Theory and Applications of Categories, Vol. 23, No. 12, 2010, pp. 243–250.

A CATEGORICAL APPROACH TO INTEGRATION

REINHARD BÖRGER

ABSTRACT. We present a general treatment of measures and integrals in certain (monoidal closed) categories. Under appropriate conditions the integral can be defined by a universal property, and the universal measure is at the same time a universal multiplicative measure. In the multiplicative case this assignment is right adjoint to the formation of the Boolean algebra of idempotents. Now coproduct preservation yields an approach to product measures.

1. Introduction

Integration is a map that assigns to each integrable function its integral. If we exclude improper integrals and admit *signed measures* (which may be negative), integration is a linear map from some *vector space* LB of *integrable functions* to $\mathbb{R}$. It is desirable that every function in this space can be integrated with respect to every bounded measure; thus we consider spaces of *bounded measurable functions.* Now we observe that mapping each set to its *indicator function* (sometimes called characteristic function) also yields a *charge* (i.e. a finitely additive set function) from B to a set of *simple functions* (i.a. measurable functions attaining only finitely many values) on B. Moreover it is a given by a *universal property.* Naturally this is not restricted to real-valued measures but also works for *vector measures.* The integral of a function x with respect to the "indicator function measure" is x itself.

Furthermore, a map $m : B \to \mathbb{R}$ has a linear extension to the vector space of all simple functions if and only if m is a charge. It is possible to introduce topologies on the spaces such that integration is the unique continuous linear extension of this map. It is convenient to work with *abstract Boolean algebras* rather than *set algebras.* Then certain analogues of function spaces can be defined by the universal property, though their elements can no longer be interpreted as functions in the usual way. So integration becomes a morphism in a suitable category rendering certain diagrams commutative. A measure should be a map from some internal Boolean algebra B over a reasonable category $\mathcal{X}$ to some object in some additive category $\mathcal{A}$; its objects should carry a vector space structure. Then the set $M(B, A)$ of measures from some (internal) Boolean algebra B

Dedicated to Dominique Bourn on the occasion of his sixtieth birthday

Received by the editors 2009-05-30 and, in revised form, 2010-04-06.

Published on 2010-04-28 in the Bourn Festschrift.

2000 Mathematics Subject Classification: 06E05 16A32, 18A15, 18A30, 18A35, 18A40, 18E05, 28A30, 28A33, 28A40, 28A45, 46G10.

Key words and phrases: internal Boolean algebra, universal measure, multiplicative measure, product measure, Boolean algebra of idempotents, symmetric monoidal closed category, cartesian closed category.

to A is in bijection with $\mathcal{A}(LB, A)$. Convergence theorems like dominated and monotone convergence should mean continuity in certain *topologies*; these should be encoded in $\mathcal{X}$. It is convenient to work in a *cartesian closed category* (e.g. *sequential spaces*); this yields nice *spaces of measurable functions* and nice *spaces of measures*. Then σ-additivity can be interpreted as finite additivity together with continuity in an appropriate topology; this leads to a special choice of the object of $\mathcal{A}$. Moreover, it is often useful to impose certain *completeness conditions* on the target spaces of the measures. In the classical situation each function this enforces inclusion of bounded measurable functions, because they are uniform limits of step functions and thus limits in the given topology.

In the preprint [3] we give a tratment in some special categories as in the author's *habilitationsschrift* [1] based on Linton's [5] approach. Here we want to reveal the categorical reasons behind the machinery and point out which conditions are necessary for which conclusions. This more flexible approach allows us also to consider other sorts of convergence and to see what holds in a more general context, e.g. for positive measures, if we consider only semi-additive categories rather than additive ones. The main categorical ideas become clear without convergence; for $\mathcal{X}$ the category of sets and $\mathcal{A}$ the category of modules over a commutative unital ring the results are presented in [2].

