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A METRIC TANGENTIAL CALCULUS

We dedicate this article to Dominique Bourn.

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Abstract. The metric jets, introduced here, generalize the jets (at order one) of Charles Ehresmann. In short, for a "good" map f (said to be "tangentiable" at a) between metric spaces, we define its metric jet tangent at a (composed of all the maps which are locally lipschitzian at a and tangent to f at a) called the "tangential" of f at a, and denoted $\mathrm{T}f_a$. So, in this metric context, we define a "new differentiability" (called "tangentiability") which extends the classical differentiability (while preserving most of its properties) to new maps, traditionally pathologic.

Introduction

First, let us mention that most of the proofs concerning the statements given here can be found in the first chapter of a paper published in arXiv (see [4]).

This paper contains a reference to two previous talks (see [2] and [3]). Our aim being a deep thought inside the fundamentals of differential calculus. Focussing on what is at the heart of the notion of differential, it is the concept of "tangency" which imposed itself in its great simplicity. Now, amazingly, this concept of tangency can be formulated without resorting to the whole traditional structure of normed vector space (here on $\mathbb{R}$), which will be denoted n.v.s.: see Section 1. It is thus the more general structure of metric space in which we work from now on, asking if it is possible to construct a meaningful "metric differential calculus". We will see that this aim has been essentially reached, even if, on the way, it required the help of an additional structure (the "transmetric" structure).

The first challenge was in the very formulation of a "differential" in a metric context: what can we replace the continuous affine maps with, though they are essential to the definition of the classical differentials? Jets (that we call metric jets in order to emphasize that the metric structure is enough to define them) will play the part of these continuous affine maps, willingly forgetting their algebraic feature. So, we introduce a "new differential" for a map f which admits a tangent at a which is locally lipschitzian at a (such a map being said to be "tangentiable" at a): it is a metric jet, tangent to f at a, called the "tangential" of f at a, and denoted $\mathrm{T}f_a$.

As is well known, jets were first introduced by C.Ehresmann in 1951 [5], in order to adapt Taylor's expansions to differential geometry; more precisely, his infinitesimal jets

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(at order one) can be seen as equivalence classes of maps of class C^1 between differentiable manifolds, under an equivalence relation of tangency. The metric jets we are proposing here are more general (being also equivalence classes of locally lipschitzian maps between metric spaces, under an analogous equivalence relation of tangency).

In analogy with the distance between two continuous affine maps, we will construct a metric to evaluate the distance between two metric jets: fixing a pair of points (a and a', respectively in the metric spaces M and M'), the set $\mathbb{J}\mathrm{et}((M,a),(M',a'))$ of the metric jets from (M,a) to (M',a') can, itself, be equipped with a metric structure!

Noticing that our distance between metric jets does not fit to speak of the distance between the tangentials of one map at two different points (these tangentials being metric jets which are tangent to this map at different points), we add a geometrical structure to our metric spaces, inspired by the translations of the usual vector space framework. These particular metric spaces are called "transmetric". In their frame, we introduce the notion of "free metric jets" (invariant by given "translation jets"). Now, if M and M' are such transmetric spaces, the set $\mathbb{J}\mathrm{et}_{\mathit{free}}(M,M')$ of the free metric jets from M to M' can be equipped with a metric structure.

Among these free metric jets, we find the free metric jets of the form $\mathrm{t}f_a$ "associated" to the tangential $\mathrm{T}f_a$ for an f supposed to be tangentiable at a. Finally, when $f : M \longrightarrow M'$ is a tangentiable map (i.e. tangentiable at every point of M), we construct its tangential $\mathrm{t}f : M \longrightarrow \mathbb{J}\mathrm{et}_{\mathit{free}}(M,M')$, whose domain and codomain are here metric spaces (since M and M' are transmetric spaces). We thus obtain a new map on which we can apply the different techniques of the new theory, as, for instance, to study the continuity or the tangentiability of this tangential $\mathrm{t}f$.

For general definitions in category theory (for instance cartesian or enriched categories), see [1].

Acknowledgements: It is a talk about Ehresmann's jets, given by Francis Borceux at the conference organised in Amiens in 2002 in honour of Andrée and Charles Ehresmann which has initiated our work. Since at that epoch we where interested, in our teaching, in what could be described uniquely with metric tools . . . hence the idea of the metric jets!

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1. The relation of tangency

The relation of tangency, which is at the heart of the differential calculus, is an essentially metric notion, since it can merely be written: $\lim_{a \neq x \to a} \frac{d(f(x),g(x))}{d(x,a)} = 0$ for two maps $f, g : M \longrightarrow M'$. However, this definition uses the term "$\lim_{a \neq x \to a}$" which makes sense solely for a point a, not isolated in M; this lead to a more general definition.

Furthermore, to make this relation of tangency compatible with the composition, we have to restrict the type of maps on which we will work. Then we will be able to define the concepts of metric jets and of tangentials.

The right definition of the notion of tangency will arise from the following equivalences, where M and M' are two metric spaces, a a fixed point in M and $f, g : M \longrightarrow M'$ two maps (a priori without any hypothesis).

1.1. PROPOSITION. *The following properties are equivalent :*

(i) $\forall \varepsilon > 0 \ \exists \eta > 0 \ \forall x \in M \qquad (d(x,a) \leq \eta \Longrightarrow d(f(x), g(x)) \leq \varepsilon d(x,a))$,

(ii) $f(a) = g(a)$ *and the map* $C : M \longrightarrow \mathbb{R}_+$ *defined by*

$$C(a) = 0 \ \ \textit{and} \ \ C(x) = \frac{d(f(x), g(x))}{d(x,a)} \ \ \forall x \neq a$$

is continuous at a,

(iii) there exists a map $c : M \longrightarrow \mathbb{R}_+$ *which is continuous at* a *and which verifies:* $c(a) = 0$ *and* $\forall x \in M \ (d(f(x), g(x)) = c(x) d(x,a))$,

(iv) there exist a neighborhood V *of* a *in* M *and a map* $c : V \longrightarrow \mathbb{R}_+$ *which is continuous at* a *and which verifies:* $c(a) = 0$ *and* $\forall x \in V \ (d(f(x), g(x)) \leq c(x) d(x,a))$.

1.2. DEFINITION. *We say that* f *and* g *are* tangent *at* a *(which is denoted by* $f \asymp_a g$*) if they verify anyone of the equivalent conditions of the above 1.1.*

1.3. REMARKS.

1. When a is not an isolated point in M (i.e $a \in \overline{M - \{a\}}$), we have: $f \asymp_a g$ iff $(f(a) = g(a)$ and $\lim_{a \neq x \to a} \frac{d(f(x), g(x))}{d(x,a)} = 0)$.
2. When a is an isolated point in M, we have $f \asymp_a g$ for any f and g verifying $f(a) = g(a)$.
3. The relation $\asymp_a$ is an equivalence relation on the set of maps from M to M' ; this relation $\asymp_a$ is called the relation of tangency at the point a.
4. If $f \asymp_a g$, then f is continuous at a iff g is continuous at a.

To study the behaviour of the relation of tangency towards composition, we consider the following situation (S) :

$$M_0 \xrightarrow{f_0} M_1 \underset{g_1}{\overset{f_1}{\rightrightarrows}} M_2 \xrightarrow{f_2} M_3$$

where M_0, M_1, M_2, M_3 are metric spaces with $a_o \in M_0$, $a_1 = f_0(a_0)$, $a_2 = f_1(a_1) = g_1(a_1)$. Under what conditions do we have one of the implications :

$$f_1 \asymp_{a_1} g_1 \implies f_1.f_0 \asymp_{a_0} g_1.f_0 \qquad \text{and} \qquad f_1 \asymp_{a_1} g_1 \implies f_2.f_1 \asymp_{a_1} f_2.g_1.$$

1.4. REMARK. The above implications are not true in general, even if the maps are continuous. Consider

$$M_0 = M_1 = M_2 = M_3 = \mathbb{R}$$

with

$$f_0 = f_2 : x \mapsto x^{1/3}\ ,\ f_1 : x \mapsto x^3 \ \text{ and } \ g_1 : x \mapsto 0$$

Even though $f_1 \asymp_0 g_1$, however $f_1.f_0 \not\asymp_0 g_1.f_0$ and $f_2.f_1 \not\asymp_0 f_2.g_1$

In order to give sufficient conditions to make the above implications true (see 1.6 below), we need the following definition:.

1.5. DEFINITION. *Let M and M' be metric spaces, $f : M \longrightarrow M'$ a map, and $a \in M$; let also k be a strictly positive real number. We say that :*

1. *f is* locally k-lipschitzian *at a (in short k-LL_a) if there exists a neighborhood V of a in M for which the restriction $f : V \longrightarrow M'$ is k-lipschitzian. f* locally lipschitzian *at a (in short LL_a) means that there exists $k > 0$ such that f is k-LL_a.*
2. *f is* k-semi-lipschitzian *at a (in short k-SL_a) if we have: $\forall x \in M\ (d(f(x), f(a)) \leq kd(x, a)))$. f* semi-lipschitzian *at a (in short SL_a) means that there exists $k > 0$ such that f is k-SL_a.*
3. *f is* locally k-semi-lipschitzian *at a (in short k-LSL_a) if, on a neighborhood V of a in M, the restriction $f : V \longrightarrow M'$ is k-SL_a. f* locally semi-lipschitzian *at a (in short LSL_a) means that there exists $k > 0$ such that f is k-LSL_a.*

Naturally, f is LL or LSL will mean that f is LL_a or LSL_a at every point $a \in M$.

