Theory and Applications of Categories, Vol. 22, No. 14, 2009, pp. 345–375.

EVENTUALLY CYCLIC SPECTRA OF PARAMETERIZED FLOWS

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Abstract.

This paper continues the work of our previous papers, *The cyclic spectrum of a Boolean flow* TAC *10* 392-419 and *Spectra of finitely generated Boolean flows* TAC *16* 434-459. We define eventually cyclic Boolean flows and the eventually cyclic spectrum of a Boolean flow. We show that this spectrum, as well as the spectra defined in our earlier papers, extend to parametrized flows on Stone spaces and on compact Hausdorff space when symbolic dynamics is used. An example shows that the cyclic spectrum for a parameterized flow is sometimes over a non-spatial locale.

1. Introduction

This paper is a continuation of our earlier papers, [Kennison, 2002] and [Kennison, 2006]. We consider flows in a category, where:

1.1. Definition. *The pair (X, t) is a* **flow** *in a category C if X is an object of C and $t : X \to X$ is a morphism, called the* **iterator**. *If (X, t) and (Y, s) are flows in C, then a* **flow homomorphism** *is a map $h : X \to Y$ for which $sh = ht$. We let* Flow(C) *denote the resulting category of flows in C.*

We are particularly interested in flows in the category of Stone spaces and in the category of Boolean algebras. While these two categories are, by the Stone duality theorem, dual to one another, our theoretical work is best carried out in the category of Boolean flows where we have constructed the cyclic and simple spectra. On the other hand, we are interested in applications to the category of Stone spaces and, more generally, in the category of compact Hausdorff space. As pointed out in [Kennison, 2006], flows in compact Hausdorff spaces can often be approximated by flows in Stone spaces by using the method of symbolic dynamics.

This paper extends the previous work by constructing an eventually cyclic spectrum and by constructing spectra for parameterized flows. We assume the reader is familiar with our previous papers, and note the following facts and conventions:

The author thanks McGill University, Concordia University, Michael Barr and Robert Raphael for support and encouragement.

Received by the editors 2008-12-18 and, in revised form, 2009-06-15.

Transmitted by Susan Niefield. Published on 2009-06-17.

2000 Mathematics Subject Classification: 06D22, 18B99, 37B99.

Key words and phrases: Parameterized Boolean flow, dynamical systems, spectrum, sheaf.

1.2. STONE DUALITY. If X is a Stone space (compact, Hausdorff and totally disconnected) then $\mathrm{Clop}(X)$ denotes the Boolean algebra of all clopen subsets of X. This sets up an equivalence between the category of Stone spaces and the dual of the category of Boolean algebras. Under this duality, a continuous function $f : X \to Y$ corresponds to the Boolean homomorphism $f^{-1} : \mathrm{Clop}(Y) \to \mathrm{Clop}(X)$. We note that f is onto (in this case, epi) if and only if f^{-1} is one-to-one (in this case mono) and f is one-to-one (mono) if and only if f^{-1} is onto (epi). So if the mono-epi factorization of $f : X \to Y$ in Stone spaces, is $f = me$ where $e : X \to Z$ and $m : Z \to Y$, then the mono-epi factorization of f^{-1} is given by $f^{-1} = e^{-1}m^{-1}$. Therefore, if Z is the image of f, then $\mathrm{Clop}(Z)$ is the image of f^{-1}.

1.3. SOME NOTATION. A flow in Stone spaces is usually denoted by (X, t) or (Y, s), almost always using "t" (and occasionally "s") to denote the iterator. A flow in Boolean algebras is usually denoted by (B, τ) or (C, τ) almost always using "τ" as the iterator.

If, by abuse of language, we say that B is a Boolean flow, then B is assumed to be a Boolean algebra, with the iterator τ left implicit.

If (X, t) is a flow in Stone spaces, then $\mathrm{Clop}(X, t)$ denotes the Boolean flow $(\mathrm{Clop}(X), t^{-1})$.

As in our previous papers, we use $\mathbf{N}$ to denote the semi-group of positive integers. We also find it convenient to use $\mathbf{N}_0$ to denote the monoid of non-negative integers.

1.4. LOCALES AND SHEAVES. Our treatment of these topics is based on [Johnstone, 1982] as summarized in [Kennison, 2006]. While some notational conventions are briefly noted in section 5, we assume the reader is familiar with [Johnstone, 1982], [Kennison, 2002] and [Kennison, 2006]. In section 5, we briefly discuss the basic results of sites and coverages, see [Johnstone, 1982, II.2.11], as this topic was not mentioned in [Kennison, 2006]. We also briefly examine the construction of the product locale, for products with Stone spaces, as needed at the end of section 4.

2. Eventually Cyclic Flows

Conceptually, a flow (X, t) is eventually cyclic if the trajectory, $(x, t(x), t^2(x), \ldots, t^n(x), \ldots)$ of an element $x \in X$ is allowed to wander around for a while before becoming part of a cycle. The cycle starts with $t^s(x)$ and has period ℓ when $t^s(x) = t^{s+\ell}(x)$. Before making a formal definition, we consider an example.

2.1. EXAMPLE. *Let $X = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and define $t : X \to X$ by $t(i) = i + 1$ for $i < 9$ and $t(9) = 6$. Then $\{6, 7, 8, 9\}$ is a cycle (called the* **loop***) which the trajectory of 0 eventually reaches. The other points, $\{0, 1, 2, 3, 4, 5\}$ form what is called the* **stem** *of X. (The terms "loop" and "stem" will be defined in greater generality, in 3.6.) Note that $t^6(x) = t^{10}(x)$ for all $x \in X$ and so $t^6 = t^{10}$ is an identity. If $(B, \tau) = \mathrm{Clop}(X, t)$ then, by Stone duality, we also have $\tau^6 = \tau^{10}$. We sometimes write this as $\tau^6 = \tau^{6+4}$ which indicates that X has a stem of length 6 and a loop of length 4.*

2.2. Definition. *A Boolean flow* (B, τ) *is* **eventually cyclic** *if for each* $b \in B$ *there exists* $(s, \ell) \in \mathbf{N}_0 \times \mathbf{N}$ *such that* $\tau^s(b) = \tau^{s+\ell}(b)$*.*

For example, if (X, t) is a flow in Stone spaces and satisfies an identity of the form $t^s = t^{s+\ell}$ for $(s, \ell) \in \mathbf{N}_0 \times \mathbf{N}$ and if $(B, \tau) = \mathrm{Clop}(X, t)$, then, by duality, $\tau^s = \tau^{s+\ell}$ so (B, τ) is eventually cyclic. But $(B, \tau) = \mathrm{Clop}(X, t)$ may be eventually cyclic even if each trajectory in X is infinite but approaches a limiting cycle. In this case, for each $x \in X$, the closure of the trajectory of x has a "stem" and a "loop" but they may each be infinitely long and involve $t^\zeta(x)$ for "transfinite" ζ. (See section 3 of this paper for details and precise definitions. We also show that X itself is a disjoint union of a stem and a loop.) A simple example of such an (X, t) with an infinite stem is given in the verification of remark 4 in 2.3, below. Other examples are in section 6. See also the examples in [Kennison, 2002] and [Kennison, 2006].

2.3. Remarks.

1. *If* (B, τ) *is cyclic, then* τ *is one-to-one and onto.*
2. *If* (B, τ) *is eventually cyclic, then* τ *need not be either one-to-one or onto.*
3. *An eventually cyclic Boolean flow,* (B, τ)*, is cyclic if and only if* τ *is one-to-one.*
4. *There exist eventually cyclic Boolean flows* (B, τ) *with* τ *onto but not one-to-one.*

We sketch the proof of the above assertions. Statement (1) is noted in [Kennison, 2002] and is easily verified. Statement (2) follows from Example 2.1, where τ is not one-to-one (because t is not onto) and τ is not onto (because t is not one-to-one). As for (3), if τ is one-to-one, then so is τ^s so if $\tau^s(b) = \tau^{s+\ell}(b)$ then $b = \tau^\ell(b)$. An example for statement (4) is given by $X = \mathbf{N} \cup \{\infty\}$, the one-point compactification of $\mathbf{N}$, with $t(n) = n + 1$ and $t(\infty) = \infty$. Let $(B, \tau) = \mathrm{Clop}(X, t)$. Then τ is onto as t is one-to-one but τ is not one-to-one as t is not onto. It is readily verified that (B, τ) is eventually cyclic and it can be shown that $\mathbf{N} \subseteq X$ is its stem.

2.4. The free profinite monoid, $\widehat{\mathbf{Z}}_{\mathbf{mon}}$, on one generator. The free profinite group on one generator, $\widehat{\mathbf{Z}}$, played an important role in [Kennison, 2002] in describing cyclic flows. For eventually cyclic flows, the monoid, $\widehat{\mathbf{Z}}_{\mathbf{mon}}$, plays a similar role.

2.5. Definition. *Note that* $\mathbf{N}_0 = \{0, 1, 2, \ldots, n, \ldots\}$ *is a monoid under addition. It is also a flow (in* Sets*) where* $t(n) = n+1$*. We say that an equivalence relation* E *on* $\mathbf{N}_0$ *is a* **flow congruence** *if* $(n, m) \in E$ *implies* $(t(n), t(m)) \in E$*. We say that* E *is a* **monoidal congruence** *if* $(n, m) \in E$ *and* $(a, b) \in E$ *imply* $(n + a, m + b) \in E$*.*

2.6. Notation. *If* E *is an equivalence relation, we write* $x \simeq y$ (mod E) *to indicate that* $(x, y) \in E$*. (If* E *is understood, we may simply write* $x \simeq y$ *for* $(x, y) \in E$*.)*

2.7. Lemma. *Let* $(\mathbf{N}_0, t)$ *be as in 2.5. An equivalence relation* E *on* $\mathbf{N}_0$ *is a flow congruence if and only if it is a monoidal congruence.*

PROOF. Clearly every monoidal congruence on $\mathbf{N}_0$ is a flow congruence, for if $(n, m) \in E$ then $(t(n), t(m)) \in E$ as $t(n) = n+1, t(m) = m+1$ and E preserves addition. Conversely, assume E is a flow congruence with $(n, m) \in E$ and $(a, b) \in E$. We have to prove $(n + a, m + b) \in E$. But then:

$$n + a = t^n(a) \simeq t^n(b) = n + b$$

A strictly similar argument shows $n + b \simeq m + b$ and the result follows. ∎

2.8. NOTATION. *Let $(s, \ell) \in \mathbf{N}_0 \times \mathbf{N}$ be given. Then:*

1. $E(s, \ell)$ denotes the smallest flow congruence on $\mathbf{N}_0$ for which $s \simeq s + \ell$.
2. $\mathbf{N}_0(s, \ell) = \mathbf{N}_0/E(s, \ell)$ denotes both a flow in Sets and a monoid. Note that the flow of Example 2.1 is isomorphic to $\mathbf{N}_0(6, 4)$.
3. We let $\triangle$ denote the **diagonal equivalence relation** for which $(n, m) \in \triangle$ only if $n = m$.
4. We write $n \simeq m \pmod{s, \ell}$ as short for $n \simeq m \pmod{E(s, \ell)}$ and $n \simeq m \pmod{\ell}$ has its usual meaning (that ℓ divides $n - m$).

2.9. LEMMA. *Assume $(s, \ell) \in \mathbf{N}_0 \times \mathbf{N}$. Then $(n, m) \in E(s, \ell)$ if and only if either $n = m$ or $n \geq s$, $m \geq s$ and $n \simeq m \pmod{\ell}$.* ∎

2.10. LEMMA. *Assume $(s, \ell) \in \mathbf{N}_0 \times \mathbf{N}$. Then $(B, \tau) = \mathrm{Clop}(\mathbf{N}_0(s, \ell), t)$ is eventually cyclic.*

PROOF. It is easy to verify that $t^s = t^{s+\ell}$ so, by Stone duality, $\tau^s = \tau^{s+\ell}$. ∎

2.11. PROPOSITION. *Let E be a flow congruence on $\mathbf{N}_0$. Then either $E = \triangle$ or $E = E(s, \ell)$ for a unique pair $(s, \ell) \in \mathbf{N}_0 \times \mathbf{N}$.*

PROOF. Assume that $E \neq \triangle$. We say that $n \in \mathbf{N}_0$ is **paired** by E if there exists $m \in \mathbf{N}_0$ with $(n, m) \in E$ but $n \neq m$. Let s be the smallest element of $\mathbf{N}_0$ which is paired by E. Let ℓ be the smallest positive element of $\mathbf{N}_0$ for which $(s, s + \ell) \in E$. Clearly $E(s, \ell) \subseteq E$. To prove that $E = E(s, \ell)$, let $(n, m) \in E$ be given. We need to show that $(n, m) \in E(s, \ell)$. We may as well assume that $n < m$. By the choice of s, we have $s \leq n$. Write $m = n + k\ell + r$ where $0 \leq r < \ell$.

We claim that $r = 0$, which will complete the proof as then $(n, m) \in E(s, \ell))$, by Lemma 2.9, and $E(s, \ell) \subseteq E$. Assume that $r \neq 0$ so $0 < r < \ell$. By the same lemma, we have $(n, n + k\ell) \in E(s, \ell) \subseteq E$ so $(n, m) \in E$ if and only if $(n + k\ell, m) \in E$, so we may as well assume that $n = n + k\ell$ and $m = n + r$.

So $(n, n+r) \in E$ and therefore $(n+i, n+i+r) \in E$ for all i. We may as well assume that $s \simeq n \pmod{\ell}$ as we can otherwise replace n by a suitably chosen $n + i$. With $s \simeq n \pmod{\ell}$ we have $(s, n) \in E(s, \ell) \subseteq E$ and $(s + r, n + r) \in E(s, \ell) \subseteq E$ (by 2.9) and, since $(n, n + r) \in E$ we get, by transitivity, $(s, s + r) \in E$ with $0 < r < \ell$ which contradicts the choice of ℓ. ∎

2.12. NOTATION. *We use the following notation and conventions when dealing with the flow congruences $E(s, \ell)$ on $\mathbf{N}_0$.*

- Notation such as (s, ℓ) or (s_1, ℓ_1) or (s_2, ℓ_2) will be assumed to refer to members of $\mathbf{N}_0 \times \mathbf{N}$ unless the contrary is explicitly stated.
- We let $q(s, \ell) : \mathbf{N}_0 \to \mathbf{N}_0(s, \ell)$ denote the obvious quotient map.
- We call $q(s, \ell)(\{0, 1, \ldots, s-1\})$ the **stem** of $\mathbf{N}_0(s, \ell)$ and the rest of $\mathbf{N}_0(s, \ell)$ its **loop**. (This is consistent with Definition 3.6).

2.13. DEFINITION. *We define $\sqsubseteq$ as the partial ordering of $\mathbf{N}_0 \times \mathbf{N}$ for which $(s_0, \ell_0) \sqsubseteq (s_1, \ell_1)$ if and only if $E(s_0, \ell_0) \subseteq E(s_1, \ell_1)$.*

We chose the $\sqsubseteq$ notation because it allows for good infs and sups (denoted by $\sqcap$ and $\sqcup$ respectively), because it suggests $\subseteq$ and because $\leq$ seems wrong since $(s_0, \ell_0) \sqsubseteq (s_1, \ell_1)$ implies that s_0 and ℓ_0 are at least as big as s_1 and ℓ_1.

