Theory and Applications of Categories, Vol. 22, No. 2, 2009, pp. 24–53.

ON DEFORMATIONS OF PASTING DIAGRAMS

D. N. YETTER

Abstract. We adapt the work of Power [14] to describe general, not-necessarily composable, not-necessarily commutative 2-categorical pasting diagrams and their composable and commutative parts. We provide a deformation theory for pasting diagrams valued in the 2-category of k-linear categories, paralleling that provided for diagrams of algebras by Gerstenhaber and Schack [9], proving the standard results. Along the way, the construction gives rise to a bicategorical analog of the homotopy G-algebras of Gerstenhaber and Voronov [10].

1. Introduction

It is the purpose of this work to describe the deformation theory of pasting diagrams of k-linear categories, functors and natural transformations. As such, it generalizes work of Gerstenhaber and Schack [9], both by giving an exposition of the well-known extension of their work on diagrams of algebras to k-linear categories and functors, and by the inclusion of natural transformations.

The present work has a number of motivations. It initially grew out of a program to extend the author's deformation theory for monoidal categories, functors and natural transformations [5, 17, 18, 19], which deforms only the structure maps, to a theory in which the composition of the category, the arrow part of the monoidal product, and the structure maps are all deformed simultaneously. That program is still in progress, and this paper is a first step in it.

It is also the first step in a program to provide a Gerstenhaber-style deformation theory for linear stacks, as pre-stacks may be considered as special instances of pasting diagrams. Consideration of not-necessarily abelian linear stacks is motivated by physical considerations in a prospective deformation quantization approach to quantum gravity (cf. [2, 3, 4]).

It is also hoped that the present work may shed light, if only by analogy, on the difficulties arising in Elgueta's deformation theory for monoidal bicategories [6].

Throughout we will consider all categories to be small, if necessary by invoking the

The author wishes to thank Kansas State University and the University of Pennsylvania for support of the sabbatical leave when this work was begun. The author also wishes to than the referee assigned by *Theory and Applications of Categories* for numerous corrections and helpful suggestions for improvements to the original manuscript.

Received by the editors 2007-09-21 and, in revised form, 2009-02-15.

Transmitted by James Stasheff. Published on 2009-02-26.

2000 Mathematics Subject Classification: Primary: 18D05, 13D03, Secondary: 18E05.

Key words and phrases: pasting diagrams, pasting schemes, deformation theory.

© D. N. Yetter, 2009. Permission to copy for private use granted.

axiom of universe. Composition will be written in diagrammatic order unless parentheses indicate functional application. Thus fg and $g(f)$ both denote the arrow obtained by following f with g, in the second case thought of as applying g to f. Throughout by abuse of notation, whenever operations defined as applying to sets, are applied to elements, the singleton set will be understood. k will denote a fixed field, and all categories and functors will be k-linear.

2. Pasting schemes and pasting diagrams: definitions

A pasting diagram is to n-categories what an ordinary diagram is to categories. A number of ways to formalize them have been developed. We will for the most part follow Power [14], whose approach mixing Street's notion of computads [15] with a geometric adaptation of Johnson's pasting schemes [11] avoids much of the combinatorial complexity of Johnson's approach. Power's description seems to be of the right generality for the present work, and we will deviate from it only to allow the description of not-necessarily composable, not-necessarily commutative diagrams. We follow Power's gentle method of exposition by initially restating the familiar in less-familiar but readily generalizable terms:

2.1. Definition. *A* 1-computad *is a (finite) directed graph . A* 1-computad morphism *is a map of directed graphs.*

Observe, in particular, that there is a forgetful functor from the category of small (or finite) categories and functors to 1-computads.

2.2. Definition. *A* 1-pasting scheme *is a finite non-empty set G equipped with an embedding to the oriented line $\mathbb{R}$. The elements (identified with their images in the line) are called* 0-cells *of the pasting scheme, and the open bounded intervals in $\mathbb{R} \setminus G$ are called the* 1-cells *of the pasting scheme.*

Denoting the sets of 0- *and* 1-*cells by G_0 and G_1 respectively, there is a function $dom : G_1 \to G_0$, (resp. $cod : G_1 \to G_0$) the* domain *(resp.* codomain *function, which assign the lesser (resp. greater) endpoint to each $1-cell$.*

The quadruple (G_0, G_1, dom, cod) defines a finite directed graph, the underlying 1-computad *of the 1-pasting scheme, which we denoted $C(G)$.*

The domain of G*, domG (resp.* codomain of G *codG) is the smallest (resp. largest) element of G_0. (These should not be confused with the domain and codomain functions.)*

2.3. Definition. *A* composable 1-pasting diagram *in a* 1-*computad H consists of a* 1-*pasting scheme and a* 1-*computad morphism $h : C(G) \to H$. The* domain *(resp.* codomain*) of the* 1-*pasting diagram is $h(domG)$ (resp. $h(codG)$). In the case where H is the underlying* 1-*computad of a category $\mathcal{A}$, we call the* 1-*pasting diagram a* labelling of G in $\mathcal{A}$. *We denoted the set of composable* 1-*pasting diagrams in a* 1-*computad H by $diags(H)$.*

2.4. Definition. *The* 1-pasting composition *of a labelling of a pasting scheme G in $\mathcal{A}$ is the arrow of $\mathcal{A}$ obtained by composing the sequence of arrows in A given by the $h(G_1)$.*

2.5. Definition. *A* general 1-pasting diagram *in a 1-computad H consists of a 1-computad G and a 1-computad morphism $h : G \to H$. The* composable parts *of a general 1-pasting diagram (G, h) are all composable 1-pasting diagrams $p : C(S) \to H$ in H that factor as $C(S) \xrightarrow{\tilde{p}} G \xrightarrow{h} H$.*

Thus far we have not really added anything to the notions of iterated composition and diagrams in a category, save to emphasize a sense that it is somehow 'more geometric' than multiplication. This heretofore needless abstraction now becomes necessary:

2.6. Definition. *A* 2-computad *consists of a 1-computad G and a set G_2 together with functions $dom : G_2 \to diags(G)$ (resp. $cod : G_2 \to diags(G)$) such that $dom(dom) = dom(cod)$ and $cod(dom) = cod(cod)$ as functions from G_2 to G_0.*

A 2-computad morphism *from (G, G_2, dom, cod) to (H, H_2, dom, cod) consists of a 1-computad morphism $f : G \to H$ and a function $f_2 : G_2 \to H_2$ such that $dom(f_2) = f(dom)$ and $cod(f_2) = f(cod)$.*

Plainly 2-computads and 2-computad morphisms form a category, and every 2-category has an underlying 2-computad obtained by forgetting the compositions.

2.7. Definition. *A* 2-pasting scheme *consists of a finite 1-computad G together with an embedding of the underlying geometric directed graph (also denoted G by abuse of notation) into the oriented plane $\mathbb{R}^2$ satisfying the following conditions:*

1. *The complement of the image of G consists of an unbounded region and finitely many open cells, which are called* faces*;*
2. *The boundary of each bounded face F is of the form $\sigma(F) \cup -\tau(F)$, where the negation indicates orientation reversal, each of $\sigma(F)$ and $\tau(F)$ are images of composable 1-pasting diagrams in G, and $dom\sigma(F) = dom\tau(F)$ (resp. $cod\sigma(F) = cod\tau(F)$); and*
3. *There exist vertices $s(G)$ and $t(G)$ in the boundary of the unbounded face such that for every vertex v, there is a composable 1-pasting diagram $h : C(H) \to G$ such that v is in the image of h, $domh = s(G)$ and $codh = t(G)$.*

It follows from these conditions that the boundary of the unbounded face E is also a union of images of composable 1-pasting diagrams $-\sigma(E) \cup \tau(E)$ (note the orientation reversal) with $dom\sigma(E) = dom\tau(E) = s(G)$ and $cod\sigma(E) = cod\tau(E) = t(G)$. We define the domain *(resp.* codomain*) of G by $domG = \sigma(E)$ (resp. $codG = \tau(E)$).*

Every 2-pasting scheme G admits an underlying 2-computad $C(G)$ in which G is the underlying 1-computad, and G_2 is the set of bounded faces, which are called 2-*cells*. The domain (resp. codomain) of a 2-cell is given by $domF = \sigma(F)$ (resp. $codF = \tau(F)$), thereby defining the maps $dom, cod : G_2 \to diags(G)$. Again these maps should not be confused with the domain and codomain of the entire pasting scheme.

2.8. Definition. *A* composable 2-pasting diagram *in a 2-computad H consists of a 2-pasting scheme and a 2-computad morphism $h : C(G) \to H$. In the case where H is the underlying 2-computad of a 2-category $\mathcal{A}$, we call the 2-pasting diagram a* labelling of G in $\mathcal{A}$. *We denoted the set of composable 2-pasting diagrams in a 2-computad H by $diags(H)$.*

In [14] Power proved

2.9. Theorem. *Every labelling of a composable 2-pasting scheme has a unique composite.*

Meaning that every iterated application of the 1- and 2-dimensional compositions to the natural transformations in a composable 2-pasting scheme which results in a single natural transformation gives the same result.

We will also need

2.10. Definition. *A* general 2-pasting diagram *in a 2-computad H consists of a 2-computad G and a 2-computad morphism $h : G \to H$. The* composable parts *of a general 2-pasting diagram (G, h) are all composable 2-pasting diagrams in H $p : C(S) \to H$ that factor as $C(S) \xrightarrow{\tilde{p}} G \xrightarrow{h} H$.*

We will not need Power's further explication of corresponding structures to describe composable pasting diagrams in higher dimensions. For our purposes, it suffices to make

2.11. Definition. *A* 3-computad *consists of a 2-computad G and as set G_3 together with functions $dom : G_3 \to diags(G)$ (resp. $cod : G_3 \to diags(G)$) such that $dom(dom) = dom(cod)$ and $cod(dom) = cod(cod)$ as functions from G_3 to $diags(G_1, G_0, cod, dom)$.*

A 3-computad morphism *from (G, G_3, dom, cod) to (H, H_3, dom, cod) consists of a 2-computad morphism $f : G \to H$ and a function $f_3 : G_2 \to H_2$ such that $dom(f_3) = f(dom)$ and $cod(f_3) = f(cod)$.*

and to observe that any 3-category admits an underlying 3-computad.

Of course, a 2-category can be regarded as a 3-category in which all 3-arrows are identities. By adopting this view, we can use general 3-pasting diagrams, targetted in the underlying 3-computad of a 2-category to specify commutativity conditions in general 2-pasting diagrams, since the presence of a 3-cell asserts the equality of its source and target.

Thus we make

2.12. Definition. *A* k-linear pasting diagram *is a 3-computad G together with a 3-computad morphism to the underlying 3-computad of $k - Cat$, the 2-category of all small k-linear categories, k-linear functors, and natural transformations.*

These are our primary objects of study. Note that we do not specify the dimension here: as part of an abstract hierarchy, these are 3-dimensional objects, but since we are working in 2-categories, they are degenerate, and in some sense still 2-dimensional.

3. Deformations of categories, functors and natural transformations: definitions and elementary results

The generalization of Gerstenhaber's deformation theory from associative algebras [7, 8] to linear categories, or 'algebroids' in the sense of Mitchell [13], is quite straight-forward, and both the one readily available source in which the construction has appeared [1] and unpublished lectures of Tsygan [16] treat it as a folk-theorem.

The deformation theory for linear functors, or for that matter commutative diagrams of linear functors, is similarly a straight-forward generalization of work of Gerstenhaber and Schack [9].

Finally, the deformation of natural transformations between undeformed functors is completely trivial, as will be seen. It is only when all three elements are combined that there is really anything new.