1.6. REMARKS. Let M and M' be metric spaces, $f : M \longrightarrow M'$ a map, and $a \in M$.

1. We have the implications: $f\ LL_a \implies f\ LSL_a$ and $f\ LSL_a \implies f$ continuous at a. The inverses of these implications are not true (see Section 3).
2. In the above situation (S), let us assume that $f_1 \asymp_{a_1} g_1$; we then have the two following implications :
   (a) $f_0\ LSL_{a_0} \implies f_1.f_0 \asymp_{a_0} g_1.f_0$,
   (b) $f_2\ LL_{a_2}$ and f_1, g_1 continuous at $a_1 \implies f_2.f_1 \asymp_{a_1} f_2.g_1$.
3. Let $g : M \longrightarrow M'$ such that $f \asymp_a g$; we then have (still by 1.1) the equivalence: $f\ LSL_a \iff g\ LSL_a$.
4. Let E, E' be two n.v.s., U an open subset of E, $a \in U$ and $f : U \longrightarrow E'$ a map; let us denote $L(E, E')$ the set of continuous linear maps from E to E'. We have the implications :
   (a) f differentiable at $a \implies f\ LSL_a$,

(b) f differentiable and $\mathrm{d}f : U \longrightarrow L(E, E')$ continuous at $a \Longrightarrow f\ LL_a$ (in particular, f of class $C^1 \implies f\ LL$).

The inverses of these implications are not true (see Section 3).

5. Let M_0, M_1, M_2 be metric spaces ; $f_0 : M_0 \longrightarrow M_1$, $f_1 : M_1 \longrightarrow M_2$ two maps, and $a_0 \in M_0$, $a_1 = f_0(a_0)$. We have the implications:

   (a) $f_0\ LSL_{a_0}$ and $f_1\ LSL_{a_1} \implies f_1.f_0\ LSL_{a_0}$.

   (b) $f_0\ LL_{a_0}$ and $f_1\ LL_{a_1} \implies f_1.f_0\ LL_{a_0}$.

6. Let M_0, M_1, M_2 be metric spaces, $a_0 \in M_0$; and also maps $f_0, g_0 : M_0 \longrightarrow M_1$, $f_1, g_1 : M_1 \longrightarrow M_2$, with $a_1 = f_0(a_0) = g_0(a_0)$. We assume that $f_0 \asymp_{a_0} g_0$ and $f_1 \asymp_{a_1} g_1$ where g_0 is LL_{a_0} and g_1 is LL_{a_1} ; then $f_1.f_0 \asymp_{a_0} g_1.g_0$.
   We could have weakened the hypothesis : $g_0\ LSL_{a_0}$ would have been enough.

7. Let M, M_0, M_1 be metric spaces, $a \in M$; and also maps $f_0, g_0 : M \longrightarrow M_0$ and $f_1, g_1 : M \longrightarrow M_1$. We have the implication:

$$f_0 \asymp_a g_0 \ \text{ and } \ f_1 \asymp_a g_1 \implies (f_0, f_1) \asymp_a (g_0, g_1)$$

8. Let M, M_0, M_1 be metric spaces, $a \in M$; and also maps $f_0 : M \longrightarrow M_0$ and $f_1 : M \longrightarrow M_1$. Then:

   (a) $f_0, f_1\ LSL_a \implies (f_0, f_1)\ LSL_a$,

   (b) $f_0, f_1\ LL_a \implies (f_0, f_1)\ LL_a$.

This implies that the categories whose objects are metric spaces and whose morphisms are maps which are LSL (resp. LL) at a point, are cartesian categories.

We conclude this section by giving a metric generalization of the mean value theorem (weakening the hypothesis of being differentiable by the one of being LSL).

1.7. Proposition. *Let M be a metric space, $[a, b]$ a compact interval of $\mathbb{R}$, $k > 0$ a fixed real number and $f : [a, b] \longrightarrow M$ a continuous map which is k-LSL_x for all x in the open interval $]a, b[$. Then we have $d(f(b), f(a)) \leq k(b - a)$.*

For the proof, we use the following well-known lemma :

1.8. Lemma. *Let $g : [a, b] \longrightarrow \mathbb{R}$ be a continuous map and k a real number such that the following property is true:*

$$\forall x \in]a, b[\ \exists x' \in]a, b]\ (x' > x \text{ and } g(x') - g(x) \leq k(x' - x)).$$

Then we have $g(b) - g(a) \leq k(b - a)$.

1.9. Corollary.

1. *Let M be a metric space, $[a, b]$ a compact interval of $\mathbb{R}$, F a finite subset of $]a, b[$; let also $f : [a, b] \longrightarrow M$ be a continuous map. Let us assume that, for all $x \in]a, b[-F$, the map f is k-LSL_x ; then $d(f(b), f(a)) \leq k(b - a)$.*

2. *Let E be a n.v.s., U an open subset of E, $a, b \in U$ such that $[a, b] \subset U$ and F a finite subset of $]a, b[$; let also M be a metric space and $f : U \longrightarrow M$ a continuous map. Let us assume that, for all $x \in]a, b[-F$, the map f is k-LSL_x ; then, we have again: $d(f(b), f(a)) \leq k\|b - a\|$.*

2. Metric jets

The metric jets (in short, the jets), which are merely equivalence classes for the relation of tangency, will play the part of the continuous affine maps of the classical differential calculus (but here, without any algebraic properties); it seems natural to equip the set of jets with a metric structure. Thanks to this metric structure, we will enrich the category of jets, between pointed metric spaces, in the category $\mathbb{M}\mathrm{et}$ (a well chosen category of metric spaces).

M and M' being metric spaces, with $a \in M$, $a' \in M'$, let us denote

$$\mathbb{LL}((M, a), (M', a'))$$

the set of maps $f : M \longrightarrow M'$ which are LL_a and which verify $f(a) = a'$. These sets $\mathbb{LL}((M, a), (M', a'))$ are the "Hom" of a category, denoted $\mathbb{LL}$, whose objects are pointed metric spaces; this category $\mathbb{LL}$ is a cartesian category (i.e. it has a final object and finite products). Now, since $\asymp_a$ is an equivalence relation on $\mathbb{LL}((M, a), (M', a'))$, we set

$$\mathbb{J}\mathrm{et}((M, a), (M', a')) = \mathbb{LL}((M, a), (M', a'))/ \asymp_a$$

2.1. Definition. *An element of* $\mathbb{J}\mathrm{et}((M, a), (M', a'))$ *is called a* jet *from* (M, a) *to* (M', a').

Let $q : \mathbb{LL}((M, a), (M', a')) \longrightarrow \mathbb{J}\mathrm{et}((M, a), (M', a'))$ be the canonical surjection. Referring to Section 1, we can compose the jets: $q(g.f) = q(g).q(f)$ when g and f are composable. So, we construct a category, denoted $\mathbb{J}\mathrm{et}$, called the category of jets, whose:

- objects are pointed metric spaces (M, a),
- morphisms $\varphi : (M, a) \longrightarrow (M', a')$ are jets (i.e. elements of $\mathbb{J}\mathrm{et}((M, a), (M', a'))$).

The previous canonical surjections extend to a functor

$$q \colon \mathbb{LL} \longrightarrow \mathbb{J}\mathrm{et}$$

(constant on the objects) which makes $\mathbb{J}\mathrm{et}$ a quotient category of $\mathbb{LL}$.

2.2. Proposition. *The functor $q : \mathbb{LL} \longrightarrow \mathbb{J}\mathrm{et}$ creates a cartesian structure on the category $\mathbb{J}\mathrm{et}$ (q being constant on the objects, it means that $\mathbb{J}\mathrm{et}$ is cartesian and q a strict morphism of cartesian categories).*

2.3. Remarks.

1. Let $\varphi : (M,a) \longrightarrow (M',a')$ be a morphism in $\mathbb{J}$et and $f \in \varphi$. If f is locally "anti-lipschitzian" at a (i.e if there exist $k > 0$ and a neighborhood V of a on which we have $d(f(x), f(y)) \geq kd(x,y)$), then φ is a monomorphism in $\mathbb{J}$et. The jet of an isometric embedding is thus a monomorphism (in particular, in the case of a metric subspace).

2. Let M be a metric space, V a neighborhood of $a \in M$. Let us set $j_a = q(j) : (V,a) \longrightarrow (M,a)$ where $j : V \hookrightarrow M$ is the canonical injection. Then, the jet j_a is an isomorphism in $\mathbb{J}$et.

Time has now come to equip the category $\mathbb{J}\text{et}((M,a),(M',a'))$ with a metric structure (where $(M,a),(M',a') \in |\mathbb{J}\text{et}|$).

First, we define $d(f,g)$ for $f,g \in \mathbb{LL}((M,a),(M',a'))$; for such f,g, we consider the map $C : M \longrightarrow \mathbb{R}_+$ defined by $C(x) = \frac{d(f(x),g(x))}{d(x,a)}$ if $x \neq a$ and $C(a) = 0$. We notice that C is bounded on a neighborhood of a: indeed, since f and g are LL_a, there exist a neighborhood V of a and a real number $k > 0$ such that the restrictions $f|_V$ and $g|_V$ are k-lipschitzian. Then, for $x \in V$, we have: $d(f(x),g(x)) \leq d(f(x),a') + d(a',g(x)) \leq d(f(x),f(a)) + d(g(a),g(x)) \leq 2kd(x,a)$, so that $C(x) \leq 2k$ for all $x \in V$.

Now, for each $r > 0$, we set $d^r(f,g) = \sup\{C(x) \,|\, x \in B'(a,r) \cap V\}$ (where $B'(a,r)$ is a closed ball; this definition does not depend on V for small r). The map $r \mapsto d^r(f,g)$ is increasing and positive, we can put: $d(f,g) = \lim_{r\to 0} d^r(f,g) = \inf_{r>0} d^r(f,g)$.