2.14. PROPOSITION.

1. *$(s_0, \ell_0) \sqsubseteq (s_1, \ell_1)$ if and only if there exists a flow homomorphism $h : \mathbf{N}_0(s_0, \ell_0) \to \mathbf{N}_0(s_1, \ell_1)$ for which $hq(s_0, \ell_0) = q(s_1, \ell_1)$ (in which case h is also a monoidal homomorphism).*
2. *$(s_0, \ell_0) \sqsubseteq (s_1, \ell_1)$ if and only if $s_0 \geq s_1$ and ℓ_0 is a multiple of ℓ_1.*
3. *The partial ordering $\sqsubseteq$ on $\mathbf{N}_0 \times \mathbf{N}$ admits infs and sups (denoted by $\sqcap$ and $\sqcup$) where:*

$$(s_0, \ell_0) \sqcap (s_1, \ell_1) = (\max(s_0, s_1), \operatorname{lcm}(\ell_0, \ell_1))$$

$$(s_0, \ell_0) \sqcup (s_1, \ell_1) = (\min(s_0, s_1), \gcd(\ell_0, \ell_1))$$

■

2.15. CONSTRUCTION OF $\widehat{\mathbf{Z}}_{\mathbf{mon}}$. We define $\widehat{\mathbf{Z}}_{\mathbf{mon}}$ as the limit of the diagram of all $\mathbf{N}_0(s, \ell)$ as (s, ℓ) varies in $\mathbf{N}_0 \times \mathbf{N}$ and of all maps $h : \mathbf{N}_0(s_0, \ell_0) \to \mathbf{N}_0(s_1, \ell_1)$ for which $hq(s_0, \ell_0) = q(s_1, \ell_1)$.

We let $p(s, \ell) : \widehat{\mathbf{Z}}_{\mathbf{mon}} \to \mathbf{N}_0(s, \ell)$ be the projection map associated with the limit. $\widehat{\mathbf{Z}}_{\mathbf{mon}}$ has a flow structure, a monoidal structure and a topological structure given by the limit topology. It is readily verified that $\widehat{\mathbf{Z}}_{\mathbf{mon}}$ is then a flow in Stone spaces.

There is a natural embedding $\mathbf{N}_0 \to \widehat{\mathbf{Z}}_{\mathbf{mon}}$ as $q(s, \ell) : \mathbf{N}_0 \to \mathbf{N}_0(s, \ell)$ is a cone over the diagram used to define $\widehat{\mathbf{Z}}_{\mathbf{mon}}$. **We will identify $\mathbf{N}_0$ with its image in $\widehat{\mathbf{Z}}_{\mathbf{mon}}$.** This has the effect of making $q(s, \ell) : \mathbf{N}_0 \to \mathbf{N}_0(s, \ell)$ the restriction of $p(s, \ell)$ to $\mathbf{N}_0$. Note also that the flow iterator, t of $\widehat{\mathbf{Z}}_{\mathbf{mon}}$, is defined by $t(\zeta) = \zeta + 1$.

2.16. REMARK. *A concrete representation of $\operatorname{Clop}(\widehat{\mathbf{Z}}_{\mathbf{mon}})$ and an alternate construction of $\widehat{\mathbf{Z}}_{\mathbf{mon}}$ appear in Example 6.3.*

2.17. LEMMA. *Recall that $\mathbf{N}_0 \subseteq \widehat{\mathbf{Z}}_{\mathbf{mon}}$. Then $\zeta \in \widehat{\mathbf{Z}}_{\mathbf{mon}}$ is in $\mathbf{N}_0$ if and only if there exists (s,ℓ) with $p(s,\ell)(\zeta)$ in the stem of $\mathbf{N}_0(s,\ell)$.*

PROOF. Suppose $\zeta \in \mathbf{N}_0 \subseteq \widehat{\mathbf{Z}}_{\mathbf{mon}}$. Choose any (s,ℓ) with $\zeta < s$. Then ζ clearly maps to the stem of $\mathbf{N}_0(s,\ell)$.

Conversely, suppose $p(s_0,\ell_0)(\zeta)$ is in the stem of $\mathbf{N}_0(s_0,\ell_0)$ for some (s_0,ℓ_0). Then there exists a unique $n \in \mathbf{N}_0$ such that $p(s_0,\ell_0)(n) = p(s_0,\ell_0)(\zeta)$ We have to show that $p(s_1,\ell_1)(n) = p(s_1,\ell_1)(\zeta)$ for all $(s_1,\ell_1) \in \mathbf{N}_0 \times \mathbf{N}$. This is immediate if $(s_0,\ell_0) \sqsubseteq (s_1,\ell_1)$ and straightforward if $(s_1,\ell_1) \sqsubseteq (s_0,\ell_0)$. The general case follows by examining $(s_0,\ell_0) \sqcap (s_1,\ell_1)$. ∎

2.18. NOTATION. *We call $\mathbf{N}_0$ the* **stem** *of $\widehat{\mathbf{Z}}_{\mathbf{mon}}$ and the rest of $\widehat{\mathbf{Z}}_{\mathbf{mon}}$ its* **loop**.

2.19. LEMMA. *The loop of $\mathbf{N}_0(s,\ell)$ is, as a monoid, isomorphic to the group $\mathbf{Z}_\ell$.*

PROOF. This is straightforward as, for $n, m \geq s$, we have $q(s,\ell)(n) = q(s,\ell)(m)$ if and only if $n \simeq m \pmod{\ell}$. ∎

2.20. NOTATION. *Further notation for $\widehat{\mathbf{Z}}_{\mathbf{mon}}$.*

- Extending a similar notation for $\mathbf{N}_0$, given $\zeta, \gamma \in \widehat{\mathbf{Z}}_{\mathbf{mon}}$, we write $\zeta \simeq \gamma \pmod{s,\ell}$ when $p(s,\ell)(\zeta) = p(s,\ell)(\gamma)$.
- Given $\zeta \in \widehat{\mathbf{Z}}_{\mathbf{mon}}$ and $(s,\ell) \in \mathbf{N}_0 \times \mathbf{N}$, we let $U(s,\ell,\zeta) = p(s,\ell)^{-1}[p(s,\ell)(\zeta)]$. It consists of all $\gamma \in \widehat{\mathbf{Z}}_{\mathbf{mon}}$ for which $\zeta \simeq \gamma \pmod{s,\ell}$.

2.21. LEMMA. *If $\zeta \simeq \gamma \pmod{s_i,\ell_i}$ for $i = 0,1$ then $\zeta \simeq \gamma \pmod{s,\ell}$ where $(s,\ell) = (s_0,\ell_0) \sqcap (s_1,\ell_1)$.*

PROOF. Let $h_0 : \mathbf{N}_0(s,\ell) \to \mathbf{N}_0(s_0,\ell_0)$ and $h_1 : \mathbf{N}_0(s,\ell) \to \mathbf{N}_0(s_1,\ell_1)$ be the obvious monoidal homomorphisms in the diagram of which $\widehat{\mathbf{Z}}_{\mathbf{mon}}$ is the limit.

Case 1: Assume that $p(s_0,\ell_0)(\zeta) = p(s_0,\ell_0)(\gamma) = w$ is in the stem of $\mathbf{N}_0(s_0,\ell_0)$. Then, as can be readily verified, there is exactly one $v \in \mathbf{N}_0(s,\ell)$ for which $h_0(v) = w$ so $p(s,\ell)(\zeta) = p(s,\ell)(\gamma) = v$.

Case 2: $p(s_1,\ell_1)(\zeta) = p(s_1,\ell_1)(\gamma)$ is in the stem of $\mathbf{N}_0(s_1,\ell_1)$. (Same as Case 1).

Case 3: $p(s_0,\ell_0)(\zeta) = p(s_0,\ell_0)(\gamma)$ is in the loop of $\mathbf{N}_0(s_0,\ell_0)$ and $p(s_1,\ell_1)(\zeta) = p(s_1,\ell_1)(\gamma)$ is in the loop of $\mathbf{N}_0(s_1,\ell_1)$. Choose $i \in \{0,1\}$ so that $s = s_i$ (as $s = \max(s_0,s_1)$). From this, it follows that $h_i : \mathbf{N}_0(s,\ell) \to \mathbf{N}_0(s_i,\ell_i)$ maps the stem of $\mathbf{N}_0(s,\ell)$ to the stem of $\mathbf{N}_0(s_i,\ell_i)$, so $p(s,\ell)(\zeta)$ and $p(s,\ell)(\gamma)$ must both lie in the loop of $\mathbf{N}_0(s,\ell)$. But the loops of $\mathbf{N}_0(s_0,\ell_0)$, $\mathbf{N}_0(s_1,\ell_1)$ and $\mathbf{N}_0(s,\ell)$ are isomorphic as monoids to $\mathbf{Z}_{\ell_0}$, $\mathbf{Z}_{\ell_1}$ and $\mathbf{Z}_\ell$. Choose $n, m \in \mathbf{N}_0$ such that $n, m \geq s$ and $\zeta \simeq n \pmod{\ell}$, $\gamma \simeq m \pmod{\ell}$. Then, $n \simeq m \pmod{\ell_0}$ and $n \simeq m \pmod{\ell_1}$, so $n \simeq m \pmod{\ell}$ as $\ell = \mathrm{lcm}(\ell_0,\ell_1)$. ∎

2.22. COROLLARY. *If $\zeta \in \widehat{\mathbf{Z}}_{\mathbf{mon}}$ then $U(s_0,\ell_0,\zeta) \cap U(s_1,\ell_1,\zeta) = U(s,\ell,\zeta)$ where $(s,\ell) = (s_0,\ell_0) \sqcap (s_1,\ell_1)$. It follows that the open sets $U(s,\ell,\zeta)$ (as s,ℓ,ζ vary) form a clopen base for the topology on $\widehat{\mathbf{Z}}_{\mathbf{mon}}$.* ∎

2.23. COROLLARY. *The subset* $\mathbf{N}_0 \subseteq \widehat{\mathbf{Z}}_{\mathbf{mon}}$ *is dense.*

PROOF. Obviously there are members of $\mathbf{N}_0$ in every basic open set of the form $U(s, \ell, \zeta)$ ■

2.24. PROPOSITION. $\widehat{\mathbf{Z}}_{\mathbf{mon}}$ *is the disjoint union* $\mathbf{N}_0 \cup \widehat{\mathbf{Z}}$ *where* $\widehat{\mathbf{Z}}$ *is isomorphic, both topologically and as a monoid, to the profinite integers. The subset* $\widehat{\mathbf{Z}} \subseteq \widehat{\mathbf{Z}}_{\mathbf{mon}}$ *is closed and the subset* $\mathbf{N}_0$ *is open and discrete.*

PROOF. Assume $\zeta \in \widehat{\mathbf{Z}}_{\mathbf{mon}}$ and $\zeta \notin \mathbf{N}_0$. Then, by Lemma 2.17, $p(s, \ell)(\zeta)$ is always in the loop of $\mathbf{N}_0(s, \ell)$ which is equivalent to $\mathbf{Z}_\ell$. It is readily verified that elements of this type form a limit of the discrete groups of the form $\mathbf{Z}_\ell$ and such a limit is isomorphic to the profinite integers. It is clear that $\widehat{\mathbf{Z}}$ is closed, as it is compact. As for $\mathbf{N}_0$, we can, for each $n \in \mathbf{N}_0$, choose $s > n$ (and any ℓ) then $U(s, \ell, n) = \{n\}$. ■

2.25. THE SUBSET $\mathbf{N}_0 \subseteq \widehat{\mathbf{Z}}_{\mathbf{mon}}$ AND ITS "SHADOW", $\widetilde{\mathbf{N}_0} \subseteq \widehat{\mathbf{Z}}_{\mathbf{mon}}$. Since $\widehat{\mathbf{Z}}_{\mathbf{mon}}$ is the disjoint union $\mathbf{N}_0 \cup \widehat{\mathbf{Z}}$ and since $\widehat{\mathbf{Z}}$, the profinite integers, contains, in a natural way, a copy of $\mathbf{N}_0$, we see that $\widehat{\mathbf{Z}}_{\mathbf{mon}}$ contains two disjoint copies of $\mathbf{N}_0$. We let $\widetilde{\mathbf{N}_0}$ denote the copy of $\mathbf{N}_0$ contained in $\widehat{\mathbf{Z}}$. That is, for $n \in \mathbf{N}_0$, we define $\tilde{n} \in \widehat{\mathbf{Z}}_{\mathbf{mon}}$ so that $p(s, \ell)(\tilde{n})$ is in the unique member of the loop of $\mathbf{N}_0(s, \ell)$ which is congruent to $n \pmod{\ell}$. Then let $\widetilde{\mathbf{N}_0} = \{\tilde{n} | n \in \mathbf{N}_0\}$. Note that every member of $\mathbf{N}_0$ is in the stem of $\widehat{\mathbf{Z}}_{\mathbf{mon}}$ and every member of $\widetilde{\mathbf{N}_0}$ is in the loop.

2.26. REMARK. *The following observations illustrate the difference between* $\mathbf{N}_0$ *and* $\widetilde{\mathbf{N}_0}$, *as subsets of* $\widehat{\mathbf{Z}}_{\mathbf{mon}}$ *and will be useful later on.*

1. If $\zeta \in \widehat{\mathbf{Z}} \subseteq \widehat{\mathbf{Z}}_{\mathbf{mon}}$ then $\zeta + \tilde{0} = \zeta$ but if $n \in \mathbf{N}_0 \subseteq \widehat{\mathbf{Z}}_{\mathbf{mon}}$ then $n + \tilde{0} = \tilde{n}$.
2. The inclusion map $I : \widehat{\mathbf{Z}} \to \widehat{\mathbf{Z}}_{\mathbf{mon}}$ is a continuous flow homomorphism and a semigroup homomorphism but not a monoidal homomorphism.
3. The map $R : \widehat{\mathbf{Z}}_{\mathbf{mon}} \to \widehat{\mathbf{Z}}$ for which $R(\zeta) = \zeta + \tilde{0}$ is a continuous flow homomorphism and a retract (a left inverse of I) and a monoidal homomorphism.

PROOF. Most of the assertions in this remark are established by examining the projections onto the monoids $\mathbf{N}_0(s, \ell)$. Note that once we are in the loop portion of $\mathbf{N}_0(s, \ell)$, adding $\tilde{0}$ has no effect, but in the stem portion of $\mathbf{N}_0(s, \ell)$, adding $\tilde{0}$ moves an element into the loop. ■

2.27. DEFINITION. *Let* S *be a set and* $T : \widehat{\mathbf{Z}}_{\mathbf{mon}} \times S \to S$ *be a function. For each* $\zeta \in \widehat{\mathbf{Z}}_{\mathbf{mon}}$, *let* $T(\zeta, \_)$ *denote the function from* S *to* S *which maps* $x \in S$ *to* $T(\zeta, x)$. *Then:*

1. We say that T is an **action** if the map $\zeta \mapsto T(\zeta, \_)$ is a monoidal homomorphism from the monoid $\widehat{\mathbf{Z}}_{\mathbf{mon}}$ to the set of functions from S to S (which is a monoid under composition). This is equivalent to the conditions that $T(0, s) = s$ for all $s \in S$ and $T(\zeta, T(\gamma, s)) = T(\zeta + \gamma, s)$ for all s, ζ, γ.