If $\mathcal{A}$ is a symmetric monoidal closed category (i.e. has an appropriate *tensor product*), there is a canonical notion of a *commutative monoid in* $\mathcal{A}$; this is a triple (A, r, u) of an $A \in \mathcal{A}$ and $\mathcal{A}$-morphisms $r : A \otimes A \to A$ ("multiplication") and $u : I \to A$ ("unit"), subject to appropriate axioms. Then the universal measure turns out to be also a *universal multiplicative measure*, i.e. the set $M^{\times}(B, A)$ of multiplicative measures from some object $B \in |\mathcal{B}|$ for the category $\mathcal{B}$ of (internal) Boolean algebras in $\mathcal{X}$ to some $A \in |\mathcal{A}|$ is in bijection with $\mathcal{B}(\hat{L}B, A)$, where $\hat{L}B$ is LB with some canonical algebra structure corresponding to *pointwise multiplication*. For valuable comments on monoidal categories the author is indebted to his deceased colleague Harald Lindner. Here a measure is called *multiplicative* if it maps the top element to the unit element of the algebra and binary intersections to products. In general, multiplicative measures are quite rare; e.g. real-valued ones correspond to ultrafilters.

But $M^{\times}(B, A)$ is also in bijection with the set of all Boolean morphisms from B to the Boolean algebra ΘA of idempotents of A. This yields an adjunction $\hat{L} \dashv \Theta$; thus the left adjoint functor $\hat{L}$ preserves colimits, in particular binary coproducts. So for the coproduct $B_0 \check{\otimes} B_1$ of two Boolean algebras B_0, B_1 we obtain an isomorphism $L(B_0 \check{\otimes} B_1) \cong LB_0 \check{\otimes} LB_1$. This allows us to define *product measures*, and we obtain a version of *Fubini's Theorem*.

2. Measures and bimeasures

In a category $\mathcal{X}$ with finite products we can define the category $\mathcal{B}$ of internal *Boolean algebras*, we assume every Boolean algebra to have a top element 1. An object $B = (\underline{B}, t_\vee, t_\wedge, t_0, t_1, t_\perp) \in |\mathcal{B}|$ is given by an object $\underline{B} \in |\mathcal{X}|$ and $\mathcal{X}$-morphisms $t_\wedge, t_\vee : \underline{B} \times \underline{B} \to \underline{B}$, $t_0, t_1 : 1 \to \underline{B}$, $t_\perp : \underline{B} \to \underline{B}$, where the last one corresponds to the *complementation* $x \mapsto x^\perp$. These morphisms should satisfy the usual axioms, i.e. for each $X \in \mathcal{X}$ the hom-set $\mathcal{X}(X, \underline{B})$ is a Boolean algebra (i.e. a complemented distributive lattice).

Now let $\mathcal{A}$ be a *semi-additive* category, i.e. a category enriched over the symmetric monoidal category of commutative monoids. Assume that $\mathcal{A}$ has finite products; note that by semi-additivity they are also finite coproducts. Let $U : \mathcal{A} \to \mathcal{X}$ be a functor that preserves finite products; think of U as a forgetful functor. This induces a canonical structure of a commutative monoid on $\mathcal{X}(X, UA)$ for all $X \in \mathcal{X}$, $A \in \mathcal{A}$. If $\mathcal{A}$ also has coequalizers, then it has all finite colimits; for additive $\mathcal{A}$ it suffices that $\mathcal{A}$ has cokernels.

For $B = (\underline{B}, t_\wedge, t_\vee, t_0, t_1, t_\perp) \in |\mathcal{B}|$, $A \in |\mathcal{A}|$, we call an $\mathcal{X}$-morphism $m : B \to UA$ an *A-valued measure on B* if it satisfies $mt_0 = m.Uo$ and $m.U(p+q) = m.t_\vee + m.t_\wedge$, where the addition on $\mathcal{X}(\underline{B}, UA)$ is as above, $p, q : A \oplus A \to A$ are the canonical projections, and $o : 0 \to A$ is the unique morphism in the semi-additive category $\mathcal{A}$ (with zero object, i.e. empty product 0). $M(B, A)$ denotes the set of all A-valued measures on B; then M is a functor from $\mathcal{B}^{\mathrm{op}} \times \mathcal{A}$ into the category of sets. Since U preserves finite products, it follows immediately that for all $A, A' \in |\mathcal{A}|$ the induced map $\mathcal{A}(A, A') \to \mathcal{X}(UA, UA')$ is a homomorphism of commutative monoids. Now for each $B \in |\mathcal{B}|$ the set-valued functor $M(B, -)$ on $\mathcal{A}$ is a finite limit of representable functors.