2.4. Proposition. *Let $d : (\mathbb{LL}((M,a),(M',a')))^2 \longrightarrow \mathbb{R}_+$ be the map defined just above.*

I. For each $f,g,h \in \mathbb{LL}((M,a),(M',a'))$, the map d verifies the following properties:

1) $d(f,g) = d(g,f)$,

2) $d(f,h) \leq d(f,g) + d(g,h)$,

3) $d(f,g) = 0 \iff f \asymp_a g$.

II. The map d factors through the quotient, giving a "true" distance on the category $\mathbb{J}\text{et}((M,a),(M',a'))$, defined by $d(q(f),q(g)) = d(f,g)$ for $f,g \in LL((M,a),(M',a'))$.

Now, we need to establish some technical properties about what we call the lipschitzian ratio of a jet (that we also need in Section 4).

2.5. Definition. *For $\varphi \in \mathbb{J}\text{et}((M,a),(M',a'))$, we set*

$$\rho(\varphi) = \inf K(\varphi)\,, \text{ where } K(\varphi) = \{k > 0 \,|\, \exists f \in \varphi,\ f \text{ is } k\text{-}LL_a\}$$

We call $\rho(\varphi)$ the lipschitzian ratio *of φ. Furthermore, we will say that φ is k-*bounded *if $\rho(\varphi) \leq k$.*

2.6. Proposition. (Properties of ρ)

1. *Let (M_0,a_0), (M_1,a_1), $(M_2,a_2) \in |\mathbb{J}\mathrm{et}|$; and also jets $\varphi_0 : (M_0,a_0) \longrightarrow (M_1,a_1)$, $\varphi_1 : (M_1,a_1) \longrightarrow (M_2,a_2)$. Then, $\rho(\varphi_1.\varphi_0) \leq \rho(\varphi_1)\rho(\varphi_0)$.*

2. *For each $\varphi \in \mathbb{J}\mathrm{et}((M,a),(M',a'))$, we have $d(\varphi, O_{aa'}) \leq \rho(\varphi)$ (where $O_{aa'} = q(\widehat{a'})$, and $\widehat{a'} : M \longrightarrow M'$ is the constant map on a').*

2.7. Definition. *We say that a jet φ is a* good jet *if the previous inequality given in 2.6.2 becomes an equality.*

2.8. Examples. In all that follows $(M,a),(M',a'),(M_i,a_i)$ are objects of $\mathbb{J}\mathrm{et}$, i.e pointed metric spaces.

0. We have $\rho(O_{aa'}) = 0$.

1. For every jet $\varphi : (M,a) \longrightarrow (M',a')$, where a or a' are isolated (respectively in M or M'), then $\rho(\varphi) = 0$.

2. If we denote $\pi_i : (M_1,a_1) \times (M_2,a_2) \longrightarrow (M_i,a_i)$ the canonical projections in $\mathbb{J}\mathrm{et}$; then $d(\pi_i, O_{aa_i}) = \rho(\pi_i) = 1$ (where $a = (a_1,a_2)$, with a_i non isolated in M_i).

3. Let M, M' be metric spaces, $f : M \longrightarrow M'$ an isometric embedding, a a non isolated point in M and $a' = f(a)$; then $d(q(f), O_{aa'}) = \rho(q(f)) = 1$. Thus, a being not isolated in M, $Id_{(M,a)}$, j_a and j_a^{-1} are good jets (see 2.3 for j_a).

2.9. Remarks. Let $(M,a),(M',a') \in \mathbb{J}\mathrm{et}$.

1. Let us assume that there exists $\varphi \in \mathbb{J}\mathrm{et}((M,a),(M'a'))$ which is an isomorphism in $\mathbb{J}\mathrm{et}$. Then, a is isolated in M iff a' is isolated in M'.

2. (M,a) is a final object in $\mathbb{J}\mathrm{et}$ iff a is isolated in M.

3. Let (M,a), (M_1,a_1), $(M_2,a_2) \in |\mathbb{J}\mathrm{et}|$; consider two jets $\varphi_1 : (M,a) \longrightarrow (M_1,a_1)$ and $\varphi_2 : (M,a) \longrightarrow (M_2,a_2)$. Then $\rho(\varphi_1,\varphi_2) = \sup_i(\rho(\varphi_i))$.

4. For each $i \in \{1,2\}$, let $(M_i,a_i),(M'_i,a'_i) \in |\mathbb{J}\mathrm{et}|$, and $\psi_i : (M_i,a_i) \longrightarrow (M'_i,a'_i)$ be two jets. Then $\rho(\psi_1 \times \psi_2) \leq \sup_i \rho(\psi_i)$.

2.10. Theorem. *Let us consider the following diagram in $\mathbb{J}\mathrm{et}$:*

$$\begin{tikzcd} (M_0,a_0) \arrow[r, shift left, "\varphi_0"] \arrow[r, shift right, "\psi_0"'] & (M_1,a_1) \arrow[r, shift left, "\varphi_1"] \arrow[r, shift right, "\psi_1"'] & (M_2,a_2) \end{tikzcd}$$

We then have the inequalities:

1. *$d(\psi_1.\psi_0, \varphi_1.\varphi_0) \leq d(\psi_1,\varphi_1)d(\psi_0,O) + \rho(\varphi_1)d(\psi_0,\varphi_0)$ (where $O = O_{a_0a_1}$: see 2.6).*

2. *$d(\psi_1.\psi_0, \varphi_1.\varphi_0) \leq d(\psi_1,\varphi_1) + d(\psi_0,\varphi_0)$ if ψ_0 and φ_1 are 1-bounded (see 2.5).*

<u>*Proof*</u> : 1. Let $i \in \{0,1\}$; and $f_i \in \varphi_i$ and $g_i \in \psi_i$. Then, there exist $k_i, k'_i > 0$ and V_i a neighborhood of a_i in M_i such that the restrictions $f_i|_{V_i}$ and $g_i|_{V_i}$ are respectively k_i-lipschitzian and k'_i-lipschitzian.

For all x near a, we have: $d(g_1.g_0(x), f_1.f_0(x)) \leq d^{k'_0 r}(g_1, f_1)d^r(g_0, \widehat{a_1})d(x, a_0) + k_1 d^r(g_0, f_0)d(x, a_0)$ (where $\widehat{a_1}$ is the constant map on a_1), which provides $d(g_1.g_0, f_1.f_0) \leq d(g_1, f_1)d(g_0, \widehat{a_1}) + k_1 d(g_0, f_0)$; hence the wanted inequality.
2. Clear. $\square$

2.11. Remarks.

1. The sets of jets being equipped with their distance, the maps:

   $$\mathbb{J}\mathrm{et}((M_0,a_0),(M_1,a_1)) \longrightarrow \mathbb{J}\ \mathrm{et}((M_0,a_0),(M_2,a_2)) : \psi \mapsto \varphi_1.\psi$$

   and

   $$\mathbb{J}\mathrm{et}((M_1,a_1),(M_2,a_2)) \longrightarrow \mathbb{J}\ \mathrm{et}((M_0,a_0),(M_2,a_2)) : \psi \mapsto \psi.\varphi_0$$

   are respectively $\rho(\varphi_1)$-lipschitzian and $d(\varphi_0, O)$-lipschitzian (where φ_0 and φ_1 are jets as in 2.10).

2. ψ_0, ψ_1, O being jets as in 2.10 (with $O = O_{a_0a_2}, O_{a_1a_2}$ or $O_{a_0a_1}$), we have the inequality: $d(\psi_1.\psi_0, O) \leq d(\psi_1, O)d(\psi_0, O)$.

3. The inequalities obtained just above and in 2.6.1 are both generalisations of the well-known inequality $\|l_1.l_0\| \leq \|l_1\|\,\|l_0\|$ for composable continuous linear maps (see 2.16 below).

2.12. Corollary.

1. *The composition of jets:*
   $\mathbb{J}\mathrm{et}((M_0,a_0),(M_1,a_1)) \times \mathbb{J}\mathrm{et}\ ((M_1,a_1),(M_2,a_2)) \xrightarrow{comp} \mathbb{J}\mathrm{et}((M_0,a_0),(M_2,a_2))$ *is LSL.*

2. *The category* $\mathbb{J}\mathrm{et}$ *can thus be enriched in the cartesian category* $\mathbb{M}\mathrm{et}$ *(whose objects are the metric spaces and whose morphisms are the locally semi-lipschitzian maps).*

3. *Let M, M' be metric spaces, V, V' be two neighborhoods, respectively of $a \in M$ and $a' \in M'$. Then, the map:*

   $$\Gamma : \mathbb{J}\mathrm{et}((V,a),(V',a')) \longrightarrow \mathbb{J}\mathrm{et}((M,a),(M',a')) : \varphi \mapsto j'_{a'}.\varphi.j_a^{-1}$$

   is an isometry (where j_a and $j'_{a'}$ have been defined in 2.3).

2.13. Proposition. *$(M,a), (M_0,a_0), (M_1,a_1)$ being objects in the category* $\mathbb{J}\mathrm{et}$*, the following canonical map can is an isometry:*

$$\begin{array}{c} \mathbb{J}\mathrm{et}((M,a),(M_0,a_0)\times(M_1,a_1)) \\ \Big\downarrow{\scriptstyle can} \\ \mathbb{J}\mathrm{et}((M,a),(M_0,a_0)) \times \mathbb{J}\mathrm{et}((M,a),(M_1,a_1)) \end{array}$$

2.14. Remark. As isometries are isomorphisms in $\mathbb{M}\mathrm{et}$, it means that $\mathbb{J}\mathrm{et}$ is an enriched cartesian category.