2. If $t : S \to S$ is a function, then T is **compatible** with t (or **compatible** if t is understood) if $T(n, \_) = t^n$ for all $n \in \mathbf{N}_0$. (If T is an action, it suffices that $T(1, \_) = t$.) If T is compatible with t, then $t^\zeta(x)$ will sometimes be used to denote $T(\zeta, x)$

3. If S is a Boolean algebra, then T is **admissible** if $T(\zeta, \_)$ is a Boolean flow homomorphism for all $\zeta \in \widehat{\mathbf{Z}}_{\mathbf{mon}}$. If S is a Stone space, then T is **admissible** if $T(\zeta, \_)$ is a continuous flow homomorphism for all $\zeta \in \widehat{\mathbf{Z}}_{\mathbf{mon}}$. It will always be clear from the context which definition of admissible applies in any specific case.

4. If S is a Boolean algebra, then T is **continuous** if $T : \widehat{\mathbf{Z}}_{\mathbf{mon}} \times S \to S$ is continuous (where $\widehat{\mathbf{Z}}_{\mathbf{mon}}$ has its limit topology, S has the discrete topology, and $\widehat{\mathbf{Z}}_{\mathbf{mon}} \times S$ has the product topology). If S is a Stone space, then T is **continuous** if $T : \widehat{\mathbf{Z}}_{\mathbf{mon}} \times S \to S$ is continuous (where $\widehat{\mathbf{Z}}_{\mathbf{mon}}$ has its limit topology, S has its Stone topology, and $\widehat{\mathbf{Z}}_{\mathbf{mon}} \times S$ has the product topology). It will always be clear from the context which definition of continuous applies in any specific case.

2.28. Lemma. *Let B be any set and $\tau : B \to B$ any function and assume $b \in B$. If $\tau^{s_1}(b) = \tau^{s_1+\ell_1}(b)$ and $(s_0, \ell_0) \sqsubseteq (s_1, \ell_1)$ then $\tau^{s_0}(b) = \tau^{s_0+\ell_0}(b)$.*

Proof. Let $E = \{(n, m) \in \mathbf{N}_0 \times \mathbf{N}_0 | \tau^n(b) = \tau^m(b)\}$. Then E is clearly a flow congruence on $\mathbf{N}_0$ and $E \neq \triangle$ so $E = E(s, \ell)$ for some $(s, \ell) \in \mathbf{N}_0 \times \mathbf{N}$. Now $(s', \ell') \in E$ if and only if $E(s', \ell') \subseteq E$ if and only if $(s', \ell') \sqsubseteq (s, \ell)$ and the result follows easily. ∎

2.29. Theorem. *Let (B, τ) be a Boolean flow. Then*

1. *(B, τ) is eventually cyclic if and only if there is a continuous, compatible action by $\widehat{\mathbf{Z}}_{\mathbf{mon}}$ on B. Moreover, such an action is necessarily admissible.*

2. *There is at most one continuous, compatible action by $\widehat{\mathbf{Z}}_{\mathbf{mon}}$ on B.*

3. *If (B, τ) and (C, τ) are eventually cyclic Boolean flows and if $h : B \to C$ is a flow homomorphism, then h preserves the $\widehat{\mathbf{Z}}_{\mathbf{mon}}$ action. So $h(\tau^\zeta(b)) = (\tau^\zeta(h(b))$ for all $\zeta \in \widehat{\mathbf{Z}}_{\mathbf{mon}}$.*

Proof. To prove (1), we first assume that (B, τ) is eventually cyclic. Let $\zeta \in \widehat{\mathbf{Z}}_{\mathbf{mon}}$ and $b \in B$ be given. Choose (s, ℓ) such that $\tau^s(b) = \tau^{s+\ell}(b)$. Choose $k \in \mathbf{N}_0$ such that $\zeta \simeq k \pmod{s, \ell}$. Then define $T(\zeta, b) = \tau^k(b)$. We claim that T is well-defined. It is clear that it does not depend on the choice of k for which $\zeta \simeq k \pmod{s, \ell}$. Also, if we assume that $\tau^{s_i}(b) = \tau^{s_i+\ell_i}(b)$ for $i = 1, 2$, we have to show that the resulting definitions of $T(\zeta, b)$ are the same. This is readily verified if the (s_i, ℓ_i) are comparable, for example, if $(s_1, \ell_1) \sqsubseteq (s_2, \ell_2)$, and the general case follows by the above lemma, using $(s_3, \ell_3) = (s_1, \ell_1) \sqcap (s_2, \ell_2)$.

It is straightforward to show that T is compatible with τ and that $T(\zeta, \_)$ is a monoidal action. Also, it is obvious that T is continuous as T is constant on neighborhoods of the form $U(s, \ell, \zeta) \times \{b\}$

We claim that any continuous, compatible action T is admissible which means we must prove that $T(\zeta, \_) : B \to B$ is a Boolean flow homomorphism for all $\zeta \in \widehat{\mathbf{Z}}_{\mathbf{mon}}$. But this is readily verified if $\zeta = n \in \mathbf{N}_0$ (as T is compatible) and the general case follows as $\mathbf{N}_0$ is dense in $\widehat{\mathbf{Z}}_{\mathbf{mon}}$ (using the fact that T is continuous).

To prove the converse, assume that $T : \widehat{\mathbf{Z}}_{\mathbf{mon}} \times B \to B$ is a continuous, compatible action. Let $b \in B$ be given. Define $\rho(b) = T(\tilde{0}, b)$. Then, as T is continuous, there exists a basic neighborhood $U = U(s, \ell, \tilde{0})$ such that $T(u, b) = \rho(b)$ for all $u \in U$. Choose $k \in \mathbf{N}_0$ so that $k \geq s$ and ℓ divides k. Then $k \in U$ so $T(k, b) = \rho(b)$. Similarly $T(k + \ell, b) = \rho(b)$. since T is compatible, we have $t^k(b) = \rho(b)$ and $t^{k+\ell}(b) = \rho(b)$ so $t^k(b) = t^{k+\ell}(b)$.

Statement (2), that T is uniquely determined by these conditions, follows as $\mathbf{N}_0$ is a dense subset of $\widehat{\mathbf{Z}}_{\mathbf{mon}}$.

To prove (3), note that if h is a flow homomorphism, then h preserves the restriction of the action T to $\mathbf{N}_0 \times B$ and the result follows as $\mathbf{N}_0$ is dense in $\widehat{\mathbf{Z}}_{\mathbf{mon}}$. ∎

2.30. Remark. *The cyclic analogue of the above theorem would say that a Boolean flow (B, τ) is cyclic if and only if B admits a continuous, compatible action by $\widehat{\mathbf{Z}}$ (a subgroup of $\widehat{\mathbf{Z}}_{\mathbf{mon}}$). The result is partially given in [Kennison, 2002, Theorem 2.1], which says that if B is cyclic, then there is such an action by $\widehat{\mathbf{Z}}$. The converse is omitted in [Kennison, 2002], but it follows by essentially the same argument as given in the above proof. Note that if T is an action on a cyclic flow B by $\widehat{\mathbf{Z}}_{\mathbf{mon}}$, then $T(\tilde{0}, b) = b$ as $\tilde{0}$ is the additive identity of $\widehat{\mathbf{Z}}$. The cyclic analogues of (2) and (3) of the above theorem are also not explicitly stated in [Kennison, 2002] but are easily proven.*

We summarize the above remark, and Remark 2.26, with the following corollary.

2.31. Corollary. *Let (B, τ) be an eventually cyclic Boolean flow so that $\tau^\zeta(b)$ is defined for all $b \in B$. Then the following statements are equivalent:*

1. *(B, τ) is cyclic.*
2. *$\tau^{\tilde{n}} = \tau^n$ for all $n \in \mathbf{N}_0$.*
3. *The action of $\widehat{\mathbf{Z}}_{\mathbf{mon}}$ on B restricts to a group action by $\widehat{\mathbf{Z}}$ on B (so that $\tau^{\tilde{0}}$ is the identity map as $\tilde{0}$ is the identity of $\widehat{\mathbf{Z}} \subseteq \widehat{\mathbf{Z}}_{\mathbf{mon}}$.)* ∎

3. From the Perspective of Stone Spaces

In this section, we explore the properties of a flow (X, t) in Stone spaces when we know that $\mathrm{Clop}(X, t)$ is a cyclic or eventually cyclic Boolean flow. We start by extending a definition from [Kennison, 2002] and proving the analogue of Theorem 2.29.

3.1. Definition. *A flow* (X, t) *in Stone spaces is said to be* **dually cyclic** *(resp.* **dually eventually cyclic***) if* $(B, \tau) = \mathrm{Clop}(X, t)$ *is a cyclic (resp. eventually cyclic) Boolean flow.*

The term "Boolean cyclic", as used in [Kennison, 2002], has been changed to dually cyclic because of the possible confusion between a "Boolean cyclic flow" (which is a flow in Stone spaces) and a "cyclic Boolean flow" (which is a flow in Boolean algebras).

3.2. Theorem. *Let* (X, t) *be a flow in Stone spaces. Then:*

1. (X, t) *is dually eventually cyclic if and only if there is a continuous, compatible, admissible action of* $\widehat{\mathbf{Z}}_{\mathbf{mon}}$ *on* X.
2. *There is at most one such action by* $\widehat{\mathbf{Z}}_{\mathbf{mon}}$ *on* X.
3. *If* (X, t) *and* (Y, t) *are both dually eventually cyclic flows in Stone spaces, and if* $h : X \to Y$ *is a continuous flow homomorphism, then* h *preserves the* $\widehat{\mathbf{Z}}_{\mathbf{mon}}$ *action. So* $h(\tau^\zeta(x)) = (\tau^\zeta(h(x))$ *for all* $\zeta \in \widehat{\mathbf{Z}}_{\mathbf{mon}}$ *and all* $x \in X$.

Proof. We let $(B, \tau) = \mathrm{Clop}(X, t)$. To prove (1), first assume that (B, τ) is eventually cyclic. The action of $\widehat{\mathbf{Z}}_{\mathbf{mon}}$ on B allows us to define $\tau^\zeta : B \to B$ for each $\zeta \in \widehat{\mathbf{Z}}_{\mathbf{mon}}$. We let $t^\zeta : X \to X$ be the continuous function determined by duality. We can then define $T : \widehat{\mathbf{Z}}_{\mathbf{mon}} \times X \to X$ by $T(\zeta, x) = t^\zeta(x)$. It follows that T is an action as the Stone duality preserves composition and $\widehat{\mathbf{Z}}_{\mathbf{mon}}$ is commutative. We must show that T is continuous. Suppose that $x \in X$ and $\zeta \in \widehat{\mathbf{Z}}_{\mathbf{mon}}$ are given, with $T(\zeta, x) = y$. Let b be a clopen neighborhood of y. We must find a neighborhood U of ζ and a clopen neighborhood c of x such that $T(U \times c) \subseteq b$. Assume that $\tau^s(b) = \tau^{s+\ell}(b)$ and that $\zeta \simeq k \pmod{s, \ell}$. Let $U = U(s, \ell, \zeta)$ and $c = \tau^k(b)$. Then $U \times c$ has the desired property.

Conversely, assume that $T : \widehat{\mathbf{Z}}_{\mathbf{mon}} \times X \to X$ is a continuous, compatible, admissible action of $\widehat{\mathbf{Z}}_{\mathbf{mon}}$ on X. Let $b \in \mathrm{Clop}(X)$ be given. Then $T^{-1}(b)$ and $T^{-1}(\neg b)$ are complimentary clopen subsets of $\widehat{\mathbf{Z}}_{\mathbf{mon}} \times X$. For each $(\zeta, x) \in \widehat{\mathbf{Z}}_{\mathbf{mon}} \times X$, we can find a basic neighborhood of the form $U(s, \ell, \zeta) \times c$ which T either maps entirely into b or entirely into $\neg b$. By compactness, we can cover $\widehat{\mathbf{Z}}_{\mathbf{mon}} \times X$ with finitely many such basic neighborhoods. By taking the inf of the $(s, \ell)'s$ that are involved, we can assume that the same (s, ℓ) is involved in the choice of each neighborhood $U(s, \ell, \zeta) \times c$. There are also finitely many clopen sets c that are involved and they clearly generate an atomic Boolean subalgebra of $\mathrm{Clop}(X)$, with atoms which will denote by $c_1, c_2, \ldots, c_n$. Then the neighborhoods are those of the form $U(s, \ell, k) \times c_i$ for $1 \le i \le n$ and $0 \le k \le (s+\ell-1)$. Defining τ^ζ as $(t^\zeta)^{-1}$, it then readily follows that $\tau^\zeta(b)$ is the union of all c_i for which $T(U(s, \ell, \zeta) \times c_i) \subseteq b$. So $\tau^\lambda(b) = \tau^\gamma(b)$ whenever $\lambda \simeq \gamma \pmod{s, \ell}$. But $s \simeq s + \ell \pmod{s, \ell}$ which implies that $\tau^s(b) = \tau^{s+\ell}(b)$.

The proofs are (2) and (3) follow, as in the proof of (2) and (3) of 2.29, because $\mathbf{N}_0$ is dense in $\widehat{\mathbf{Z}}_{\mathbf{mon}}$. ∎

3.3. Definition. *Let (B, τ) be a Boolean flow. Following [Kennison, 2002], we define $b \in B$ to be* **periodic** *if there exists $n \in \mathbf{N}$ with $\tau^n(b) = b$. We let* $\mathrm{Loop}(B, \tau)$ *denote the subset of all periodic elements of B.*

We further define $b \in B$ to be **eventually periodic** *if there exist $(s, \ell) \in \mathbf{N}_0 \times \mathbf{N}$ such that $\tau^s(b) = \tau^{s+\ell}(b)$. We let* $\mathrm{ECLoop}(B, \tau)$ *denote the subset of all eventually periodic elements of B.*

3.4. Proposition. *Let (B, τ) be a Boolean flow Then:*

1. $\mathrm{Loop}(B, \tau)$ *is a cyclic subflow of B.*
2. *The inclusion of* $\mathrm{Loop}(B, \tau)$ *in (B, τ) is a coreflection of (B, τ) into the full subcategory of cyclic flows.*
3. *The inclusion of* $\mathrm{ECLoop}(B, \tau)$ *in (B, τ) is a coreflection of (B, τ) into the full subcategory of eventually cyclic flows.*
4. *Assume that (B, τ) is an eventually cyclic Boolean flow so that τ^ζ is defined for every $\zeta \in \widehat{\mathbf{Z}}_{\mathbf{mon}}$. Let $\rho = \tau^{\tilde{0}}$. Then ρ is a flow homomorphism which retracts (B, τ) onto* $\mathrm{Loop}(B, \tau)$ *(that is, ρ is a left inverse of the inclusion $i : \mathrm{Loop}(B, \tau) \subseteq (B, \tau)$). Moreover, ρ is the unique flow homomorphism retraction onto* $\mathrm{Loop}(B)$.

