H K}
\end{aligned}$$

In the same way, one shows that the coherence isomorphisms α, λ, ρ and γ of the tensor product in $\mathbf{SMod}_S$ respect $\sim$, and descend to dagger isomorphisms in $\mathbf{HMod}_S$. For example:

$$\begin{aligned}\langle \lambda(s \otimes h) \mid h' \rangle_H &= \langle sh \mid h' \rangle_H \\ &= s^{\ddagger} \langle h \mid h' \rangle_H \\ &= \langle s \mid 1 \rangle_S \cdot \langle h \mid h' \rangle_H \\ &= \langle s \otimes h \mid 1 \otimes h' \rangle_{S \otimes_H H} \\ &= \langle s \otimes h \mid \lambda^{-1}(h') \rangle_{S \otimes_H H},\end{aligned}$$

so $\lambda^{\dagger} = \lambda^{-1}$. A routine check shows that $(\otimes_H, S)$ makes $\mathbf{HMod}_S$ into a symmetric monoidal category.

Finally, let us verify that these tensor products descend to $\mathbf{sHMod}_S$ when S is multiplicatively cancellative. Suppose $0 = \langle [h \otimes k]_\sim \mid [h \otimes k]_\sim \rangle_{H \otimes_H K} = \langle h \mid h \rangle_H \cdot \langle k \mid k \rangle_K$. Then since S is multiplicatively cancellative, either $\langle h \mid h \rangle_H = 0$ or $\langle k \mid k \rangle_H = 0$. Since H and K are assumed strict, this means that either $h = 0$ or $k = 0$. In both cases we conclude $[h \otimes k]_\sim = 0$, so that $H \otimes_H K$ is indeed strict. ∎

Now suppose $\mathbf{H}$ is a nontrivial[2] locally small pre-Hilbert category with monoidal unit I. Then $S = \mathbf{H}(I, I)$ is a commutative involutive semiring, and $\mathbf{H}$ is enriched over $\mathbf{SMod}_S$. Explicitly, the zero morphism is the unique one that factors through the zero object, the sum $f + g$ of two morphisms $f, g \colon X \to Y$ is given by

$$X \xrightarrow{\Delta} X \oplus X \xrightarrow{f \oplus g} Y \oplus Y \xrightarrow{\nabla} Y,$$

and the multiplication of a morphism $f \colon X \to Y$ with a scalar $s \colon I \to I$ is determined by

$$X \xrightarrow{\cong} I \otimes X \xrightarrow{s \otimes f} I \otimes Y \xrightarrow{\cong} Y.$$

The scalar multiplication works more generally for symmetric monoidal category [Abramsky, 2005]. The fact that the above provides an enrichment in $\mathbf{SMod}_S$ (and that this enrichment is furthermore functorial) is proved in [Heunen, 2008]. Hence there is a functor $\mathbf{H}(I, -) \colon \mathbf{H} \to \mathbf{SMod}_S$. If I is a generator, this functor is faithful. We will now show that this functor in fact factors through $\mathbf{sHMod}_S$.

3.7. Lemma. *Let $\mathbf{H}$ be a nontrivial locally small pre-Hilbert category. Denote by I its monoidal unit. Then $S = \mathbf{H}(I, I)$ is a commutative involutive semiring, and S^+ is zerosumfree. When moreover I is simple, S is multiplicatively cancellative.*

Proof. For the proof that S is a semiring we refer to [Heunen, 2008]. If I is simple, [Vicary, 2008, 3.5] shows that S is multiplicatively cancellative, and [Vicary, 2008, 3.10] shows that S^+ is zerosumfree in any case. ∎

[2]The unique trivial semiring S with $0 = 1$ is sometimes excluded from consideration by convention. For example, fields usually require $0 \neq 1$ by definition. In our case, the semiring S is trivial iff the category $\mathbf{H}$ is trivial, *i.e.* when $\mathbf{H}$ is the one-morphism (and hence one-object) category. For this reason many results in this paper assume $\mathbf{H}$ to be nontrivial, but the main result, Theorem 6.3, holds regardlessly.

3.8. Theorem. *Let* $\mathbf{H}$ *be a nontrivial locally small pre-Hilbert category. Denote its monoidal unit by* I. *There is a functor* $\mathbf{H}(I,-)\colon \mathbf{H} \to \mathbf{sHMod}_S$ *for* $S = \mathbf{H}(I,I)$. *It preserves* $\dagger$, $\oplus$, *and kernels. It is monoidal when* I *is simple. It is faithful when* I *is a generator.*

Proof. We have to put an S-valued inner product on $\mathbf{H}(I,X)$. Inspired by Lemma 3.4, we define $\langle - \,|\, - \rangle\colon \mathbf{H}(I,X)^{\ddagger} \otimes \mathbf{H}(I,X) \to \mathbf{H}(I,I)$ by (linear extension of) $\langle x \,|\, y \rangle = x^{\dagger} \circ y$ for $x, y \in \mathbf{H}(I,X)$. The Yoneda lemma shows that its power transpose $x \mapsto x^{\dagger} \circ (-)$ is a monomorphism. Thus $\mathbf{H}(I,X)$ is a Hilbert S-semimodule. A fortiori, [Vicary, 2008, 2.11] shows that it is a strict Hilbert semimodule.