2.1. THEOREM. *If the functor U from a semi-additive category $\mathcal{A}$ with finite colomits into a category $\mathcal{X}$ has a left adjoint $F : \mathcal{X} \to \mathcal{A}$, then for each $B = (\underline{B}, t_\wedge, t_\vee, t_0, t_1, t_\perp) \in \mathcal{B}$ the functor $M(B, -)$ is represented by the codomain LB of the joint coequalizer of the pairs $\mathrm{id}_{\underline{B} \oplus \underline{B}}, (t_\vee + t_\wedge) : \underline{B} \oplus \underline{B} \to \underline{B}$ and $o, t_0 : 0 \to \underline{B}$. In particular, $M(B, -)$ is representable, and $L : \mathcal{B} \to \mathcal{A}$ is a functor.*

PROOF. By adjunction, for each $X \in |\mathcal{X}|$ the functor $\mathcal{X}(X, U(-))$ is represented by $FX \in |\mathcal{A}|$. Therefore the above colimit in $\mathcal{A}$ induces a limit of representable functors, that yields that $M(B, -)$ is represented by LB. The functoriality of F follows from the Yoneda Lemma. ■

FB can be viewed as a *space of integrable functions* on B. Representability of $M(B, -)$ can be interpreted as a universal property of the measure $\chi_B \in M(B, LB)$ that corresponds to the identity morphism on B. Then $\chi_B(x)$ plays the role of the *indicator function* of x. The induced map $LB \to A$ can be viewed as *integration* with respect to m; usually it is a continuous linear extension; a suitable completeness condition allows to extend the integral to more general measurable functions than just step functions. σ-Additivity means the same as finite additivity plus continuity in a reasonable sequential topology. So it might be appropriate to work in a category $\mathcal{X}$ of topological spaces. This approach automatically leads to vector integration, since $A = \mathbb{R}$ is not necessary. Observe that integration along χ_B is the identity morphism.

Now we *enrich* the structure of $M(B, A)$. First observe that $M(B, A)$ is isomorphic to $\mathcal{A}(LB, A)$ and a commutative submonoid of $\mathcal{X}(\underline{B}, UA) \cong \mathcal{A}(FB, A)$. If $\mathcal{A}$ is an additive category, $M(B, A)$ is even always an abelian group. If $\mathcal{X}$ is cartesian closed, we can define an *object of measures* as a subobject of the cartesian power $(UA)^{\underline{B}} \in |\mathcal{X}|$; this subobject is mapped to $M(B, A) \subset \mathcal{X}(\underline{B}, UA)$ by U if U is represented by the terminal object. Now assume that we have a *symmetric monoidal structure* on $\mathcal{A}$ with a functor $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, a *unit object* $I \in |\mathcal{A}|$, natural *associativity morphisms* $\alpha(A_0, A_1, A_2) : (A_0 \otimes A_1) \otimes A_2 \to A_0 \otimes (A_1 \otimes A_2)$ for all $A_0, A_1, A_2 \in |\mathcal{A}|$, *unit isomorphisms* $\rho A : A \otimes I \to A$, $\lambda A : I \otimes A \to A$ and a *symmetry* $\sigma(A_0, A_1) : A_0 \otimes A_1 \to A_1 \otimes A_0$ for $A_0, A_1 \in |\mathcal{A}|$. Then we can use the above isomorphism and obtain a *function space object* $[\hat{L}B, A]$ in $\mathcal{A}$.