We conclude this section with a come back to vectorial considerations.

2.15. Proposition.

1. *Let M be a metric space (with $a \in M$) and E a n.v.s. Then, we can canonically equip the sets $\mathbb{LL}((M,a),(E,0))$ and $\mathbb{J}\mathrm{et}((M,a),(E,0))$ with vectorial space structures, making linear the canonical surjection $q : \mathbb{LL}((M,a),(E,0)) \longrightarrow \mathbb{J}\mathrm{et}((M,a),(E,0))$. Besides, the distance on $\mathbb{J}\mathrm{et}((M,a),(E,0))$ derives from a norm (providing a structure of n.v.s. on $\mathbb{J}\mathrm{et}((M,a),(E,0))$).*

2. *Let M, M' be metric spaces, $a \in M$, $a' \in M'$; let also $\varphi \in \mathbb{J}\mathrm{et}((M',a'),(M,a))$ and E a n.v.s. Then, the map $\widetilde{\varphi} : \mathbb{J}\mathrm{et}((M,a),(E,0)) \longrightarrow \mathbb{J}\mathrm{et}((M',a'),(E,0)) : \psi \mapsto \psi.\varphi$ is linear and continuous.*

3. *E and E' being n.v.s., the canonical map : $j : L(E,E') \longrightarrow \mathbb{J}\mathrm{et}((E,0),(E',0))$: $l \mapsto q(l)$ is a linear isometric embedding ($L(E,E')$ being equipped with the norm $\|l\| = \sup_{\|x\|\leq 1} \|l(x)\|$).*

2.16. Corollary. *If $l : E \longrightarrow E'$ is a continuous linear map, then $q(l)$ is a good jet; more precisely, we have the equalities: $d(q(l),O) = \rho(q(l)) = \|l\|$. Here, we can replace "linear" by "affine".*

3. Tangentiability

In this new context, the notion of tangentiability plays the part of the one of differentiability, of which it keeps lots of properties. This section gives some examples and counter-examples to understand and visualize this new notion.

3.1. Definition. *Let $f : M \longrightarrow M'$ be a map between metric spaces and $a \in M$. We say that f is* tangentiable *at a (in short $Tang_a$) if there exists a map $g : M \longrightarrow M'$ which is LL_a such that $g \succ\!\!\prec_a f$.*

When f is $Tang_a$, we set $\mathrm{T}f_a = \{g : M \longrightarrow M' \,|\, g \succ\!\!\prec_a f\,;\; g\ LL_a\}$; $\mathrm{T}f_a$ is a jet $(M,a) \longrightarrow (M', f(a))$, said tangent to f at a, and that we can call the tangential *of f at a (not to be mixed up with $\mathrm{t}f_a$, defined in a special context (see Section 5)).*

3.2. Proposition. *The inverses of the following implications are not true (see 3.7).*

1. *We have the implications: $f\ LL_a \implies f\ Tang_a$ and $f\ Tang_a \implies f\ LSL_a \implies$ continuous at a.*

2. *Let E, E' be n.v.s., U an open subset of E, $a \in U$ and $f : U \longrightarrow E'$ a map. We have the implication: f differentiable at a $\implies f$ tangentiable at a.*

3.3. Remarks.

1. f is LL_a iff f is $Tang_a$ with $f \in \mathrm{T}f_a$ (then $\mathrm{T}f_a = q(f)$); in particular, $\mathrm{T}(Id_M)_a = q(Id_M) = Id_{(M,a)}$. Moreover, for every jet $\varphi : (M, a) \longrightarrow (M', a')$, we have $\varphi = \mathrm{T}g_a$ for every $g \in \varphi$.

2. Actually, f is differentiable at a iff f is $Tang_a$ where its tangential $\mathrm{T}f_a$ at a possesses an affine map. For such a differential map, it is the unique affine map Af_a defined by $Af_a(x) = f(a) + \mathrm{d}f_a(x - a)$(the translate at a of its differential $\mathrm{d}f_a$); thus $\mathrm{T}f_a = q(Af_a)$.

3.4. Proposition. (properties of the tangential)

1. *Let M, M', M'' be metric spaces, $f : M \longrightarrow M'$, $g : M' \longrightarrow M''$ two maps, and $a \in M$, $a' = f(a)$. If f is $Tang_a$ and g is $Tang_{a'}$, then $g.f$ is $Tang_a$ and we have $\mathrm{T}(g.f)_a = \mathrm{T}g_{a'}.\mathrm{T}f_a$.*

2. *Let M, M_0, M_1 be metric spaces, $f_0 : M \longrightarrow M_0$, $f_1 : M \longrightarrow M_1$ be two maps, and $a \in M$. If f_0 and f_1 are tangentiable at a, then $(f_0, f_1) : M \longrightarrow M_0 \times M_1$ is tangentiable at a and we have $\mathrm{T}(f_0, f_1)_a = (\mathrm{T}f_{0a}, \mathrm{T}f_{1a})$.*

3. *Let M_0, M_1, M'_0, M'_1 be four metric spaces, $f_0 : M_0 \longrightarrow M'_0$ and $f_1 : M_1 \longrightarrow M'_1$ two maps, and $a_0 \in M_0$, $a_1 \in M_1$. If f_0 is $Tang_{a_0}$ and f_1 $Tang_{a_1}$, then $f_0 \times f_1 : M_0 \times M_1 \longrightarrow M'_0 \times M'_1$ is $Tang_{(a_0,a_1)}$ and we have $\mathrm{T}(f_0 \times f_1)_{(a_0,a_1)} = \mathrm{T}f_{0a_0} \times \mathrm{T}f_{1a_1}$.*

3.5. Examples and counter-examples. All the maps considered below are functions $\mathbb{R} \longrightarrow \mathbb{R}$, where $\mathbb{R}$ is equipped with its usual structure of normed vector space (these different functions give counter-examples to the inverses of the implications given in 3.2).

1. Consider $f_0(x) = x^{1/3}$; this function is continuous but not LSL_0.

2. Consider $f_1(x) = x \sin \frac{1}{x}$ if $x \neq 0$ and $f_1(0) = 0$; this function is obviously LSL_0, however not $Tang_0$: indeed, if f_1 was $Tang_0$, there would exist a fonction $g : \mathbb{R} \longrightarrow \mathbb{R}$ and a neighborhood V of 0 such that $g \asymp_0 f_1$ and $g|_V$ is k-lipschitzian for a $k > 0$. Let us consider the two sequences of reals defined by $x_n = 1/2n\pi$ and $y_n = 1/(4n+1)\frac{\pi}{2}$; they verify $\lim_n |\frac{f_1(x_n)-g(x_n)}{x_n}| = 0 = \lim_n |\frac{f_1(y_n)-g(y_n)}{y_n}|$, so that $\lim_n 2\pi n g(x_n) = 0$ and $\lim_n (4n+1)\frac{\pi}{2} g(y_n) = 1$. Now, since $g|_V$ is k-lipschitzian, we have $|\frac{g(x_n)-g(y_n)}{x_n-y_n}| \leq k$ for n big enough, which is equivalent to $|\frac{4n+1}{n}(2n\pi g(x_n)) - 4((4n+1)\frac{\pi}{2} g(y_n))| \leq \frac{k}{n}$. It remains to do $n \to +\infty$ which leads to a contradiction.

3. Consider $f_2(x) = x^2 \sin \frac{1}{x^2}$ if $x \neq 0$ and $f_2(0) = 0$; this function is $Tang_0$ (since it is differentiable at 0); however not LL_0 (because $lim_{k \to +\infty} f'_2(\frac{1}{\sqrt{2k\pi}}) = -\infty$).

4. Consider $\vartheta(x) = |x|$; this function is $Tang_0$ (it is 1-lipschitzian!), however not differentiable at 0.

Let us now consider two metric spaces M, M', and two maps$f, g : M \longrightarrow M'$ which are $Tang_a$ where $a \in M$. As for general jets (see Section 2), we can speak of the distance between the two jets $\mathrm{T}f_a$, $\mathrm{T}g_a$. The question is: do we still have $d(\mathrm{T}f_a,\mathrm{T}g_a) = d(f, g)$?, provided that we can define $d(f, g)$ for maps f and g which are only $Tang_a$:

So, let M, M' be metric spaces, $a \in M$, $a' \in M'$ and $f, g : M \longrightarrow M'$ two maps which are $Tang_a$ and which verify $f(a) = g(a) = a'$. Let us consider $f_1 \in \mathrm{T}f_a$ and $g_1 \in \mathrm{T}g_a$; then $f_1 \rightarrowtail\!\!\!\!\leftarrowtail_a f$ and $g_1 \rightarrowtail\!\!\!\!\leftarrowtail_a g$; so that there exists a neighborhood V of a on which we have $d(f_1(x), f(x)) \leq d(x, a)$ and $d(g_1(x), g(x)) \leq d(x, a)$. Furthermore, since f_1 and g_1 are LL_a, we know that there exists also a neighborhood W of a on which the map $x \mapsto \frac{d(f_1(x),g_1(x))}{d(x,a)}$, if $x \neq a$, is bounded (let us say by R). Now, if we take $x \in V_1 = V \cap W$, $x \neq a$, we obtain: $\frac{d(f(x),g(x))}{d(x,a)} \leq \frac{d(f(x),f_1(x))}{d(x,a)} + \frac{d(f_1(x),g_1(x))}{d(x,a)} + \frac{d(g_1(x),g(x))}{d(x,a)} \leq 2 + R$. So, the map $C(x) = \frac{d(f(x),g(x))}{d(x,a)}$, if $x \neq a$ and $C(a) = 0$, is still bounded on V_1. Thus, for $r > 0$, we can again set $d^r(f, g) = \sup\{C(x)|x \in V_1 \cap B'(a, r)\}$ and finally again $d(f, g) = \lim_{r\to 0} d^r(f, g) = \inf_{r>0} d^r(f, g)$.