Moreover, the image of a morphism $f\colon X \to Y$ of $\mathbf{H}$ under $\mathbf{H}(I,-)$ is indeed a morphism of $\mathbf{sHMod}_S$, that is, it is adjointable, since

$$\langle f \circ x \,|\, y \rangle_{\mathbf{H}(I,Y)} = (f \circ x)^{\dagger} \circ y = x^{\dagger} \circ f^{\dagger} \circ y = \langle x \,|\, f^{\dagger} \circ y \rangle_{\mathbf{H}(I,X)}$$

for $x \in \mathbf{H}(I,X)$ and $y \in \mathbf{H}(I,Y)$. This also shows that $\mathbf{H}(I,-)$ preserves $\dagger$. Also, by definition of product we have $\mathbf{H}(I, X \oplus Y) \cong \mathbf{H}(I,X) \oplus \mathbf{H}(I,Y)$, so the functor $\mathbf{H}(I,-)$ preserves $\oplus$.

To show that $\mathbf{H}(I,-)$ preserves kernels, suppose that $k = \ker(f)\colon K \rightarrowtail X$ is a kernel of $f\colon X \to Y$ in $\mathbf{H}$. We have to show that $\mathbf{H}(I,k) = k \circ (-)\colon \mathbf{H}(I,K) \rightarrowtail \mathbf{H}(I,X)$ is a kernel of $\mathbf{H}(I,f) = f \circ (-)\colon \mathbf{H}(I,X) \to \mathbf{H}(I,Y)$ in $\mathbf{sHMod}_S$. First of all, one indeed has $\mathbf{H}(I,f) \circ \mathbf{H}(I,k) = \mathbf{H}(I, f \circ k) = 0$. Now suppose that $l\colon Z \to \mathbf{H}(I,X)$ also satisfies $\mathbf{H}(I,f) \circ l = 0$. That is, for all $z \in Z$, we have $f \circ (l(z)) = 0$. Since k is a kernel, for each $z \in Z$ there is a unique $m_z\colon I \to K$ with $l(z) = k \circ m_z$. Define a function $m\colon Z \to \mathbf{H}(I,K)$ by $m(z) = m_z$. This is a well-defined module morphism, since l is; for example,

$$k \circ m_{z+z'} = l(z+z') = l(z) + l(z') = (k \circ m_z) + (k \circ m_{z'}) = k \circ (m_z + m_{z'}),$$

so that $m(z+z') = m(z) + m(z')$ because k is mono. In fact, m is the unique module morphism satisfying $l = \mathbf{H}(I,k) \circ m$. Since k is a dagger mono, we have $m = \mathbf{H}(I,k^{\dagger}) \circ l$. So as a composition of adjointable module morphisms m is a well-defined morphism of $\mathbf{sHMod}_S$. Thus $\mathbf{H}(I,k)$ is indeed a kernel of $\mathbf{H}(I,f)$, and $\mathbf{H}(I,-)$ preserves kernels.

If I is simple then $\mathbf{sHMod}_S$ is monoidal. To show that $\mathbf{H}(I,-)$ is a monoidal functor we must give a natural transformation $\varphi_{X,Y}\colon \mathbf{H}(I,X) \otimes \mathbf{H}(I,Y) \to \mathbf{H}(I, X \otimes Y)$ and a morphism $\psi\colon S \to \mathbf{H}(I,I)$. Since $S = \mathbf{H}(I,I)$, we can simply take $\psi = \mathrm{id}$. Define φ by mapping $x \otimes y$ for $x\colon I \to X$ and $y\colon I \to Y$ to

$$I \xrightarrow{\cong} I \otimes I \xrightarrow{x \otimes y} X \otimes Y.$$

It is easily seen that φ and ψ make the required coherence diagrams commute, and hence $\mathbf{H}(I,-)$ is a monoidal functor. ∎

4. The scalar field

This section shows that the scalars in a pre-Hilbert category whose monoidal unit is a simple generator necessarily form an involutive field. First, we need a factorisation result, which is interesting in its own right.

4.1. LEMMA. *In any dagger category:*

(a) *A dagger mono which is epic is a dagger isomorphism.*

(b) *If both gf and f are dagger epic, so is g.*

(c) *If m and n are dagger monic, and f is an isomorphism with $nf = m$, then f is a dagger isomorphism.*

PROOF. For (a), notice that $ff^\dagger = \mathrm{id}$ implies $ff^\dagger f = f$, from which $f^\dagger f = \mathrm{id}$ follows from the assumption that f is epi. For (b): $gg^\dagger = gff^\dagger g = gf(gf)^\dagger = \mathrm{id}$. Finally, consider (c). If f is isomorphism, in particular it is epi. If both nf and n are dagger mono, then so is f, by (b). Hence by (a), f is dagger isomorphism. ∎

4.2. LEMMA. *In any pre-Hilbert category, a morphism m is mono iff* $\ker(m) = 0$.

PROOF. Suppose $\ker(m) = 0$. Let u, v satisfy $mu = mv$. Put q to be the dagger coequaliser of u and v. Since q is dagger epic, $q = \mathrm{coker}(w)$ for some w. As $mu = mv$, m factors through q as $m = nq$. Then $mw = nqw = n0 = 0$, so w factors through $\ker(m)$ as $w = \ker(m) \circ p$ for some p. But since $\ker(m) = 0$, $w = 0$. So q is a dagger isomorphism, and in particular mono. Hence, from $qu = qv$ follows $u = v$. Thus m is mono.

$$\begin{tikzcd}
\cdot \arrow[dr, "\ker(m)"] & \cdot \arrow[l, dashed, "p"'] \arrow[d, "w"] & \\
\cdot \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & \cdot \arrow[r, "m"] \arrow[d, "q"] & \cdot \\
& \cdot \arrow[ur, dashed, "n"'] &
\end{tikzcd}$$

Conversely, if m is mono, $\ker(m) = 0$ follows from $m \circ \ker(m) = 0 = m \circ 0$. ∎

4.3. LEMMA. *Any morphism in a pre-Hilbert category can be factored as a dagger epi followed by a mono. This factorisation is unique up to a unique dagger isomorphism.*