By a *symmetric monoidal functor* we mean a functor $U : \mathcal{A} \to \mathcal{X}$ together with a natural transformation β that assigns to each $(A_0, A_1) \in |\mathcal{A} \times \mathcal{A}|$ an $\mathcal{X}$-morphism $\beta(A_0, A_1) : UA_0 \times UA_1 \to U(A_0 \otimes A_1)$ and also with an $\mathcal{X}$-morphism $j : T \to UI$, where $T \in |\mathcal{X}|$ is the terminal object (i.e. the unit object in the cartesian symmetric monoidal (closed) structure), subject to the following conditions: $U\alpha(A_0, A_1, A_2).\beta(A_0 \otimes A_1, A_2).\beta(A_0, A_1) \times \mathrm{id}_{UA_2} = \beta(A_0, A_1 \otimes A_2).\mathrm{id}_{UA_0} \otimes \beta(A_1, A_2).t$ for $A_0, A_1, A_2 \in |\mathcal{A}|$ and the canonical associativity isomorphism $t : (UA_0 \times UA_1) \times UA_2 \to UA_0 \times (UA_1 \times UA_2)$. Moreover, $U\lambda A.\beta(I, A).j \times \mathrm{id}_{UA}$ and $U\rho A.\beta(A, I).\mathrm{id}_{UA} \times j$ are the canonical projection isomorphisms $T \times UA \to UA$ and $UA \times T \to UA$ for all $A \in |\mathcal{A}|$; moreover, we assume $U\sigma(A_0, A_1).\beta(A_0, A_1) = \beta(A_1, A_0).s$ for $A_0, A_1 \in |\mathcal{A}|$ and the canonical interchange isomorphism $s : A_0 \times A_1 \to A_1 \times A_0$. This can be done analogously for an arbitrary symmetric monoidal structure on $\mathcal{X}$ (not only for the cartesian one). A symmetric monoidal functor always induces a functor between the corresponding categories of commutative monoids; in our case a commutative monoid (A, r, u) in $\mathcal{A}$ (with $A \in |\mathcal{A}|$, $r : A \otimes A \to A$, $u : I \to A$) is mapped to the commutative monoid $(UA, Ur.\beta(A, A), Uu.j)$.

For a left adjoint $F \dashv U$ we obtain $\mathcal{A}$-morphisms $F(X_0 \times X_1) \to F(UFX_0 \times UFX_1) \to FU(X_0 \otimes X_1) \to X_0 \otimes X_1$, where the first arrow is given by the units $X_i \to UFX_i$; the second one is $F\beta(FX_0, FX_1)$, and the third one is the counit. Moreover, the $\mathcal{A}$-morphism $Fj : FT \to FUI$ followed by the counit $FUI \to I$ yields an $\mathcal{A}$-morphism $FT \to I$. In this way, F becomes a *symmetric comonoidal* functor, i.e. the analogues of the above equations hold in $\mathcal{A}^{\mathrm{op}}$. If the above natural morphisms $F(X_0 \times FX_1) \to FX_0 \otimes FX_1$ and $FT \to I$ are isomorphisms, their inverses yield a symmetric monoidal structure on F, hence F and U are both symmetric monoidal functors. In this case we call $F \dashv U$ a *symmetric monoidal adjunction*; it induces an adjunction between the corresponding categories of monoids (cf. [4]). Note that β need not be a natural isomorphism, and j need not be invertible either.

For $B_0, B_1 \in \mathcal{B}$ and $A \in \mathcal{A}$ an element of $M(B_0, [LB_1, A])$ corresponds to an $\mathcal{A}$-morphism $LB_0 \to [LB_1, A]$ by adjunction; the monoidal structure on $\mathcal{A}$ yields a corresponding $\mathcal{A}$-morphism $LB_0 \otimes LB_1 \to A$. If the above joint coequalizers $c_i : FB_i \to LB_i$ exist for $i \in \{0, 1\}$, c_i corresponds to $\chi_{B_i} : B_i \to ULB_i$. Composing $U[L(c_0 \otimes c_1), A]$ with our above $\mathcal{A}$-morphism $B_0 \to U[LB_0 \otimes LB_1, A]$ yields another $\mathcal{A}$-morphism $LB_0 \otimes LB_1 \to A$.