3.6. Proposition. *If $f, g : M \longrightarrow M'$ are $Tang_a$ where $a \in M$, we have $d(\mathrm{T}f_a, \mathrm{T}g_a) = d(f, g)$, this d being defined just above.*

4. Transmetric spaces

In order to define a tangential map $\mathrm{t}f : M \longrightarrow$ "$\mathbb{J}\mathrm{et}(M, M')$" for a tangentiable map $f : M \longrightarrow M'$ (see Section 5), we need to define first such a set "$\mathbb{J}\mathrm{et}(M, M')$" equipped with an adequate distance. We have succeeded in it, assuming that M and M' are transmetric spaces and introducing new jets called free metric jets (in opposition to the metric jets we have used up to now) whose set will be denoted $\mathbb{J}\mathrm{et}_{free}(M, M')$; this set being a good candidate for our unknown "$\mathbb{J}\mathrm{et}(M, M')$".

4.1. Definition. *A* transmetric space *is a metric space M, supposed to be non empty, equipped with a functor $\gamma : Gr(M) \longrightarrow \mathbb{J}et$ (where the category $\mathbb{J}\mathrm{et}$ has been defined at the beginning of Section 2, and $Gr(M)$ is the groupoid associated to the undiscrete equivalence relation on M; thus $|Gr(M)| = M$ and $Hom(a, b) = \{(a, b)\}$ for all $a, b \in M$) verifying:*

- for every $a \in M$, $\gamma(a) = (M, a)$,

- for every morphism $(a, b) : a \longrightarrow b$ in $Gr(M)$, the jet $\gamma(a, b) : (M, a) \longrightarrow (M, b)$ is 1-bounded (i.e. verifies $\rho(\gamma(a, b)) \leq 1$: see Section 2); thus invertible in $\mathbb{J}\mathrm{et}$.

Before giving some examples, let us give the following special case:

4.2. Definition. *A* left isometric group *is a metric space G equipped with a group structure verifying the following condition: $\forall g, g_0, g_1 \in G \quad (d(g.g_0, g.g_1) = d(g_0, g_1))$.*

4.3. Remarks.

1. In an equivalent manner, in 4.2 we could assume only that $d(g.g_0, g.g_1) \leq d(g_0, g_1)$ for all $g, g_0, g_1 \in G$.

2. Let G be a left isometric group; then, for all $g \in G$, the map $G \longrightarrow G : g' \mapsto g.g'$ is isometric.

4.4. PROPOSITION. *Every left isometric group G can be equipped with a canonical structure of transmetric space.*

Proof : Here, γ is the composite $Gr(G) \xrightarrow{\theta} \mathbb{LL} \xrightarrow{q} \mathbb{J}\text{et}$, where θ is the functor defined by $\theta(g) = (G, g)$ and $\theta(g_0, g_1)$ is the morphism $(G, g_0) \longrightarrow (G, g_1)$ in $\mathbb{LL}$ which assignes $g_1.g_0^{-1}.g$ to g. $\square$

4.5. EXAMPLES.

1. Here are examples of transmetric spaces which are even left isometric groups:
   (a) Every n.v.s. is a left isometric (additive) group: here, $\gamma(a, b) = q(\theta(a, b))$, where $\theta(a, b)$ is the translation $x \mapsto b - a + x$.
   (b) The multiplicative group $S^1 = \{z \in \mathbb{C} |\, |z| = 1\}$ is also a left isometric group.
   (c) E being an euclidian space, the orthogonal group $O(E)$, equipped with its operator norm, is a left isometric group.
   (d) The additive subgroups of $\mathbb{R}$ (as, for example $\mathbb{Q}$) are left isometric groups whuch are not n.v.s.
2. Every non empty discrete space has a unique structure of transmetric space; conversely, if a transmetric space possesses an isolated point, it is a discrete space.
3. The 4.6 below will provide a lot of transmetric spaces which are not left isometric groups.

4.6. PROPOSITION.

1. *Let M_0 and M_1 be transmetric spaces; then $M_0 \times M_1$ has a canonical stucture of transmetric space.*
2. *Let M be a transmetric space and $U \neq \emptyset$, an open subset of M; then, U has a canonical structure of transmetric space.*

Proof : 1. The functor $\gamma : Gr(M_0 \times M_1) \longrightarrow \mathbb{J}\text{et}$ is defined by $\gamma((a_0, a_1), (b_0, b_1)) = \gamma_0(a_0, b_0) \times \gamma_1(a_1, b_1)$, where γ_0 and γ_1 give the transmetric structures on M_0 and M_1.
2. The functor $\breve{\gamma} : Gr(U) \longrightarrow \mathbb{J}\text{et}$ is defined by $\breve{\gamma}(a) = (U, a)$ and $\breve{\gamma}(a, b) = j_b^{-1}.\gamma(a, b).j_a : (U, a) \longrightarrow (U, b)$. $\square$

4.7. Definition. *Let M, M' be transmetric spaces; a map $f : M \longrightarrow M'$ is called a* morphism of transmetric spaces *if it is a tangentiable map (at every point of M) such that, for every $a, b \in M$, the following diagram commutes in the category* $\mathbb{J}\mathrm{et}$*:*

$$\begin{tikzcd}
(M,a) \arrow[r, "\mathrm{T}f_a"] \arrow[d, "\gamma(a,b)"'] & (M', f(a)) \arrow[d, "\gamma(f(a),f(b))"] \\
(M,b) \arrow[r, "\mathrm{T}f_b"'] & (M', f(b))
\end{tikzcd}$$

We will denote $\mathbb{T}\mathrm{rans}$ *the category whose objects are the transmetric spaces and whose morphisms are the morphisms of transmetric spaces.*

4.8. Examples. (of such morphisms)

1. Every continuous affine map $f : E \longrightarrow E'$ (between n.v.s.).
2. $Id_M : M \longrightarrow M$ and the canonical injection $j : U \hookrightarrow M$, if $U \neq \emptyset$, is an open subset of a transmetric space M.
3. Every constant map (on $c \in M'$) $\widehat{c} : M \longrightarrow M'$ between transmetric spaces; it is the case for the unique map $!_M : M \longrightarrow \mathbb{I}$ (where M is a transmetric space and $\mathbb{I} = \{0\}$).
4. The canonical projections $p_i : M_0 \times M_1 \longrightarrow M_i$, where M_0 and M_1 are transmetric spaces.

4.9. Proposition. *The category* $\mathbb{T}\mathrm{rans}$ *is a cartesian category, and the forgetful functor* $\mathbb{T}\mathrm{rans} \longrightarrow \mathbb{E}\mathrm{ns}$ *is a morphism of cartesian categories.*

We are now ready to construct the set $\mathbb{J}\mathrm{et}_{\mathit{free}}(M, M')$ of free metric jets, when M and M' are transmetric spaces. First, we consider the set $J(M, M') = \coprod_{(a,a') \in M \times M'} \mathbb{J}\mathrm{et}((M, a), (M', a'))$ on which we define the following equivalence relation: $(\varphi, a, a') \sim (\psi, b, b')$ if the following diagram commutes in the category $\mathbb{J}\mathrm{et}$:

$$\begin{tikzcd}
(M,a) \arrow[r, "\varphi"] \arrow[d, "\gamma(a,b)"'] & (M', a') \arrow[d, "\gamma(a',b')"] \\
(M,b) \arrow[r, "\psi"'] & (M', b')
\end{tikzcd}$$

Then, we set $\mathbb{J}\mathrm{et}_{\mathit{free}}(M, M') = J(M, M')/ \sim$.

4.10. DEFINITION. *The elements of* $\mathbb{J}\mathrm{et}_{\mathit{free}}(M, M')$ *are called* free metric jets *(in short,* free jets*, here from* M *to* M'*).*

If $q : J(M, M') \longrightarrow \mathbb{J}\mathrm{et}_{\mathit{free}}(M, M')$ is the canonical surjection, we set $[\varphi, a, a'] = q(\varphi, a, a')$ when $(\varphi, a, a') \in J(M, M')$.

We are going to canonically equip $\mathbb{J}\mathrm{et}_{\mathit{free}}(M, M')$ with a structure of metric space. First, for $(\varphi, a, a'), (\psi, b, b') \in J(M, M')$, we denote $d((\varphi, a, a'), (\psi, b, b'))$ the distance between the jets $\gamma(a', b').\varphi$ and $\psi.\gamma(a, b)$ (see Section 2). Using the properties of the lipschitzian ratio and the theorem of Section 2, we prove:

4.11. PROPOSITION.

1. *For each* $(\varphi, a, a'), (\psi, b, b'), (\xi, c, c') \in J(M, M')$*, we have the following properties:*
   - *(a)* $d((\varphi, a, a'), (\psi, b, b')) = d((\psi, b, b'), (\varphi, a, a'))$,
   - *(b)* $d((\varphi, a, a'), (\xi, c, c')) \leq d((\varphi, a, a'), (\psi, b, b')) + d((\psi, b, b'), (\xi, c, c'))$,
   - *(c)* $d((\varphi, a, a'), (\psi, b, b')) = 0 \iff (\varphi, a, a') \sim (\psi, b, b')$.
2. *The map* $d : (J(M, M'))^2 \longrightarrow \mathbb{R}_+$ *factors through the quotient, giving a "true" distance on* $\mathbb{J}\mathrm{et}_{\mathit{free}}(M, M')$*, defined by* $d([\varphi, a, a'], [\psi, b, b']) = d((\varphi, a, a'), (\psi, b, b'))$ *for all* $(\varphi, a, a'), (\psi, b, b') \in J(M, M')$.