Proof. Let a morphism f be given. Put $k = \ker(f)$ and $e = \mathrm{coker}(k)$. Since $fk = 0$ (as $k = \ker(f)$), f factors through $e(= \mathrm{coker}(k))$ as $f = me$.

$$
\begin{tikzcd}
& {} \arrow[dl, "l"'] \arrow[d, "h"] & \\
{} \arrow[r, "k"'] & {} \arrow[r, "f"] \arrow[d, "e"'] & {} \\
{} \arrow[r, "g"'] & {} \arrow[ur, "m"] \arrow[r, shift right, "q"'] & {} \arrow[u, "r"'] \arrow[l, shift right, "s"']
\end{tikzcd}
$$

We have to show that m is mono. Let g be such that $mg = 0$. By Lemma 4.2 it suffices to show that $g = 0$. Since $mg = 0$, m factors through $q = \mathrm{coker}(g)$ as $m = rq$. Now qe is a dagger epi, being the composite of two dagger epis. So $qe = \mathrm{coker}(h)$ for some h. Since $fh = rqeh = r0 = 0$, h factors through $k(= \ker(f))$ as $h = kl$. Finally $eh = ekl = 0l = 0$, so e factors through $qe = \mathrm{coker}(h)$ as $q = sqe$. But since e is (dagger) epic, this means $sq = \mathrm{id}$, whence q is mono. It follows from $qg = 0$ that $g = 0$, and the factorisation is established. Finally, by Lemma 4.1(c), the factorisation is unique up to a dagger isomorphism. ∎

We just showed that any Hilbert category has a factorisation system consisting of monos and dagger epis. Equivalently, it has a factorisation system of epis and dagger monos. Indeed, if we can factor $f^\dagger$ as an dagger epi followed by a mono, then taking the daggers of those, we find that $f^{\dagger\dagger} = f$ factors as an epi followed by a dagger mono. The combination of both factorisations yields that every morphism can be written as a dagger epi, followed by a monic epimorphism, followed by a dagger mono; this can be thought of as a generalisation of *polar decomposition*.

Recall that a *semifield* is a commutative semiring in which every nonzero element has a multiplicative inverse. Notice that the scalars in the embedding theorem for Abelian categories do not necessarily have multiplicative inverses.

4.4. Lemma. *If* $\mathbf{H}$ *is a nontrivial pre-Hilbert category with simple monoidal unit* I*, then* $S = \mathbf{H}(I, I)$ *is a semifield.*

Proof. We will show that S is a semifield by proving that any $s \in S$ is either zero or isomorphism. Factorise s as $s = me$ for a dagger mono $m\colon \mathrm{Im}(s) \rightarrowtail I$ and an epi $e\colon I \twoheadrightarrow \mathrm{Im}(s)$. Since I is simple, either m is zero or m is isomorphism. If $m = 0$ then $s = 0$. If m is isomorphism, then s is epi, so $s^\dagger$ is mono. Again, either $s^\dagger = 0$, in which case $s = 0$, or $s^\dagger$ is isomorphism. In this last case s is also isomorphism. ∎

The following lemma shows that every scalar also has an additive inverse. This is always the case for the scalars in the embedding theorem for Abelian categories, but the usual proof of this fact is denied to us because epic monomorphisms are not necessarily isomorphisms in a pre-Hilbert category (see Appendix A).

4.5. Lemma. *If* $\mathbf{H}$ *is a nontrivial pre-Hilbert category whose monoidal unit* I *is a simple generator, then* $S = \mathbf{H}(I, I)$ *is a field.*

Proof. Applying [Golan, 1999, 4.34] to the previous lemma yields that S is either zerosumfree, or a field. Assume, towards a contradiction, that S is zerosumfree. We will show that the kernel of the codiagonal $\nabla = [\mathrm{id}, \mathrm{id}] \colon I \oplus I \to I$ is zero. Suppose $\nabla \circ \langle x, y \rangle = x + y = 0$ for $x, y \colon X \to I$. Then for all $z \colon I \to X$ we have $\nabla \circ \langle x, y \rangle \circ z = 0 \circ z = 0$, *i.e.* $xz + yz = 0$. Since S is assumed zerosumfree hence $xz = yz = 0$, so $\langle x, y \rangle \circ z = 0$. Because I is a generator then $\langle x, y \rangle = 0$. Thus $\ker(\nabla) = 0$. But then, by Lemma 4.2, ∇ is mono, whence $\kappa_1 = \kappa_2$, which is a contradiction. ∎

Collecting the previous results about the scalars in a pre-Hilbert category yields Theorem 4.8 below. It uses a well-known characterisation of subfields of the complex numbers, that we recall in the following two lemmas.

4.6. Lemma. *[Grillet, 2007, Theorem 4.4] Any field of characteristic zero and at most continuum cardinality can be embedded in an algebraically closed field of characteristic zero and continuum cardinality.* ∎

4.7. Lemma. *[Chang & Keisler, 1990, Proposition 1.4.10] All algebraically closed fields of characteristic zero and continuum cardinality are isomorphic.* ∎

4.8. Theorem. *If* $\mathbf{H}$ *is a nontrivial pre-Hilbert category whose monoidal unit* I *is a simple generator, then* $S = \mathbf{H}(I, I)$ *is an involutive field of characteristic zero of at most continuum cardinality, with* S^+ *zerosumfree.*

Consequently, there is a monomorphism $\mathbf{H}(I, I) \rightarrowtail \mathbb{C}$ *of fields. However, it does not necessarily preserve the involution.*

