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HOMOLOGY OF N-FOLD GROUPOIDS

Dedicated to Dominique Bourn on the occasion of his sixtieth birthday

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ABSTRACT. Any semi-abelian category $\mathcal{A}$ appears, via the discrete functor, as a full replete reflective subcategory of the semi-abelian category of internal groupoids in $\mathcal{A}$. This allows one to study the homology of n-fold internal groupoids with coefficients in a semi-abelian category $\mathcal{A}$, and to compute explicit higher Hopf formulae. The crucial concept making such computations possible is the notion of protoadditive functor, which can be seen as a natural generalisation of the notion of additive functor.

Introduction

In recent years a new approach to non-abelian homological algebra is being developed in the context of homological [1] and of semi-abelian categories [23]. A crucial role in the definition of these categories is played by the notion of "protomodularity" discovered by Bourn [5], which is a fundamental property the categories of groups, rings, algebras, crossed modules, topological groups and other similar structures all have in common.

In a semi-abelian category $\mathcal{A}$, it has been possible to develop a theory of higher central extensions [15, 14], a generalisation of the theory by Janelidze and Kelly [22] also inspired by [20], allowing one to define and study a natural notion of semi-abelian homology of a given object G in $\mathcal{A}$, where the "coefficients" can now be taken in any Birkhoff subcategory $\mathcal{B}$ of $\mathcal{A}$. The corresponding "homology objects", which can be expressed in terms of higher Hopf formulae involving generalised commutators depending on the choice of $\mathcal{B}$, include, in particular, the formulae giving the homology groups of a group discovered by Brown and Ellis [10]: for this, it suffices to choose for $\mathcal{A}$ the category Gp of groups and for $\mathcal{B}$ its Birkhoff subcategory Ab of abelian groups. Other examples have been studied in [15], obtaining, for instance, the new formulae describing the homology of a precrossed module with coefficients in the category XMod of crossed modules.

Once the general theory has been settled, the crucial step in order to make the Hopf formulae precise in specific examples consists in finding an explicit characterisation of the higher central extensions. In some cases, it is not easy to find such a characterisation:

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accordingly, it is natural to look for some additional "good" properties of the reflector from $\mathcal{A}$ to $\mathcal{B}$ which could be helpful to effectively compute the commutators appearing in the formulae of the homology objects.

In this paper we introduce and begin to study a property which is useful for this purpose, namely the notion of *protoadditive functor*, which can be seen as a generalisation of the classical notion of additive functor: when $\mathcal{A}$ and $\mathcal{B}$ are pointed protomodular categories, a functor $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ is protoadditive if it preserves split short exact sequences. The protoadditivity for a functor $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ is thus equivalent to its additivity whenever both $\mathcal{A}$ and $\mathcal{B}$ are additive categories.

When a Birkhoff reflector $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ is protoadditive, it is quite simple to characterise the higher central extensions. We examine this fact here in detail by looking at a chain of Birkhoff adjunctions in which all the left adjoints are protoadditive functors, as we show in the first section. If $\mathcal{A}$ is semi-abelian, the category $\mathsf{Gpd}(\mathcal{A})$ of internal groupoids and functors is semi-abelian [7], and this implies that the same remains true for the category $\mathsf{Gpd}^n(\mathcal{A}) = \mathsf{Gpd}(\mathsf{Gpd}^{n-1}(\mathcal{A}))$ of n-fold internal groupoids. Consequently, it is meaningful to investigate the semi-abelian homology of an n-fold internal groupoid with coefficients in $\mathcal{A}$ by considering the chain of adjunctions