Proof. The proofs of (1), (2), (3) are straightforward. As for (4), note that $\rho^2 = \rho$ so, letting C be the image of ρ, we see that ρ retracts B onto C. By Theorem 2.29, ρ is a Boolean homomorphism and a flow homomorphism because $\rho\tau = \tau^{\tilde{0}+1} = \tau\rho$. It follows that C is a subflow and, by using Corollary 2.31(2), we readily see that C is the largest cyclic subflow of B, so $C = \mathrm{Loop}(B)$.

To prove uniqueness, suppose $\rho' : B \to \mathrm{Loop}(B)$ is another flow homomorphism and retraction. Then by Theorem 2.29, ρ' preserves the action of $\widehat{\mathbf{Z}}_{\mathbf{mon}}$ so ρ' preserves $\rho = \tau^{\tilde{0}}(b)$ so $\rho'\rho = \rho\rho'$. But $\rho(\rho'(b)) = \rho'(b)$ as ρ is a retraction onto the range of ρ' and, similarly $\rho'(\rho(b)) = \rho(b)$ so $\rho'(b) = \rho(\rho'(b)) = \rho'(\rho(b)) = \rho(b)$. ∎

3.5. Corollary.

1. *In the category of Boolean flows, the cyclic flows are closed under subflows, quotient flows and colimits.*
2. *In the category of Boolean flows, the eventually cyclic flows are closed under subflows, quotient flows and colimits.*
3. *In the category of flows in Stone spaces, the dually cyclic flows are closed under subflows, quotient flows and limits.*
4. *In the category of flows in Stone spaces, the dually eventually cyclic flows are closed under subflows, quotient flows and limits.*
5. *The flow $(\widehat{\mathbf{Z}}_{\mathbf{mon}}, t)$ in Stone spaces, where $t(\zeta) = \zeta + 1$, is dually eventually cyclic.*

PROOF. (1), (2): The closures under quotient flows and subflows are easily proven. The closure under colimits follows as the cyclic flows (resp. the eventually cyclic flows) are coreflective.

(3), (4): These follow by Stone duality.

(5) This follows from (4) as $\widehat{\mathbf{Z}}_{\mathbf{mon}}$ is a limit of dually eventually cyclic flows. ∎

We note that an alternate proof that $\widehat{\mathbf{Z}}_{\mathbf{mon}}$ is dually eventually cyclic follows from Theorem 3.2 as $\widehat{\mathbf{Z}}_{\mathbf{mon}}$ clearly acts on itself in a continuous, compatible manner.

3.6. DEFINITION. *Let (X, t) be a dually eventually cyclic flow in Stone spaces and let $R : X \to X$ be $t^{\tilde{0}}$. Define* $\mathrm{Loop}(X)$ *to be the image of R.*

We define $\mathrm{Stem}(X)$ *as the complement of* $\mathrm{Loop}(X)$

3.7. LEMMA. *Suppose that (X, t) is a dually eventually cyclic flow in Stone spaces and let $A \subseteq X$ be a closed subflow. Then A is a dually cyclic subflow if and only if $(t)^{\tilde{0}}$, when restricted to A, is the identity.*

PROOF. Assume that $A \subseteq X$ is dually cyclic. The restriction $t|A$ is the iterator of A and, since A is dually cyclic, $(t|A)^{\tilde{0}}$ is, by 2.31, the identity on A. But the inclusion $(A, t|A) \to (X, t)$ preserves the action of $\widehat{\mathbf{Z}}_{\mathbf{mon}}$ so $(t|A)^{\tilde{0}}$ is the restriction of $t^{\tilde{0}}$. The proof of the converse is similar. ∎

3.8. THEOREM. *Let (X, t) be a dually eventually cyclic flow in Stone spaces. Then* $\mathrm{Loop}(X)$ *is the largest dually cyclic closed subflow of X and R (defined above) retracts X onto* $\mathrm{Loop}(X)$*. Moreover, R is the unique continuous flow homomorphism which is a retraction onto* $\mathrm{Loop}(X)$

Also, if $\zeta \in \widehat{\mathbf{Z}}$, the loop of $\widehat{\mathbf{Z}}_{\mathbf{mon}}$, then $t^{\zeta}(x) \in \mathrm{Loop}(X)$ for all $x \in X$. It follows that if $x \in \mathrm{Stem}(X)$ then the boundary of the trajectory of x lies in $\mathrm{Loop}(X)$. In general, if $h : X \to Y$ is a continuous flow homomorphism, where X and Y are both dually eventually cyclic, then h maps $\mathrm{Loop}(X)$ to $\mathrm{Loop}(Y)$.

PROOF. Factor $R : X \to X$ as $R = (I)R_0$ where $I : \mathrm{Loop}(X) \subseteq X$ is the inclusion and $R_0 : X \to \mathrm{Loop}(X)$ is defined so that $R_0(x) = R(x)$. Similarly, the map $\rho = \tau^{\tilde{0}} : B \to B$ has a mono-epi factorization as $B \to \mathrm{Loop}(B) \subseteq B$. Under the Stone duality ρ corresponds to R and, as explained in 1.2, the mono-epi factorization is preserved, so $\mathrm{Clop}(\mathrm{Loop}(X)) = \mathrm{Loop}(B)$. This proves that $\mathrm{Loop}(X)$ is dually cyclic and that $x \in \mathrm{Loop}(X)$ if and only if $x = R(x)$. By the lemma, it easily follows that $\mathrm{Loop}(X)$ is the largest dually cyclic subflow of X.

If $r : X \to \mathrm{Loop}(X)$ is another left inverse continuous flow to the inclusion $\mathrm{Loop}(X) \subseteq X$, then the analogue of the argument given in 3.4, applies here to show that $r = R$.

The final statements easily follow from the fact that $(t^{\tilde{0}})(t^{\zeta}) = t^{\zeta}$ when ζ is in $\widehat{\mathbf{Z}}$, the loop of $\widehat{\mathbf{Z}}_{\mathbf{mon}}$ and the fact that h preserves $t^{\tilde{0}}$. ∎

3.9. REMARK. *The argument in the above proof that R is the unique retraction onto* $\mathrm{Loop}(X)$ *is essentially the same as the argument for the analogous statement about* $\mathrm{Loop}(B)$*, but the statements, both of which show that certain types of left inverses are unique, are not duals. (The dual of a statement which says that a left inverse is unique would be a statement that a right inverse is unique.) We leave it to the reader to formulate the duals of these statements.*

3.10. EXAMPLE. *The subsets* $\mathrm{Stem}(\mathbf{N}_0(s,\ell))$ *and* $\mathrm{Stem}(\widehat{\mathbf{Z}}_{\mathbf{mon}})$ *and the subflows* $\mathrm{Loop}(\mathbf{N}_0(s,\ell))$ *and* $\mathrm{Loop}(\widehat{\mathbf{Z}}_{\mathbf{mon}})$ *are what were previously defined as the stems and loops of these flows.*

3.11. REMARK. In general, when (B,τ) is not necessarily eventually cyclic, then $\mathrm{Loop}(B)$ is a cyclic subflow of B, so, when $B = \mathrm{Clop}(X)$, we could define $\mathrm{Loop}(X)$ as a cyclic **quotient** of X (and the largest such quotient). It is only when B is eventually cyclic that we can assert that $\mathrm{Loop}(X)$ would also be a closed cyclic subflow of X (and the largest such subflow). Example 6.1 shows that X may have a largest closed cyclic subflow which fails to be a retract, so X cannot be dually eventually cyclic. Example 6.2 shows that X may fail to have a largest closed cyclic subflow, which again shows that X cannot be dually eventually cyclic.

We end this section with the following potentially useful proposition.

3.12. PROPOSITION. *Let (X,t) be a flow in Stone spaces and let* $(B,\tau) = \mathrm{Clop}(X,t)$*. Suppose that $\mathcal{S}$ is a family of clopen subsets which forms a subbase for the topology on X such that for each $b \in \mathcal{S}$, there exists (s,ℓ) with $\tau^s(b) = \tau^{s+\ell}(b)$. Then (B,τ) is eventually cyclic.*

PROOF. Let $\mathcal{F}$ be the set of all $b \in B$ for which there exists (s,ℓ) with $\tau^s(b) = \tau^{s+\ell}(b)$. Suppose $b,c \in \mathcal{F}$ are given. Choose (s_0,ℓ_0) and (s_1,ℓ_1) such that $\tau_0^s(b) = \tau^{s_0+\ell_0}(b)$ and $\tau_1^s(c) = \tau^{s_1+\ell_1}(c)$. Let $(s,\ell) = (s_0,\ell_0) \sqcap (s_1,\ell_1)$. Then it follows from lemma 2.28 that $\tau^s(b) = \tau^{s+\ell}(b)$ and $\tau^s(c) = \tau^{s+\ell}(c)$. Since τ is as Boolean homomorphism, it follows that $\tau^s(b \wedge c) = \tau^{s+\ell}(b \wedge c)$ and $\tau^s(b \vee c) = \tau^{s+\ell}(b \vee c)$. From this it follows that $\mathcal{F}$ is closed under the formation of finite intersections and unions. Since $\mathcal{S} \subseteq \mathcal{F}$, we see that $\mathcal{F}$ is a base for the topology on X. Since each $b \in B$ is open in X it is a union of basic opens. Since b is compact, it is a finite union of basic opens and so is in $\mathcal{F}$. ∎

4. Parameterized Flows

There are many examples of flows in compact Hausdorff spaces that depend on parameters. The main idea of this section is that such a parameterized flow corresponds to a sheaf of Boolean flows. The precise statement of this idea is given in Propositions 4.2, 4.3 and 4.5 and in Remark 4.7 which discusses extensions of these propositions.

4.1. Definition. *Let X be a compact Hausdorff space and P be a topological space of "parameters". By a* **parameterized flow** *on X with parameters in P, we mean a continuous function $t : P \times X \to X$. For each $p \in P$ we define $t_p : X \to X$ by $t_p(x) = t(p, x)$ and the parameterized flow t can be written as $\{(X, t_p)|p \in P\}$. We also let $\pi_P : P \times X \to P$ and $\pi_X : P \times X \to X$ denote the projection maps.*

4.2. Proposition. *Suppose $t : P \times X \to X$ is a parameterized flow on X, as above. Assume that X is a Stone space. Let $B = \mathrm{Clop}(X)$ and, for each $p \in P$, let $\tau_p : B \to B$ be defined as $(t_p)^{-1}$. Consider the sheaf of Boolean algebras over P which is constantly equal to B (so it is represented by the local homeomorphism $B \times P \to P$). Then this sheaf of Boolean algebras can be extended to a sheaf of Boolean flows whose stalk over $p \in P$ is the flow (B, τ_p).*

Proof. Let b and c be clopen subsets of X such that $\tau_p(b) = c$ for some $p \in P$. To prove the proposition, we must show there exists a neighborhood U of p such that $\tau_u(b) = c$ for all $u \in U$. Note that $\tau_u(b) = c$ if and only if $t_u(c) \subseteq b$ and $t_u(\neg c) \subseteq \neg b$ (where $\neg$ denotes the complement).

We first show that there exists a neighborhood U_1 of p such that $t_u(c) \subseteq b$ for all $u \in U_1$. Let:

$$A = \{(u, x) \in P \times X | t(u, x) \notin b \text{ and } x \in c\}$$

Then A is a closed subset of $P \times X$ as $A = t^{-1}(\neg b)) \cap {\pi_X}^{-1}(c)$. Since X is compact, the map $\pi_p : P \times X \to P$ is a closed mapping, so $\pi_P(A)$ is a closed subset of P. Clearly $p \notin \pi_P(A)$, as $c = {t_p}^{-1}(b)$. Let U_1 be any neighborhood of p which misses $\pi_p(A)$.

A strictly similar proof shows that there exists a neighborhood U_2 of p such that $t_u(\neg c) \subseteq \neg b$ for all $u \in U_2$. Now let $U = U_1 \cap U_2$. ∎

4.3. Proposition. *Conversely, if P is a topological space and if (B, τ) is a sheaf of Boolean flows over P where B (ignoring τ) is a constant sheaf, constantly equal to* $\mathrm{Clop}(X)$ *for some Stone space X, then there exists a parameterized flow $t : P \times X \to X$ such that the stalk of (B, τ) is (B, τ_p) where $\tau_p = {t_p}^{-1}$ for each $p \in P$.*

Proof. Let τ_p be the action of τ on the stalk $B_p = B$ over $p \in P$. Then, by duality, $\tau_p = {t_p}^{-1}$ for a continuous $t_p : X \to X$. Define $t : P \times X \to X$ by $t(p, x) = t_p(x)$. It remains to show that t is continuous. Suppose $(p, x) \in P \times X$ and $t(p, x) = y$. Let b be any clopen neighborhood of y. It suffices to find a neighborhood of (p, x) which t maps into b. Let $c = {t_p}^{-1}(b) = \tau_p(b)$. By the sheaf condition, there exists a neighborhood U of p such that $c = \tau_u(b)$ for all $u \in U$. Then $U \times c$ is the desired neighborhood of (p, x). ∎

4.4. Review of symbolic dynamics. Recall, as discussed in [Kennison, 2006], that if $t : X \to X$ is a flow in compact Hausdorff spaces and if $A_0, A_1, \ldots A_n$ is a finite collection of closed subsets of X, with $X = \bigcup A_i$, we can then use symbolic dynamics to approximate the flow (X, t) by a flow on a Stone space $\widehat{X}$. To do this, we let $\Sigma = \{0, 1, \ldots, n\}$ be the set of all "symbols" and say that $x \in X$ is associated with σ in $\Sigma^{\mathbf{N}_0}$ if $t^n(x) \in A_{\sigma(n)}$ for all $n \in \mathbf{N}_0$. Then $\widehat{X}$ is the subspace of all $\sigma \in \Sigma^{\mathbf{N}_0}$ which are associated with at least one $x \in X$. We still use t to denote the iterator on $\widehat{X}$ where it is defined by:

$$t(\sigma(0), \sigma(1), \ldots, \sigma(n), \ldots) = (\sigma(1), \ldots, \sigma(n), \ldots)$$

4.5. Proposition. *Let $t : P \times X \to X$ be a parameterized flow on X (where X is compact, Hausdorff but not necessarily totally disconnected). Let $A_0, A_1, \ldots A_n$ be a finite collection of closed subsets of X, with $X = \bigcup A_i$ and let $\Sigma = \{0, 1, \ldots, n\}$ be as above. For each $p \in P$, let $\widehat{X}_p$ denote the subspace of $\Sigma^{\mathbf{N}_0}$ constructed from the flow (X, t_p) as above. Let $B_p = \mathrm{Clop}(\widehat{X}_p)$ and let $\tau_p = {t_p}^{-1}$. Then there exists a sheaf of Boolean flows over P, whose stalk over p is (B_p, τ_p) (as constructed in the proof below).*