Proof. To establish characteristic zero, we have to prove that for all scalars $s \colon I \to I$ the property $s + \cdots + s = 0$ implies $s = 0$, where the sum contains n copies of s, for all $n \in \{1, 2, 3, \ldots\}$.. So suppose that $s + \cdots + s = 0$. By definition, $s + \cdots + s = \nabla^n \circ (s \oplus \cdots \oplus s) \circ \Delta^n = \nabla^n \circ \Delta^n \circ s$, where $\nabla^n = [\mathrm{id}]_{i=1}^n \colon \bigoplus_{i=1}^n I \to I$ and $\Delta^n = \langle \mathrm{id} \rangle_{i=1}^n \colon I \to \bigoplus_{i=1}^n I$ are the n-fold (co)diagonals. But $0 \neq \nabla^n \circ \Delta^n = (\Delta^n)^\dagger \circ \Delta^n$ by Lemma 2.11 of [Vicary, 2008], which states that $x^\dagger x = 0$ implies $x = 0$ for every $x \colon I \to X$. Since S is a field by Lemma 4.5, this means that $s = 0$. ∎

This theorem is of interest to reconstruction programmes, that try to derive major results of quantum theory from simpler mathematical assumptions, for among the things to be reconstructed are the scalars. For example, [Solèr, 1995] shows that if an orthomodular pre-Hilbert space is infinite dimensional, then the base field is either $\mathbb{R}$ or $\mathbb{C}$, and the space is a Hilbert space.

With a scalar field, we can sharpen the preservation of finite biproducts and kernels of Theorem 3.8 to preservation of all finite limits. Since $\mathbf{H}(I, -)$ preserves the dagger, it hence also preserves all finite colimits. (In other terms: $\mathbf{H}(I, -)$ is exact.)

4.9. Corollary. *The functor $\mathbf{H}(I,-)\colon \mathbf{H} \to \mathbf{sHMod}_{\mathbf{H}(I,I)}$ preserves all finite limits and all finite colimits, for any pre-Hilbert category $\mathbf{H}$ whose monoidal unit I is a simple generator.*

Proof. One easily checks that $F = \mathbf{H}(I,-)$ is an $\mathbf{Ab}$-functor, *i.e.* that $(f+g)\circ(-) = (f\circ -)+(g\circ -)$ [Heunen, 2008]. Hence, F preserves equalisers:

$$F(\mathrm{eq}(f,g)) = F(\ker(f-g)) = \ker(F(f-g)) = \ker(Ff - Fg) = \mathrm{eq}(Ff, Fg).$$

Since we already know from Theorem 3.8 that F preserves finite products, we can conclude that it preserves all finite limits. And because F preserves the self-duality †, it also preserves all finite colimits. ■

5. Extension of scalars

The main idea underlying this section is to exploit Theorem 4.8. We will construct a functor $\mathbf{HMod}_R \to \mathbf{HMod}_S$ given a morphism $R \to S$ of commutative involutive semirings, and apply it to the above $\mathbf{H}(I,I) \to \mathbb{C}$. This is called *extension of scalars*, and is well known in the setting of modules (see *e.g.* [Ash, 2000, 10.8.8]). Let us first consider in some more detail the construction on semimodules.

Let R and S be commutative semirings, and $f\colon R \to S$ a homomorphism of semirings. Then any S-semimodule M can be considered an R-semimodule M_R by defining scalar multiplication $r\cdot m$ in M_R in terms of scalar multiplication of M by $f(r)\cdot m$. In particular, we can regard S as an R-semimodule. Hence it makes sense to look at $S\otimes_R M$. Somewhat more precisely, we can view S as a left-S-right-R-bisemimodule, and M as a left-R-semimodule. Hence $S\otimes_R M$ becomes a left-S-semimodule (see [Golan, 1999]). This construction induces a functor $f^*\colon \mathbf{SMod}_R \to \mathbf{SMod}_S$, acting on morphisms g as $\mathrm{id}\otimes_R g$. It is easily seen to be strong monoidal and to preserve biproducts and kernels.

Now let us change to the setting where R and S are involutive semirings, $f\colon R\to S$ is a morphism of involutive semirings, and we consider Hilbert semimodules instead of semimodules. The next theorem shows that this construction lifts to a functor $f^*\colon \mathbf{sHMod}_R \to \mathbf{sHMod}_S$ (under some conditions on S and f). Moreover, the fact that any S-semimodule can be seen as an R-semimodule via f immediately induces another functor $f_*\colon \mathbf{SMod}_S \to \mathbf{SMod}_R$. This one is called *restriction of scalars*. In fact, f_* is right adjoint to f^* [Borceux, 1994, vol 1, 3.1.6e]. However, since we do not know how to fashion an sesquilinear R-valued form out of an S-valued one in general, it seems impossible to construct an adjoint functor $f_*\colon \mathbf{sHMod}_S \to \mathbf{sHMod}_R$.

5.1. Theorem. *Let R be a commutative involutive semiring, S be a multiplicatively cancellative commutative involutive ring, and $f\colon R \rightarrowtail S$ be a monomorphism of involutive semirings. There is a faithful functor $f^*\colon \mathbf{sHMod}_R \to \mathbf{sHMod}_S$ that preserves †. If R is multiplicatively cancellative, then f^* is strong monoidal. If both R^+ and S^+ are zerosumfree, then f^* also preserves $\oplus$.*

Proof. Let M be a strict Hilbert R-semimodule. Defining the carrier of f^*M to be $S \otimes_R M$ turns it into an S-semimodule as before. Furnish it with

$$\langle s \otimes m \,|\, s' \otimes m' \rangle_{f^*M} = s^\ddagger \cdot s' \cdot f(\langle m \,|\, m' \rangle_M).$$

Assume $0 = \langle s \otimes m \,|\, s \otimes m \rangle_{f^*M} = s^\ddagger s f(\langle m \,|\, m \rangle_M)$. Since S is multiplicatively cancellative, either $s = 0$ or $f(\langle m \,|\, m \rangle_M) = 0$. In the former case immediately $s \otimes m = 0$. In the latter case $\langle m \,|\, m \rangle_M = 0$ since f is injective, and because M is strict $m = 0$, whence $s \otimes m = 0$. Since S is a ring, this implies that f^*M is a strict Hilbert S-semimodule. For if $\langle x \,|\, - \rangle_{f^*M} = \langle y \,|\, - \rangle_{f^*M}$ then $\langle x - y \,|\, - \rangle_{f^*M} = 0$, so in particular $\langle x - y \,|\, x - y \rangle_{f^*M} = 0$. Hence $x - y = 0$ and $x = y$.