To fix notation, we review the basic elements of the theory:

3.1. Definition. *A* deformation $\hat{\mathcal{C}}$ of a k-linear category $\mathcal{C}$ *is an R-linear category (i.e. a category enriched in R-modules), for R a unital commutative local k-algebra, with maximal ideal $\mathfrak{m}$, whose objects are those of $\mathcal{C}$, with $\hat{\mathcal{C}}(X,Y) = \mathcal{C}(X,Y) \otimes_k R$, whose composition and identity arrows reduce modulo $\mathfrak{m}$ to those of $\mathcal{C}$.*

For R as above, an $\mathfrak{m}$-adic deformation $\hat{\mathcal{C}}$ of a k-linear category $\mathcal{C}$ *is a category enriched in the category of $\mathfrak{m}$-adically complete R-modules with obvious the monoidal structure given $\hat{\otimes}_R$, the $\mathfrak{m}$-adic completion of $\otimes_R$, whose objects are those of $\mathcal{C}$, with $\hat{\mathcal{C}}(X,Y) = \mathcal{C}(X,Y)\hat{\otimes}_k R$, whose composition and identity arrows reduce modulo $\mathfrak{m}$ to those of $\mathcal{C}$.*

Two deformations in either sense are equivalent *if there is an isomorphism of categories between them that reduces to the identity functor modulo $\mathfrak{m}$. We refer to such an isomorphism as* an equivalence of deformations.

The trivial deformation *is the deformation in which the composition on the original category is simply extended by bilinearity, or by bilinearity and continuity in the $\mathfrak{m}$-adic case, while* a trivial deformation *is one equivalent to the trivial deformation.*

Throughout we will be concerned only with n^{th}*-order deformations* in which $R = k[\epsilon]/\langle\epsilon^{n+1}\rangle$ and *formal deformations*, $\langle\epsilon\rangle$-adically complete deformations with respect to $R = k[[\epsilon]]$. Collectively, we will refer to these as *1-parameter deformations*.

In these cases the composition in $\hat{\mathcal{C}}$ has the form

$$f \star g = \sum \mu^{(j)}(f,g)\epsilon^j$$

while identity maps in $\hat{\mathcal{C}}$ are of the form

$$\hat{1}_A = \sum \iota^{(j)}(A)\epsilon^j$$

where $\mu^{(0)}(f,g) = fg$, and $\iota^{(0)}(A) = 1_A$, the sums being bounded for n^{th}-order deformations, and extending to infinity for formal deformations. In this case, the trivial deformation has $\mu^{(j)} \equiv 0$ and $\iota^{(j)} \equiv 0$ for $j > 0$.

The definition then translates into equational conditions on the $\mu^{(i)}$'s and $\iota^{(j)}$'s:

3.2. PROPOSITION. *The coefficients $\mu^{(i)}(f,g)$ and $\iota^{(j)}(A)$ in a 1-parameter deformation of a category $\mathcal{C}$ satisfy*

$$\begin{aligned}
\mu^{(0)}(f,g) &= fg && (1)\\
\iota^{(0)}(A) &= 1_A && (2)\\
\sum_{i=0}^{n} \mu^{(i)}(\iota^{(n-i)}(s(f)), f) &= 0 \text{ for } n>0 && (3)\\
\sum_{i=0}^{n} \mu^{(i)}(f, \iota^{(n-i)}(t(f)) &= 0 \text{ for } n>0 && (4)\\
\sum_{i=0}^{n} \mu^{(n-i)}(\mu^{(i)}(a,b),c) &= \sum_{i=0}^{n} \mu^{(n-i)}(a,\mu^{(i)}(b,c)), && (5)
\end{aligned}$$

and a family of coefficients $\{\mu^{(i)} | i = 1, \ldots\}$ defines a 1-parameter deformation if and only if it satisfies these equations.

PROOF. 1 and 2 are the requirement that the the deformation reduce to the identity modulo $\mathfrak{m} = \langle\epsilon\rangle$. By trivial calculations 3 and 4 are seen to be the preservation of identities and 5 the preservation of associativity. ■

Here, of course, there is an upper bound in the indices of the $\mu^{(i)}$'s in the case of an n^{th}-order deformation, and no bound in the case of a formal deformation. We will not bother to note this again in the discussion below of deformations of functors and natural transformations.

Equivalences between 1-parameter deformations, similarly can be characterized in terms of coefficients.

In particular, we have

3.3. PROPOSITION. *Given two 1-parameter deformations of $\mathcal{C}$, $\hat{\mathcal{C}}_1$ and $\hat{\mathcal{C}}_2$, with compositions given by $f \star_1 g = \sum \mu_1^{(i)}(f,g)\epsilon^i$ and $f \star_2 g = \sum \mu_1^{(i)}(f,g)\epsilon^i$ respectively, and identity maps given by $\hat{1}_{1A} = \sum \iota_1^{(j)}(A)\epsilon^j$ and $\hat{1}_{2A} = \sum \iota_2^{(j)}(A)\epsilon^j$ an equivalence Φ is given by a functor given on objects by the identity, and on arrows by*

$$\Phi(f) = \sum \Phi^{(i)}(f)\epsilon^i,$$

where

$$\begin{aligned}
\Phi^{(0)}(f) &= f && (6)\\
\sum_{j=0}^{i} \Phi^{(j)}(\iota_1^{(i-j)}(A)) &= \iota_2^{(i)}(A) \text{ for } i>0 \text{ and all } A \in Ob(\mathcal{C}) && (7)\\
\sum_{j=0}^{i} \Phi^{(j)}(\mu_1^{(i-j)}(f,g)) &= \sum_{k+l+m=i} \mu_2^{(k)}(\Phi^{(l)}(f), \Phi^{(m)}(g)) && (8)
\end{aligned}$$

Moreover, a family of assignments of parallel arrows to arrows in $\mathcal{C}$ defines an equivalence of 1-parameter deformations if and only if it satisfies equations 6-8.

PROOF. Equation 6 is the requirement that Φ reduce to the identity functor modulo $\mathfrak{m}$, equations 7 and 8 are the preservation of identity arrows and composition respectively. Preservation of sources and targets is trivial, and invertibility follows immediately from the reduction to the identity modulo $\mathfrak{m}$. ■

Classical discussions of deformations of unital associative algebras sometimes omit or gloss over the question of deforming the identity element. There is good reason for this, which holds with equal force in the many objects setting:

3.4. THEOREM. *If $\hat{\mathcal{C}}$ is a 1-parameter deformation of a k-linear category $\mathcal{C}$, with deformed composition $f \star g = \sum \mu^{(i)}(f,g)\epsilon^i$ and deformed identities $\hat{1}_A = \sum \iota^{(j)}(A)\epsilon^j$, there exists an equivalent deformation $\tilde{\mathcal{C}}$ with deformed composition $f * g = \sum \tilde{\mu}^{(i)}(f,g)\epsilon^i$ and identity arrows equal to the undeformed identities 1_A.*

PROOF. We begin by constructing the functor from $\hat{\mathcal{C}}$ to $\tilde{\mathcal{C}}$: map each object to itself, and each map g to $1_{s(g)} \star g$. Plainly $\hat{1}_A$ is mapped to 1_A.

Now observe that in $\hat{\mathcal{C}}$, 1_A is an automorphism of A: the coefficients $\kappa^{(i)}$ of its inverse 1_A^{-1} can be found inductively from the conditions $\kappa^{(0)} = 1_A$ and $\iota^{(n)} = \sum_{i=0}^{n} \mu^{(i)}(\kappa^{n-i}, 1_A)$.

We can now define the composition on $\tilde{\mathcal{C}}$. For $f : A \to B$ and $g : B \to C$, let $f * g = f \star 1_B^{-1} \star g$. It is immediate by construction that 1_A is an identity for $*$. Associativity of $*$ follows from the associativity of $\star$, as does the fact that the assignment of maps at the beginning of the proof is a functor, since $1_A \star (f \star g) = (1_A \star f) \star 1_B^{-1} \star (1_B \star g)$.

It is easy to see that reducing modulo $\langle \epsilon \rangle$ gives the identity functor of $\mathcal{C}$, and that the inverse functor is given on arrows by $f \mapsto 1_A^{-1} \star f$.

Likewise it is easy to see that $\tilde{\mu}^{(n)} = \sum_{i+j+k=n} \mu^{(k)}(\mu^j(f, \kappa^i), g)$. ■

For functors the natural notion of deformation is given by

3.5. DEFINITION. *A* deformation $\hat{F}$ of a k-linear functor $F : \mathcal{C} \to \mathcal{D}$ *is a triple $(\hat{\mathcal{C}}, \hat{\mathcal{D}}, \hat{F})$, where $\hat{\mathcal{C}}$ (resp. $\hat{\mathcal{D}}$) is a deformation of $\mathcal{C}$ (resp. $\mathcal{D}$) over the same local ring R, and $\hat{F}$ is a functor enriched in the category of R-modules or $\mathfrak{m}$-adically complete R-modules, as appropriate, from $\hat{\mathcal{C}}$ to $\hat{\mathcal{D}}$ that reduces modulo $\mathfrak{m}$ to F.*

As for categories, in the case of 1-parameter deformations, the definition of a deformation a k-linear functor is equivalent to a family of equational conditions on the coefficients of powers of ϵ:

3.6. PROPOSITION. *If $(\hat{\mathcal{C}}, \hat{\mathcal{D}}, \hat{F})$ is a 1-parameter deformation of a functor $F : \mathcal{C} \to \mathcal{D}$ with the composition in $\hat{\mathcal{C}}$ (resp. $\hat{\mathcal{D}}$) given by $f \star g = \sum \mu^{(i)}(f,g)\epsilon^i$ (resp. $f * g = \sum \nu^{(i)}(f,g)\epsilon^i$) and identities given by $\hat{1}_A = \sum \iota^{(i)}(A)\epsilon^i$ (resp. $\hat{1}_A = \sum \lambda^{(i)}(A)\epsilon^i$) and $\hat{F}$ given on arrows by $\hat{F}(f) = \sum F^{(i)}(f)\epsilon^i$, then the $\mu^{(i)}$'s and $\nu^{(i)}$'s satisfy equations 1-5, and moreover*

$$F^{(0)} = F \tag{9}$$

$$\sum_{i+j=n} F^{(i)}(\iota^{(j)}(A)) = \lambda^{(n)}(F(A)) \text{ for } n > 0 \text{ and all } x \in Ob(\mathcal{C}) \tag{10}$$

$$\sum_{i+j=n, i,j\geq 0} F^{(i)}(\mu^{(j)}(f,g)) = \sum_{k+l+m=n\, k,l,m\geq 0} \nu^{(k)}(F^{(l)}(f), F^{(m)}(g)), \tag{11}$$

and the families of coefficients define a 1-parameter deformation if and only if they satisfy these equations.

PROOF. 9 is the requirement that $\hat{F}$ reduce modulo $\mathfrak{m} = \langle\epsilon\rangle$ to F. In the presence of 9 10 is equivalent to the preservation of identity arrows by $\hat{F}$, while 11 is equivalent to the preservation of (the deformed) composition. ■

Observe that an equivalence of deformations of a k-linear category $\mathcal{C}$ is simply a deformation of the identity functor that has the two deformations of $\mathcal{C}$ as source and target.