Proof. Since $\widehat{X}_p$ is a subflow of $\Sigma^{\mathbf{N}_0}$, it follows by duality that B_p is a quotient flow of $B = \mathrm{Clop}(\Sigma^{\mathbf{N}_0})$. For each $p \in P$, define I_p as the ideal of all $b \in B$ for which $b \cap \widehat{X}_p = \emptyset$. (This is the ideal for which $B_p = B/I_p$.) We claim that if $b \in I_p$ then there exists U, a neighborhood of p such that $b \in I_u$ for all $u \in U$ (and the proof readily follows from this claim). We need the following lemma to continue our proof:

4.6. Lemma. *Let $t : P \times X \to X$ be a parameterized flow on the compact Hausdorff space X and let $n \in \mathbf{N}_0$ be given. Then the map $f_n : P \times X \to X$ defined by $f_n(p, x) = (t_p)^n(x)$ is continuous.*

Proof of the Lemma. Define $\theta : P \times X \to P \times X$ by $\theta(p, x) = (p, t(p, x))$. Then θ is clearly continuous and observe that $f_n = \pi_X \theta^n$. ∎

Proof of the Proposition, concluded. We assume that $b \in I_p$ and first consider the case where b is a basic clopen subset of $\Sigma^{\mathbf{N}_0}$, meaning that there exist distinct integers, $n_1, \ldots, n_k \in \mathbf{N}_0$ and "symbols" $s_1, \ldots, s_k \in \Sigma$ such that:

$$b = {\pi_{n_1}}^{-1}(s_1) \cap {\pi_{n_2}}^{-1}(s_2) \cap \ldots \cap {\pi_{n_k}}^{-1}(s_k)$$

Define $g : P \times X \to X^k$ by:

$$g(p, x) = ({t_p}^{n_1}(x), {t_p}^{n_2}(x), \ldots, {t_p}^{n_k}(x))$$

Then g is continuous, by the above lemma. Let $A = g^{-1}(A_{s_1} \times \ldots \times A_{s_k})$. It is clear that $p \notin \pi_P(A)$ for if $(p, x) \in A$ then x is associated, via t_p, with a member of b contradicting the assumption that $b \in I_p$. But since X is compact, it follows that π_P is a closed mapping, so $\pi_P(A)$ is a closed subset of P so there exists a neighborhood U of p with $U \cap \pi_P(A) = \emptyset$. Then U is the desired neighborhood.

In the general case, the clopen set b is a union of basic clopen subsets. Since b is compact, it is a finite union of such clopens, say $b = b_1 \cup b_2 \cup \ldots \cup b_n$. Since we are still assuming that $b \in I_p$ it follows that each $b_i \in I_p$ so there exists U_i, a neighborhood of p with the above property. Then $U = U_1 \cap U_2 \cap \ldots \cap U_n$ has the desired property. ∎

4.7. Remark. *The constructions in Propositions 4.2, 4.3 and 4.5 still apply even when P is a locale, even though the resulting sheaves of Boolean flows need not be determined by the behavior on the stalks.*

We sketch the proof of this remark. We note that a flow on X parameterized by a locale P is given by a locale map $t : P \times X \to X$ where $P \times X$ is the product of the locale P with $\mathcal{O}(X)$, see [Johnstone, 1982]. An element of the product locale $P \times X$ (or coproduct frame) can be defined as a family $\mathcal{J}$ of pairs (U, c) where $U \in P$ and $c \in \mathrm{Clop}(X)$ in the same way that, if P is spatial, an open subset W of $P \times X$ can be determined by the family of all such (U, c) with $U \times c \subseteq W$. The conditions on the family $\mathcal{J}$ are that:

1. $\mathcal{J}$ is downward closed ($(U, c) \in \mathcal{J}, V \subseteq U$ and $d \subseteq c$ imply $(V, d) \in \mathcal{J}$).
2. If $(U_\alpha, c) \in \mathcal{J}$ for all α, then $(\bigvee U_\alpha, c) \in \mathcal{J}$
3. If $c = \bigvee c_\alpha$ is a **finite** sup for which $(U, c_\alpha) \in \mathcal{J}$ for all α, then $(U, c) \in \mathcal{J}$.

A family $\mathcal{J}$ satisfying the above conditions will be called a **coverage ideal** as this is an example of a C-ideal, as defined in [Johnstone, 1982, II.2.11] and reviewed briefly in the next section of this paper. We note that condition (3) can be restated as requiring that for all $U \in P$, we have $(U, \emptyset) \in \mathcal{J}$ and if $(U, c) \in \mathcal{J}$ and $(U, d) \in \mathcal{J}$ then $(U, c \vee d) \in \mathcal{J}$. The family of all coverage ideals, ordered by inclusion, forms a frame, which is $P \times X$.

We observe that if $\mathcal{J}$ and $\mathcal{K}$ are coverage ideals, then $\mathcal{J} \vee \mathcal{K}$, the smallest coverage ideal containing $\mathcal{J} \cup \mathcal{K}$, has the useful property that $(U, a) \in \mathcal{J} \vee \mathcal{K}$ if and only if there exists an **admitting family** $\{U_\alpha, c_\alpha, d_\alpha\}$ with $(U_\alpha, c_\alpha) \in \mathcal{J}, (U_\alpha, d_\alpha) \in \mathcal{K}$ and $U = \bigvee U_\alpha$ and, $a \leq c_\alpha \vee d_\alpha$ for all α. (Note that if $\{U_\alpha, c_\alpha, d_\alpha\}$ admits (U, a) and $\{U'_\alpha, c'_\alpha, d'_\alpha\}$ admits (U, a'), then $\{U_\alpha \wedge U_\beta, c_\alpha \vee c'_\beta, d_\alpha \vee d'_\beta\}$ admits $(U, a \vee a')$.)

To generalize Proposition 4.2, suppose that $t : P \times X \to X$ is a locale map, where X is a Stone space (so t is a frame homomorphism from $\mathcal{O}(X)$ to $P \times X$). For each clopen b of X, we have elements, $t(b)$ and $t(\neg b)$, of $P \times X$. Since these elements are complimentary, we see that $(\top_P, X)$ is in the coverage ideal $t(b) \vee t(\neg b)$, and, by the above observation on $\mathcal{J} \vee \mathcal{K}$, we can write $\top_P = \bigvee U_\alpha$ where there are clopens c_α, d_α of X such that $c_\alpha \vee d_\alpha = X$ and each $(U_\alpha, c_\alpha) \in t(b)$ while each $(U_\alpha, d_\alpha) \in t(\neg b)$. It follows that $c_\alpha = \neg d_\alpha$. Let B be the constant sheaf over P obtained from the Boolean algebra $\mathrm{Clop}(X)$ in Sets and define τ so that $\tau(b) = c_\alpha$ on U_α.

(This result reduces to Proposition 4.2 when P is a spatial locale because the Stone space X is locally compact, so the product locale coincides with the locale of the product topology, see [Johnstone, 1982, page 61].)

The proof of the converse result, Proposition 4.3, goes through to the localic case in similar fashion. To define $t : P \times X \to X$ we need to define the frame homomorphism from the opens of X to $P \times X$ and we can reverse the steps in the above argument.

As for extending Proposition 4.5 to the localic case, suppose X is compact Hausdorff and that $A_0, A_1, \ldots A_n$ is a finite collection of closed subsets of X, with $X = \bigcup A_i$. Let

B be the sheaf of Boolean flows over P generated by the presheaf which is constantly $\mathrm{Clop}(\Sigma^{\mathbf{N}_0})$. We need to define the flow ideal I so that the sheaf we want to construct is B/I. To do this, we must indicate when b is in I for each clopen subset b of $\Sigma^{\mathbf{N}_0}$. Suppose that b is a basic clopen, then, as in the proof of 4.5, we can find a closed subset $A \subseteq X^k$ such that $b \in I_U$ for U an open subset of P, if and only if ${\pi_P}^{-1}(U) \subseteq g^{-1}(X^k - A)$. A straightforward argument shows that this gives us the definition of I.

5. Sheaves of Boolean Flows and their Spectra

We assume the reader is familiar with locales and sheaves as in [Johnstone, 1982] and [Kennison, 2006]. So, for us, a frame is the same thing as a locale except that a locale map from L to M is a frame homomorphism from M to L. If X is a topological space, then $\mathcal{O}(X)$ denotes its frame of open sets. (We may denote this locale by $\mathcal{O}(X)$ or by X depending on which notation seems to fit the context best.) A locale is **spatial** if it is equivalent to $\mathcal{O}(X)$ for some topological space X.

If B is a sheaf over L, then an element of $B(u)$ is sometimes called a **section** (over u). If b and c are sections over u, then $\|b = c\|$ is defined as the largest element $v \in L$ with $v \leq u$ such that $b|v = c|v$, see [Kennison, 2006]. (Note that we use $b|v$ to denote the restriction of b to v.)

If L is a locale, then $\mathrm{Sh}(L)$ denotes the category of sheaves over L. If $f : L \to M$ is a frame homomorphism, then the functor $f_* : \mathrm{Sh}(M) \to \mathrm{Sh}(L)$ is defined so that $f_*(C)(u) = C(f(u))$. The functor $f^* : \mathrm{Sh}(L) \to \mathrm{Sh}(M)$ is the left adjoint of f_*.

As in [Kennison, 2006], an object of the **category of Boolean flows over locales** is a pair (B, L) where B is a Boolean flow over L. We say $(\theta, f) : (B, L) \to (C, M)$ is a morphism if $f : L \to M$ is a **frame** homomorphism and $\theta : f^*(B) \to C$ is a Boolean flow homomorphism over M. The composition of $(\theta, f) : (B, L) \to (C, M)$ with $(\psi, g) : (C, M) \to (D, N)$ is $(\psi g^*(\theta), fg)$.

5.1. Definition. *Let (B, L) be a Boolean flow over L. Let $\mathrm{Loop}_0(B)$ be the sub-presheaf of B for which $\mathrm{Loop}_0(B)(u) = \mathrm{Loop}(B(u))$ and let $\mathrm{Loop}(B)$ be the subsheaf of B generated by $\mathrm{Loop}_0(B)$. Then (B, L) is a* **cyclic sheaf of Boolean flows** *if $\mathrm{Loop}(B) = B$.*

Similarly, let $\mathrm{ECLoop}_0(B)$ be the sub-presheaf of B for which $\mathrm{ECLoop}_0(B)(u) = \mathrm{ECLoop}(B(u))$ and let $\mathrm{ECLoop}(B)$ be the subsheaf of B generated by $\mathrm{ECLoop}_0(B)$. Then (B, L) is an **eventually cyclic sheaf of Boolean flows** *if $\mathrm{ECLoop}(B) = B$.*

5.2. Lemma. *Let (B, L) be a Boolean flow over L. Then (B, L) is cyclic if and only if, for all $b \in B(u)$:*

$$u = \bigvee\{\|b = \tau^n(b)\| \,|n \in \mathbf{N}\}.$$

Also (B, L) is eventually cyclic if and only if, for all $b \in B(u)$:

$$u = \bigvee\{\left\|\tau^s(b) = \tau^{s+\ell}(b)\right\| \,|(s, \ell) \in \mathbf{N}_0 \times \mathbf{N}\}.$$

■

There is a similar condition for B being a simple sheaf of Boolean flows, as defined in [Kennison, 2006] and this will be recalled later in this section.

5.3. Proposition. *Let (B, L) be an eventually cyclic Boolean flow over L. Then there is a unique action by $\widehat{\mathbf{Z}}_{\mathbf{mon}}$ (by sheaf morphisms which are also flow homomorphisms) which is consistent with the action of $\widehat{\mathbf{Z}}_{\mathbf{mon}}$ on $\mathrm{ECLoop}_0(B)$.*

Proof. By Theorem 2.29, the monoid $\widehat{\mathbf{Z}}_{\mathbf{mon}}$ acts on each $\mathrm{ECLoop}_0(u)$ and, using part (3) of that theorem, it readily follows that these actions patch together to get the desired action on $\mathrm{ECLoop}(B) = B$. ∎

5.4. Some Open Questions. Even if B is eventually cyclic as a sheaf of Boolean flows, it does not follow that each $B(u)$ is eventually cyclic, since $\mathrm{ECLoop}_0(B)$ need not equal $\mathrm{ECLoop}(B)$. So the action by $\widehat{\mathbf{Z}}_{\mathbf{mon}}$ need not be continuous, as then $B(u)$ would have to be eventually cyclic by Theorem 2.29. We have the following questions, which are similar to ones pursued in [Kennison, 2006] with only partial success:

- When B is an eventually cyclic sheaf of Boolean flows over L and $u \in L$ how do we classify the resulting action by $\widehat{\mathbf{Z}}_{\mathbf{mon}}$ on $B(u)$ which seems natural but is not necessarily continuous?
- Furthermore, suppose $B(u) = \mathrm{Clop}(X)$ (where X may depend on u). We can use the possibly non-continuous action T by $\widehat{\mathbf{Z}}_{\mathbf{mon}}$ to extend Definition 3.6 by letting the "pseudo-loop" of X be the image of $t^{\tilde{0}} = T(\tilde{0}, \_)$. What are the properties of this pseudo-loop?

We proceed to construct various spectra for a Boolean flow over a locale. In view of the previous section, this allows us to define spectra for a parameterized flow. As in [Kennison, 2006], our construction starts with the construction of the universal quotient flow but our approach uses locales defined by generators and relations. Constructions of this type are discussed in [Johnstone, 1982] but not in [Kennison, 2006] so we review them briefly.

5.5. Defining Locales by Generators and Relations. Suppose we want to construct the frame generated by a set of elements on which some equations, involving infs and sups, are imposed. We can readily find the meet-semilattice generated by the elements (where a meet-semilattice is a partially ordered set, with finite infs). We need to impose some equations of the form $\bigvee\{a_i\} = a$. When this happens, we say that $\{a_i\}$ forms a "covering" of a. To be more precise, we establish some notation and make a formal definition.

5.6. Notation. *Let A be a meet-semilattice. Then:*

1. For $a \in A$, we let $\downarrow(a)$ denote $\{b \in A | b \leq a\}$.
2. For $S \subseteq\downarrow(a)$ and $b \in A$, we let $S \wedge b$ denote$\{s \wedge b | s \in S\}$.

5.7. DEFINITION. *Let A be a meet-semilattice. We say that C is a* **coverage** *on A if C is a function such that $C(a)$ is, for all $a \in A$, a family of subsets of $\downarrow(a)$ with the* **meet-stability** *property that $S \in C(a)$ and $b \leq a$ imply $S \wedge b \in C(b)$. The members of $C(a)$ are called "coverings" of a.*

We say that (A, C) is a **site** *if C is a coverage on A.*

5.8. DEFINITION. *Let (A, C) be a site. A subset $\mathcal{J} \subseteq A$ is a C-***ideal** *if and only if:*

1. *$a \in \mathcal{J}$ and $b \leq a$ imply $b \in \mathcal{J}$*
2. *$S \in C(a)$ and $S \subseteq \mathcal{J}$ imply $a \in \mathcal{J}$.*

The set of all C-ideals is denoted by $C\text{-}\mathrm{Idl}(A)$ which, when partially ordered by inclusion, forms a frame, see [Johnstone, 1982, II.2.11].