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift right, "D"'] \arrow[r, phantom, "\perp"] & \mathsf{Gpd}(\mathcal{A}) \arrow[l, shift right, "\pi_0"'] \arrow[r, shift right, "D^2"'] \arrow[r, phantom, "\perp"] & \mathsf{Gpd}^2(\mathcal{A}) \arrow[l, shift right, "\pi_0^2"'] \quad \dots \quad \mathsf{Gpd}^{n-1}(\mathcal{A}) \arrow[r, shift right, "D^n"'] \arrow[r, phantom, "\perp"] & \mathsf{Gpd}^n(\mathcal{A}) \arrow[l, shift right, "\pi_0^n"'] \quad \dots \end{tikzcd} \qquad (1)$$

where the D^i's and the π_0^i's are the higher order "discrete" and "connected components" functors, respectively. Observe that when the coefficients are taken in the semi-abelian category $\mathcal{A} = \mathsf{Gp}$ of groups, the objects of $\mathsf{Gpd}^n(\mathcal{A})$ can be seen as the n-cat-groups considered in [24]. The n-fold internal groupoids in the category of groups form an interesting algebraic category, which has been intensively investigated in recent years, as it shares many remarkable properties with the category of groups.

In the second section we obtain, by induction, a precise characterisation of the k-fold central extensions of the n-fold internal groupoids relative to (1) (Theorem 2.12), yielding a "higher extension version" of a result in [17] (recalled here below as Corollary 2.8). In the last section we first show that the category $\mathsf{Gpd}^n(\mathcal{A})$ of internal n-fold groupoids has enough regular projectives whenever $\mathcal{A}$ has enough regular projectives (Proposition 3.2): this is due to the fact that the forgetful functor $U\colon \mathsf{Gpd}^n(\mathcal{A}) \longrightarrow \mathcal{A}$ sending an n-fold internal groupoid to its "object of n-fold arrows" is monadic and preserves regular epimorphisms (Proposition 3.1). Finally, we give explicit Hopf formulae for the homology of an n-fold internal groupoid (Theorem 3.3).

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1. Internal groupoids and protoadditive functors

Let $\mathcal{A}$ be a semi-abelian category [23]. This means that $\mathcal{A}$ is

- finitely complete and finitely cocomplete ;
- pointed, with zero object 0 ;
- Barr exact ;
- Bourn protomodular.

Recall that, in the presence of a zero object, the property of protomodularity of $\mathcal{A}$ can be expressed by asking the validity of the split short five lemma : in any commutative diagram of split short exact sequences in $\mathcal{A}$

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d, "u"'] & A \arrow[r, shift right] \arrow[d, "v"'] & B \arrow[l, shift right] \arrow[r] \arrow[d, "w"] & 0 \\ 0 \arrow[r] & K' \arrow[r] & A' \arrow[r, shift right] & B' \arrow[l, shift right] \arrow[r] & 0 \end{tikzcd}$$

the arrow v is an isomorphism whenever u and w are isomorphisms.

We denote an (internal) groupoid in $\mathcal{A}$ by $\mathbf{A} = (A, A_0, m_A, d_A, c_A, i_A)$ or, more briefly, by $\mathbf{A}$: it can be pictured as a diagram in $\mathcal{A}$ of the form

$$\begin{tikzcd} A \times_{A_0} A \arrow[r, "m_A"] & A \arrow[r, shift left=2, "d_A"] \arrow[r, shift right=2, "c_A"'] & A_0, \arrow[l, "i_A"'] \end{tikzcd} \tag{2}$$

where A_0 represents the "object of objects", A the "object of arrows", $A \times_{A_0} A$ the "object of composable arrows", d_A the "domain", c_A the "codomain", i_A the "identity", and m_A the "composition". Of course, these arrows have to satisfy the usual commutativity conditions expressing, internally, the fact that $\mathbf{A}$ is a groupoid, as explained, for instance, in [25]. A groupoid is connected if the canonical arrow $(d_A, c_A): A \to A_0 \times A_0$ is a regular epimorphism. When $\mathcal{A}$ is a semi-abelian category, the category $\mathsf{Gpd}(\mathcal{A})$ of groupoids in $\mathcal{A}$ and (internal) functors is again semi-abelian [7]; furthermore, the groupoid structure on any (internal) reflexive graph is unique, when it exists, and the forgetful functor from $\mathsf{Gpd}(\mathcal{A})$ to the category $\mathsf{RG}(\mathcal{A})$ of reflexive graphs in $\mathcal{A}$ is full [11].