There are evident notions of equivalence between deformations of k-linear functors:

3.7. DEFINITION. *Two deformations $(\hat{\mathcal{C}}, \hat{\mathcal{D}}, \hat{F})$ and $(\tilde{\mathcal{C}}, \tilde{\mathcal{D}}, \tilde{F})$ are* strongly equivalent *if $\hat{\mathcal{C}} = \tilde{\mathcal{C}}$, $\hat{\mathcal{D}} = \tilde{\mathcal{D}}$, and there exists a natural isomorphism $\phi : \hat{F} \Rightarrow \tilde{F}$ which reduces to Id_F modulo $\mathfrak{m}$.*

Two deformations $(\hat{\mathcal{C}}, \hat{\mathcal{D}}, \hat{F})$ and $(\tilde{\mathcal{C}}, \tilde{\mathcal{D}}, \tilde{F})$ are weakly equivalent *if there exist equivalences of deformations of categories, $\Gamma : \hat{\mathcal{C}} \to \tilde{\mathcal{C}}$ and $\Delta : \hat{\mathcal{D}} \to \tilde{\mathcal{D}}$, and a natural isomorphism $\phi : \hat{F}\Delta \Rightarrow \Gamma\tilde{F}$, which reduces modulo $\mathfrak{m}$ to Id_F.*

For 1-parameter deformations, each type of equivalence can be characterized by equations on coefficients of powers of ϵ. We give the more general case of weak equivalence, as strong equivalence is simply specialization to the case where Γ and Δ are identity functors:

3.8. PROPOSITION. *If (Γ, Δ, ϕ) is a (weak) equivalence between 1-parameter deformations $(\hat{\mathcal{C}}_1, \hat{\mathcal{D}}_1, \hat{F}_1)$ and $(\hat{\mathcal{C}}_2, \hat{\mathcal{D}}_2, \hat{F}_2)$ of $F : \mathcal{C} \to \mathcal{D}$, with $\Gamma(f) = \sum \Gamma^{(i)}(f)\epsilon^i$, $\Delta(f) = \sum \Delta^{(i)}(f)\epsilon^i$, and $\phi_x = \sum \phi_x^{(i)}\epsilon^i$, then the $\Gamma^{(i)}$'s (resp. $\Delta^{(i)}$'s) and the coefficients $\mu_n^{(i)}$ (resp. $\nu_n^{(i)}$) defining the composition on $\hat{\mathcal{C}}_n$ (resp. $\hat{\mathcal{D}}_n$) $n = 1, 2$ satisfy equations 6-8* mutatis mutandis, *and*

$$\phi_x^{(0)} = 1_x \text{ for all } x \in Ob(\mathcal{C}) \tag{12}$$

$$\sum_{i+j+k=n} \Delta^{(i)}(F_1^{(j)}(f))\phi_y^{(k)} = \sum_{p+q+r=n} \phi_x^{(p)} F_2^{(q)}(\Gamma^{(r)}(f)) \tag{13}$$

$$\text{for all } n \geq 0 \text{ and all } f : x \to y \in Arr(\mathcal{C}).$$

Moreover a family of coefficients defines a weak equivalence of 1-parameter deformations if and only if it satisfies the given conditions.

PROOF. The conditions not involving ϕ must hold by Proposition 3.3. Equation 12 is the requirement that ϕ reduce to the identity natural transformation modulo $\mathfrak{m}$. Equation 13 is naturality. ■

Theorem 3.4 can be extended to functors:

3.9. THEOREM. *If $F : \mathcal{A} \to \mathcal{B}$ is a functor, and $(\hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{F})$ is a deformation, then there is a weakly equivalent deformation in which the identity arrows of $\mathcal{A}$ and $\mathcal{B}$ are undeformed.*

PROOF. Construct equivalent deformations $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$ of the categories as in Theorem 3.4. Let $\Phi_{\mathcal{X}} : \hat{\mathcal{X}} \to \tilde{\mathcal{X}}$, $\Psi_{\mathcal{X}} : \tilde{\mathcal{X}} \to \hat{\mathcal{X}}$, $\sigma_{\mathcal{X}} : \Phi_{\mathcal{X}}\Psi_{\mathcal{X}} \Rightarrow Id_{\hat{\mathcal{C}}}$, and $\tau_{\mathcal{X}} : \Psi_{\mathcal{X}}\Phi_{\mathcal{X}} \Rightarrow Id_{\tilde{\mathcal{C}}}$ define the equivalence of categories for $\mathcal{X} = \mathcal{A}, \mathcal{B}$.

Then $\tilde{F} := \Psi_A \hat{F} \Phi_B$ gives the deformed functor from $\tilde{\mathcal{A}}$ to $\tilde{\mathcal{B}}$, and $\Phi_{\mathcal{B}}(F(\sigma_{\mathcal{A}}))$ gives the necessary natural isomorphism. (Recall our convention that compositions are in diagrammatic order unless parentheses indicate 'evaluation at'.) ■

Finally for natural transformations, we make

3.10. DEFINITION. *A* deformation $\hat{\sigma}$ of a natural transformation $\sigma : F \Rightarrow G$ *(for functors $F, G : \mathcal{C} \to \mathcal{D}$ is a five-tuple $(\hat{\mathcal{C}}, \hat{\mathcal{D}}, \hat{F}, \hat{G}, \hat{\sigma})$, where $\hat{\mathcal{C}}$ (resp. $\hat{\mathcal{D}}, \hat{F}, \hat{G}$) is a deformation of $\mathcal{C}$ (resp. $\mathcal{D}, F, G$), and $\hat{\sigma}$ is a natural transformation from $\hat{F}$ to $\hat{G}$, which reduces modulo $\mathfrak{m}$ to σ.*

For deformations of natural transformations, there are three obvious notions of equivalence:

3.11. DEFINITION. *Two deformations $(\hat{\mathcal{C}}, \hat{\mathcal{D}}, \hat{F}, \hat{G}, \hat{\sigma})$ and $(\tilde{\mathcal{C}}, \tilde{\mathcal{D}}, \tilde{F}, \tilde{G}, \tilde{\sigma})$ of a natural transformation σ are* strongly equivalent *if $(\hat{\mathcal{C}}, \hat{\mathcal{D}}, \hat{F}, \hat{G}) = (\tilde{\mathcal{C}}, \tilde{\mathcal{D}}, \tilde{F}, \tilde{G})$.*

Two deformations $(\hat{\mathcal{C}}, \hat{\mathcal{D}}, \hat{F}, \hat{G}, \hat{\sigma})$ and $(\tilde{\mathcal{C}}, \tilde{\mathcal{D}}, \tilde{F}, \tilde{G}, \tilde{\sigma})$ are equivalent *if $\hat{\mathcal{C}} = \tilde{\mathcal{C}}$, $\hat{\mathcal{D}} = \tilde{\mathcal{D}}$, and $\hat{F}$ (resp. $\hat{G}$) is strongly equivalent to $\tilde{F}$ (resp. $\tilde{G}$) by a strong equivalence of deformations of functors ϕ (resp. γ), and, moreover, $\hat{\sigma}\gamma = \phi\tilde{\sigma}$).*

Finally, two deformations $(\hat{\mathcal{C}}, \hat{\mathcal{D}}, \hat{F}, \hat{G}, \hat{\sigma})$ and $(\tilde{\mathcal{C}}, \tilde{\mathcal{D}}, \tilde{F}, \tilde{G}, \hat{\sigma})$ are weakly equivalent *if there exist equivalences of deformations of categories, $\Gamma : \hat{C} \to \tilde{C}$ and $\Delta : \hat{D} \to \tilde{D}$, and a natural isomorphisms $\phi : \hat{F}\Delta \Rightarrow \Gamma\tilde{F}$ (resp. $\psi : \hat{G}\Delta \Rightarrow \Gamma\tilde{G}$), which reduces modulo $\mathfrak{m}$ to Id_F (resp. Id_G), and, moreover $\hat{\sigma}\psi = \phi\tilde{\sigma}$).*

Here again, in the case of 1-parameter deformations, the definition is equivalent to a family of equational conditions on the coefficients of ϵ:

3.12. PROPOSITION. *If $(\hat{\mathcal{C}}, \hat{\mathcal{D}}, \hat{F}, \hat{G}, \hat{\sigma})$ is a 1-parameter deformation of a natural transformation $\sigma : F \Rightarrow G$ with the composition on $\hat{\mathcal{C}}$ (resp. $\hat{\mathcal{D}}$) given by $f \star g = \sum \mu^{(i)}(f, g)\epsilon^i$ (resp. $f \star g = \sum \nu^{(i)}(f, g)\epsilon^i$), $\hat{F}$ (resp. $\hat{G}$) given on arrows by $\hat{F}(f) = \sum F^{(i)}(f)\epsilon^i$ (resp. $\hat{G}(f) = \sum G^{(i)}(f)\epsilon^i$), and $\hat{\sigma}_x = \sum \sigma_x^{(i)}\epsilon^i$ then the $\mu^{(i)}$'s and $\nu^{(i)}$'s satisfy equations 1-5, they and the $F^{(i)}$'s and $G^{(i)}$'s satisfy 9-11, and moreover*

$$\sum_{i+j+k=n\, i,j,j\geq 0} \nu^{(i)}(F^{(j)}(f), \sigma_y^{(k)}) \quad = \quad \sum_{p+q+r=n\, p,q,r\geq 0} \nu^{(p)}(\sigma_x^{(q)}, G^{(r)}(f)) \tag{14}$$

for all $f : x \to y$ in $\mathcal{C}$.

Moreover families of coefficients define a 1-parameter deformation if and only if they satisfy these conditions.

PROOF. The conditions not involving the $\sigma^{(i)}$'s must hold by earlier propositions. Equation 14 is simply the naturality of $\hat{\sigma}$. ■

And finally, equivalences between 1-parameter deformations of natural transformations can be reduced to equations on the coefficients:

3.13. PROPOSITION. *If $(\Gamma, \Delta, \phi, \psi)$ is a weak equivalence between two deformations of a natural transformation $\sigma : F \Rightarrow G$ between functors $F, G : \mathcal{C} \to \mathcal{D}$, $(\hat{\mathcal{C}}_1, \hat{\mathcal{D}}_1, \hat{F}_1, \hat{G}_1, \hat{\sigma}_1)$ and $(\hat{\mathcal{C}}_2, \hat{\mathcal{D}}_2, \hat{F}_2, \hat{G}_2, \hat{\sigma}_2)$, with $\Gamma(f) = \sum \Gamma^{(i)}(f)\epsilon^i$, $\Delta(f) = \sum \Delta^{(i)}(f)\epsilon^i$, $\phi_x = \sum \phi_x^{(i)}\epsilon^i$, $\psi_x = \sum \psi_x^{(i)}\epsilon^i$, then the coefficients defining the compositions on the $\hat{\mathcal{C}}_n$'s and $\hat{\mathcal{D}}_n$'s $(n = 1, 2)$ together with the $\Gamma^{(i)}$'s and the $\Delta(i)$'s, the $F_n^{(i)}$'s $(n = 1, 2)$ (resp. $G_n^{(i)}$'s $(n = 1, 2)$) and the $\phi^{(i)}$'s (resp. the $\psi^{(i)}$'s) satisfy the conditions of Proposition 3.8* mutatis mutandis*, and moreover*

$$\sum_{i+j=n} \phi_x^{(i)} \sigma_{2\,x}^{(j)} = \sum_{p+q=n} \sigma_{1\,x}^{(p)} \psi_x^{(q)} \tag{15}$$

for all n and $x \in Ob(\mathcal{C})$.

Moreover, families of coefficients define a weak equivalence of deformations of natural transformations if and only if they satisfy these conditions.

PROOF. All but the last condition must hold by Proposition 3.8 and the definition of weak equivalence. The last is simply the condition that $\sigma_1 \psi = \phi \sigma_2$. ■

Again, up to weak equivalence, we may assume that identity maps have been left undeformed, as the following follows by the construction of Theorem 3.9, and a little 2-categorical diagram chase around a 'square pillow' with two squares and two bigons as faces, the equivalences from Theorem 3.9 as the seams between the square faces, the square faces being the natural transformation in the weak equivalences between the deformations of F and G, and one bigon being the original deformation of σ:

3.14. THEOREM. *If $(\hat{\mathcal{C}}, \hat{\mathcal{D}}, \hat{F}, \hat{G}, \hat{\sigma})$ is a deformation of a natural transformation $\sigma : F \Rightarrow G$ for $F, G : \mathcal{C} \to \mathcal{D}$, then there exists a weakly equivalent deformation of σ in which the identity maps of both $\mathcal{C}$ and $\mathcal{D}$ are undeformed.*

Having arrived at this point, it is clear what a deformation of a 3-cell is: it is simply a deformation of the equal bounding 2-cells.

In the case where the diagram is simply a 'bigonal pillow', that is a 3-computad with a single 3-cell with single 2-cells as source and target, there is nothing more to be said. We will see, however, once we consider pasting of deformations, and deformations of pasting diagrams, that in general the matter is not trivial.

4. The Hochshild cohomology of k-linear categories, functors and natural transformations: definitions and elementary results

As was the case in [9], the cohomology appropriate to objects turns out to be a special case of that appropriate to arrows, though with groups in different cohomological dimensions playing corresponding role. Here, however, we can apply this observation twice.