Moreover, the image of a morphism $g \colon M \to M'$ of $\mathbf{sHMod}_R$ under f^* is a morphism of $\mathbf{sHMod}_S$, as its adjoint is $\mathrm{id} \otimes g^\dagger$:

$$\begin{aligned}
\langle (\mathrm{id} \otimes g)(s \otimes m) \,|\, s' \otimes m' \rangle_{f^*M'} &= \langle s \otimes g(m) \,|\, s' \otimes m' \rangle_{f^*M'} \\
&= s^\ddagger s' f(\langle g(m) \,|\, m' \rangle'_M) \\
&= s^\ddagger s' f(\langle m \,|\, g^\dagger(m') \rangle_M) \\
&= \langle s \otimes m \,|\, s' \otimes g^\dagger(m') \rangle_{f^*M} \\
&= \langle s \otimes m \,|\, (\mathrm{id} \otimes g^\dagger)(s' \otimes m') \rangle_{f^*M}.
\end{aligned}$$

Obviously, f^* is faithful, and preserves $\dagger$. If dagger biproducts are available, then f^* preserves them, since biproducts distribute over tensor products. If dagger tensor products are available, showing that f^* preserves them comes down to giving an isomorphism $S \to S \otimes_R R$ and a natural isomorphism $(S \otimes_R X) \otimes_S (S \otimes_R Y) \to S \otimes_R (X \otimes_R Y)$. The obvious candidates for these satisfy the coherence diagrams, making f^* strong monoidal. ∎

5.2. Corollary. *Let S be an involutive field of characteristic zero and at most continuum cardinality. Then there is a strong monoidal, faithful functor $\mathbf{sHMod}_S \to \mathbf{sHMod}_{\mathbb{C}}$ that preserves all finite limits and finite colimits, and preserves $\dagger$ up to an isomorphism of the base field.*

Proof. The only claim that does not follow from previous results is the statement about preservation of finite (co)limits. This comes down to a calculation in the well-studied situation of module theory [Ash, 2000, Exercise 10.8.5]. ∎

Note that the extension of scalars functor f^* of the previous theorem is full if and only if f is a regular epimorphism, *i.e.* iff f is surjective. To see this, consider the inclusion $f \colon \mathbb{N} \hookrightarrow \mathbb{Z}$. This is obviously not surjective. Now, $\mathbf{SMod}_{\mathbb{N}}$ can be identified with the category $\mathbf{cMon}$ of commutative monoids, and $\mathbf{SMod}_{\mathbb{Z}}$ can be identified with the category $\mathbf{Ab}$ of Abelian groups. Under this identification, $f^* \colon \mathbf{cMon} \to \mathbf{Ab}$ sends an object $X \in \mathbf{cMon}$ to $X \coprod X$, with inverses being provided by swapping the two terms X. For a morphism g, $f^*(g)$ sends (x, x') to (gx, gx'). Consider $h \colon X \coprod X \to X \coprod X$, determined by $h(x, x') = (x', x)$. If $h = f^*(g)$ for some g, then $(x', x) = h(x, x') = f^*g(x, x') = (gx, gx')$, so $gx = x'$ and $gx' = x$ for all $x, x' \in X$. Hence g must be constant, contradicting $h = f^*g$. Hence f^* is not full.

6. Completion

Up to now we have concerned ourselves with algebraic structure only. To arrive at the category of Hilbert spaces and continuous linear maps, some analysis comes into play. Looking back at Definition 3.1, we see that a strict Hilbert $\mathbb{C}$-semimodule is just a pre-Hilbert space, *i.e.* a complex vector space with a positive definite sesquilinear form on it. Any pre-Hilbert space X can be completed to a Hilbert space $\widehat{X}$ into which it densely embeds [Reed & Simon, 1972, I.3].

A morphism $g\colon X \to Y$ of $\mathbf{sHMod}_{\mathbb{C}}$ amounts to a linear transformation between pre-Hilbert spaces that has an adjoint. So $\mathbf{sHMod}_{\mathbb{C}} = \mathbf{preHilb}$. However, these morphisms need not necessarily be bounded, and only bounded adjointable morphisms can be extended to the completion $\widehat{X}$ of their domain [Reed & Simon, 1972, I.7]. Therefore, we impose another axiom on the morphisms of $\mathbf{H}$ to ensure this. Basically, we rephrase the usual definition of boundedness of a function between Banach spaces for morphism between Hilbert semimodules. Recall from Lemma 4.8 that the scalars $S = \mathbf{H}(I, I)$ in a pre-Hilbert category $\mathbf{H}$ are always an involutive field, with S^+ zerosumfree. Hence S is ordered by $r \leq s$ iff $r + p = s$ for some $p \in S^+$. We use this ordering to define boundedness of a morphism in $\mathbf{H}$, together with the norm induced by the canonical bilinear form $\langle f \mid g \rangle = f^\dagger \circ g$.