In the present article we shall be interested in studying the higher central extensions and the homology arising from the adjunction

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift right, "D"'] \arrow[r, phantom, "\perp"] & \mathsf{Gpd}(\mathcal{A}), \arrow[l, shift right, "\pi_0"'] \end{tikzcd} \tag{3}$$

where $D: \mathcal{A} \longrightarrow \mathsf{Gpd}(\mathcal{A})$ is the discrete functor associating, with any object $A \in \mathcal{A}$, the discrete equivalence relation on A, and the functor $\pi_0: \mathsf{Gpd}(\mathcal{A}) \longrightarrow \mathcal{A}$ is the connected components functor. This functor π_0 sends a groupoid $\mathbf{A}$ as in (2) to the object $\pi_0(\mathbf{A})$ in $\mathcal{A}$ given by the coequalizer of d_A and c_A.

Note that D restricts to an equivalence $\mathcal{A} \cong \mathsf{Dis}(\mathcal{A})$, where $\mathsf{Dis}(\mathcal{A})$ is the full subcategory of $\mathsf{Gpd}(\mathcal{A})$ whose objects are the discrete equivalence relations. This category

$\mathsf{Dis}(\mathcal{A})$ is clearly closed in $\mathsf{Gpd}(\mathcal{A})$ under subobjects and regular quotients, therefore it is a Birkhoff subcategory of $\mathsf{Gpd}(\mathcal{A})$.

The adjunction (3) determines a torsion theory $(\mathcal{T}, \mathcal{F})$ (in the sense of [8]) in the semi-abelian category $\mathsf{Gpd}(\mathcal{A})$. Indeed, the subcategory $\mathcal{F} = \mathsf{Dis}(\mathcal{A})$ of $\mathsf{Gpd}(\mathcal{A})$ is the "torsion-free subcategory", while the "torsion subcategory" $\mathcal{T} = \mathsf{ConnGpd}(\mathcal{A})$ is the category of connected groupoids in $\mathcal{A}$. To see this, it suffices to observe that any groupoid $\mathbf{A}$ determines a short exact sequence

$$0 \longrightarrow \Gamma_{\mathbf{A}}(0) \longrightarrow \mathbf{A} \xrightarrow{\eta_{\mathbf{A}}} D\pi_0(\mathbf{A}) \longrightarrow 0\ ,$$

where $\eta_{\mathbf{A}}\colon \mathbf{A} \longrightarrow D\pi_0(\mathbf{A})$ is the $\mathbf{A}$-component of the unit of the adjunction (3) and $D\pi_0(\mathbf{A})$ is then the discrete equivalence relation on the coequalizer of d_A and c_A, while the kernel $\Gamma_{\mathbf{A}}(0)$ of $\eta_{\mathbf{A}}$ is a connected groupoid, as one can easily see by checking that $\Gamma_{\mathbf{A}}(0)$ is precisely the full subgroupoid of $\mathbf{A}$ given by the "connected component of 0". Furthermore, any functor from a connected groupoid to a discrete groupoid clearly is the zero arrow, so that also the second axiom $\mathsf{Hom}_{\mathsf{Gpd}(\mathcal{A})}(\mathcal{T}, \mathcal{F}) = \{0\}$ in the definition of torsion theory is satisfied. The fact that $\pi_0\colon \mathsf{Gpd}(\mathcal{A}) \longrightarrow \mathcal{A}$ determines a torsion-free reflection could also be deduced from the results in [4] and [8].

The reflection in (3) will be shown to have the following important property, which will be defined in the context of pointed protomodular categories in the sense of Bourn [5].

1.1. Definition. *Let $\mathcal{A}$ and $\mathcal{B}$ be pointed protomodular categories. A functor $F\colon \mathcal{A} \to \mathcal{B}$ is a* protoadditive *functor if it preserves split short exact sequences: if*

$$0 \longrightarrow K \longrightarrow A \underset{\longrightarrow}{\overset{\longleftarrow}{}} B \longrightarrow 0 \tag{4}$$

is a split short exact sequence in $\mathcal{A}$, then

$$0 \longrightarrow F(K) \longrightarrow F(A) \underset{\longrightarrow}{\overset{\longleftarrow}{}} F(B) \longrightarrow 0$$

is a split short exact sequence in $\mathcal{B}$.