It turns out that the obvious Hochschild complex for a natural transformation depends only on its source and target, we thus begin with

4.1. Definition. *The* Hochschild complex of a parallel pair of functors,

$$F, G : \mathcal{A} \to \mathcal{B}$$

has cochain groups given by

$$C^n(F,G) := \prod_{x_0,\ldots,x_n \in Ob(\mathcal{A})} \mathrm{Hom}_k(\mathcal{A}(x_0,x_1) \otimes \ldots \otimes \mathcal{A}(x_{n-1},x_n), \mathcal{B}(F(x_0),G(x_n))),$$

and for $n = 0$, noting that k is the empty tensor product,

$$C^0(F,G) := \prod_{x_0 \in Ob(\mathcal{A})} \mathrm{Hom}_k(k, \mathcal{B}(F(x_0),G(x_0)))$$

with coboundary given by

$$\begin{aligned}
&\delta\psi(f_0 \otimes \ldots \otimes f_n) := \\
&\quad F(f_0)\phi(f_1 \otimes \ldots \otimes f_n) + \\
&\quad \sum_{i=1}^{n} (-1)^i \phi(f_0 \otimes \ldots \otimes f_{i-1}f_i \otimes \ldots \otimes f_n) + \\
&\quad (-1)^{n+1}\phi(f_0 \otimes \ldots \otimes f_{n-1})G(f_n)
\end{aligned}$$

$\delta^2 = 0$ by the usual calculation.

The impression that natural transformations themselves are forgotten in this definition is deceptive. In fact, in this context naturality itself turns out to be a cohomological condition:

4.2. Proposition. *A natural transformation from F to G is a 0-cocycle in $C^\bullet(F,G)$, or equivalently a 0-dimensional cohomology class.*

Proof. Observe that

$$C^0(F,G) = \prod_{x \in Ob(\mathcal{A})} \mathrm{Hom}_k(k, \mathcal{B}(F(x),G(x))).$$

(In the sources, one has an empty tensor products of hom-spaces in $\mathcal{A}$, since there is no "next" element object, while in the target one has the hom-space from the image of the first object under F to the last, here the same object, under G.)

Thus a 0-cochain is an assignment of an arrow $\phi_x : F(x) \to G(x)$ for each object x of $\mathcal{A}$.

The cocycle condition then is

$$0 = \delta(\phi)(f : x_0 \to x_1) = F(f)\phi_{x_1} - \phi_{x_0}G(f),$$

that is, the naturality of ϕ. Since $C^{-1}(F, G) = 0$, the 0-cohomology classes and 0-cocycles can be identified. ■

We will also be interested in the subcomplex of normalized Hochschild cochains–those which vanish whenever one of the arguments is an identity arrow. We denote this subcomplex by $\bar{C}^\bullet(F, G)$.

Theorem 3.4 has a cohomological analog:

4.3. Theorem. *The normalized Hochschild complex $\bar{\mathcal{C}}^\bullet(F, G)$ is a chain deformation retract of the Hochschild complex $\mathcal{C}^\bullet(F, G)$.*

Proof. The result follows from the same trick used in [12].

Call a cochain ϕ i-normalized if $\phi(f_0, \ldots, f_{n-1})$ is zero whenever f_j is an identity arrow for any $j \leq i$.

The i-normalized cochains from a subcomplex $\mathcal{C}^\bullet_i(F, G)$ of $\mathcal{C}^\bullet(F, G)$, and satisfy

$$\mathcal{C}^\bullet_{i+1}(F, G) \subset \mathcal{C}^\bullet_i(F, G),$$

and

$$\cap_{i=0}^{\infty} \mathcal{C}^\bullet_i(F, G) = \bar{\mathcal{C}}^\bullet(F, G).$$

For $k \geq 0$, define maps $s^k : \mathcal{C}^n(F, G) \to \mathcal{C}^{n-1}(F, G)$ by

$$s^k(\phi)(f_1, \ldots f_{n-1}) = \begin{cases} 0 & \text{if } k > n \\ \phi(f_1, \ldots, f_k, 1_{t(f_k)}, f_{k+1}, \ldots, f_{n-1}) & \text{if } k \leq n \end{cases}$$

Let $h^k(\phi) := \phi - \delta(s^k(\phi) - s^k(d(\phi))$.

Then h^0 and the inclusion $i_0 : \mathcal{C}^\bullet_0(F, G) \to \mathcal{C}^\bullet(F, G)$ form a chain deformation retraction of the whole complex onto the subcomplex, with s^0 as the homotopy from $i_0 h^0$ to the identity of $\mathcal{C}^\bullet(F, G)$. And, h^k and the inclusion of $i_k : \mathcal{C}^\bullet_{k-1}(F, G) \to \mathcal{C}^\bullet_k(F, G)$ form a chain deformation retraction, with s^k as the homotopy from $i_k h^k$ to the identity of $\mathcal{C}^\bullet_{k-1}(F, G)$.

Note that h^k is the identity map on $\mathcal{C}^n(F, G)$ for $n < k$. Thus, the formal infinite composition $\bar{h} = h^0 h^1 h^2 \ldots$ is finite in each dimension, and gives a chain deformation retraction of $\mathcal{C}^\bullet(F, G)$ onto $\bar{\mathcal{C}}^\bullet(F, G)$, with $s^0 + h^0 s^1 + h^0 h^1 s^2 + \ldots$ as the homotopy from $\bar{i}\bar{h}$ to the identity of $\mathcal{C}^\bullet(F, G)$, where $\bar{i}$ is the inclusion of the subcomplex of normalized cochains. ■

We also make

4.4. Definition. *The* Hochschild complex of a functor F *is*

$$(C^\bullet(F), \delta) := (C^\bullet(F, F), \delta).$$

The Hochschild complex of a k-linear category $\mathcal{C}$ *is*

$$(C^\bullet(\mathcal{C}), \delta) := (C^\bullet(Id_\mathcal{C}, Id_\mathcal{C}), \delta).$$

The subcomplexes of normalized cochains $\bar{C}^\bullet(F)$ and $\bar{C}^\bullet(\mathcal{C})$ are defined in the obvious way.

Observe that this definition agrees with that given in [1] and in the special case of algebras with that given in [7, 8].

Not surprisingly, the Hochschild complexes admit a rich algebraic structure generalizing that discovered by Gerstenhaber [7, 8] on the Hochschild complex of an associative algebra (cf. also [10]).

First, given three parallel functors $F, G, H : \mathcal{A} \to \mathcal{B}$, there is a cup-product, or more properly a cup-product-like 2-dimensional (in the sense of bicategory theory) composition, $\cup : C^n(F, G) \otimes C^m(G, H) \to C^{n+m}(F, H)$ given by

$$\phi \cup \psi(f_1 \otimes \ldots \otimes f_{n+m}) := (-1)^{nm} \phi(f_1 \otimes \ldots \otimes f_n)\psi(f_{n+1} \otimes \ldots \otimes f_{n+m})$$

Second, given functors $G, H : \mathcal{B} \to \mathcal{C}$ and $F_0, \ldots, F_n : \mathcal{A} \to \mathcal{B}$, there is a brace-like 1-dimensional composition (cf. [10])

$$\begin{aligned} &-\{-, \ldots, -\} : \\ &C^K(G, H) \otimes C^{k_1}(F_0, F_1) \otimes \ldots \otimes C^{k_n}(F_{n-1}, F_n) \to \\ &\qquad\qquad C^{K+k_1+\ldots k_n - n}(G(F_0), H(F_n)) \end{aligned}$$

given by

$$\begin{aligned} &\phi\{\psi_1, \ldots, \psi_n\}(f_1, \ldots, f_N) := \\ &\quad \sum (-1)^\epsilon \phi(F_0(f_1), \ldots, \psi_1(f_{l_1+1}, \ldots, f_{l_1+k_1}), F_1(f_{l_1+k_1+1}), \\ &\quad \ldots \psi_2(f_{l_2+1}, \ldots, f_{l_2+k_2}), \ldots, \psi_n(f_{l_n+1}, \ldots, f_{l_n+k_n}), F_n(f_{l_n+k_n+1}), \ldots, F_n(f_N)) \end{aligned}$$

where $N = K + k_1 + \ldots k_n - n$ and in each term $\epsilon = \sum_{i=1}^n (k_i - 1)l_i$, where l_i is the total number of inputs occurring before ψ_i, and the outer sum ranges over all insertions of the ψ_i's, in the given order, with any number, including zero, of the arguments preceding ψ_1 (resp. between ψ_i and ψ_{i+1}, following ψ_n) with F_0 (resp. F_i, F_n) applied, and a total of N arguments, including both those inside and outside of the ψ_i's.

In what follows, we will have call to consider a special instance of the brace-like 1-composition: given a pair of parallel functors $F, G : \mathcal{A} \to \mathcal{B}$, there is a map (of graded vector spaces, though not of cochain complexes)

$$-\{-\} : C^{\bullet+1}(\mathcal{B}) \otimes C^0(F, G) \to C^\bullet(F, G)$$

The brace-like 1-composition and cup-product-like 2-composition satisfy the identities given in Gerstenhaber and Voronov [10] whenever sources and targets are in agreement so that both sides of the equation are defined. Also note that if $n > K$ $\phi\{\psi_1, \ldots \psi_n\} = 0$, since the outer sum in the definition is empty.

Of course, in the many objects setting, it is not in general possible to reverse the order of $x\{y\}$, except in special cases, nor to add the two orderings, without yet more conditions. The most general case in which the usual construction gives rise to a differential graded Lie algebra structure on $C^\bullet(F, G)$ is that where both F and G are idempotent endofunctors on some category. This includes, of course, the special case $C^\bullet(Id_\mathcal{C}, Id_\mathcal{C}) = C^\bullet(\mathcal{C})$.

The circumstance for typical pairs of parallel functors, however, suggests that the view that deformation theory is governed by a differential graded Lie algebra ought to be generalized to regard the brace algebra structure as more fundamental. In view of this observation, we will consider analogs of the Maurer-Cartan equation for functors and natural transformations expressed in terms of the brace-like composition rather than seeking a formulation in terms of a dg-Lie algebra or L^∞-algebra structure.

There are also a number of rather obvious cochain maps between these complexes induced by the usual 2-categorical operations:

4.5. Proposition. *If $F, G : \mathcal{C} \to \mathcal{D}$, and $H : \mathcal{D} \to \mathcal{E}$ are functors, then there is a cochain map $H_*(-) : C^\bullet(F, G) \to C^\bullet(H(F), H(G))$ given by*

$$H_*(\phi)(f_1, \ldots, f_n) := H(\phi(f_1, \ldots, f_n)).$$

Similarly if $F : \mathcal{C} \to \mathcal{D}$ and $G, H : \mathcal{D} \to \mathcal{E}$ are functors, there is a cochain map $F^*(-) = -(F^\bullet) : C^\bullet(G, H) \to C^\bullet(G(F), H(F))$ given by

$$F^*(\phi)(f_1, \ldots, f_n) = \phi(F^\bullet)(f_1, \ldots, f_n) := \phi(F(f_1), \ldots, F(f_n)).$$

Proof. When it is remembered that all functors are linear, and that the analog of the left and right actions on the bimodule of coefficients involve the source and target functors of the pair, both statements follow by trivial calculations. ■

Both of these cochain maps can be expressed in terms of the brace-like 1-composition: for any functor $K : \mathcal{X} \to \mathcal{Y}$ the action of K on the hom vectorspaces gives a family of linear maps $\mathcal{X}(x, y) \to \mathcal{Y}(K(x), K(y))$, defining a 1-cochain $\mathbb{K} \in C^1(K, K)$.

We then have

$$H_*(\phi) = \mathbb{H}\{\phi\}$$

and

$$F^*(\phi) = \phi\{\mathbb{F}, \ldots, \mathbb{F}\}.$$

Notice in the special case where $\mathcal{C} = \mathcal{D}$ and $F = G = Id_\mathcal{C}$, H_* defines a cochain map $H_* : C^\bullet(\mathcal{C}) \to C^\bullet(H)$, while in the case where $\mathcal{D} = \mathcal{E}$ and $G = H = Id_\mathcal{D}$, F^* defines a cochain map $F^* : C^\bullet(\mathcal{D}) \to C^\bullet(F)$.