6.1. Definition. *Let* $\mathbf{H}$ *be a symmetric dagger monoidal dagger category with dagger biproducts. A scalar* $M\colon I \to I$ *is said to* bound *a morphism* $g\colon X \to Y$ *when* $x^\dagger g^\dagger g x \leq M^\dagger x^\dagger x M$ *for all* $x\colon I \to X$. *A morphism is called* bounded *when it has a bound. A* Hilbert category *is a pre-Hilbert category whose morphisms are bounded.*

In particular, a morphism $g\colon X \to Y$ in $\mathbf{sHMod}_S$ is bounded when there is an $M \in S$ satisfying $\langle g(x) \mid g(x) \rangle \leq M^\dagger M \langle x \mid x \rangle$ for all $x \in X$.

Almost by definition, the functor $\mathbf{H}(I, -)$ of Theorem 3.8 preserves boundedness of morphisms when $\mathbf{H}$ is a Hilbert category. The following lemma shows that also the extension of scalars of Theorem 5.1 preserves boundedness. It is noteworthy that a combinatorial condition (boundedness) on the category $\mathbf{H}$ ensures an analytic property (continuity) of its image in $\mathbf{sHMod}_{\mathbb{C}}$, as we never even assumed a topology on the scalar field, let alone assuming completeness.

6.2. Lemma. *Let* $f\colon R \to S$ *be a morphism of involutive semirings. If* $g\colon X \to Y$ *is bounded in* $\mathbf{sHMod}_R$*, then* $f^*(g)$ *is bounded in* $\mathbf{sHMod}_S$.

Proof. First, notice that $f\colon R \to S$ preserves the canonical order: if $r \leq r'$, say $r + t^\ddagger t = r'$ for $r, r', t \in R$, then $f(r) + f(t)^\dagger f(t) = f(r + t^\dagger t) = f(r')$, so $f(r) \leq f(r')$.

Suppose $\langle g(x) \mid g(x) \rangle_Y \leq M^\ddagger M \langle x \mid x \rangle_X$ for all $x \in X$ and some $M \in R$. Then $f(\langle g(x) \mid g(x) \rangle_Y) \leq f(M^\ddagger M \langle x \mid x \rangle_X) = f(M)^\ddagger f(M) f(\langle x \mid x \rangle_X)$ for $x \in X$. Hence for

$s \in S$:

$$\begin{aligned}\langle f^*g(s\otimes x) \,|\, f^*g(s\otimes x)\rangle_{f^*Y} &= \langle (\mathrm{id}\otimes g)(s\otimes x) \,|\, (\mathrm{id}\otimes g)(s\otimes x)\rangle_{f^*Y} \\ &= \langle s\otimes g(x) \,|\, s\otimes g(x)\rangle_{f^*Y} \\ &= s^\ddagger s f(\langle g(x) \,|\, g(x)\rangle_Y) \\ &\leq s^\ddagger s f(M)^\ddagger f(M) f(\langle x \,|\, x\rangle_X) \\ &= f(M)^\ddagger f(M)\langle s\otimes x \,|\, s\otimes x\rangle_{f^*X}.\end{aligned}$$

Because elements of the form $s \otimes x$ form a basis for $f^*X = S \otimes_R X$, we thus have

$$\langle f^*g(z) \,|\, f^*g(z)\rangle_{f^*Y} \leq f(M)^\ddagger f(M)\langle z \,|\, z\rangle_{f^*X}$$

for all $z \in f^*X$. In other words: f^*g is bounded (namely, by $f(M)$). ∎

Combining this section with Theorems 3.8 and 4.8, Corollary 5.2 and Lemma 6.2 now results in our main theorem. Notice that the completion preserves biproducts and kernels and thus equalisers, and so preserves all finite limits and colimits.

6.3. Theorem. *Any locally small Hilbert category* $\mathbf{H}$ *whose monoidal unit is a simple generator has a monoidal embedding into the category* **Hilb** *of Hilbert spaces and continuous linear maps that preserves* † *(up to an isomorphism of the base field) and all finite limits and finite colimits.*

Proof. The only thing left to prove is the case that $\mathbf{H}$ is trivial. But if $\mathbf{H}$ is a one-morphism Hilbert category, its one object must be the zero object, and its one morphism must be the zero morphism. Hence sending this to the zero-dimensional Hilbert space yields a faithful monoidal functor that preserves † and $\oplus$, trivially preserving all (co)limits. ∎

To finish, notice that the embedding of the Hilbert category **Hilb** into itself thus constructed is (isomorphic to) the identity functor.

7. Conclusion

Let us conclude by discussing several further issues.

7.1. Dimension. The embedding of Theorem 6.3 is strong monoidal (*i.e.* preserves $\otimes$) if the canonical (coherent) morphism is an isomorphism

$$\mathbf{H}(I, X) \otimes \mathbf{H}(I, Y) \cong \mathbf{H}(I, X \otimes Y),$$

where the tensor product in the left-hand side is that of (strict) Hilbert semimodules. This is a quite natural restriction, as it prevents degenerate cases like $\otimes = \oplus$. Under this condition, the embedding preserves compact objects [Heunen, 2008]. This means that compact objects correspond to finite-dimensional Hilbert spaces under the embedding in question. Our embedding theorem also shows that every Hilbert category embeds into a C*-category [Ghez, Lima & Roberts, 1985]. This relates to representation theory. Compare *e.g.* [Doplicher & Roberts, 1989], who establish a correspondence between a compact group and its categories of finite-dimensional, continuous, unitary representations; the latter category is characterised by axioms comparable to those of pre-Hilbert categories, with moreover every object being compact.