On the other hand, if $\mathcal{X}$ is cartesian closed, $U[LB_1, A]$ is canonically equivalent to a subobject of $(UA)^{\underline{B}_1}$ the $\mathcal{X}$-morphism $\hat{w} : \underline{B}_0 \to U[LB_1, A]$ yields a morphism $\underline{B}_0 \to (UA)^{\underline{B}_1}$; by cartesian closedness this corresponds to some $w : \underline{B}_0 \times \underline{B}_1 \to UA$. Since $\hat{w}$ is a measure, we get $w.(t_{\vee,0} + t_{\wedge,0}) \times \mathrm{id}_{\underline{B}_1} = w.(p_0 + q_0) \times \mathrm{id}_{\underline{B}_1}$ and $wt_{0,0} = o$; we might say roughly that b is a measure in the first component. Since the codomain of $\hat{w}$ is the object $[L\hat{B}_1, A]$ of measures, we also obtain $b.(\mathrm{id}_{\underline{B}_0} \times (t_{\vee,1} + t_{\wedge,1})) = w.(\mathrm{id}_{B_0} \times (p_1 + q_1))$ and $bt_{0,1} = o$, i.e. w is also a measure in the second component (for $B_i = (\underline{B}_i, t_{\wedge,i}, t_{\vee,i}, t_{0,i}, t_{1,i}, t_{\perp,i})$. So we call w a *bimeasure* in this case. Conversely, if w is a bimeasure, i.e. satisfies the above equations, we get the morphisms $\hat{w}$ as above and then the morphism $LB_0 \otimes LB_1 \to A$. A bimeasure $w : B_0 \times B_1 \to \mathbb{R}$ with values in $[0,1]$ may indicate the joint probability of some event from B_0 and some event from B_1; it describes their dependence. For independent Boolean algebras B_0, B_1 and probability measures $m_0 : B_0 \to \mathbb{R}$, $m_1 : B_1 \to \mathbb{R}$ we have $w(x_0, x_1) = m_0(x_0)m_1(x_1)$ for all $x_0 \in X_0$, $x_1 \in X_1$.

2.2. Lemma. *Assume that $F \dashv U : \mathcal{A} \to \mathcal{X}$ is a monoidal adjunction for the cartesian structure on a cartesian closed category $\mathcal{X}$ with finite limits and that $\mathcal{A}$ is semi-additive and has finite colimits. Then the set-valued functor that assigns to each pair of Boolean algebras B_0, B_1 in $\mathcal{B}$ the set of A-valued bimeasures for each $A \in \mathcal{A}$ is represented by $LB_0 \times LB_1$.*

3. Multiplicative measures and idempotents

If we have two Boolean algebras $B_0, B_1 \in |\mathcal{B}|$ (of events), whose dependence is described by a bimeasure w, it may be useful to have common Boolean algebra and a probability measure on it. One important case is independence, (i.e. $w(x,y) = m_0(x)m_1(y)$ for some $m_i \in M(B_i, \mathbb{R})$); the other extremal case is $B_0 = B_1 =: B$ and $w(x,y) := m(x \wedge y)$ for some probability measure $m : B \to \mathbb{R}$. It turns out that $mt_\wedge : B \times B \to UA$ is an A-valued bimeasure on B for all $B = (\underline{B}, t_\wedge, t_\vee, t_0, t, 1, t_\perp) \in |\mathcal{B}|$. In particular, for the universal measure $\chi_B : B \to LB$, it corresponds to some $r : LB \otimes LB \to LB$ by 2.2. Moreover, for the terminal object T of $\mathcal{X}$ the $\mathcal{X}$-morphism $t_1 : T \to \underline{B}$ corresponds to some $\mathcal{A}$-morphism $I \cong FT \to FB$ by adjunction; composition with the canonical morphism $u : FB \to LB$. By a direct check we see that $\hat{L}B := (LB, r, u)$ is even a *monoid* in the symmetric monoidal category $\mathcal{A}$; this follows from associativity and the unit law of $\wedge$ in Boolean algebras.