Of particular interest is the cochain map

$$F^*(p_2) - F_*(p_1) : C^\bullet(\mathcal{C}) \oplus C^\bullet(\mathcal{D}) \to C^\bullet(F)$$

the cone over which will occur in the classification of deformations of functors.

Equally trivial is the proof of

4.6. Proposition. *If $\tau : F_1 \Rightarrow F_2$ is a natural transformation, then post- (resp. pre-) composition by τ induces a cochain map $\tau^* : C^\bullet(F_2, G) \to C^\bullet(F_1, G)$ (resp. $\tau_* : C^\bullet(G, F_1) \to C^\bullet(G, F_2)$ for any functor G.*

These cochain maps can be expressed in terms of the cup-product-like 2-composition:

$$\tau^*(\phi) = \tau \cup \phi$$

and

$$\tau_*(\phi) = \phi \cup \tau.$$

These cochain maps, however, are less important than one induced by a natural transformation $\sigma : F \Rightarrow G$ from the cone over the map

$$\begin{bmatrix} F_* & -F^* \\ G_* & -G^* \end{bmatrix} : C^\bullet(\mathcal{A}) \oplus C^\bullet(\mathcal{B}) \to C^\bullet(F) \oplus C^\bullet(G).$$

to $C^\bullet(F, G)$.

Let $C^\bullet(\mathcal{A} \overset{F}{G} \mathcal{B})$ denote this cone, so the cochain groups are

$$C^\bullet(\mathcal{A} \overset{F}{G} \mathcal{B}) := C^{\bullet+1}(\mathcal{A}) \oplus C^{\bullet+1}(\mathcal{B}) \oplus C^\bullet(F) \oplus C^\bullet(G)$$

with coboundary operators given by

$$d_{\mathcal{A}\overset{F}{G}\mathcal{B}} = \begin{bmatrix} -d_{\mathcal{A}} & 0 & 0 & 0 \\ 0 & -d_{\mathcal{B}} & 0 & 0 \\ -F_* & F^* & d_F & 0 \\ -G_* & G^* & 0 & d_G \end{bmatrix}$$

We then have:

4.7. Proposition. *Let $\sigma : F \Rightarrow G$ be a natural transformation, then*

$$\sigma\ddagger := \begin{bmatrix} 0 & (-)\{\sigma\} & \sigma_* & \sigma^* \end{bmatrix} : C^\bullet(\mathcal{A} \overset{F}{G} \mathcal{B}) \to C^\bullet(F, G)$$

is a cochain map.

Proof. Recall that a natural transformation is a 0-cocycle in $C^\bullet(F, G)$, and thus we can use the brace-like 1-composition to define the second entry.

Using subscripts to distinguish between the various coboundaries, we then have

$$d_{F,G}\left(\sigma \ddagger \left(\left[\begin{array}{c}\phi\\ \psi\\ \upsilon\\ \omega\end{array}\right]\right)\right) = d_{F,G}(\psi\{\sigma\}) + d_{F,G}(\sigma_*(\upsilon)) - d_{F,G}(\sigma^*(\omega))$$

and

$$\begin{aligned}\sigma \ddagger \left(d_{\substack{F\\ \mathcal{A}G\mathcal{B}}}\left(\left[\begin{array}{c}\phi\\ \psi\\ \upsilon\\ \omega\end{array}\right]\right)\right) &= -d_{\mathcal{B}}(\psi)\{\sigma\} - \sigma_*(F_*(\phi)) + \sigma_*(F^*(\psi))\\ &\quad +\sigma_*(d_F(\upsilon) + \sigma^*(G_*(\phi)) - \sigma^*(G^*(\psi)) - \sigma^*(d_G(\omega),\end{aligned}$$

which we must show are equal.

The terms involving ϕ in the second expression cancel by the naturality of σ, while the terms involving υ (resp. ω) in the two expressions are equal since σ_* (resp. σ^*) is a cochain map. It thus remains only to show that the terms involving ψ agree.

Expanding $d_{\mathcal{B}}(\psi)\{\sigma\}(f,g)$ for $X \xrightarrow{f} Y \xrightarrow{g} Z$ gives

$$\begin{gathered}\sigma_X\psi(G(f),G(g) - \psi(\sigma_X G(f),G(g)) + \psi(\sigma_X,G(fg)) - \psi(\sigma_X,G(f))G(g)\\ -F(f)\psi(\sigma_Y,G(g)) + \psi(F(f)\sigma_Y,G(g)) - \psi(F(f),\sigma_Y(G(g)) + \psi(F(f),\sigma_Y)G(g)\\ +F(f)\psi(F(g),\sigma_Z) - \psi(F(fg),\sigma_Z) + \psi(F(f),F(g)\sigma_Z) - \psi(F(f),F(g))\sigma_Z\end{gathered}$$

The terms in which one of the arguments of ψ consists of an instance of σ composed with another map (in either order) cancel in pairs by the naturality of σ. The first and last term are $\sigma^*(G^*(\psi))(f,g)$ and $-\sigma_*(F^*(\psi))(f,g)$ respectively. The remaining terms are easily seen to be $-d_{F,G}(\psi\{\sigma\})(f,g)$, yielding the desired result. ■

Because of its importance to the deformation theory of natural transformations, we will denote the cone on $\sigma\ddagger$ by $\mathfrak{C}^\bullet(\sigma)$, likewise we will denote the cone on $F^*(p_2) - F_*(p_1)$ by $\mathfrak{C}^\bullet(F)$, and for completeness, we will let $\mathfrak{C}^\bullet(\mathcal{C})$ be another notation for the Hocschild complex of $\mathcal{C}$.

More explicitly,

$$\mathfrak{C}^\bullet(\sigma) = C^{\bullet+2}(\mathcal{A}) \oplus C^{\bullet+2}(\mathcal{B}) \oplus C^{\bullet+1}(F) \oplus C^{\bullet+1}(G) \oplus C^\bullet(F,G)$$

with coboundary operators given by

$$\mathfrak{d}_\sigma = \left[\begin{array}{ccccc} d_{\mathcal{A}} & 0 & 0 & 0 & 0\\ 0 & d_{\mathcal{B}} & 0 & 0 & 0\\ F_* & -F^* & -d_F & 0 & 0\\ G_* & -G^* & 0 & -d_G & 0\\ 0 & -(-)\{\sigma\} & -\sigma_* & -\sigma^* & d_{F,G}\end{array}\right].$$

While

$$\mathfrak{C}^\bullet(F) = C^{\bullet+1}(\mathcal{A}) \oplus C^{\bullet+1}(\mathcal{B}) \oplus C^\bullet(F)$$

with coboundary operators given by

$$\mathfrak{d}_F = \begin{bmatrix} -d_{\mathcal{A}} & 0 & 0 \\ 0 & -d_{\mathcal{B}} & 0 \\ F_* & -F^* & d_F \end{bmatrix}.$$

We will refer to $\mathfrak{C}^\bullet(\sigma)$ (resp. $\mathfrak{C}^\bullet(F)$, $\mathfrak{C}^\bullet(\mathcal{C})$) as *the deformation complex of the natural transformation (resp. functor, category)*, and denote its cohomology groups by $\mathfrak{H}^\bullet(\sigma)$ (resp. $\mathfrak{H}^\bullet(F)$, $\mathfrak{H}^\bullet(\mathcal{C})$).

It is the purpose of the next section to justify these names by showing that first order deformations are classified up to equivalence by the expected cohomology group, and that the obstructions to extending a deformation to higher order all lie in the next cohomological dimension.

Finally, we introduce a deformation complex for a 3-cell $1_\sigma : \sigma \equiv\!\!\rangle \sigma$. It might seem reasonable to have this simply be $\mathfrak{C}^\bullet(\sigma)$ again. However, for reasons which will become clear once begin considering pasting diagrams in general, it will be better to use a weakly equivalent complex:

First let $\hat{\mathfrak{C}}^\bullet(\sigma)$ denote the cone on

$$i_1(\sigma\ddagger) + i_2(\sigma\ddagger) : C^\bullet(\mathcal{A} \overset{F}{G} \mathcal{B}) \to C^\bullet(F,G) \oplus C^\bullet(F,G),$$

Then $\mathfrak{C}^\bullet(1_\sigma)$ is the cone on

$$\begin{aligned} &p_5 - p_6 : \hat{\mathfrak{C}}^\bullet(\sigma) = \\ &\quad C^{\bullet+2}(\mathcal{A}) \oplus C^{\bullet+2}(\mathcal{B}) \oplus C^{\bullet+1}(G) \oplus C^{\bullet+1}(F) \oplus C^\bullet(F,G) \oplus C^\bullet(F,G) \\ &\quad \to C^\bullet(F,G). \end{aligned}$$

Observe that this is plainly weakly equivalent to $\mathfrak{C}^\bullet(\sigma)$.

5. First order deformations without pasting and cohomology

Let us begin by considering the first order case of the equational conditions defining 1-parameter deformations. For categories and functors, we obtain the obvious generalization of the results of Gerstenhaber and Schack [9] to the many-objects case:

For categories, equations 1-5 become

$$\mu^{(0)}(f,g) = fg \tag{16}$$

$$\mu^{(1)}(1_{s(f)}, f) = 0 \tag{17}$$

$$\mu^{(1)}(f, 1_{t(f)}) = 0 \tag{18}$$

$$\mu^{(1)}(a,b)c + \mu^{(1)}(ab,c) = a\mu^{(1)}(b,c) + \mu^{(1)}(a,bc) \tag{19}$$

Equation 19, as expected, says that $\mu^{(1)}$ is a Hochschild 2-cocycle, while equations 17 and 18 require it to be normalized in the obvious sense.

The equations defining an equivalence of such deformations, 6-8 become

$$\begin{aligned} \Phi^{(0)}(f) &= f && (20)\\ \Phi^{(1)}(1_x) &= 0 \text{ for all } x \in Ob(\mathcal{C}) && (21)\\ \mu_1^{(1)}(f,g) + \Phi^{(1)}(fg) &= \mu_2^{(1)}(f,g) + \Phi^{(1)}(f)g + f\Phi^{(1)}(g) && (22) \end{aligned}$$

That is, two first order deformations defined by $\mu_1^{(1)}$ and $\mu_2^{(1)}$ are equivalent exactly when the $\mu_i^{(1)}$'s are cohomologous.

So we establish the folk theorem generalizing the classical result of Gerstenhaber:

5.1. Theorem. *The first order deformations of a k-linear category $\mathcal{C}$ are classified up to equivalence by $\mathfrak{H}^2(\mathcal{C})$, the second Hochschild cohomology of the category.*

Similarly, equations 9-11 become

$$\begin{aligned} F^{(0)} &= F && (23)\\ F^{(1)}(1_x) &= 0 \text{ for all } x \in Ob(\mathcal{C}) && (24)\\ F(\mu^{(1)}(f,g)) + F^{(1)}(fg) &= \\ &\quad \nu^{(1)}(F(f),F(g)) + F^{(1)}(f)F(g) + F(f)F^{(1)}(g) && (25) \end{aligned}$$

Again, as expected, in the case where the deformations of the source and target categories are both trivial (i.e. $\mu^{(1)} \equiv 0$ and $\nu^{(1)} \equiv 0$), equation 25 says that $F^{(1)}$ is a Hochschild 1-cocycle, while in the general case it cobounds $F_*(\mu^{(1)}) - F^*(\nu^{(1)})$. Or, put another way, $(\mu^{(1)}, \nu^{(1)}, F^{(1)})$ are a 1-cocycle in the cone on $F^*(p_2) - F_*(p_1)$.

Weak equivalence between two first order deformations $(\hat{\mathcal{C}}_i, \hat{\mathcal{D}}_i, \hat{F}_i)$, $i = 1, 2$, of $F : \mathcal{C} \to \mathcal{D}$, then consists of equivalences of first order deformations of categories $\Gamma = Id_{\mathcal{C}} + \Gamma^{(1)}\epsilon$ from $\hat{\mathcal{C}}_1$ to $\hat{\mathcal{C}}_2$, and $\Delta = Id_{\mathcal{D}} + \Delta^{(1)}\epsilon$ from $\hat{\mathcal{D}}_1$ to $\hat{\mathcal{D}}_2$, together with ϕ given by $\phi_x = 1_x + \phi_x^{(1)}$ satisfying

$$\Delta^{(1)}(F(f)) + F_1^{(1)}(f) + F(f)\phi_y^{(1)} = \phi_x^{(1)}F(f) + F_2^{(1)}(f) + F(\Gamma^{(1)}(f)) \qquad (26)$$

for all $f : x \to y \in Arr(\mathcal{C})$.