We define $g : A \to C\text{-}\mathrm{Idl}(A)$ so that $g(a)$ is the smallest C-ideal containing $\downarrow(a)$.

5.9. DEFINITION. *Let (A, C) be a site. Then an (A, C)-***presheaf** *is a functor $F : A^{op} \to$ Sets. Given such a presheaf F and a covering $S \in C(a)$, we say that $\{x_s \in F(s) | s \in S\}$ is an S-***compatible family** *if for all $s_1, s_2 \in S$ the elements x_{s_1}, x_{s_2} have the same restriction to $s_1 \wedge s_2$.*

We say that F satisfies the **sheaf condition** *(resp. the* **separation condition***) with respect to $S \in C(a)$ if for every S-compatible family, $\{x_s\}$, there exists a unique (resp. at most one) $x \in F(a)$ for which $x|s = x_s$ for all $s \in S$.*

*The presheaf F is an (A, C)-***sheaf** *(resp. an (A, C)-***separated** *presheaf) if F is a presheaf which satisfies the sheaf condition (resp. the separation condition) for all $a \in A$ and all $S \in C(a)$.*

5.10. REMARK. *We will use the following properties of sites:*

1. The site (A, C) generates the frame $C\text{-}\mathrm{Idl}(A)$ where the meet-preserving map $g : A \to C\text{-}\mathrm{Idl}(A)$ converts coverings to sups and satisfies the universal property for such maps.
2. Every (A, C)-sheaf extends to a sheaf over $C\text{-}\mathrm{Idl}(A)$. An (A, C)-sheaf of Boolean flows (meaning that $F(a)$ is a Boolean flow for all $a \in A$ and the restrictions are flow homomorphisms) extends to a Boolean flow over $C\text{-}\mathrm{Idl}(A)$.
3. If F is an (A, C)-sheaf of Boolean flows, then a subsheaf I is a flow ideal of F if $I(a) \subseteq F(a)$ is a flow ideal for all $a \in A$. In this case, we let F/I be the (A, C)-sheaf generated by the presheaf which maps $a \in A$ to $F(a)/I(a)$, The universal map from this presheaf to the sheaf it generates is then a one-to-one map from $F(a)/I(a) \to (F/I)(a)$

The proofs of the assertions in this remark can be obtained from [Johnstone, 1982] and the references listed there. One approach, to (3), is to extend the site (A, C) to a Grothendieck site, as discussed in [Johnstone, 2002, Proposition 2.1.9] or to use [Borceux, 1994, Proposition 3.2.12].

5.11. Definition. *In what follows, if* (B, τ) *is a Boolean flow, then it is convenient to define*

$$\sigma : B \to B \text{ by } \sigma(b) = b \vee \tau(b).$$

5.12. Heuristics for constructing the universal quotient flow. As shown in [Kennison, 2006], the universal quotient flow for a Boolean flow (in Sets) is a sheaf over $\mathcal{W}$, the space of all flow ideals of B. A base for the topology on $\mathcal{W}$ is given by the collection $\{N(b)|b \in B\}$, where $N(b) = \{I \in \mathcal{W}|b \in I\}$. Since these basic opens generate the frame $\mathcal{O}(\mathcal{W})$, we can think of the elements of B as generating the locale $\mathcal{O}(\mathcal{W})$ where each element b corresponds to the basic open $N(b)$. But the given ordering on B is wrong because if $b \leq c$ then $N(b) \supseteq N(c)$.

So, for $b, c \in B$, we define $b \sqsubseteq c$ if and only if $b \geq c$ then the meet is given by $b \sqcap c = b \vee c$. To account for the fact that $N(b) \subseteq N(\tau(b))$ we make $\{b \sqcap \tau(b)\}$ a one-point covering of b. More generally, we require $\{c\}$ to be a one-point covering of b whenever $\sigma(b) \sqsubseteq c \sqsubseteq b$. Note that $b \sqcap \tau(b) = \sigma(b)$

We leave it as an exercise for the interested reader to verify that this defines a site and the frame it generates is isomorphic to $\mathcal{O}(\mathcal{W})$. (This result also follows by comparing the universal quotient as constructed in [Kennison, 2006] to the construction given below.)

5.13. Construction of the universal flow quotient. Let B be a Boolean flow over a locale L. We want to construct a quotient flow (B^0, L^0) through which all other quotient flows factor uniquely, as in [Kennison, 2006].

We start by describing L^0 in terms of "generators and relations".

5.14. Definition. *Let* (B, L) *be a Boolean flow over a locale. Define*

$$A = \{(u, b)|u \in L, b \in B(u)\}$$

Define a partial ordering on A *so that* $(u, b) \sqsubseteq (v, c)$ *if and only if* $u \leq v$ *and* $b \geq c|u$. *Then* A *is a meet semi-lattice with top element* $(\top, 0)$ *and infs given by* $(u, b) \sqcap (v, c) = (u \wedge v, b' \vee c')$ *where* b' *and* c' *are the restrictions of* b *and* c *to* $u \wedge v$.

We also define a coverage C *on* A *so that* $\{(u, c)\}$ *is a one-point coverage of* (u, b) *whenever* $b \leq c \leq b \vee \tau(b)$ *and* (u, b) *is covered by* $\{(u_i, b_i)\}$ *whenever* $u = \bigvee u_i$ *and* $b_i = b|u_i$. *We let* $L^0 = C$*-Idl and let* $g : A \to L^0$ *map the generators into* L^0. *We define a frame homomorphism* $h : L \to L^0$ *for which* $h(u) = [u, 0]$ *where* $[u, 0] = g(u, 0)$. *We let* $\sqsubseteq$ *denote the order relation on* L^0.

Note that the elements $[u, b]$ *generate* L^0 *in the sense that every other element is a sup of elements of the form* $[u, b]$.

5.15. Definition. *Let* (B, L) *and* (A, C) *and* L^0 *be as above. For each* $u \in L$ *and each* $b \in B(u)$, *let* $I(b)$ *be the smallest flow ideal of* $B(u)$ *which contains* b.

Let $B' : A^{op} \to \mathrm{Sets}$ *be the presheaf for which* $B'(u, b) = B(u)/I(b)$, *with the obvious restrictions. We define* B^0 *as the* (A, C)*-sheaf generated by the* (A, C)*-presheaf* B'. *We let* $\nu : B' \to B^0$ *be the universal map.*

We further define $\overline{B}$ to be the (A,C)-presheaf for which $\overline{B}(u,b) = B(u)$ with the obvious restrictions. We let $q : \overline{B} \to B'$ be the map for which $q_{(u,b)} : B(u) \to B(u)/I(b)$ is the quotient map (note that $B(u) = \overline{B}(u,b)$ and $B(u)/I(b) = B'(u,b)$).

5.16. LEMMA. *The (A,C)-presheaf $\overline{B}$ is a sheaf.* ■

5.17. LEMMA. *$c \in I(b)$ if and only if $c \le \sigma^n(b)$ for some $n \in \mathbf{N}$.*

PROOF. This is proven in [Kennison, 2006] and we repeat the simple proof here. Clearly if $c \le \sigma^n(b)$ for some $n \in \mathbf{N}$ then $c \in I(b)$. Conversely, it is readily shown that the set of all such elements c forms a flow ideal containing b. ■

5.18. LEMMA. *Let $\eta : \overline{B} \to B^0$ be defined as νq where q is defined above and ν is the universal map sending B' to its associated sheaf, B^0. Let $b, c \in B(u)$ be given. Then $\eta_{[u,b]}(c) = 0$ if and only if we can write $u = \bigvee\{u_i\}$ so that for each i there exists $n_i \in \mathbf{N}$ with $c_i \le \sigma^{n_i}(b_i)$ (where $c_i = c|u_i$ and $b_i = b|u_i$).*

PROOF. We define a subfunctor I of $\overline{B}$ so that $I(u,b)$ is the flow ideal $I(b) \subseteq B(u)$. Define another subfunctor $\tilde{I}$ so that $I \subseteq \tilde{I} \subseteq \overline{B}$ and that $c \in \tilde{I}(b)$ if and only if we can write $u = \bigvee\{u_i\}$ so that $c_i \in I(b_i)$ (where $c_i = c|u_i$ and $b_i = b|u_i$). It is readily shown that $\tilde{I}$ is an (A,C)-sheaf and is the subsheaf of $\overline{B}$ generated by I. It follows that $\overline{B}/\tilde{I}$ is an (A,C)-separated presheaf and, since $B' = \overline{B}/I$, it is readily shown that $\eta_{(u,b)}(c) = 0$ if and only if $c \in \tilde{I}(b)$. The lemma follows from Remark 5.10(3). ■

5.19. COROLLARY. *The natural map $B(u) \to B'[u,0] \to B^0[u,0]$ is one-to-one as $\tilde{I}(0) = \{0\}$.* ■

5.20. LEMMA. *Given $B \in \mathrm{Sh}(M)$, let $f^0(B)$ be the presheaf for which $f^0(B)(v)$ is the set of pairs (x,u) for which $x \in B(u)$ and $f(u) \ge v$ with the understanding that $(x_1,u_1) = (x_2,u_2)$ whenever there exists $w \le u_1 \wedge u_2$ with $f(w) \ge v$ and $x_1|w = x_2|w$. Then $f^*(B)$ is the sheafification of $f^0(B)$.*

PROOF. We sketch the proof of this result. Recall that if $C \in \mathrm{Sh}(M)$, then $f_*(C) \in \mathrm{Sh}(L)$ is defined so that $f_*(C)(u) = C(f(U))$ and $f^* : \mathrm{Sh}(L) \to \mathrm{Sh}(M)$ is defined as the left adjoint of f_*. So we must show there is a natural bijective correspondence between natural transformations $\lambda : f^*(B) \to C$ and $\hat{\lambda} : B \to f_*(C)$.

Let $\lambda : f^*(B) \to C$ be given. to define $\hat{\lambda}$, we must define $\hat{\lambda}_u(b)$ whenever $b \in B(u)$. But then (b,u) represents an element of $f^0(B)(u)$ hence of $f^*(B)(u)$ and we define $\hat{\lambda}_u(b) = \lambda_{f(u)}(b,u)$. In the other direction, given $\hat{\lambda} : B \to f_*(C)$, we define $\lambda : f^*(B) \to C$ so that $\lambda_v(b,u) = \hat{\lambda}_u(b)|v$. The remaining details are a bit tedious but straightforward. ■

5.21. NOTATION. *Let $f : L \to M$ be a frame homomorphism. Then by the adjointness between f^* and f_*, for every natural transformation $\lambda : f^*(B) \to C$ there is a* **transpose** *$\hat{\lambda} : B \to f_*(C)$. We use this "hat" notation to denote the transpose.*

5.22. LEMMA. *The (A, C)-sheaf B^0 extends to a sheaf (also denoted by B^0) on L^0.*

The (A, C)-sheaf $\overline{B}$ extends to a sheaf (also denoted by $\overline{B}$) on L^0.

The (A, C)-natural transformation $\eta : \overline{B} \to B^0$ extends to a natural transformation (also denoted by η) between the extensions of these sheaves to L^0.

Recall that $h : L \to L^0$ is a frame homomorphism. Then $h^*(B) = \overline{B}$ so $\eta : h^*(B) \to B^0$ is a sheaf morphism.

PROOF. The only difficult part is showing that $h^*(B) = \overline{B}$. By the above lemma, let (x, v) represent an element of $h^*(B)[u, b]$ with $x \in B(v)$ and $h(v) \geq [u, b]$. Recall that (x_1, v_1) is identified with (x_2, v_2) if there exists $w \leq v_1 \wedge v_2$ with $h(w) \geq [u, b]$ and x_1, x_2 having equal restrictions to w. So each (x, v) is equivalent to its restriction $(x|u, u)$ and it readily follows that $h^*(B)(u, b)$ corresponds to $B(u) = \overline{B}[u, b]$. The extension to other elements of L^0 is straightforward as they are all sups of elements of the form $[u, b]$. ∎

5.23. THEOREM. *The morphism $(\eta, h) : (B, L)$ to (B^0, L^0) is the universal quotient flow of (B, L).*

PROOF. First, it follows from the above lemma that (η, h) is a morphism of flows over locales. We need to show that it is a localic quotient map and that every other such quotient map from (B, L) factors in the appropriate way.

By the construction of $h^*(B)$, we see, for all $(u, b) \in L^0$, that $h^*(B)(u, b)$ contains the image of $\eta_u : B(u) \to B^0(u, 0)$ which then restricts to an element in $B^0(u, b)$ so it contains $B(u)/I(b)$ which is $B'(u, b)$. But the elements in $B'(u, b)$ generate the sheaf B^0, so $\eta : h^*(B) \to B^0$ is an epimorphism.

Suppose $(\lambda, m) : (B, L) \to (F, M)$ is another quotient flow of (B, L). So F is a Boolean flow over the locale M and $m : L \to M$ is a **frame** homomorphism and $\lambda : m^*(B) \to F$ is a sheaf epimorphism. We have to show there is a unique map $(\overline{\lambda}, \overline{m}) : (B^0, L^0) \to (F, M)$ with $\overline{\lambda} : \overline{m}^*(B^0) \to (F, M)$ an isomorphism such that the following diagram commutes:

$$\begin{tikzcd}
(B, L) \arrow[rr, "{(\eta,h)}"] \arrow[dr, "{(\lambda,m)}"'] & & (B^0, L^0) \arrow[dl, dashed, "{(\overline{\lambda},\overline{m})}"] \\
& (F, M) &
\end{tikzcd}$$

We need some lemmas.

5.24. LEMMA. *Given $b \in B(u)$ we let $\overline{b}$ denote the corresponding element of $B^0[u, 0]$. Assume that $\alpha \sqsubseteq [u, 0]$ in L^0. Then $\overline{b}|\alpha = 0$ if and only if $\alpha \sqsubseteq [u, b]$.*

PROOF. Clearly $\overline{b}|[u, b] = 0$ so one direction is clear. Conversely, assume that $\overline{b}|\alpha = 0$. We first consider the case where $\alpha = [u, c]$ for some $c \in B(u)$. Since $b|[u, c] = 0$ it follows that b must be in $\tilde{I}(c)$ as the map $B'(u)/\tilde{I}(c) \to B^0[u, c]$ is one-to-one. But $b \in \tilde{I}(c)$ means we can write $u = \bigvee\{u_i\}$ with $b_i \leq \sigma^{n_i}(c_i)$ for some $n_i \in \mathbf{N}$ where $b_i = b|u_i$ and $c_i = c|u_i$.

Now since $b_i \le \sigma^{n_i}(c_i)$ then $[u_i, \sigma^{n_i}(c_i)] \sqsubseteq [u_i, b_i]$. But $[u_i, c_i] = [u_i, \sigma^{n_i}(c_i)] \sqsubseteq [u_i, b_i]$ and, by taking sups, we get $[u, c] \sqsubseteq [u, b]$ or $\alpha \sqsubseteq [u, b]$.