Any additive functor between additive categories (with pullbacks) is protoadditive. In particular any reflective subcategory of an additive category has a protoadditive reflector. Of course, any left exact functor between pointed protomodular categories is protoadditive. In the homological and semi-abelian contexts any protolocalisation in the sense of [3] gives a protoadditive functor.

The principal example of a protoadditive functor considered in the present article will be the connected components functor $\pi_0\colon \mathsf{Gpd}(\mathcal{A}) \longrightarrow \mathcal{A}$, with $\mathcal{A}$ semi-abelian: remark that this is not a protolocalisation, since π_0 does not preserve kernels.

1.2. Theorem. *Let $\mathcal{A}$ be a semi-abelian category. Then $\pi_0\colon \mathsf{Gpd}(\mathcal{A}) \longrightarrow \mathcal{A}$ is a protoadditive functor.*

Proof. As explained in [4], the adjunction (3) can be decomposed into the composite of two adjunctions as in the diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, phantom, "\perp"] & \mathsf{Eq}(\mathcal{A}) \arrow[l, shift right, "\pi_0"'] \arrow[r, phantom, "\perp"] & \mathsf{Gpd}(\mathcal{A}) \arrow[l, shift right, "\mathsf{Supp}"'] \end{tikzcd}$$

with $D\colon \mathcal{A} \to \mathsf{Eq}(\mathcal{A})$ and $H\colon \mathsf{Eq}(\mathcal{A}) \to \mathsf{Gpd}(\mathcal{A})$ the lower arrows,

where $\mathsf{Eq}(\mathcal{A})$ is the category of (internal) equivalence relations in $\mathcal{A}$, $H\colon \mathsf{Eq}(\mathcal{A}) \to \mathsf{Gpd}(\mathcal{A})$ is the inclusion and its left adjoint $\mathsf{Supp}\colon \mathsf{Gpd}(\mathcal{A}) \to \mathsf{Eq}(\mathcal{A})$ is the "support" functor, sending a groupoid $\mathbf{A} = (A, A_0, m_A, d_A, c_A, i_A)$ as in (2) to the equivalence relation

$$\begin{tikzcd} \mathsf{Supp}\,(\mathbf{A}) \arrow[r, shift left=2, "\delta_A"] \arrow[r, shift right=2, "\gamma_A"'] & A_0 \arrow[l, "\iota_A" description] \end{tikzcd},$$

where $\mathsf{Supp}\,(\mathbf{A})$ is the "regular image" of the arrow $(d_A, c_A)\colon A \to A_0 \times A_0$. Consequently, in order to show that $\pi_0\colon \mathsf{Gpd}(\mathcal{A}) \to \mathcal{A}$ is a protoadditive functor, it suffices to prove that both the left adjoints $\mathsf{Supp}\colon \mathsf{Gpd}(\mathcal{A}) \to \mathsf{Eq}(\mathcal{A})$ and $\pi_0\colon \mathsf{Eq}(\mathcal{A}) \to \mathcal{A}$ preserve split short exact sequences, and this is what we are now going to check.

First, by applying the functor Supp to a split short exact sequence in $\mathsf{Gpd}(\mathcal{A})$