It is easy to see in the case of a strong equivalence, where $\Delta^{(1)}$ and $\Gamma^{(1)}$ are both zero, that this says ϕ cobounds the difference of $F_1^{(1)}$ and $F_2^{(1)}$. Recalling a result from Gerstenhaber and Schack [9] makes it obvious what is happening in the general case: $(\Gamma^{(1)}, \Delta^{(1)}, \phi^{(1)})$ cobounds the difference $(\mu_2^{(1)}, \nu_2^{(1)}, F_2^{(1)}) - (\mu_1^{(1)}, \nu_1^{(1)}, F_1^{(1)})$ in the cone on $F^*(p_2) - F_*(p_1)$. So we have

5.2. Theorem. *The first order deformations of a k-linear functor $F : \mathcal{C} \to \mathcal{D}$ are classified up to weak equivalence by the first cohomology $\mathfrak{H}^1(F)$ of the deformation complex of F.*

Again this is the obvious generalization of the classical result for algebras (or rather algebra homomorphisms).

For deformations $(\hat{\mathcal{C}}, \hat{\mathcal{D}}, \hat{F}, \hat{G}, \hat{\sigma})$ of a natural transformation $\sigma : F \Rightarrow G$, for $F, G : \mathcal{C} \to \mathcal{D}$, equation 14 becomes

$$\begin{aligned} \nu^{(1)}(F(f), \sigma_y) + F^{(1)}(f)\sigma_y + F(f)\sigma_y^{(1)} = \\ \nu^{(1)}(\sigma_x, G(f)) + \sigma_x^{(1)}G(f) + \sigma_x G^{(1)}(f) \end{aligned} \tag{27}$$

Here the cohomological interpretation of this equation is not so clear. However, once the somewhat baroque definition of the cochain map $\sigma\ddagger$ is recalled, it is easy to see that this equation is the additional requirement on the fifth coordinate to ensure that $(\mu^{(1)}, \nu^{(1)}, F^{(1)}, G^{(1)}, \sigma^{(1)})$ be a 0-cocycle in the cone on $\sigma\ddagger$.

Similarly, equation 15 reduces to

$$\phi_x^{(1)}\sigma_x + \sigma_{2x}^{(1)} \quad = \quad \sigma_x\psi_x^{(1)} + \sigma_{1x}^{(1)} \tag{28}$$

This is the condition on the $C^0(F, G)$ coordinate for $(\Gamma^{(1)}, \Delta^{(1)}, \psi^{(1)}, \phi^{(1)}, 0)$ to cobound the difference of the $(\mu_i^{(1)}, \nu_i^{(1)}, G_i^{(1)}, F_i^{(1)}, \sigma_i^{(1)})$ $(i = 1, 2)$. The other conditions requiring the other coordinates to give equivalences of the deformations source and target functors and categories give the remaining conditions, so we have

5.3. Theorem. *The first order deformations of $\sigma : F \to G$ are classified up to weak equivalence by the zeroth cohomology $\mathfrak{H}^0(\sigma)$ of the deformation complex of σ.*

6. Higher order deformations and obstructions without pasting

Equations 5, 11, and 14 are the crucial defining conditions on deformations of categories, functors and natural transformations, respectively. The other equations are either normalization conditions to ensure correct behavior on identity arrows, conditions on the zeroth order term to ensure reduction to what is being deformed, or conditions inherited from the deformation of a source or target.

In each case, we can solve the equation of index n to separate the terms involving coefficients of index n from the other terms.

Equation 5 gives

$$\begin{aligned} &\sum_{i+j=n \; 0\leq i,j<n} \mu^{(i)}(a, \mu^{(j)}(b, c)) - \mu^{(i)}(\mu^{(j)}(a, b), c) \\ &= a\mu^{(n)}(b, c) - \mu^{(n)}(ab, c) + \mu^{(n)}(a, bc) - \mu^{(n)}(a, b)c \qquad (29) \\ &= \delta\mu^{(n)}(a, b, c) \qquad (30) \end{aligned}$$

As expected, the left-hand side is the same formula as Gerstenhaber's obstructions for the deformation theory of associative algebras, and the condition that the next term satisfies is that it cobounds the obstruction. Letting $\langle\epsilon\rangle$ be the maximal ideal of $k[\epsilon]/\langle\epsilon^n\rangle$ or $k[[\epsilon]]$ and $\otimes$ denote $\otimes_k$ or its $\langle\epsilon\rangle$-adic completion, as appropriate, these conditions for all n can be neatly packaged into the single condition that

$$\bar{\mu} = \sum \mu^{(i)}\epsilon^i \in C^2(\mathcal{C}) \otimes \langle\epsilon\rangle$$

satisfies the Maurer-Cartan equation

$$\delta(\bar{\mu}) = \frac{1}{2}[\bar{\mu}, \bar{\mu}],$$

where the (graded) Lie bracket is the obvious generalization of that given for algebras by Gerstenhaber, and δ is the extension of the coboundary operator by linearity (and continuity), or equivalently the equation

$$\delta(\bar{\mu}) = \bar{\mu}\{\bar{\mu}\}.$$

The presence of sources and targets is no impediment to the same proof as in [7] that each obstruction is always a 3-cocycle.

Similarly solving equation 11 to separate index n terms gives

$$\begin{aligned}
&\sum_{i+j=n,\, 0\leq i,j<n} F^{(i)}(\mu^{(j)}(f,g)) - \sum_{k+l+m=n\; 0\leq k,l,m<n} \nu^{(k)}(F^{(l)}(f), F^{(m)}(g)) \\
&= \quad F(f)F^{(n)}(g) - F^{(n)}(fg) + F^{(n)}(f)F(g) \\
&\qquad - F(\mu^{(n)}(f,g) + \nu^{(n)}(F(f), F(g))
\end{aligned} \tag{31}$$

The right-hand side, is, of course, the $C^2(F)$ summand of $\delta(\mu^{(n)}, \nu^{(n)}, F^{(n)})(f,g)$ in $\mathfrak{C}^2(F)$.

Letting $\bar{\mu}$ be defined as above, and $\bar{\nu}$ analogously, with $\bar{F} := \sum F^{(i)}\epsilon^i \in C^1(F) \otimes \langle\epsilon\rangle$, allows us collect all of the equations into the condition

$$\delta(\bar{F}, \bar{\mu}, \bar{\nu}) = (\bar{F}\{\bar{\mu}\} - \bar{\nu}\{\bar{F}\} - \bar{F}\bar{F} - \bar{\nu}\{\bar{F}, \bar{F}\}, \bar{\mu}\{\bar{\mu}\}, \bar{\nu}\{\bar{\nu}\}),$$

where

$$\delta(\bar{F}, \bar{\mu}, \bar{\nu}) = (\delta(\bar{F}) + F(\bar{\mu}) - \bar{\nu}(F,F), \delta(\bar{\mu}), \delta(\bar{\nu}))$$

since we are in the cone on $F^*(p_2) - F_*(p_1)$. Notice that since, $\bar{\mu}\{\bar{\mu}\} = \frac{1}{2}[\bar{\mu}, \bar{\mu}]$, and similarly for $\bar{\nu}$, the last two coordinates are simple restating the Maurer-Cartan equation for the deformation of the source and target categories.

Finally, solving equation 14 to separate index n terms gives

$$\begin{aligned}\sum_{i+j+k=n\,0\leq i,j,j<n} & \nu^{(i)}(F^{(j)}(f),\sigma_y^{(k)}) - \nu^{(i)}(\sigma_x^{(j)},G^{(k)}(f)) \\ = \quad & \nu^{(n)}(\sigma,G(f)) - \nu^{(n)}(F(f),\sigma) \\ & +\sigma^{(n)}G(f) - F(f)\sigma^{(n)} + \sigma G^{(n)}(f) - F^{(n)}(f)\sigma \end{aligned} \tag{32}$$

Here it is easy to verify that the right-hand side is the $C^1(F,G)$ summand of

$$\delta(\mu^{(n)},\nu^{(n)},F^{(n)},G^{(n)},\sigma^{(n)})$$

in $\mathfrak{C}^1(\sigma)$.

Again all instances of this equation can be collected into a single equation relating the differential and the cup-like and brace-like compositions: letting $\bar{\mu},\bar{\nu}$, and $\bar{F}$ be as above, and $\bar{G}$ be defined similarly, and letting $\bar{\sigma} = \sigma^{(i)}\epsilon^i$, this equation becomes

$$p_5(\delta(\bar{\sigma},\bar{F},\bar{G},\bar{\mu},\bar{\nu})) = \delta(\bar{\sigma}) - \bar{\nu}\{\bar{F},\sigma\} - \bar{\nu}\{\sigma,\bar{G}\} - \bar{\nu}\{\bar{\sigma}\} + \bar{F}\bar{\sigma} - \bar{\sigma}\bar{G} - \bar{\nu}\{\bar{F},\bar{\sigma}\} - \bar{\nu}\{\bar{\sigma},\bar{G}\}$$

The conditions on the other coordinates are those given previously.

Equations 29, 31 and 32, then identify the obstructions to extending an $n-1^{st}$ order deformation to an n^{th} order deformation of a category, functor or natural transformation, respectively.

As expected, we have

6.1. Theorem. *The obstruction*

$$\omega^n_{\mathcal{A}} := \sum_{i+j=n\,0\leq i,j<n} \mu^{(i)}(a,\mu^{(j)}(b,c)) - \mu^{(i)}(\mu^{(j)}(a,b),c)$$

(resp.

$$\omega^n_F := (\omega^n_A,\omega^n_B,\upsilon^n_F)$$

where the first and second coordinates are obstructions to deforming the categories $\mathcal{A}$ and $\mathcal{B}$ respectively, and

$$\upsilon^n_F := \sum_{i+j=n,\,0\leq i,j<n} F^{(i)}(\mu^{(j)}(f,g)) - \sum_{k+l+m=n\,0\leq k,l,m<n} \nu^{(k)}(F^{(l)}(f),F^{(m)}(g)),$$

$$\omega^n_\sigma := (\omega^n_A,\omega^n_B,\upsilon^n_F,\upsilon^n_G,\upsilon^n_\sigma)$$

where the first four coordinates are as described above and

$$\upsilon^n_\sigma := \sum_{i+j+k=n\,0\leq i,j,j<n} \nu^{(i)}(F^{(j)}(f),\sigma_y^{(k)}) - \nu^{(i)}(\sigma_x^{(j)},G^{(k)}(f)) \ )$$

is closed in $\mathfrak{C}^\bullet(\mathcal{A})$ *(resp.* $\mathfrak{C}^\bullet(F)$, $\mathfrak{C}^\bullet(\sigma)$*).*

PROOF. The statements concerning the obstructions in the case of categories (resp. functors) follows by the same proof given by Gerstenhaber [7, 8] in the case of algebras (resp. Gerstenhaber and Schack [9] in the case of algebra homomorphisms), as does the vanishing of all but the last coordinate in $\delta(\omega^n_\sigma)$.

It thus remains only to show that the last coordinate of $\delta(\omega^n_{\mathcal{A}}, \omega^n_{\mathcal{B}}, \omega^n_G, \omega^n_F, \omega^n_\sigma)$ vanishes.