Corollary 4.9 opens the way to *diagram chasing* (see *e.g.* [Borceux, 1994, vol 2, Section 1.9]): to prove that a diagram commutes in a pre-Hilbert category, it suffices to prove this in pre-Hilbert spaces, where one has access to actual elements. As discussed above, when $\mathbf{H}$ is compact, and the embedding $\mathbf{H} \to \mathbf{preHilb}$ is strong monoidal, then the embedding takes values in the category of finite-dimensional pre-Hilbert spaces. The latter coincides with the category of finite-dimensional Hilbert spaces (since every finite-dimensional pre-Hilbert space is Cauchy complete). This partly explains the main claim in [Selinger, 2008], namely that an equation holds in all dagger traced symmetric monoidal categories if and only if it holds in finite-dimensional Hilbert spaces.

7.2. Functor categories. We have used the assumption that the monoidal unit is simple in an essential way. But if $\mathbf{H}$ is a pre-Hilbert category whose monoidal unit is simple, and $\mathbf{C}$ is any nontrivial small category, then the functor category $[\mathbf{C}, \mathbf{H}]$ is a pre-Hilbert category, albeit one whose monoidal unit is not simple anymore. Perhaps the embedding theorem can be extended to this example. The conjecture would be that any pre-Hilbert category whose monoidal unit is a generator (but not necessarily simple), embeds into a functor category $[\mathbf{C}, \mathbf{preHilb}]$ for some category $\mathbf{C}$. This requires reconstructing $\mathbf{C}$ from $\mathrm{Sub}(I)$.

Likewise, it would be preferable to be able to drop the condition that the monoidal unit be a generator. To accomplish this, one would need to find a dagger preserving embedding of a given pre-Hilbert category into a pre-Hilbert category with a finite set of generators. In the Abelian case, this can be done by moving from $\mathbf{C}$ to $[\mathbf{C}, \mathbf{Ab}]$, in which $\coprod_{X \in \mathbf{C}} \mathbf{C}(X, -)$ is a generator. But in the setting of Hilbert categories there is no analogon of $\mathbf{Ab}$. Also, Hilbert categories tend not to have infinite coproducts.

7.3. Topology. Our axiomatisation allowed inner product spaces over $\mathbb{Q}$ as a (pre-)Hilbert category. Additional axioms, enforcing the base field to be (Cauchy) complete and hence (isomorphic to) the real or complex numbers, could perhaps play a role in topologising the above to yield an embedding into sheaves of Hilbert spaces. A forthcoming paper studies subobjects in a (pre-)Hilbert category, showing that quantum logic is just an incarnation of categorical logic. But this is also interesting in relation to [Amemiya & Araki, 1966], which shows that a pre-Hilbert space is complete if and only if its lattice of closed subspaces is orthomodular.

7.4. Fullness. A natural question is under what conditions the embedding is full. Imitating the answer for the embedding of Abelian categories, we can only obtain the following partial result, since Hilbert categories need not have infinite coproducts, as opposed to **Ab**. An object X in a pre-Hilbert category $\mathbf{H}$ with monoidal unit I is said to be *finitely generated* when there is a dagger epi $\bigoplus_{i\in I} I \longrightarrow\!\!\triangleright X$ for some finite set I.

7.5. THEOREM. *The embedding of Theorem 3.8 is full when every object in* $\mathbf{H}$ *is finitely generated.*

PROOF. We have to prove that $\mathbf{H}(I,-)$'s action on morphisms, which we temporarily denote $T\colon \mathbf{H}(X,Y) \to \mathbf{sHMod}_S(\mathbf{H}(I,X),\mathbf{H}(I,Y))$, is surjective when X is finitely generated. Let $\Phi\colon \mathbf{H}(I,X) \to \mathbf{H}(I,Y)$ in $\mathbf{sHMod}_S$. We must find $\varphi\colon X \to Y$ in $\mathbf{H}$ such that $\Phi(x) = \varphi \circ x$ for all $x\colon I \to X$ in $\mathbf{H}$. Suppose first that $X = I$. Then $\Phi(x) = \Phi(\mathrm{id}_I \circ x) = \Phi(\mathrm{id}_I) \circ x$ since Φ is a morphism of S-semimodules. So $\varphi = \Phi(\mathrm{id}_I)$ satisfies $\Phi(x) = \varphi \circ x$ for all $x\colon I \to X$ in $\mathbf{H}$. In general, if X is finitely generated, there is a finite set I and a dagger epi $p\colon \bigoplus_{i\in I} I \longrightarrow\!\!\triangleright X$. Denote by Φ_i the composite morphism

$$\mathbf{H}(I,I) \xrightarrow{T(\kappa_i)} \mathbf{H}(I,\textstyle\bigoplus_{i\in I} I) \xrightarrow{T(p)} \mathbf{H}(I,X) \xrightarrow{\Phi} \mathbf{H}(I,Y) \quad \text{in } \mathbf{sHMod}_S.$$

By the previous case ($X = I$), for each $i \in I$ there is $\varphi_i \in \mathbf{H}(I,Y)$ such that $\Phi_i(x) = \varphi_i \circ x$ for all $x \in S$. Define $\bar\varphi = [\varphi_i]_{i\in I}\colon \bigoplus_{i\in I} I \to Y$, and $\bar\Phi = \Phi \circ T(p)\colon \mathbf{H}(I, \bigoplus_{i\in I} I) \to \mathbf{H}(I,Y)$. Then, for $x \in \mathbf{H}(I, \bigoplus_{i\in I} I)$:

$$\begin{aligned}\bar\Phi(x) &= \Phi(p\circ x) = \Phi(p \circ (\sum_{i\in I} \kappa_i \circ \pi_i) \circ x) = \sum_{i\in I} \Phi(p \circ \kappa_i \circ \pi_i \circ x)\\ &= \sum_{i\in I} \Phi_i(\pi_i \circ x) = \sum_{i\in I} \varphi_i \circ \pi_i \circ x = \bar\varphi \circ x.\end{aligned}$$

Since p is a dagger epi, it is a cokernel, say $p = \mathrm{coker}(f)$. Now

$$\bar\varphi \circ f = \bar\Phi(f) = \Phi(p \circ f) = \Phi(0) = 0,$$

so there is a (unique) $\varphi\colon X \to Y$ with $\bar\varphi = \varphi \circ p$. Finally, for $x\colon G \to X$,

$$\Phi(x) = \Phi(p \circ p^\dagger \circ x) = \bar\Phi(p^\dagger \circ x) = \bar\varphi \circ p^\dagger \circ x = \varphi \circ p \circ p^\dagger \circ x = \varphi \circ x.$$

■

A. The category of Hilbert spaces

We denote the category of Hilbert spaces and continuous linear transformations by **Hilb**. First, we show that **Hilb** is actually a Hilbert category. Subsequently, we prove that it is not an Abelian category.