4.12. REMARK. Let M, M' be transmetric spaces; then $\mathbb{J}\mathrm{et}_{\mathit{free}}(M, M')$ possesses a particular element $[O_{aa'}, a, a']$ denoted O: for $[O_{aa'}, a, a']$ does not depend on the choice of $(a, a') \in M \times M'$; this free jet O will be called the free zero of $\mathbb{J}\mathrm{et}_{\mathit{free}}(M, M')$. In the same way, $\mathbb{J}\mathrm{et}_{\mathit{free}}(M, M)$ possesses also a free identity, denoted I_M, which is equal to $[Id_{(M,x)}, x, x]$.

4.13. PROPOSITION. *Let* M, M' *be transmetric spaces,* $a, b \in M$ *and* $a', b' \in M'$.

1. *If* $c(\varphi) = (\varphi, a, a')$*, then the composite*

$$can : \mathbb{J}\mathrm{et}((M, a), (M', a')) \xrightarrow{c} J(M, M') \xrightarrow{q} \mathbb{J}\mathrm{et}_{\mathit{free}}(M, M')$$

is an isometry.

2. *Let us set* $\Omega(\varphi) = \gamma(a', b').\varphi.\gamma(b, a)$*; this defines a map*

$$\Omega : \mathbb{J}\mathrm{et}((M, a), (M', a')) \longrightarrow \mathbb{J}\mathrm{et}((M, b), (M', b'))$$

which is an isometry, and the following diagram commutes:

$$\begin{tikzcd}
\mathbb{J}\mathrm{et}((M, a), (M', a')) \arrow[rr, "\Omega"] \arrow[dr, "can"'] & & \mathbb{J}\mathrm{et}((M, b), (M', b')) \arrow[dl, "can"] \\
& \mathbb{J}\mathrm{et}_{\mathit{free}}(M, M') &
\end{tikzcd}$$

Let now M_0, M_1, M_2 be three transmetric spaces; we consider the map

$$comp : J(M_0, M_1) \times J(M_1, M_2) \longrightarrow J(M_0, M_2)$$

defined by $(\varphi_1, b_1, b_2).(\varphi_0, a_0, a_1) = (\varphi_1.\gamma(a_1, b_1).\varphi_0, a_0, b_2)$ (as in the diagram below):

$$(M_0, a_0) \xrightarrow{\varphi_0} (M_1, a_1) \xrightarrow{\gamma(a_1,b_1)} (M_1, b_1) \xrightarrow{\varphi_1} (M_2, b_2)$$

4.14. PROPOSITION.

1. *This map comp factors through the quotient:*

$$\begin{tikzcd} J(M_0, M_1) \times J(M_1, M_2) \arrow[r, "comp"] \arrow[d, "q\times q"'] & J(M_0, M_2) \arrow[d, "q"] \\ \mathbb{J}\mathrm{et}_{\mathit{free}}(M_0, M_1) \times \mathbb{J}\mathrm{et}_{\mathit{free}}(M_1, M_2) \arrow[r, "comp"'] & \mathbb{J}\mathrm{et}_{\mathit{free}}(M_0, M_2) \end{tikzcd}$$

(and we will write $[\varphi_1, b_1, b_2].[\varphi_0, a_0, a_1] = [\varphi_1.\gamma(a_1, b_1).\varphi_0, a_0, b_2]$*, which gives simply* $[\varphi_1, b_1, b_2].[\varphi_0, a_0, a_1] \overset{\star}{=} [\varphi_1.\varphi_0, a_0, b_2]$ *when* $a_1 = b_1$*).*

2. *The composition comp* : $\mathbb{J}\mathrm{et}_{\mathit{free}}(M_0, M_1) \times \mathbb{J}\mathrm{et}_{\mathit{free}}(M_1, M_2) \longrightarrow \mathbb{J}\mathrm{et}_{\mathit{free}}(M_0, M_2)$*, defined just above, is LSL.*

So, we construct a new category, enriched in $\mathbb{M}\mathrm{et}$, denoted $\mathbb{J}\mathrm{et}_{\mathit{free}}$, called the category of free jets,whose:
- objects are the transmetric spaces M,
- "Hom" are the metric spaces $\mathbb{J}\mathrm{et}_{\mathit{free}}(M, M')$,
- identity $\mathbb{I} \longrightarrow \mathbb{J}\mathrm{et}_{\mathit{free}}(M, M)$, is the map giving the free identity $I_M = [Id_{(M,a)}, a, a]$,
- composition $\mathbb{J}\mathrm{et}_{\mathit{free}}(M, M') \times \mathbb{J}\mathrm{et}_{\mathit{free}}(M', M'') \longrightarrow \mathbb{J}\mathrm{et}_{\mathit{free}}(M, M'')$ is the previous *comp*.

4.15. DEFINITION. *We will denote* $\mathbb{J}\mathrm{et}'$ *the full subcategory of the category* $\mathbb{J}\mathrm{et}$ *whose objects are the pointed transmetric spaces. Just as* $\mathbb{J}\mathrm{et}$ *(see Section 2), it is a category enriched in* $\mathbb{M}\mathrm{et}$*.*

4.16. PROPOSITION.

1. *Actually, we have a forgetful enriched functor* $U : \mathbb{J}\mathrm{et}' \longrightarrow \mathbb{J}\mathrm{et}$*.*

2. *We have a functor* $can' : \mathbb{J}\mathrm{et}' \longrightarrow \mathbb{J}\mathrm{et}_{\mathit{free}}$*, defined by* $can'(M, a) = M$ *and, for* $\varphi : (M, a) \longrightarrow (M', a')$*, by* $can'(\varphi) = can(\varphi) = [\varphi, a, a']$*; then, this functor is enriched in* $\mathbb{M}\mathrm{et}$*.*

We now come back to cartesian considerations.

4.17. REMARK. By definition, $f : M \longrightarrow M'$ is a morphism of transmetric spaces iff we have $(\mathrm{T}f_x, x, f(x)) \sim (\mathrm{T}f_y, y, f(y))$ for all $x, y \in M$; so that the free jet $[\mathrm{T}f_x, x, f(x)]$ is independent on the choice of $x \in M$; we denote $\kappa(f)$ this element of $\mathbb{J}\mathrm{et}_{\mathit{free}}(M, M')$. In particular, if $c \in M'$, we have seen that the constant map $\widehat{c}$ is a morphism of transmetric spaces and that $\kappa(\widehat{c}) = O$.

4.18. Proposition.

1. *The map* $\mathbb{T}\mathrm{rans}(M,M') \longrightarrow \mathbb{J}\mathrm{et}_{\mathit{free}}(M,M') : f \mapsto \kappa(f)$ *extends to a functor* $\kappa : \mathbb{T}\mathrm{rans} \longrightarrow \mathbb{J}\mathrm{et}_{\mathit{free}}$ *which is constant on the objects.*

2. *This functor* $\kappa : \mathbb{T}\mathrm{rans} \longrightarrow \mathbb{J}\mathrm{et}_{\mathit{free}}$ *creates a cartesian structure on the category* $\mathbb{J}\mathrm{et}_{\mathit{free}}$ *(*κ *being constant on the objects, it means that* $\mathbb{J}\mathrm{et}_{\mathit{free}}$ *is cartesian and* κ *a strict morphism of cartesian categories).*

3. *The functor* $can' : \mathbb{J}\mathrm{et}' \longrightarrow \mathbb{J}\mathrm{et}_{\mathit{free}}$ *is a morphism of cartesian categories.*

4. M, M_0, M_1 *being transmetric spaces, the canonical map*

$$can : \mathbb{J}\mathrm{et}_{\mathit{free}}(M, M_0 \times M_1) \longrightarrow \mathbb{J}\mathrm{et}_{\mathit{free}}(M, M_0) \times \mathbb{J}\mathrm{et}_{\mathit{free}}(M, M_1)$$

is an isometry (like its analogue for $\mathbb{J}et$*).*

4.19. Proposition. *Let* M, M' *be two transmetric spaces, and* U, U' *two non empty open subsets of* M *and* M' *respectively (with* $j : U \hookrightarrow M$ *and* $j' : U' \hookrightarrow M'$ *the canonical injections).*

1. $\kappa(j) : U \longrightarrow M$ *is an isomorphism in* $\mathbb{J}\mathrm{et}_{\mathit{free}}$.

2. *The canonical map* $\Gamma : \mathbb{J}\mathrm{et}_{\mathit{free}}(U, U') \longrightarrow \mathbb{J}\mathrm{et}_{\mathit{free}}(M, M') : \Phi \mapsto \kappa(j').\Phi.\kappa(j)^{-1}$ *is an isometry.*

4.20. Proposition.

1. *Let* M, M' *be two transmetric spaces. Then the map* $J(M, M') \longrightarrow \mathbb{R}_+ : (\varphi, a, a') \mapsto \rho(\varphi)$ *factors through the quotient; we still denote* $\rho : \mathbb{J}\mathrm{et}_{\mathit{free}}(M, M') \longrightarrow \mathbb{R}_+$ *this factorization. (see Section 2 for the definition of the lipschitzian ratio of a jet).*

2. *Let* M_0, M_1, M_2 *be three transmetric spaces; then, if* $\Phi_0 : M_0 \longrightarrow M_1$ *and* $\Phi_1 : M_1 \longrightarrow M_2$ *are free jets, we have* $\rho(\Phi_1.\Phi_0) \le \rho(\Phi_1)\rho(\Phi_0)$.