The coboundary is

$$\begin{aligned}
&\delta_{F,G}(\upsilon^n_\sigma) - \sum_{i+j=n} \nu^{(i)}\{\nu^{(j)}\}\{\sigma\} - \\
&\qquad \sigma_x \left[\sum_{i+j=n} G^{(i)}(\mu^{(j)}(f,g)) - \sum_{i+j+k=n} \nu^{(i)}(G^{(j)}(f), G^{(k)}(g)) \right] + \\
&\qquad \left[\sum_{i+j=n} F^{(i)}(\mu^{(j)}(f,g)) - \sum_{i+j+k=n} \nu^{(i)}(F^{(j)}(f), F^{(k)}(g)) \right] \sigma_z
\end{aligned}$$

The key to the straightforward but tedious calculation which shows this vanishes is to immediately rewrite $\sum_{i+j=n} G^{(i)}(\mu^{(j)}(f,g))$ using equation 14 (or equivalently 32), cancel the terms involving $F^{(i)}$'s with those already in the original expression, then rewrite the terms still involving $\mu^{(j)}$'s using equation 11 (or 31).

Using equations 29, 31, and 32 the number of terms can be steadily reduced (though at four points in the calculation as carried out by the author, 32 must be used to replace three sums by three others). At two points the naturality of σ must be used. ■

The reader intent on recovering the complete calculation for him- or herself is advised to first carry out the case of $n = 2$, where equations 29, 31, and 32 are simply cocycle conditions. ■

7. Deformations induced by single compositions

Since deformations of categories, functors and natural transformations are themselves categories, functors and natural transformations, it is clear that the usual operations in the 2-category $k[[\epsilon]] - cat$ or $k[\epsilon]/\langle\epsilon^2\rangle - cat$ induce operations on the deformations, and in particular, that deformations of the parts of a composable pasting diagram induce a deformation of its pasting composition.

Relatively trivial calculations establish formulas for the induced deformation in the case of individual compositions:

7.1. PROPOSITION. *If* $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ *is a composition of* k*-linear functors with composite* $\Phi : \mathcal{A} \to \mathcal{C}$*, and* $\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}}, \tilde{F}$*, and* $\tilde{G}$ *are deformations of its parts, with* $\tilde{F} = \sum F^{(i)}\epsilon^i$ *and* $\tilde{G} = \sum G^{(i)}\epsilon^i$*, then*

$$\Phi^{(i)} := \sum_{j+k=i} G^{(j)}(F^{(k)})$$

are the terms of a deformation of Φ*, called* the induced deformation of Φ.

7.2. Proposition. *If* $F : \mathcal{A} \to \mathcal{B}$ *is a* k*-linear functor, and* $\sigma : G \Rightarrow H$ *is a natural transformation between* k*-linear functors* $G, H : \mathcal{B} \to \mathcal{C}$*, and* $\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}}, \tilde{F}, \tilde{G}, \tilde{H}$*, and* $\tilde{\sigma}$ *are deformations of its parts, with* $\tilde{\sigma}_X = \sum \sigma_X^{(i)} \epsilon^i$*, then* $\tilde{\tau} = \sum \sigma_{F(X)}^{(i)} \epsilon^i$ *is a deformation of* $\tau = \sigma_F$ *as a natural transformation from the induced deformation of* $G(F)$ *to the induced deformation of* $H(F)$.

7.3. Proposition. *If* $F, G : \mathcal{A} \to \mathcal{B}$ *are* k*-linear functors, and* $\sigma : F \Rightarrow G$ *is a natural transformation between them, and* $H : \mathcal{B} \to \mathcal{C}$ *is a* k*-linear functor and* $\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}}, \tilde{F}, \tilde{G}, \tilde{H}$*, and* $\tilde{\sigma}$ *are deformations of its parts, with* $\tilde{\sigma}_X = \sum \sigma_X^{(i)} \epsilon^i$*, and* $\tilde{F} = \sum F^{(i)} \epsilon^i$*, then* $\tilde{\tau} = \sum \tau^{(i)} \epsilon^i$ *given by*

$$\tau_X^{(i)} := \sum_{j+k=i} H^{(j)}(\sigma_X^{(k)})$$

is a deformation of $H(\sigma)$ *as a natural transformation from the induced deformation of* $H(F)$ *to the induced deformation of* $H(G)$.

Finally, we have

7.4. Proposition. *If* $F, G, H : \mathcal{A} \to \mathcal{B}$ *are* k*-linear functors, and* $\sigma : F \Rightarrow G$ *and* $\tau : G \Rightarrow H$ *are natural transformations with composite* $\phi : F \Rightarrow G$*, and* $\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \tilde{F}, \tilde{G}, \tilde{H}, \tilde{\sigma}$*, and* $\tilde{\tau}$ *are deformations of the parts, then*

$$\phi^{(i)} := \sum_{k+j+l=i} \nu^{(k)}(\sigma^{(j)}, \tau^{(l)}) \epsilon^i$$

defines a deformation of the composite ϕ.

Observe that in each proposition, the terms of the induced deformation can be expressed in terms of the brace-like 1-composition, being $G^{(j)}\{F^{(k)}\}$, $\sigma^{(i)}\{F\}$, $F^{(j)}\{\sigma^{(k)}\}$ and $\nu^{(k)}\{\sigma^{(j)}, \tau^{(l)}\}$, respectively.

It follows from Power's Pasting Theorem that deformations of all parts of a composable pasting diagram induce a deformation of the composite.

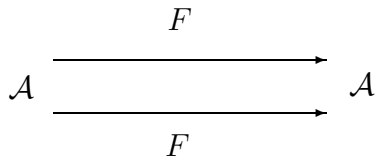
8. The cohomology of k-linear pasting diagrams

What remains now is to fit the parts introduced thus far together. To do this we must return to the description of a pasting diagram.

First, we should note that not just the image of the diagram, the categories, functors and natural transformations involved, but the 'shape' of the diagram will matter a great

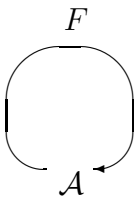
deal: even if the labeling of the pasting scheme includes coincidences, the different copies of the same category, functor, or natural transformation may be deformed independently.

Thus, for instance, in the diagram



a deformation will involve two, not necessarily equal, deformations of the category $\mathcal{A}$, and two, not necessarily equal, deformations of the functor F.

If we were interested, instead, in a deformation of the endofunctor F, as an endofunctor, that is a single deformation of $\mathcal{A}$, and a deformation of F in which both source and target are deformed by this deformation, we would, instead be studying the deformations of the diagram



In light of this discussion, it should be clear that the relevant cochain complex for the deformation of an entire pasting diagram will arise by iterated mapping cone constructions from cochain complexes associated to the labels of the various cells of the diagram.

In particular, the groups of cochains for a diagram

$$D : G \to 3 - computad(k - Cat),$$

are given by

$$\begin{aligned}\mathfrak{C}^\bullet(D) \;=\; & \bigoplus_{v\in G_0} C^{\bullet+3}(D(v)) \oplus \bigoplus_{e\in G_1} C^{\bullet+2}(D(e)) \oplus \\ & \bigoplus_{f\in G_2} C^{\bullet+1}(dom(D(f)), cod(D(f)) \oplus \\ & \bigoplus_{s\in G_3} C^{\bullet}(dom(cod(D(s))), cod(cod(D(s))))\end{aligned}$$

Here, of course each $D(v)$ is a linear category, each $D(e)$ is a functor, as are $cod(D(f))$, $dom(D(f))$, $cod(cod(D(s))) = cod(dom(D(s)))$, and $dom(cod(D(s))) = dom(dom(D(s)))$, these last four being the composition of the $D(e)$'s along the codomain of a 2-arrow (resp. the domain of a 2-arrow, the common codomain of the domain and codomain of a 3-cell,

and the common domain of the domain and codomain of a 3-cell). (Recall 3-cells have composable pasting diagrams as domain and codomain, and merely assert the equality of the composites.)

Notice here we are abusing notation somewhat by not explicitly applying $comp(-)$ to the domains and codomains of 2- and twice iterated domains and codomains of 3-cells, which properly are composable 1-pasting schemes. This should cause no confusion, as the Hochschild complexes are only defined for pairs of parallel 1-arrows, not for composable 1-pasting schemes. We retain this convention throughout what follows.

The tricky thing is to succinctly describe the coboundary maps. The dimension shifts hint at the construction: the coboundary maps will arise from an iterated mapping cone construction.

First we need

8.1. Proposition. *If D is a composable 1-pasting diagram, and F is a 1-arrow therein, then the pre- and post-composition cochain maps of Proposition 4.5 induce a unique cochain map*

$$\wp_D^F : C^\bullet(F) \to C^\bullet(comp(D)),$$

where $comp(D)$ denotes the composition of the arrows in D.

Similarly if D is a 1-pasting diagram which is the union of two composable 1-pasting diagrams D_F and D_G, which are identical except for the label on one element of the underlying G_1, which is F and G respectively, then the pre- and post-composition cochain maps of Proposition 4.5 induce a unique cochain map

$$\wp_D^{F,G} : C^\bullet(F, G) \to C^\bullet(comp(D_F), comp(D_G)).$$

Proof. The map is constructed simply by iterated application of the cochain maps of Proposition 4.5. It is unique by associativity of 1-composition. ■

Similarly, with the quite non-trivial proof of uniqueness in Power's Pasting Theorem [14] replacing the rather trivial proof that associativity implies uniqueness of iterated compositions, we have

8.2. Proposition. *If D is a composable 2-pasting diagram, and σ is a 2-arrow therein, the pre- and post-composition cochain maps of Propositions 4.5 and 4.6 induce a unique cochain map*

$$\wp_D^\sigma : C^\bullet(comp(dom(\sigma)), comp(cod(\sigma))) \to C^\bullet(comp(dom(D)), comp(cod(D)))$$

Armed with these results, we can now proceed to construct the coboundaries:

First, observe that there is a cochain map

$$\kappa_1 := \sum_{e \in G_1} \iota_e(D(e)^*(p_{cod(e)})) - \sum_{e \in G_1} \iota_e(D(e)_*(p_{dom(e)})) : \bigoplus_{v \in G_0} C^\bullet(D(v)) \to \bigoplus_{e \in G_1} C^\bullet(D(e)).$$

The components of this map, are, of course, the cochain maps arising in the construction of $\mathfrak{C}^\bullet(D(e))$ as in Section 5, and the fact that it is a cochain map follows from this.

The cone on this map, which we denote $\mathfrak{C}_1^\bullet(D)$ has cochain groups

$$\bigoplus_{v\in G_0} C^{\bullet+1}(D(v)) \oplus \bigoplus_{e\in G_1} C^\bullet(D(e))$$

This construction is simply a replication for each 1-cell of the pasting diagram of that given in Section 5. 1-cocycles in $\mathfrak{C}_1^\bullet(D)$ classify simultaneous deformation of all objects and functors in the diagram (recalling our earlier warning that occurrences of a functor or category at a different place in the diagram may be deformed differently.

For 2-cells, we cannot simply replicate the earlier construction, because in general the source and target are composable 1-pasting diagrams, rather than 1-cells. We therefore proceed in two steps, the second of which corresponds to replicating the construction in Section 5, while the first involves the maps $\wp_D^F$.

We then have

8.3. Proposition. *For any pasting diagram G,*

$$\wp := \sum_{v\in G_0} i_v(p_v) + \sum_{e\in\Delta\in diag_1(G)} i_\Delta(\wp^{D(e)}_{D(\Delta)}(p_e)) : \mathfrak{C}_1^\bullet(D) \to$$

$$\bigoplus_{v\in G_0} C^{\bullet+1}(D(v)) \oplus \bigoplus_{\Delta\in diag_1(G)} C^\bullet(comp(D(\Delta))),$$

is a cochain map, where $diag_1(G)$ denotes the set of composable 1-pasting diagrams in G. It assigns to a 1-cocycle in $\mathfrak{C}_1^\bullet(D)$ which names a deformation of all categories and functors in the diagram, a 1-cocycle in the target complex whose Δ summand gives the deformations of the domain and codomain categories of $D(\Delta)$ and the deformation of the $comp(D(\Delta))$ induce by the deformations of the composed functors.