First, there is a dagger in **Hilb**, by the Riesz representation theorem. The dagger of a morphism $f\colon X \to Y$ is its adjoint, *i.e.* the unique $f^\dagger$ satisfying

$$\langle f(x) \mid y\rangle_Y = \langle x \mid f^\dagger(y)\rangle_X.$$

It is also well-known that **Hilb** has finite dagger biproducts: $X \oplus Y$ is carried by the direct sum of the underlying vector spaces, with inner product

$$\langle (x, y) \,|\, (x', y') \rangle_{X \oplus Y} = \langle x \,|\, x' \rangle_X + \langle y \,|\, y' \rangle_Y.$$

Furthermore, **Hilb** has kernels: the kernel of $f\colon X \to Y$ is (the inclusion of) $\{x \in X \mid f(x) = 0\}$. Since $\ker(f)$ is in fact a closed subspace, its inclusion is isometric. That is, **Hilb** in fact has dagger kernels. Consequently $\ker(g - f)$ is a dagger equaliser of f and g in **Hilb**.

We now turn to the requirement that every dagger mono be a dagger kernel.

A.1. LEMMA. *The monomorphisms in* **Hilb** *are the injective continuous linear transformations.*

PROOF. If m is injective, then it is obviously mono. Conversely, suppose that $m\colon X \rightarrowtail Y$ is mono. Let $x, x' \in X$ satisfy $m(x) = m(x')$. Define $f\colon \mathbb{C} \to X$ by (continuous linear extension of) $f(1) = x$, and $g\colon \mathbb{C} \to X$ by (continuous linear extension of) $g(1) = x'$. Then $mf = mg$, whence $f = g$ and $x = x'$. Hence m is injective. ∎

Recall that Hilbert spaces have orthogonal projections, that is: if X is a Hilbert space, and $U \subseteq X$ a closed subspace, then every $x \in X$ can be written as $x = u + u'$ for unique $u \in U$ and $u' \in U^\perp$, where

$$U^\perp = \{x \in X \mid \forall_{u \in U}. \langle u \,|\, x \rangle = 0\}. \tag{2}$$

The function that assigns to x the above unique u is a morphism $X \to U$, the *orthogonal projection* of X onto its closed subspace U.

A.2. PROPOSITION. *In Hilb, every dagger mono is a dagger kernel.*

PROOF. Let $m\colon M \rightarrowtail X$ be a dagger mono. In particular, m is a split mono, and hence its image is closed [Aubin, 2000, 4.5.2]. So, without loss of generality, we can assume that m is the inclusion of a closed subspace $M \subseteq X$. But then m is the dagger kernel of the orthogonal projection of X onto M. ∎

All in all, **Hilb** is a Hilbert category. So is its full subcategory **fdHilb** of finite-dimensional Hilbert categories. Also, if $\mathbf{C}$ is a small category and $\mathbf{H}$ a Hilbert category, then $[\mathbf{C}, \mathbf{H}]$ is again a Hilbert category.

Since **Hilb** has biproducts, kernels and cokernels, it is a pre-Abelian category. But the behaviour of epis prevents it from being an Abelian category.

A.3. LEMMA. *The epimorphisms in* **Hilb** *are the continuous linear transformations with dense image.*

PROOF. Let $e\colon X \to Y$ satisfy $\overline{e(X)} = Y$, and $f, g\colon Y \to Z$ satisfy $fe = ge$. Let $y \in Y$, say $y = \lim_n e(x_n)$. Then

$$f(y) = f(\lim_n e(x_n)) = \lim_n f(e(x_n)) = \lim_n g(e(x_n)) = g(\lim_n e(x_n)) = g(y).$$

So $f = g$, whence e is epi.

Conversely, suppose that $e\colon X \twoheadrightarrow Y$ is epi. Then $\overline{e(X)}$ is a closed subspace of Y, so that $Y/\overline{e(X)}$ is again a Hilbert space, and the projection $p\colon Y \to Y/\overline{e(X)}$ is continuous and linear. Consider also $q\colon Y \to Y/\overline{e(X)}$ defined by $q(y) = 0$. Then $pe = qe$, whence $p = q$, and $\overline{e(X)} = Y$. ■

From this, we can conclude that **Hilb** is not an Abelian category, since it is not balanced: there are monic epimorphisms that are not isomorphic. In other words, there are injections that have dense image but are not surjective. For example, $f\colon \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ defined by $f(e_n) = \frac{1}{n}e_n$ is not surjective, as $\sum_n \frac{1}{n}e_n$ is not in its range. But it is injective, self-adjoint, and hence also has dense image.

Another way to see that **Hilb** is not an Abelian category is to assert that the inclusion of a nonclosed subspace is mono, but cannot be a kernel since these are closed.

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Institute for Computing and Information Sciences,
Institute for Mathematics, Astrophysics and Particle Physics,
Radboud University Nijmegen,
Postbus 9010, 6500 GL Nijmegen,
the Netherlands
Email: `heunen@math.ru.nl`

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