Again, we conclude this paragraph with vectorial considerations.

4.21. Proposition.

1. *Let* M *be a transmetric space and* E *a n.v.s. Then, the metric space* $\mathbb{J}\mathrm{et}_{\mathit{free}}(M, E)$ *has also a canonical structure of n.v.s. (its distance defined in 4.11 derives from a norm). Besides, for every* $a \in M$, *the map* $can : \mathbb{J}\mathrm{et}((M, a), (E, 0)) \longrightarrow \mathbb{J}\mathrm{et}_{\mathit{free}}(M, E)$ *(see 4.13) is a linear isometry.*

2. *Let* M *be a transmetric space and* E *a n.v.s. Let also* $\Phi \in \mathbb{J}\mathrm{et}_{\mathit{free}}(M', M)$. *Then, the map* $\widetilde{\Phi} : \mathbb{J}\mathrm{et}_{\mathit{free}}(M, E) \longrightarrow \mathbb{J}\mathrm{et}_{\mathit{free}}(M', E) : \Psi \mapsto \Psi.\Phi$ *is linear and continuous.*

5. Tangential

Untill now, we have only spoken of the tangential $\mathrm{T}f_a$, at a, of a map $f : M \longrightarrow M'$, tangentiable at a (see Section 3). ¿From now on, we will speak of the tangential $\mathrm{t}f : M \longrightarrow \mathbb{J}\mathrm{et}_{\mathit{free}}(M, M')$ when $f : M \longrightarrow M'$ is a tangentiable map between transmetric spaces (as we speak of the differential $\mathrm{d}f : U \longrightarrow L(E, E')$ for a differentiable map $f : U \longrightarrow E'$ when U is an open subset of E, a n.v.s., just as E').

First, consider M, M' two transmetric spaces and $f : M \longrightarrow M'$ a tangentiable map at the point $x \in M$; then, we set $\mathrm{t}f_x = [\mathrm{T}f_x, x, f(x)]$.

Now, if $f : M \longrightarrow M'$ is tangentiable (at every point in M), we can define the map $\mathrm{t}f : M \longrightarrow \mathbb{J}\mathrm{et}_{\mathit{free}}(M, M') : x \mapsto \mathrm{t}f_x$ (in fact, this map $\mathrm{t}f$ is the following composite $M \xrightarrow{\mathrm{T}f} J(M, M') \xrightarrow{q} \mathbb{J}\mathrm{et}_{\mathit{free}}(M, M')$, where $J(M, M')$, $\mathbb{J}\mathrm{et}_{\mathit{free}}(M, M')$ and q have been defined in Section 4, and where $\mathrm{T}f(x) = (\mathrm{T}f_x, x, f(x))$.

5.1. Definition. *The map* $\mathrm{t}f$ *defined just above (for a tangentiable map* f *between transmetric spaces) will be called the* tangential *of* f.

5.2. Proposition. *Let* $f : M \longrightarrow M'$ *be a tangentiable map between transmetric spaces. Then, the map* $\mathrm{t}f : M \longrightarrow \mathbb{J}\mathrm{et}_{\mathit{free}}(M, M')$ *is constant iff* f *is a morphism of transmetric spaces; in this case we have* $\mathrm{t}f_x = \kappa(f)$ *for every* $x \in M$.

Let us now begin with the particular vectorial context: we assume here that E, E' are n.v.s. and U a non empty open subset of E; consider then the following composite:

$$J : L(E, E') \xrightarrow{j} \mathbb{J}\mathrm{et}((E, 0), (E', 0)) \xrightarrow{can} \mathbb{J}\mathrm{et}_{\mathit{free}}(E, E') \xrightarrow{\Gamma^{-1}} \mathbb{J}\mathrm{et}_{\mathit{free}}(U, E')$$

See Section 2 for j and Section 4 for *can* and Γ. We set $Im(J) = J(L(E, E'))$, the image of J in $\mathbb{J}\mathrm{et}_{\mathit{free}}(U, E')$.

5.3. Remark. The map $J : L(E, E') \longrightarrow \mathbb{J}\mathrm{et}_{\mathit{free}}(U, E')$ defined just above is a linear isometric embedding.

5.4. Proposition.

1. *Let* $f : U \longrightarrow E'$ *be a tangentiable map and* $a \in U$*; then,* f *is differentiable at* a *iff* $\mathrm{t}f_a \in Im(J)$*; in this case, we have* $\mathrm{t}f_a = J(\mathrm{d}f_a)$.

2. *Let* $f : U \longrightarrow E'$ *a differentiable map; then* f *is tangentiable and the following diagram commutes:*

$$\begin{tikzcd} & U \arrow[dl, "\mathrm{d}f"'] \arrow[dr, "\mathrm{t}f"] & \\ L(E, E') \arrow[rr, "J"'] & & \mathbb{J}\mathrm{et}_{\mathit{free}}(U, E') \end{tikzcd}$$

The proof of 5.4.1 uses the result of the following lemma:

5.5. Lemma. *Let* $[\varphi, a, b] \in \mathbb{J}\mathrm{et}_{\mathit{free}}(U, E')$. *Then,* $[\varphi, a, b] \in Im(J)$ *iff there exists* $l \in L(E, E')$ *such that* $Al|_U \in \varphi$*, where* $Al(x) = b + l(x - a)$*; and we have* $J(l) = [\varphi, a, b]$.

We now come back to the general transmetric context.

5.6. Proposition. *Let* M, M', M'' *be three transmetric spaces;* $f : M \longrightarrow M'$*,* $g : M' \longrightarrow M''$ *two maps. We assume that* f *and* g *are tangentiable and that their tangentials* $\mathrm{t}f : M \longrightarrow \mathbb{J}\mathrm{et}_{\mathit{free}}(M, M')$ *and* $\mathrm{t}g : M' \longrightarrow \mathbb{J}\mathrm{et}_{\mathit{free}}(M', M'')$ *are continuous, respectively at* $a \in M$ *and* $a' = f(a) \in M'$*. Then the map* $\mathrm{t}(g.f) : M \longrightarrow \mathbb{J}\mathrm{et}_{\mathit{free}}(M, M'')$ *is well defined and continuous at* a*, and we have* $\mathrm{t}(g.f)_a = \mathrm{t}g_{a'}.\mathrm{t}f_a$.

<u>*Proof*</u> : The continuity of the map $\mathrm{t}(g.f)$ comes from the following composite:

$$M \xrightarrow{(Id,f)} M \times M' \xrightarrow{\mathrm{t}f \times \mathrm{t}g} \mathbb{J}\mathrm{et}_{\mathit{free}}(M, M') \times \mathbb{J}\mathrm{et}_{\mathit{free}}(M', M'') \xrightarrow{comp} \mathbb{J}\mathrm{et}_{\mathit{free}}(M, M'')$$

□

5.7. Proposition. *Let* M, M_0, M_1 *be transmetric spaces, with* $a \in M$ *and* $f_0 : M \longrightarrow M_0$*,* $f_1 : M \longrightarrow M_1$ *two maps. We assume that, for each* $i \in \{0, 1\}$*,* $f_i : M \longrightarrow M_i$ *is tangentiable and that its tangential* $\mathrm{t}f_i : M \longrightarrow \mathbb{J}\mathrm{et}_{\mathit{free}}(M, M_i)$ *is continuous at* a*. Then, the map*
$\mathrm{t}(f_0, f_1) : M \longrightarrow \mathbb{J}\mathrm{et}_{\mathit{free}}(M, M_0 \times M_1)$ *is well defined and continuous at* a*, and we have* $\mathrm{t}(f_0, f_1)_a = (\mathrm{t}f_{0a}, \mathrm{t}f_{1a})$.

<u>*Proof*</u> : The continuity of the map $\mathrm{t}(f_0, f_1)$ comes from the following composite:

$$M \xrightarrow{(\mathrm{t}f_0,\mathrm{t}f_1)} \mathbb{J}\mathrm{et}_{\mathit{free}}(M, M_0) \times \mathbb{J}\mathrm{et}_{\mathit{free}}(M, M_1) \xrightarrow{can^{-1}} \mathbb{J}\mathrm{et}_{\mathit{free}}(M, M_0 \times M_1)$$

□

5.8. Proposition. *Let* E *be a n.v.s.,* U *an open subset of* E*,* $a, b \in U$ *such that* $[a, b] \subset U$ *and* F *a finite subset of* $]a, b[$*; let also* M *be a transmetric space and* $f : U \longrightarrow M$ *a continuous map. Now, if we assume that, for all* $x \in]a, b[-F$*, the map* f *is* $Tang_x$ *and that* $d(tf_x, O) \leq k$ *(where* O *is the free zero of 4.12 and* k *a fixed positive real number), then we have* $d(f(b), f(a)) \leq k||b - a||$.