Next suppose $\alpha = [v, c]$. Since $\alpha \le [u, 0]$ we have $v \le u$ so the above argument applies by restricting everything to v.

In general, α is a sup of elements of the form $[v, c]$ and we can apply the argument to each $[v, c]$. ∎

5.25. Lemma. *Let (B, L) and $(\eta, h) : (B, L) \to (B^0, L^0)$ and $(\lambda, m) : (B, L) \to (F, M)$ be as in the above diagram. Let $b \in B(u)$ be given, and, as in the above lemma, let $\overline{b}$ denote the corresponding element of $B^0[u, 0]$. If $(\overline{\lambda}, \overline{m})$ exists as indicated in the above diagram, then $\overline{m}[u, b] = \|\overline{b} = 0\|$.*

Proof. Assume that $\overline{\lambda}$ and $\overline{m}$ exist. Let $v = \overline{m}[u, b]$ and $w = \|\overline{b} = 0\|$. Consider $\overline{\lambda}_v : \overline{m}^*(B^0(v)) \to F(v)$. By Lemma 5.20, the elements of $\overline{m}^*(B^0(v))$ are represented by pairs (x, α) with $\alpha \in L^0$, $\overline{m}(\alpha) \ge v$, $x \in B^0(\alpha)$. Also (x_1, α_1) is identified with (x_2, α_2) if there exists $\beta \sqsubseteq \alpha_1 \wedge \alpha_2$ with $h(\beta) \sqsupseteq [u, b]$ and x_1, x_2 having equal restrictions to β.

Now $\overline{b} \in B^0[u, 0]$ restricts to 0 in $B^0[u, b]$ and $\overline{m}(u, b) \ge v$ (in fact $\overline{m}(u, b) = v$) so $\overline{b}$ restricts to 0 in $F(v)$ so $v \le \|\overline{b} = 0\| = w$.

On the other hand, by definition of w, we see that the image of $\overline{b}$ when restricted to $\overline{m}^*(B^0)$ then mapped by λ must be 0. Since λ_w is one-to-one, we see that $\overline{b}$ restricts to 0 in $\overline{m}^*(B^0)$. So there exists $\alpha \in L^0$ such that $\overline{b}|\alpha = 0$ and $\overline{m}(\alpha) \ge w$. But $\alpha \sqsubseteq [u, b]$ (by Lemma 5.24) so $\overline{m}[u, b] \ge \overline{m}(\alpha) \ge w$. But $v = \overline{m}[u, b]$ so $v \ge w$. ∎

Proof of the theorem. Let $(\eta, h) : (B, L) \to (B^0, L^0)$ and $(\lambda, m) : (B, L) \to (F, M)$ be as in the above diagram, with λ a sheaf epimorphism over M. We must show there exists a unique $(\overline{\lambda}, \overline{m}) : (B^0, L^0) \to (F, M)$ with $\overline{\lambda} : \overline{m}^*(B^0) \to (F, M)$ an isomorphism such that $(\overline{\lambda}, \overline{m})(\eta, h) = (\lambda, m)$.

We first show the uniqueness. the above lemma shows that $\overline{m}$ is uniquely determined by the condition that $\overline{m}[u, b] = \|\overline{b} = 0\|$ as the elements $[u, b]$ generate L^0.

We next note that $\overline{\lambda}$ is determined by $\hat{\overline{\lambda}} : B^0 \to m_*(F)$. It suffices to show that $\hat{\overline{\lambda}}_{[u,b]}$ is determined as the lower arrow of the following commutative diagram:

$$\begin{tikzcd}
B^0[u, 0] \arrow[r, "\hat{\overline{\lambda}}_{[u,0]}"] \arrow[d] & F(m(u)) \arrow[d] \\
B(u)/\tilde{I}(u) \arrow[r, "\hat{\overline{\lambda}}_{[u,b]}"'] & F(\|\overline{b} = 0\|)
\end{tikzcd}$$

Note that the upper arrow, $\hat{\overline{\lambda}}_{[u,0]}$, is determined by the given map λ. The vertical arrows are restrictions and the one on the left is onto, so the lower arrow is uniquely determined. Since $\hat{\overline{\lambda}}_{[u,b]}$ is determined, we see that $\hat{\overline{\lambda}}_\alpha$ is determined for all $\alpha \in L^0$ as every such α is a sup of elements of the form $[u,b]$. Therefore $\hat{\hat{\lambda}}$ is determined and it determines $\overline{\lambda}$.

As for the existence of $(\overline{\lambda}, \overline{m})$, we define them in the only possible way and show that they have the desired properties. So we can define $\hat{\hat{\lambda}}$ using the above diagram, noting that $\tilde{I}$ will get mapped to 0 by the restriction to $F(\|\overline{b}=0\|)$. From $\hat{\hat{\lambda}}$ we can define $\overline{\lambda}$. To show that $\overline{\lambda}$ is an isomorphism, it suffices to show that it is one-to-one (it has to be onto as λ is). So suppose $v \in M$ and consider $\overline{\lambda}_v : \overline{m}^* B^0(v) \to F(v)$. Let $\overline{\lambda}_v(x) = 0$. We have to show that $x = 0$. By Lemma 5.20, assume x is represented by (α, y) with $\alpha \in L^0$ and $y \in B^0(\alpha)$ where $\overline{m}(\alpha) \geq v$. By writing $\alpha = \bigvee [u_i, v_i]$, we can reduce to the case where $\alpha = [u,b]$ for $u \in L$ and $b \in B(u)$. Since y is obtained by patching elements of $B(v)/\tilde{I}(b)$ we may reduce to the case where x is represented by $([u,b],c)$ for $c \in B/\tilde{I}(b)$ and $\overline{m}[u,b] \geq v$ and $\overline{\lambda}(c) = 0$. But if $v \leq \overline{m}[u,b]$, then $v \leq \overline{m}[u, b \vee c] = [u,b] \wedge [u,c]$. So x is represented by $([u, b \vee c], c)$ and $c \simeq 0$ modulo $\tilde{I}(b \vee c)$ so $x = 0$ in $\overline{m}^*(B)(v)$.

The other details are straightforward. ∎

5.26. Construction of the cyclic spectrum of (B,L). Let (B,L) be a Boolean flow over a locale and let (B^0, L^0) be its universal quotient flow. Then as shown in [Kennison, 2006, Proposition 3.23], there is a largest sublocale of L^0 for which the restriction of B^0 is cyclic and this sublocale, together with the restriction of B^0, is the cyclic spectrum of (B,L). An examination of this sublocale readily shows it is generated by forcing $\bigvee\{\|b = \tau^n(b)\| \,|n \in \mathbf{N}\}$ to be u for all $b \in B(u)$. This could be done by adding suitable coverings to the site (A,C) used above.

5.27. Construction of the eventually cyclic spectrum of (B,L). The analysis used above extends in a completely straightforward way to give us the eventually cyclic spectrum. Proposition 3.23 of [Kennison, 2006] readily extends to give us the largest sublocale for which the restriction of B^0 is eventually cyclic. Again, we could simply add new covers to the site (A,C) used above by requiring, for all $b \in B(u)$, that $\{\|\tau^s(b) = \tau^{s+\ell(b)}\|\}$ be a covering of $(u,0)$ (and enough other coverings to give us meet-stability, as in 5.7). This gives us the spectrum directly, without first constructing the universal quotient flow.

5.28. Making the spectrum non-trivial. A Boolean flow over a locale is **non-trivial** if $\|0=1\| = \perp$. We can force any flow to be non-trivial by requiring that the empty family cover $\|0=1\|$. We generally recommend that this be done to avoid giving the spectrum superfluous elements. (Doing this gives us the "non-trivial cyclic spectrum" or the "non-trivial eventually cyclic spectrum".)

5.29. Construction of the simple spectrum of (B,L). Simple Boolean flows are defined in [Kennison, 2006]. They must be non-trivial and satisfy the condition that for all $b \in B(u)$ we have:

$$u = \|b=0\| \vee [\bigvee_k \|\sigma^k(b) = 1\|]$$

(We note that "k - Exp(b)" is used in [Kennison, 2006] for $\sigma^k(b)$.)

We can construct the simple spectrum of (B, L) by restricting the universal quotient flow to the largest sublocale for which it is non-trivial and satisfies the above condition. Again, we could alternatively do this by adding some coverings to the site (A, C).

We generally find the description of these spectra in terms of generators and relations to be both theoretically and computationally useful. Sometimes it is interesting to know that the resulting locale is spatial, as in the following proposition.

5.30. PROPOSITION. *Let (B, L) be a Boolean flow over a locale and let (B^0, L^0) be its universal quotient. Then L^0 is spatial if L is.*

PROOF. Assume $L = \mathcal{O}(P)$ where P is a topological space. Let B_p be the stalk of B over $p \in P$. Let $\mathcal{W}_p$ be the set of all flow ideals of B_p. If $b \in B(u)$, for u an open neighborhood of $p \in P$, let b_p denote the element of the stalk B_p represented by b. We define a new space Q where:

$$Q = \{(p, I) | p \in P, I \in \mathcal{W}_p\}$$

If u is open in P and $b \in B(u)$ we let:

$$N(u, b) = \{(q, J) | q \in u, b_q \in J\}$$

A straightforward verification shows that the family $\{N(u, b)\}$ is a base for a topology on Q. To show that $\mathcal{O}(Q)$ is isomorphic to L^0, we use the fact, given in [Johnstone, 1982], that L^0, as a frame, is the set of all C-ideals of A. If $U \in \mathcal{O}(Q)$ then we define:

$$\Phi(U) = \{(u, b) | N(u, b) \subseteq U\}$$

and if $\mathcal{J}$ is a C-ideal of A, we define:

$$\Theta(\mathcal{J}) = \bigcup \{N(u, b) | (u, b) \in \mathcal{J}\}$$

It is readily shown that Φ and Θ are well-defined frame homomorphisms and inverses of each other. The only difficulty is in showing that $\Phi(\Theta(\mathcal{J})) = \mathcal{J}$. To prove this, assume:

$$N(v, c) \subseteq \bigcup \{N(u, b) | (u, b) \in \mathcal{J}\}$$

We must then show that $(v, c) \in \mathcal{J}$. Let $p \in v$ be arbitrary and let I be the smallest ideal of B_p which contains c_p. Then $(p, I) \in N(v, c)$ so there exists $(u, b) \in \mathcal{J}$ such that $(p, c_p) \in N(u, b)$. We may as well assume that $u \leq v$ (otherwise replace u by $u \wedge v$). So $p \in u$ and $b_p \in I$. But, by Lemma 5.17, this implies that $b_p \leq \sigma^n(c_p)$ (for some n). We also may as well assume that $u = \|b \leq \sigma^n(c)\| = \|b \wedge \sigma^n(c) = b\|$ (otherwise replace u by $\|b \wedge \sigma^n(c) = b\|$). Then from the covering property of $\mathcal{J}$ we readily see that $(u, c|u) \in \mathcal{J}$ and the set of all such $(u, c|u)$ covers (v, c). ∎

6. Examples and Observations

6.1. EXAMPLE. *Let* $X = \mathbf{Z} \cup \{-\infty, \infty\}$ *with* $t(x) = x+1$, $t(-\infty) = -\infty$ *and* $t(\infty) = \infty$). *Let* $B = \mathrm{Clop}(X)$.

1. X has a largest closed cyclic subflow, namely $\{-\infty, \infty\}$, but there is no continuous flow homomorphism retracting X onto $\{-\infty, \infty\}$. So X cannot be dually eventually cyclic.

2. The universal quotient of B is a sheaf over the space $\mathcal{W}$ of all flow ideals of B, which is, by Stone duality, equivalent to the set of all closed subflows of X. To get a non-trivial spectrum, we eliminate the empty subflow (corresponding to B as an ideal of itself) and, letting $[n, \infty] = \{i | n \leq i \leq \infty\}$, we can list the closed, non-empty subflows of X as:

$$\{\infty\}, \quad \{-\infty\}, \quad \{-\infty, \infty\}, \quad [n, \infty], \quad \{-\infty\} \cup [n, \infty], \quad X$$

   (where n is allowed to vary in $\mathbf{Z}$.) All of these subflows are readily seen to be dually eventually cyclic except for X itself. The eventually cyclic non-trivial spectrum is $(B_j, \mathcal{O}(X)_j)$ where j is a nucleus representing the largest sublocale of $\mathcal{O}(\mathcal{W})$ for which the restriction of B is eventually cyclic. But this sublocale cannot contain the point corresponding to X itself and the largest sublocale not containing this point is equivalent to the space $\mathcal{W} - \{\{0\}\}$ as the zero ideal corresponds to the subflow X itself. A direct verification shows that the restriction to this subspace is eventually cyclic.

3. The non-trivial, cyclic spectrum of the above flow is a sheaf over a locale on the space with three points, corresponding to $\{\infty\}, \{-\infty\}, \{-\infty, \infty\}$. We leave it to the reader to verify this and compute the non-discrete topology on this space.

6.2. EXAMPLE. *Let* S *be the disjoint union of the flows* $\{Z_p\}$ *where* p *varies in the set of all primes. Let* $t : S \to S$ *be defined as* $t(x) = x + 1$ *on each* Z_p. *Let* X *be the coproduct, in Stone spaces, of these flows, then* X *is the Stone-Čech compactification of* S. *The extension of* t *to* X *is then* βt, *but, by abuse of language, we will denote* βt *by* t.

1. As discussed in [Kennison, 2002, p.395,407], X is not dually cyclic.

2. X does not have a largest closed cyclic subflow as such a subflow would have to contain each copy of Z_p but the union of all of these is dense in X, so X is not dually eventually cyclic.

3. B can be represented as the global sections of a cyclic Boolean flow over a locale, in fact over a discrete topological space which has a point p for each prime number and has stalk equal to $\mathrm{Clop}(Z_p)$ at p. So $\widehat{\mathbf{Z}}$ acts on B, and also acts on X. But these actions are not continuous.