Now assume that $\mathcal{A}$ is even a *symmetric monoidal category* and that $F \dashv U$ is a *symmetric monoidal adjunction*; then $\hat{L}$ is a functor from $\mathcal{B}$ into the category $\mathcal{R}$ of commutative monoids in $\mathcal{A}$. For $B \in |\mathcal{B}|$ and $\hat{A} = (A, r, u) \in |\mathcal{R}|$ we call an $m \in \mathcal{X}(\underline{B}, U\underline{A})$ a *multiplicative $\hat{A}$-valued measure on* B if it is an A-valued measure on B and satisfies $mt_\wedge = r.(m \otimes m)$ and $mt_1 = uj$ for the canonical isomorphism $j : LT \to I$. In the classical case of set-functions these equations mean $m(x \wedge y) = m(x)m(y)$ for all $x, y \in \underline{B}$ and $m(1) = 1$. Then we denote the space of all $\hat{A}_0$-valued multiplicative measures on B by $M^\times(B, \hat{A})$. Multiplicative measures do not play an important role in practice; e.g. multiplicative real-valued measures correspond to ultrafilters. But the universal measure $\chi_B : B \to LB$ is always multiplicative if the space LB is equipped with *pointwise multiplication*: The indicator function of the intersection of two sets is the product of the indicator functions, and the constant function with value is a unit element.

3.1. Lemma. *Let $\mathcal{A}$ be a semi-additive symmetric monoidal closed category with finite colimits; let $\mathcal{X}$ be a cartesian closed category with finite limits, and let $F \dashv U : \mathcal{A} \to \mathcal{X}$ be a symmetric monoidal adjunction. Then for each Boolean algebra in $\mathcal{B}$ the functor that* *assigns to each monoid $\hat{A} \in |\mathcal{R}|$ the set $M^\times(B, \hat{A})$ of multiplicative $\hat{A}$-valued measures on B is represented by $\hat{L}B$.*

Proof. For $m \in M^\times(B, A)$, $\hat{A} = (A, r, u) \in \mathcal{R}$, and the induced $\mathcal{A}$-morphism $l : LB \to A$ with $l\chi_B = m$ we obtain $l\hat{r} = r.(l \otimes l)$ and $l\hat{u} = u$ for $\hat{L}B = (LB, \hat{r}, \hat{u})$ by multiplicativity of m. Hence l is even (induced by) a monoid homomorphism, i.e. some $\mathcal{R}$-morphism $\hat{L}B \to A$. ∎

The functor $M^\times$ (a opposed to M) is also *representable* in the *first argument* if $\mathcal{A}$ is even an *additive category* (i.e. admits *subtraction*). Since our adjunction $F \dashv U$ is a symmetric monoidal adjunction, U transforms an object of $\mathcal{R}$, i.e. a commutative monoid in $\mathcal{A}$, into a commutative monoid in the cartesian closed category $\mathcal{X}$, hence a commutative monoid in $\mathcal{X}$ in the usual Linton-Lawvere universally algebraic sense. Moreover the right-adjoint functor U preserves limits, in particular finite products; since $\mathcal{A}$ is an additive category, this leads to an addition on the corresponding $\mathcal{X}$-object. This addition and the above monoid multiplication together yield an *internal commutative unital ring* in $\mathcal{X}$.

In a commutative unital ring R, the idempotent elements (i.e. elements $x \in R$ with $x^2 = x$) form a Boolean algebra with 0 and 1 as in R, the meet as multiplication; the join given by $x \vee y := x + y - xy$ and the complement given by $x^\perp := 1 - x$. For $A \in |\mathcal{R}|$ as above, these operations also yield $\mathcal{X}$-morphisms for the internal ring structure on UA. Moreover, we can restrict them to the object of idempotents; this can be constructed as the equalizer of the identity morphism of UA and $r.U\beta(A, A).d$ where $D : UA \to UA \times UA$ is the diagonal. This yields an internal Boolean algebra in $\mathcal{X}$, i.e. an object $\Theta A \in |\mathcal{B}|$. Then $\Theta : \mathcal{R} \to \mathcal{B}$ is a functor.

3.2. Theorem. *For an additive symmetric monoidal closed category $\mathcal{A}$ with finite colimits, a cartesian closed category $\mathcal{X}$ with finite limits and a symmetric monoidal adjunction $F \dashv U : \mathcal{A} \to \mathcal{X}$ and for each commutative unital ring A in (the given symmetric monoidal structure of) $\mathcal{A}$ the set-valued functor that maps each internal Boolean algebra B in $\mathcal{X}$ to $M^\times(B, A)$ is representable (by the internal Boolean algebra ΘA of idempotents).*