<u>*Proof*</u> : Comes from 1.7 and the following lemma:

5.9. Lemma. *Let* $f : M \longrightarrow M'$ *be a map,* $a \in M$ *and* $b = f(a)$*. We assume that* f *is* $Tang_a$ *and that* $d(\mathrm{T}f_a, O_{ab}) < k$*. Then,* f *is* k*-*LSL_a *(thanks to 3.6).*

We now give a generalization of 1.6.4.(b):

5.10. Theorem. *Let* E *and* M *be respectively a n.v.s. and a transmetric space,* U *a non empty open subset of* E *and* $f : U \longrightarrow M$ *a tangentiable map such that* $\mathrm{t}f : U \longrightarrow \mathbb{J}\mathrm{et}_{\mathit{free}}(U, M)$ *is continuous at* $a \in U$*. Then,* f *is* LL_a *and we have* $d(\mathrm{t}f_a, O) = \rho(\mathrm{t}f_a)$*; i.e* $\mathrm{t}f_a$ *is a* good free jet *(the analogue, for the free jets, of what we have previously defined for the jets in Section 2).*

Proof : First, thanks to 2.6.2 and by definition of the lipschitzian ratio ρ for jets or free metric jets (see 2.5 and 4.25), we have $d(\mathrm{t}f_a, O) = d(\mathrm{T}f_a, O_{af(a)}) \leq \rho(\mathrm{T}f_a) = \rho(\mathrm{t}f_a)$.

Now, let $\bar{f} : U \longrightarrow \mathbb{R}$ be the following composite:

$$U \xrightarrow{\mathrm{t}f} \mathbb{J}\mathrm{et}_{free}(U, M) \xrightarrow{d(-,O)} \mathbb{R}$$

The map $\bar{f}$ is continuous at a, by composition. Let us fix $\varepsilon > 0$; then, there exists $r > 0$ such that $B(a, r) \subset U$ and $\bar{f}(x) < \bar{f}(a) + \varepsilon$ for all $x \in B(a, r)$. Let us set $R = \bar{f}(a) + \varepsilon$; then, for all $x \in B(a, r)$, we have $d(\mathrm{t}f_x, O) = \bar{f}(x) < R$, so that (thanks to 5.8), the restriction $f|_{B(a,r)}$ is R-lipschitzian (since $B(a, r)$ is convex). Thus f is LL_a, so that $\mathrm{T}f_a = q(f)$. Furthermore, we have $\rho(\mathrm{t}f_a) = \rho(\mathrm{T}f_a) = \rho(q(f)) \leq R = \bar{f}(a) + \varepsilon$; we have thus obtained $\rho(\mathrm{t}f_a) \leq d(\mathrm{t}f_a, O) + \varepsilon$ for all $\varepsilon > 0$, so that $\rho(\mathrm{t}f_a) \leq d(\mathrm{t}f_a, O)$. □

5.11. Proposition. *Let E be a n.v.s. and U a non empty open subset of E, and M a transmetric space.*

1. *If U is convex, then every morphism of transmetric spaces $f : U \longrightarrow M$ is R-lipschitzian (where $R = d(\kappa(f), O)$; see Section 4 for $\kappa(f)$). Hence, $d(\kappa(f), O) = \rho(\kappa(f))$; i.e $\kappa(f)$ is a good free jet (this result is true for every continuous affine map).*

2. *If U is not convex, f is R-LL on U, and $\kappa(f)$ is still a good free jet.*

Proof : 1. We use 5.8, since, for all $x \in U$, we have $d(\mathrm{t}f_x, O) = d(\kappa(f), O) = R$; thus f is R-lipschitzian on U (which is convex), so that $\rho(\kappa(f)) \leq R$. For the inverse inequality, we use 2.6.2.

2. Because every open subset of a n.v.s. is locally convex. □

5.12. Definition. *Let M, M' be transmetric spaces and $f : M \longrightarrow M'$ a map. We say that f is* continuously tangentiable *if f is tangentiable and if its tangential $\mathrm{t}f : M \longrightarrow \mathbb{J}\mathrm{et}_{free}(M, M')$ is continuous.*

5.13. Proposition. *The continuously tangentiable maps are stable under composition and under pairs.*

5.14. Examples. (of continuously tangentiable maps)

1. Every morphism of transmetric spaces.

2. Every map which is of class C^1.

5.15. Theorem. *Let E, E' be n.v.s. where E' is complete, U a non empty open subset of E and $f : U \longrightarrow E'$ a continuously tangentiable map which is also differentiable at every point of a dense subset D of U. Then, f is of class C^1.*

Proof : Referring to 5.4.1, we know that, a priori, the set of the points of U at which f is differentiable is nothing but $\overset{-1}{\mathrm{t}f}\,(Im(J))$; thus, we must first show that this set is equal to U. Since, by hypothesis, we have the inclusions $D \subset \overset{-1}{\mathrm{t}f}\,(Im(J)) \subset U$ with D dense in U, we have just to prove that $\overset{-1}{\mathrm{t}f}\,(Im(J))$ is closed in U. Actually, since $Im(J)$ is complete (just as the n.v.s. $L(E,E')$, for $J : L(E,E') \longrightarrow \mathbb{J}\mathrm{et}_{free}(U,E')$ is an isometric embedding), it is closed in $\mathbb{J}\mathrm{et}_{free}(U,E')$, so that $\overset{-1}{\mathrm{t}f}\,(Im(J))$ is also closed in U (since $\mathrm{t}f : U \longrightarrow \mathbb{J}\mathrm{et}_{free}(U,E')$ is continuous); f is thus differentiable on U. Finally, the fact that the map $\mathrm{d}f : U \longrightarrow L(E,E')$ is continuous comes from the fact it factors through $\mathrm{t}f$ (see the commutative triangle in 5.4.2. □

5.16. Example. The function $\vartheta : \mathbb{R} \longrightarrow \mathbb{R} : x \mapsto |x|$ is tangentiable (for it is lipschitzian); however, it cannot be continuously tangentiable on $\mathbb{R}$, since it is differentiable on $\mathbb{R}^*$ and not of class C^1 on $\mathbb{R}$.

We now give a generalization of the fact that the composite of two maps of class C^2 is also of class C^2.

5.17. Theorem. *Let M, M' be transmetric spaces, E a n.v.s. and U a non empty open subset of E. Let also $f : M \longrightarrow U$ and $g : U \longrightarrow M'$ be two continuously tangentiable maps. We assume that $\mathrm{t}f : M \longrightarrow \mathbb{J}\mathrm{et}_{free}(M,U)$ is $Tang_a$ where $a \in M$ and that $\mathrm{t}g : U \longrightarrow \mathbb{J}\mathrm{et}_{free}(U,M')$ is $LL_{a'}$ where $a' = f(a) \in U$. Then, $\mathrm{t}(g.f) : M \longrightarrow \mathbb{J}\mathrm{et}_{free}(M,M')$ is $Tang_a$.*

Proof : Since $g : U \longrightarrow M'$ is continuously tangentiable, we know (by 5.10) that $d(\mathrm{t}g_y, O) = \rho(\mathrm{t}g_y)$ for all $y \in U$. We now need the following lemma:

5.18. Lemma.

1. *Let $(M_1,a_1),(M_2,a_2) \in |\mathbb{J}\mathrm{et}|$ and $Jeg((M_1,a_1),(M_2,a_2))$ be the set of good jets, i.e $\{\varphi \in \mathbb{J}\mathrm{et}\,((M_1,a_1),(M_2,a_2))\,|\,d(\varphi, O_{a_1a_2}) = \rho(\varphi)\}$. Then the following restriction is LL: $\mathbb{J}\mathrm{et}((M_0,a_0),(M_1,a_1)) \times Jeg((M_1,a_1),(M_2,a_2)) \xrightarrow{comp} \mathbb{J}\mathrm{et}((M_0,a_0),(M_2,a_2))$.*
2. *Let M_0, M_1, M_2 be transmetric spaces; if $Jeg_{free}(M_1,M_2) = \{\Phi \in \mathbb{J}\mathrm{et}_{free}(M_1,M_2)\,|\,d(\Phi,O) = \rho(\Phi)\}$. Then the following restriction is LL: $\mathbb{J}\mathrm{et}_{free}(M_0,M_1) \times Jeg_{free}(M_1,M_2) \xrightarrow{comp} \mathbb{J}\mathrm{et}_{free}(M_0,M_2)$.*

Thus, coming back to the proof of 5.17, we deduce that $\mathrm{t}g_y \in Jeg_{free}(U,M')$. We can then consider the restriction $\breve{\mathrm{t}g} : U \longrightarrow Jeg_{free}(U,M')$ of $\mathrm{t}g$. Besides, knowing that $g.f$ is continuously tangentiable with $\mathrm{t}(g.f)_x = \mathrm{t}g_{f(x)}.\mathrm{t}f_x$ for all $x \in M$, we can write its tangential $\mathrm{t}(g.f)$ as the composite:

$$M \xrightarrow{(Id,f)} M \times U \xrightarrow{\mathrm{t}f \times \breve{\mathrm{t}g}} \mathbb{J}\mathrm{et}_{free}(M,U) \times Jeg_{free}(U,M') \xrightarrow{comp} \mathbb{J}\mathrm{et}_{free}(M,M')$$

So that we only have to justify the tangentiability, at the right point, of every forementionned map: by Section 3, (Id,f) is $Tang_a$ and $\mathrm{t}f \times \breve{\mathrm{t}g}$ is $Tang_{(a,a')}$; and *comp* is $LL_{(\mathrm{t}f_a,\mathrm{t}g_{a'})}$ (by lemma 5.18), thus $Tang_{(\mathrm{t}f_a,\mathrm{t}g_{a'})}$. Their composite $\mathrm{t}(g.f)$ is thus $Tang_a$. □

Here are some *open problems* that we submit to the sagacity of the reader (after having gone through this paper)

1. We have seen, in the examples given in 4.8, that every continuous affine map between n.v.s. is a morphism of transmetric spaces. The question is: are there any other morphisms of transmetric spaces in such a vectorial context?

2. Still in the case of the n.v.s., does there exist continuously tangentiable maps which are not of class C^1 (see 5.14 and 5.15)?

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