Proof. That the first two coordinates of each summand behave correctly is immediate. That the third coordinate in each summand behaves correctly follows from the fact that the maps of Proposition 8.1 are cochain maps, and easily verified the cancellation of terms involving images of cochains associated to an intermediate category in the composable pasting diagram Δ. ■

And by replicating the construction of Section 5, there is a cochain map

$$\kappa_2 := \sum_{\sigma\in G_2} i_\sigma(D(\sigma) \ddagger (\pi_\sigma)) :$$
$$\bigoplus_{v\in G_0} C^{\bullet+1}(D(v)) \oplus \bigoplus_{\Delta\in diag_1(G)} C^\bullet(comp(D(\Delta))) \to$$
$$\bigoplus_{\sigma\in G_2} C^\bullet(dom(D(\sigma))), cod(D(\sigma)))),$$

where π_σ is the projection from the source onto its summands constituting

$$C^\bullet(dom(cod(D(\sigma))) \overset{dom(D(\sigma))}{cod(D(\sigma))} cod(cod(D(\sigma)))).$$

The cone on the composite $\kappa_2(\wp)$, which we denote $\mathfrak{C}_2^\bullet(D)$ then has cochain groups

$$\bigoplus_{v\in G_0} C^{\bullet+2}(D(v)) \oplus \bigoplus_{e\in G_1} C^{\bullet+1}(D(e)) \oplus \bigoplus_{f\in G_2} C^\bullet(dom(D(f)), cod(D(f)).$$

We now need one last cone construction to allow the presence of 3-cells in a pasting diagram to enforce equality between its source and target:

Define a cochain map

$$\kappa_3 := \sum_{c\in G_3,\ f\in cod(c)} \iota_c(\wp^{D(f)}_{cod(c)}(p_f)) - \sum_{c\in G_3,\ f\in dom(c)} \iota_c(\wp^{D(f)}_{dom(c)}(p_f)) :$$
$$\mathfrak{C}_2^\bullet(D) \to \bigoplus_{c\in G_3} C^\bullet(dom(cod(D(s))), cod(cod(D(s)))).$$

The cone on this map is then the desired cochain complex, which we denote $\mathfrak{C}^\bullet(D)$, and call *the deformation complex of the pasting scheme D*.

Observe that it is this last step that obliged us to define $\mathfrak{C}^\bullet(1_\sigma)$ to be a cone on the difference of two projections, rather than simply $\mathfrak{C}^\bullet(\sigma)$: in the context of a pasting scheme, the source and target of a 3-cell are, in general, pasting compositions of natural transformations, which must each be deformed. The only convenient artifice within the cohomological framework for enforcing equality of the induced deformations on the two sides of a 3-cell is to combine the maps $\wp^\sigma_D$ of Proposition 8.2 with the last cone construction of Section 4.

We then have

8.4. Theorem. *First order deformations of the pasting diagram D are classified up to equivalence by $\mathfrak{H}^{-1}(D)$, the negative-first cohomology of the deformation complex $\mathfrak{C}^\bullet(D)$.*

and

8.5. Theorem. *Given an $n-1^{st}$-order deformation of a pasting diagram D, there is a cocycle in $\mathfrak{C}^0(D)$, each direct summand of which is given by the formula for the obstruction to deforming the label on the cell of the computad indexing the direct summand given in Theorem 6.1. This cocycle is the obstruction to extending the deformation to an n^{th} order deformation, and if it vanishes in cohomology, any 0-cochain cobounding it gives the degree n term of an n^{th} order deformation extending the given deformation.*

Notice that the somewhat strange cohomological dimensions are correct: the cohomological dimension in the cone corresponds to the cohomological dimension in the groups associated to the 3-cells of the 3-computad, so the -1-cocycles in the cone have coordinates which are a -1-cochain for each 3-cell (necessarily 0, indicating equality between

its source and target), and, as expected, a 0-cochain for each 2-cell, a 1-cochain on each 1-cell, and a 2-cochain on each 0-cell, collectively satisfying the cocycle condition for the iterated mapping cone.

PROOFS: Both results would follow immediately from Theorems 5.1, 5.2, 5.3 and 6.1 and Propositions 7.1 through 7.4, (albeit with the cohomological dimension shifted up 1) were it not for the presence of 3-cells indicating commutative parts of the pasting diagram. However, the presence of 3-cells, their own labels being undeformable, enforces the equality of the induced deformations on their source and target, since a 0-cocycle (in this case the difference between either the induced first-order deformations of the source and target, or the difference of the cocycles induced by the obstruction on each face of the source and target is trivial in cohomology if and only if it is zero, there being only the zero -1-cochain to cobound it. ■

9. Prospects

As regards the first motivation for this paper: in work in progress, a doctoral student under the author is attempting the construction of a cohomology theory governing the simultaneous deformation of the composition, arrow-part of the monoidal product, and structure maps of a monoidal category. It appears that the deformations are governed by the total complex of a 'multicomplex'–a bigraded object which 'looks like a spectral sequence with all the pages smashed together'. The theory of the present paper gives the (0,1)-differentials in one direction, while the (1,0)-differentials are given by the differentials of [18, 19].

As regards the second motivation: it is easy to see that k-linear stacks are a special case of pasting diagrams, indeed, k-linear pre-stacks are more or less the same thing as k-linear pasting diagrams. It is clear that deformation of a pre-stack (as a pasting diagram) cannot create new descent data, since any descent data in an order n deformation must be descent data at all lower orders, as quotienting by powers of ϵ will preserve commutativity. On the other hand, in general deformation can destroy commutativity, and thus descent data. It appears natural conjecture that any effective descent data which is not destroyed by a given deformation remains effective in the deformation.

The author plans to prove this conjecture in subsequent work.

References

[1] M. Anel. Moduli of linear categories: examples of higher stacks. lecture at IHES, 26-04-2006.

[2] J. Baez and J. Barrett. The quantum tetrahedron in 3 and 4 dimensions. *Adv. Theor. Math. Phys.*, 4:815–850, 1999.

[3] L. Crane. Categorical geometry and the mathematical foundations of quantum gravity. arXiv:gr-qc/06021202, 2006.

[4] L. Crane. What is the mathematical structure of quantum spacetime? arXiv:0706.4452, 2007.

[5] L. Crane and D.N. Yetter. Deformations of (bi)tensor categories. *Cahier de Topologie et Géometrie Differentielle Catégorique*, 1998.

[6] J. Elgueta. Cohomology and deformation theory of monoidal 2-categories, i. *Adv. in Math.*, 182:204–277, 2004.

[7] M. Gerstenhaber. The cohomology of an associative ring. *Ann. of Math.*, 78(2):267–288, 1963.

[8] M. Gerstenhaber. On the deformation of rings and algebras. *Ann. of Math.*, 79(1):59–103, 1964.

[9] M. Gerstenhaber and S.D Schack. On the deformation of algebra morphisms and diagrams. *Trans. Amer. Math. Soc.*, 279:1–50, 1983.

[10] M. Gerstenhaber and A.A. Voronov. Homotopy g-algebras and moduli space operad. *Int. Math. Res. Notices*, 3:141–153, 1995.

[11] Michael Johnson. *Pasting Diagrams in n-Categories with Applications to Coherence Theorems and Categories of Paths*. PhD thesis, University of Sydney, 1988. Univ. of Sydney Research Report 88-14.

[12] A. Lazarev. Hochschild cohomology and moduli spaces of strongly homotopy associative algebras. *Homology Homotopy Appl.*, 5(1):73–100, 2003.

[13] B. Mitchell. Rings with several objects. *Adv. in Math.*, 8:1–161, 1972.

[14] J. Power. An n-categorical pasting theorem. In et al Carboni, A., editor, *Category Theory*, pages 326–358. Springer-Verlag, Berlin, 1991. SLNM no. 1488.

[15] R. Street. The algebra of oriented simplexes. *Journal of Pure and Applied Algebra*, 49:283–335, 1987.

[16] B. Tsygan. Deformation quantization of gerbes. unpublished lecture at University of Pennsylvania, 22-03-2006.

[17] D.N. Yetter. Braided deformations of monoidal categories and vassiliev invariants. In E. Getzler and M. Kapranov, editors, *Higher Category Theory*, pages 117–134. American Mathematical Society, 1998. A.M.S. Contemp. Math. vol. 230.

[18] D.N. Yetter. *Functorial Knot Theory: Categories of Tangles, Coherence, Categorical Deformations, and Topological Incariants*. World Scientific, Singapore, 2001. Series on Knots and Everything, vol. 26.

[19] D.N. Yetter. Abelian categories of modules over a (lax) monoidal functor. *Adv. in Math.*, 174:266–309, 2003.

Department of Mathematics
Kansas State University
Manhattan, KS 66506
Email: dyetter@math.ksu.edu

This article may be accessed at http://www.tac.mta.ca/tac/ or by anonymous ftp at ftp://ftp.tac.mta.ca/pub/tac/html/volumes/22/2/22-02.{dvi,ps,pdf}

THEORY AND APPLICATIONS OF CATEGORIES (ISSN 1201-561X) will disseminate articles that significantly advance the study of categorical algebra or methods, or that make significant new contributions to mathematical science using categorical methods. The scope of the journal includes: all areas of pure category theory, including higher dimensional categories; applications of category theory to algebra, geometry and topology and other areas of mathematics; applications of category theory to computer science, physics and other mathematical sciences; contributions to scientific knowledge that make use of categorical methods.
Articles appearing in the journal have been carefully and critically refereed under the responsibility of members of the Editorial Board. Only papers judged to be both significant and excellent are accepted for publication.
Full text of the journal is freely available in .dvi, Postscript and PDF from the journal's server at `http://www.tac.mta.ca/tac/` and by ftp. It is archived electronically and in printed paper format.

SUBSCRIPTION INFORMATION. Individual subscribers receive abstracts of articles by e-mail as they are published. To subscribe, send e-mail to `tac@mta.ca` including a full name and postal address. For institutional subscription, send enquiries to the Managing Editor, Robert Rosebrugh, `rrosebrugh@mta.ca`.

INFORMATION FOR AUTHORS. The typesetting language of the journal is TEX, and LATEX2e strongly encouraged. Articles should be submitted by e-mail directly to a Transmitting Editor. Please obtain detailed information on submission format and style files at `http://www.tac.mta.ca/tac/`.

MANAGING EDITOR. Robert Rosebrugh, Mount Allison University: `rrosebrugh@mta.ca`

TEXNICAL EDITOR. Michael Barr, McGill University: `barr@math.mcgill.ca`

ASSISTANT TEX EDITOR. Gavin Seal, McGill University: `gavin_seal@fastmail.fm`

TRANSMITTING EDITORS.
Richard Blute, Université d' Ottawa: `rblute@uottawa.ca`
Lawrence Breen, Université de Paris 13: `breen@math.univ-paris13.fr`
Ronald Brown, University of North Wales: `ronnie.profbrown (at) btinternet.com`
Aurelio Carboni, Università dell Insubria: `aurelio.carboni@uninsubria.it`
Valeria de Paiva, Cuill Inc.: `valeria@cuill.com`
Ezra Getzler, Northwestern University: `getzler(at)northwestern(dot)edu`
Martin Hyland, University of Cambridge: `M.Hyland@dpmms.cam.ac.uk`
P. T. Johnstone, University of Cambridge: `ptj@dpmms.cam.ac.uk`
Anders Kock, University of Aarhus: `kock@imf.au.dk`
Stephen Lack, University of Western Sydney: `s.lack@uws.edu.au`
F. William Lawvere, State University of New York at Buffalo: `wlawvere@acsu.buffalo.edu`
Jean-Louis Loday, Université de Strasbourg: `loday@math.u-strasbg.fr`
Ieke Moerdijk, University of Utrecht: `moerdijk@math.uu.nl`
Susan Niefield, Union College: `niefiels@union.edu`
Robert Paré, Dalhousie University: `pare@mathstat.dal.ca`
Jiri Rosicky, Masaryk University: `rosicky@math.muni.cz`
Brooke Shipley, University of Illinois at Chicago: `bshipley@math.uic.edu`
James Stasheff, University of North Carolina: `jds@math.unc.edu`
Ross Street, Macquarie University: `street@math.mq.edu.au`
Walter Tholen, York University: `tholen@mathstat.yorku.ca`
Myles Tierney, Rutgers University: `tierney@math.rutgers.edu`
Robert F. C. Walters, University of Insubria: `robert.walters@uninsubria.it`
R. J. Wood, Dalhousie University: `rjwood@mathstat.dal.ca`