Proof. It has to be shown that every multiplicative A-valued measure M on B induces a $\mathcal{B}$-morphism (i.e. an internal Boolean homomorphism) $B \to \Theta A$. This is shown as in ordinary rings; all elements are idempotent, because multiplicativity of the measure implies $m(x)^2 = m(x \wedge x) = m(x)$ for all x in the Boolean algebra. ∎

If $2 = 1+1$ is invertible in a commutative unital ring R, then an R-valued charge m (i.e. an additive set function) with $m(1) = 1$ on a Boolean algebra B is already multiplicative if it attains only idempotent values. Indeed, for $x, y \in \underline{B}$ with $x \wedge y = 0$ we obtain $m(x) + m(y) = m(x \vee y) = m(x \vee y)^2 = (m(x) + m(y))^2 = m(x)^2 + 2m(x)m(y) + m(y)^2 = m(x) + m(y) + 2m(x)m(y)$, hence $2m(x)m(y) = 0$ an thus $m(x)m(y) = 0$ by invertibility of 2. For arbitrary $x, y \in \underline{B}$ this leads to $m(x)m(y) = (m(x \wedge y) + m(x \wedge y^\perp))(m(x \wedge y) + m(x^\perp \wedge y)) = m(x \wedge y)^2 + m(x \wedge y)m(x \wedge y^\perp) + m(x^\perp \wedge y)m(x \wedge y) + m(x^\perp \wedge y)m(x \wedge y^\perp) = m(x \wedge y) + 0 + 0 + 0 = m(x \wedge y)$ because $x \wedge y \wedge x \wedge y^\perp = x^\perp \wedge y \wedge x \wedge y = x^\perp \wedge y \wedge x \wedge y^\perp = 0$. By internalization we see that an A-valued measure with $mt_1 = uj$ as above is multiplicative if $\mathrm{id}_A + \mathrm{id}_A$ is invertible in $\mathcal{A}(A, A)$.

Combining the above natural isomorphisms $\mathcal{B}(B, \Theta A) \cong M^\times(B, A) \cong \mathcal{A}(\hat{L}B, A)$, we immediately see:

3.3. THEOREM. *For an additive symmetric monoidal closed category $\mathcal{A}$ with finite colimits, a cartesian closed category $\mathcal{X}$ with finite limits and a symmetric monoidal adjunction $F \dashv U : \mathcal{A} \to \mathcal{X}$, the functor $\Theta : \mathcal{R} \to \mathcal{B}$ that assigns to every internal commutative unital ring A in $\mathcal{A}$ the internal Boolean algebra ΘA of idempotents is right adjoint to the functor $\hat{L} : \mathcal{B} \to \mathcal{R}$.*

So the left adjoint functor $\hat{L} : \mathcal{B} \to \mathcal{R}$ preserves colimits, in particular binary coproducts. But it is well-known that binary coproducts in $\mathcal{R}$ can be constructed as *tensor products*. The tensor product $A_0 \otimes A_1$ of two $\mathcal{R}$-objects (i.e. monoids in $\mathcal{A}$) $\hat{A}_0 = (A_0, r_0, u_0)$ and $\hat{A}_1 = (A_1, r_1, u_1)$ is defined by $\hat{A}_0 \otimes \hat{A}_1 := (A_0 \otimes A_1, (r_0 \otimes r_1).s, (u_0 \otimes u_1).(\lambda I)^{-1})$, where $s : (A_0 \otimes A_1) \otimes (A_0 \otimes A_1) \to (A_0 \otimes A_0) \otimes (A_1 \otimes A_1)$ is the canonical interchange isomorphism from the symmetric monoidal structure that interchanges the two middle arguments; remember $\lambda I = \rho I : I \otimes I \to I$. The underlying $\mathcal{A}$-morphisms $A_0, A_1 \to A_0 \otimes A_1$ of the coproduct injections are the composites $A_0 \to A_0 \otimes I \to A_0 \otimes A_1$ and $A_1 \to I \otimes A_1 \to A_0 \otimes A_1$, where the left-hand morphisms are the coherence isomorphisms from the monoidal structure and the right-hand ones are $\mathrm{id}_{A_0} \otimes u_1$ and $u_0 \otimes \mathrm{id}_{A_1}$. Combination with 2.2 yields the following