6.3. EXAMPLE. *(Concrete representations of flows on $\beta\mathbf{N}_0$, $\widehat{\mathbf{Z}}_{\mathbf{mon}}$ and $\widehat{\mathbf{Z}}$.)*

1. *Let $t : \mathbf{N}_0 \to \mathbf{N}_0$ be defined by $t(x) = x + 1$. Let B be the Boolean algebra of all subsets of $\mathbf{N}_0$ and let $\tau : B \to B$ be t^{-1}. Then (B, τ) is a Boolean flow. The corresponding flow in Stone Space is $(\beta\mathbf{N}_0, \beta t)$ where β denotes the Stone-Čech compactification. This is the free flow in Stone spaces generated by a single point. (cf. Example 6.7)*

2. *Let $B_{ec} = \mathrm{ECLoop}(B)$, where B is as given above. Then $b \in B_{ec}$ means that there exists $(s, \ell) \in \mathbf{N}_0 \times \mathbf{N}$ such that for $n \geq s$ we have $n \in B_{ec}$ if and only if $n+\ell \in B_{ec}$. Then $\mathrm{Clop}(\widehat{\mathbf{Z}}_{\mathbf{mon}}) \simeq (B_{ec}, \tau)$. Moreover $\widehat{\mathbf{Z}}_{\mathbf{mon}}$ is equivalent to the quotient of the Stone-Čech compactification of $\mathbf{N}_0$ where two ultrafilters on $\mathbf{N}_0$ are identified when they contain the same elements of B_{ec}.*

   In other words, the points of $\widehat{\mathbf{Z}}_{\mathbf{mon}}$ correspond to maximal filters of sets in B_{ec}. Given $n \in \mathbf{N}_0$, there are two associated maximal filters. One is the principal filter generated by $\{n\}$ (so b is in this filter if and only if $n \in b$). The other is generated by sets of the form:

   $$b(n, s, \ell) = \{m \in \mathbf{N}_0 | m \geq s, \text{ and } m \simeq n (\mathrm{mod}\ \ell)$$

   This maximal filter corresponds to $\tilde{n}$ in $\widetilde{\mathbf{N}_0}$, the **shadow** *of $\mathbf{N}_0 \subseteq \widehat{\mathbf{Z}}_{\mathbf{mon}}$.*

3. *Let $B_{cyc} = \mathrm{Loop}(B)$. Then $b \in B_{cyc}$ means that there exists ℓ such that $x \in b$ if and only if $x + \ell \in b$. In this case, $\mathrm{Clop}(B_{cyc}) \simeq \mathrm{Clop}(\widehat{\mathbf{Z}})$.*

PROOF. We will only sketch the proof of (2). The other proofs are similar. Since $\mathbf{N}_0$ is dense in $\widehat{\mathbf{Z}}_{\mathbf{mon}}$, the clopens of $\widehat{\mathbf{Z}}_{\mathbf{mon}}$ are determined by their intersections with $\mathbf{N}_0$. Every clopen b of $\widehat{\mathbf{Z}}_{\mathbf{mon}}$ satisfies $\tau^s(b) = \tau^{s+\ell}(b)$ for some $(s, \ell) \in \mathbf{N}_0 \times \mathbf{N}$ it follows that the intersection of b with $\mathbf{N}_0$ lies in B_{ec}. Conversely, every $b \in B_{ec}$ is of the form $p(s, \ell)^{-1}(c) \cap \mathbf{N}_0$ for c a clopen of $\mathbf{N}_0(s, \ell)$ and the remaining details are straightforward. ∎

6.4. OBSERVATION. *Let B be a Boolean flow. Then $\mathcal{W}$, the space of all flow ideals of B and its subspace, $\mathcal{W}_{cyc}$ of all $I \in \mathcal{W}$ with B/I cyclic, are both sober spaces.*

PROOF. Recall that $\{N(b) | b \in B\}$ is a base for the topology on $\mathcal{W}$ where $N(b) = \{I \in \mathcal{W} | b \in I\}$. Assume that $F \subseteq \mathcal{W}$ is a closed irreducible subset, meaning that F is non-empty and whenever $F \subseteq F_1 \cup F_2$ for F_1, F_2 closed in $\mathcal{W}$, we have either $F \subseteq F_1$ or $F \subseteq F_2$. We need to find an $I_0 \in \mathcal{W}$ such that F is the closure of the one-point set $\{I_0\}$, which is $\{I \in \mathcal{W} | I \subseteq I_0\}$.

Let $I_0 = \bigcup\{I | I \in F\}$. It is readily shown that I_0 is a flow ideal. For example, assume $b, c \in I_0$ but $b \vee c \notin I_0$. Then $F \subseteq (\mathcal{W} - N(b)) \cup (\mathcal{W} - N(c))$ but neither $F \subseteq (\mathcal{W} - N(b))$ nor $F \subseteq (\mathcal{W} - N(b))$, contradicting the irreducibility of F.

It readily follows that $I_0 \in F$ from the fact that F is closed and, similarly, that the closure of $\{I_0\}$ is a subset of F. But, by construction, every $I \in F$ is a subset of I_0, so F is the closure of $\{I_0\}$. Finally, I_0 is clearly unique as $\mathcal{W}$ is a T_0-space.

The same argument applies to $\mathcal{W}_{cyc}$. ∎

6.5. Observation. *If A is an algebra for any finitary algebraic theory and $\mathcal{W}$ is the space of all congruences on A, topologized so that $\{N(a,b)\}$ is a subbase, where $N(a,b) = \{E \in \mathcal{W} | (a,b) \in E\}$, then $\mathcal{W}$ is a sober space and so is any subspace consisting of all $E \in \mathcal{W}$ for which A/E satisfies a geometric condition.*

The argument given for 6.4 applies.

6.6. Observation. *In [Kennison, 2002], we defined a cyclic flow for any "well-behaved" category in terms of equalizers. In much the same way, we could also define an eventually cyclic flow for any such category. It would still be the case that the cyclic flows in* Flow(C) *as well as the eventually cyclic flows would form full, coreflective subcategories, as the arguments given for Proposition 3.4 easily extend to the general case.*

6.7. Observation. *If $\Sigma = \{0,1\}$ then, as observed in [Kennison, 2006], the Boolean flow* $\mathrm{Clop}(\Sigma^{\mathbf{N}_0})$ *is generated by the element $g = {\pi_0}^{-1}(1)$. In fact,* $\mathrm{Clop}(\Sigma^{\mathbf{N}_0})$ *is the free Boolean flow on one generator.*

This can be proven directly or by noting that if A is any Boolean algebra, then the coproduct of $\mathbf{N}_0$ copies of A is the free Boolean flow generated by A (this is a special case of an observation in [Lawvere, 1986]) and $\Sigma = \{0,1\}$ is the free Boolean algebra on one generator. Note that Clop converts the power of $\mathbf{N}_0$ copies of Σ in Stone spaces into the copower of $\mathbf{N}_0$ copies of $\mathrm{Clop}(\Sigma)$.

It similarly follows that if $\Sigma = \{1, 2, \ldots, 2^n\}$, then $\mathrm{Clop}(\Sigma^{\mathbf{N}_0})$ is the free Boolean flow on n generators. Also if Σ has n elements, then $\mathrm{Clop}(\Sigma^{\mathbf{N}_0})$ is the Boolean flow generated by $b_1, b_2, \ldots, b_n$ subject to the conditions that $b_i \wedge b_j = 0$ for $i \neq j$ and $\bigvee\{b_i\} = 1$.

6.8. Example. [Irrational Rotations of the Circle] *Let $X = [0,2]$ with the points 0 and 2 identified, so that X is a circle. (We use the interval $[0,2]$ to avoid the fractions that would arise if we used $[0,1]$.) We can think of X as $\mathbf{R}/2\mathbf{Z}$ (the reals modulo the even integers) so that X has a useful modulo 2 addition. Let P be the irrationals in $[0,1]$. Define $t : P \times X \to X$ by $t(p,x) = t_p(x) = x + p$ (with the addition modulo $2\mathbf{Z}$). Let $A_0 = [0,1]$, $A_1 = [1,2]$ be closed subsets of X and use symbolic dynamics, as in Proposition 4.5 to get a sheaf of Boolean flows over P. Then:*

1. *The simple spectrum of $(B, \mathcal{O}(P))$ is $(B, \mathcal{O}(P))$ itself.*

2. *The non-trivial cyclic spectrum of $(B, \mathcal{O}(P))$ is $(B_j, \mathcal{O}(P)_j)$ where j is a nucleus on $\mathcal{O}(P)$ for which u is dense in $j(u)$. (So the sublocale j is at at least as large as the double-negation sublocale, for which the nucleus is given by the interior of the closure).*

3. *The non-trivial eventually cyclic spectrum of $(B, \mathcal{O}(P))$ coincides with its non-trivial cyclic spectrum.*

4. *For $u \in \mathcal{O}(P)$ the natural map $\hat{\eta}_u : B(u) \to B_j(u)$ is one-to-one (where j is as above).*

PROOF. (1) The universal quotient flow for $(B, \mathcal{O}(P))$ is spatial in view of Proposition 5.30. The flow $\mathrm{Clop}(X, t_p)$ is simple for each p because, as is well-known, the only non-empty closed subflow of X is X itself. The requirement that a simple flow be non-trivial eliminates the places where $0 = 1$ and, by examining the proof of Proposition 5.30, we see that we get $(B, \mathcal{O}(P))$.

(2) We will prove this after stating some lemmas.

(3) The iterator, t_p, has an inverse for each p, so the Boolean iterator has an inverse and will be one-to-one so all eventually cyclic quotients are cyclic, and the arguments used for (2) will apply.

(4) Corollary 6.11 says that if a section $b \neq 0$ at p, then b is non-zero in a neighborhood of p and this leads to the result, in view of the nature of the nucleus j. ∎

6.9. LEMMA. *Let X and P be as above and let $\widehat{X}_p \subseteq 2^{\mathbf{N}_0}$ be the family of all $\{0,1\}$-sequences which are compatible with some $x \in X$ under t_p. Let $\pi_{(n)} : 2^{\mathbf{N}_0} \to 2^{n+1}$ be the projection onto the first $n+1$ coordinates. Let $s = (s_0, s_1, \ldots, s_n)$ be given and let $b = \pi_{(n)}^{-1}(s)$. Let $c(k)$ be the number of* **symbol changes** *in $(s_0, s_1, \ldots, s_k)$ (that is, $c(k)$ is the number of times $s_i \neq s_{i+1}$ for $0 \leq i < k$.) Then $b \neq 0$ in B_p if and only if, for all $k, m \in \{0, 1, \ldots, n\}$ we have:*

(1) $c(k) - 1 \leq kp \leq c(k) + 1$

(2) $c(m) - c(k) - 1 \leq (m-k)p \leq c(m) - c(k) + 1$

PROOF. (In this proof, we work with addition on $\mathbf{R}$, not $\mathbf{R}/2\mathbf{Z}$.) Assume that $b \neq 0$ in B_p. Then, since $0 < p < 1$, we see that $c(k)$ is the number of times the orbit of x crosses a boundary separating A_0 and A_1. So if x crosses $c(k)$ boundaries in going from x to $(t_p)^k(x)$ then $(t_p)^k(x) - x = kp$ is approximately $c(k)$ (as A_0 and A_1 both have length 1). The precise information relating to kp to $c(k)$ is readily seen to be inequality (1). (We have to a bit careful if x lies on a boundary point, but the inequalities are still valid).

Inequality (2) follows in a similar way, for $k \leq m$ as $c(m) - c(k)$ is the number of boundary crossings from $(t_p)^k(x)$ to $(t_p)^m(x)$. If $k > m$ then use the above argument with k and m reversed and multiply the resulting inequality by -1. (We note that inequality (2) is sharper than the inequality we get by combining inequalities of form (1) for k and m.)

As for the converse, assume that p satisfies all inequalities of the form (1) and (2). We want to find an $x \in [0, 2]$ which is compatible with an element of b.

Case 1: Assume $s_0 = 0$. So now we need to find $x \in [0, 1]$ which is compatible with a member of b. The conditions that $(t_p)^k(x) \in A_{s_k}$ reduce to:

(3) $c(k) \leq (x + kp) \leq c(k) + 1$

We let $L(k) = c(k) - kp$ and $R(k) = c(k) + 1 - kp$ and rewrite (3) as

(3′) $L(k) \leq x \leq R(k)$

We need to show that this set of inequalities is consistent, or that $L(k) \leq R(m)$ for all k, m. But the second half of inequality (2) readily implies that $c(k) + (m-k)p \leq c(m) + 1$ and this leads to $L(k) \leq R(m)$. So the inequalities (3′) have a simultaneous solution, but we need to find a solution x in $[0,1]$. It suffices to show that $L(k) \leq 1$ and $0 \leq R(k)$. But both of these inequalities readily follow from inequality (1).

Case 2: Assume $s_0 = 1$. A similar argument works here. Alternatively, replace $s = (s_0, s_1, \ldots, s_n)$ by $1 - s = (1 - s_0, 1 - s_1, \ldots, 1 - s_n)$ (which does not affect the values of $c(k)$) then if x is compatible with $1 - s$, we see that $x - 1$ is compatible with s. ∎

6.10. COROLLARY. *Let b be as above. If $b \neq 0$ in B_p then $b \neq 0$ in a neighborhood of p.*

PROOF. Inequalities (1) and (2) are strict (so that $\leq$ can be replaced by $<$) as p is irrational and the other terms are rational. So the inequality holds in an open set. ∎

6.11. COROLLARY. *The above corollary applies to any clopen of $2^{\mathbf{N}_0}$.*

Let b be a basic clopen, meaning a finite intersection of subbasic clopens of the form $(\pi_i)^{-1}(s)$. Let n be the largest value of i that appears in the intersection. Then b is a finite union of clopens of the form $\pi_{(n)}{}^{-1}(s)$ for $s \in 2^{n+1}$. Then every clopen is a finite union of basic clopens and the result easily follows. ∎

PROOF OF (2) OF EXAMPLE 6.8. It suffices to show that if j is the double-negation nucleus, then $(B_j, \mathcal{O}(P)_j)$ is cyclic. We can extend the parameter space P to all elements (including the rationals) in (0,1], because the double-negation nucleus on the extended space will be the same. We must show that $\bigvee \|\tau^n(b) = b\|$ represents the top element of $\mathcal{O}(P)_j$. But $\bigvee \|\tau^n(b) = b\|$ contains all the rationals so it is a dense open set. ∎

PROOF OF (4) OF EXAMPLE 6.8. This follows from Corollary 6.11. ∎

In the above example, we note that we do not want to allow $p = 0$ as then t is the identity (which would be cyclic) but when we apply symbolic dynamics to this case, a point on the boundary of A_0 and A_1 is compatible with every element of $2^{\mathbf{N}_0}$. (This is one case where symbolic dynamics is very misleading.)

Also, nothing is gained by extending P to the interval $(0, 2)$ as one can show that $\widehat{X}_p = \widehat{X}_{p-1}$ (we leave the details to the reader). If we used three sets, A_0, A_1, A_2 for symbolic dynamics, we could distinguish p from $p - 1$.

SOME OPEN QUESTIONS.

1. Can we characterize those Boolean flows which are equivalent to the global sections for a cyclic Boolean sheaf over a locale? (This question leads to the questions raised in 5.4.)

2. Let X be the Stone space of Example 6.2. What are the cyclic and eventually cyclic spectra of $\mathrm{Clop}(X)$? In particular, are there any closed, dually cyclic subflows of X that lie in the "outgrowth", in this case, $X - S$?

3. Is the cyclic spectrum of an ordinary Boolean flow (that is, a Boolean flow in the category of Sets) always spatial?

4. Same question, but for the eventually cyclic spectrum or the simple spectrum.

5. What is the simple spectrum of $\mathrm{Clop}(2^{\mathbf{N}_0}, t)$? (This question was pursued in [Kennison, 2006], but only partial results were obtained.)

6. Suppose $(B, \tau) = \mathrm{Clop}(X, t)$. If the trajectory of each $x \in X$ is finite, must (B, τ) be eventually cyclic? In general, what conditions on (B, τ) are necessary, or sufficient, for (X, t) to have finite trajectories?

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