3.4. THEOREM. *For an additive symmetric monoidal closed category $\mathcal{A}$ with finite colimits, a cartesian closed category $\mathcal{X}$ with finite limits and a symmetric monoidal adjunction $F \dashv U : \mathcal{A} \to \mathcal{X}$ and for a coproduct $B_0 \otimes B_1$ of internal Boolean algebras in $\mathcal{X}$ there is a monoid isomorphism $\hat{L}(B_0 \otimes B_1) \cong \hat{L}B_0 \otimes \hat{L}B_1$ (in particular an isomorphism $L(B_0 \otimes B_1) \cong LB_0 \otimes LB_1$ in $\mathcal{A}$), natural in both arguments. For each $A \in |\mathcal{A}|$, the set of A-valued bimeasures on B_0, B_1 is in natural bijection with $M(B_0 \otimes B_1, A)$.*

By induction this can be extended to finitely many internal Boolean algebras. If $\mathcal{X}$ and $\mathcal{A}$ have filtered colimits, they even yield product measures of more than two measures; observe that $\otimes$ preserves the involved colimits in $\mathcal{A}$ by monoidal closedness. If we consider two measures $m_0 \in M(B_0, A)$, $m_1 \in M(B_1, A)$ for some $(A, r, u) \in \mathcal{R}$, the bimeasure $r.(m_0 \times m_1) : B_0 \times B_1 \to A$ induces a *product measure* on $B_0 \otimes B_1$; this can be extended to finitely many measures as above, if $\mathcal{A}$ has filtered colimits (which are preserved by all $- \otimes A$ because $\mathcal{A}$ is monoidal closed).

References

[1] R. Börger, Ein kategorieller Zugang zur Integrationstheorie, Habilitationsschrift, FernUniversität in Hagen, 1989.

[2] R. Börger, Fubini's Theorem from a categorical viewpoint, Category Theory at Work, Heldermann, Berlin 1991, 367-375.

[3] R. Börger, What is an integral?, Hagen 2010 (preprint).

[4] G.M. Kelly, Doctrinal adjunction, Proceedings Sydney Category Sminar, 1972/1973, Lecture Notes in Mathematic, Springer, Berlin 420 (1974), 257-280.

[5] F.E.J. Linton, Functorial measure theory, Proceedings of the Confrence on Functional Analysis, Irvine 1966, Thompson, Washington D.C. (1968), 26-49.

Fakultät für Mathematik und Informatik, FernUniversität, 58084 Hagen, Germany
Email: `Reinhard.Boerger@FernUni-Hagen.de`

This article may be accessed at `http://www.tac.mta.ca/tac/` or by anonymous ftp at `ftp://ftp.tac.mta.ca/pub/tac/html/volumes/23/12/23-12.{dvi,ps,pdf}`

THEORY AND APPLICATIONS OF CATEGORIES (ISSN 1201-561X) will disseminate articles that significantly advance the study of categorical algebra or methods, or that make significant new contributions to mathematical science using categorical methods. The scope of the journal includes: all areas of pure category theory, including higher dimensional categories; applications of category theory to algebra, geometry and topology and other areas of mathematics; applications of category theory to computer science, physics and other mathematical sciences; contributions to scientific knowledge that make use of categorical methods.
Articles appearing in the journal have been carefully and critically refereed under the responsibility of members of the Editorial Board. Only papers judged to be both significant and excellent are accepted for publication.
Full text of the journal is freely available in .dvi, Postscript and PDF from the journal's server at http://www.tac.mta.ca/tac/ and by ftp. It is archived electronically and in printed paper format.

SUBSCRIPTION INFORMATION. Individual subscribers receive abstracts of articles by e-mail as they are published. To subscribe, send e-mail to tac@mta.ca including a full name and postal address. For institutional subscription, send enquiries to the Managing Editor, Robert Rosebrugh, rrosebrugh@mta.ca.

INFORMATION FOR AUTHORS. The typesetting language of the journal is TEX, and LATEX2e strongly encouraged. Articles should be submitted by e-mail directly to a Transmitting Editor. Please obtain detailed information on submission format and style files at http://www.tac.mta.ca/tac/.