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{K} \arrow[r, "\mathbf{k}"] & \mathbf{A} \arrow[r, shift right, "\mathbf{f}"'] & \mathbf{B} \arrow[l, shift right, "\mathbf{s}"'] \arrow[r] & 0, \end{tikzcd}$$

one gets the following commutative diagram in $\mathcal{A}$:

$$\begin{tikzcd} & \mathsf{Supp}\,(\mathbf{K}) \arrow[r, "\mathsf{Supp}\,(\mathbf{k})"] \arrow[d, shift right, "\delta_K"'] \arrow[d, shift left, "\gamma_K"] & \mathsf{Supp}\,(\mathbf{A}) \arrow[r, shift right, "\mathsf{Supp}\,(\mathbf{f})"'] \arrow[d, shift right, "\delta_A"'] \arrow[d, shift left, "\gamma_A"] & \mathsf{Supp}\,(\mathbf{B}) \arrow[l, shift right, "\mathsf{Supp}\,(\mathbf{s})"'] \arrow[d, shift right, "\delta_B"'] \arrow[d, shift left, "\gamma_B"] & \\ 0 \arrow[r] & K_0 \arrow[r, "k_0"'] & A_0 \arrow[r, shift right, "f_0"'] & B_0 \arrow[l, shift right, "s_0"'] \arrow[r] & 0. \end{tikzcd} \tag{5}$$

It suffices to check that the upper sequence in (5) is left exact. But this follows from the

3×3-Lemma [6] applied to the following commutative diagram in $\mathcal{A}$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K[\eta_{\mathbf{K}}] \arrow[r] \arrow[d] & K[\eta_{\mathbf{A}}] \arrow[r, shift right] \arrow[d] & K[\eta_{\mathbf{B}}] \arrow[r] \arrow[l, shift right] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r, "k"] \arrow[d, "\eta_K"'] & A \arrow[r, shift right, "f"'] \arrow[d, "\eta_A"'] & B \arrow[r] \arrow[l, shift right, "s"'] \arrow[d, "\eta_B"] & 0 \\
 & \mathsf{Supp}\,(\mathbf{K}) \arrow[r, "\mathsf{Supp}\,(\mathbf{k})"'] \arrow[d] & \mathsf{Supp}\,(\mathbf{A}) \arrow[r, shift right, "\mathsf{Supp}\,(\mathbf{f})"'] \arrow[d] & \mathsf{Supp}\,(\mathbf{B}) \arrow[l, shift right, "\mathsf{Supp}\,(\mathbf{s})"'] \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where η is the ("object of arrows component" of the) unit of the adjunction $\mathsf{Supp} \dashv H$. Note that the upper sequence is indeed short exact: it can also be obtained by vertically taking kernels in the following diagram of split short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "k"] \arrow[d, "(\delta_K, \gamma_K)"'] & A \arrow[r, shift right, "f"'] \arrow[d, "(\delta_A, \gamma_A)"'] & B \arrow[r] \arrow[l, shift right, "s"'] \arrow[d, "(\delta_B, \gamma_B)"] & 0 \\
0 \arrow[r] & K_0 \times K_0 \arrow[r, "k_0 \times k_0"'] & A_0 \times A_0 \arrow[r, shift right, "f_0 \times f_0"'] & B_0 \times B_0 \arrow[r] \arrow[l, shift right, "s_0 \times s_0"'] & 0.
\end{tikzcd}$$

Secondly, consider a split short exact sequence in $\mathsf{Eq}(\mathcal{A})$:

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r, "k"] \arrow[d, shift right, "\delta_R"'] \arrow[d, shift left, "\gamma_R"] & S \arrow[r, shift right, "f"'] \arrow[d, shift right, "\delta_S"'] \arrow[d, shift left, "\gamma_S"] & T \arrow[r] \arrow[l, shift right, "s"'] \arrow[d, shift right, "\delta_T"'] \arrow[d, shift left, "\gamma_T"] & 0 \\
0 \arrow[r] & A_0 \arrow[r, "k_0"'] & B_0 \arrow[r, shift right, "f_0"'] & C_0 \arrow[r] \arrow[l, shift right, "s_0"'] & 0.
\end{tikzcd}$$

The functor $\pi_0\colon \mathsf{Eq}(\mathcal{A}) \to \mathcal{A}$ then induces the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K[\delta_R] \arrow[r, "\overline{k}"] \arrow[d, "\gamma_R \circ \mathsf{Ker}\,(\delta_R)"'] & K[\delta_S] \arrow[r, shift right, "\overline{f}"'] \arrow[d, "\gamma_S \circ \mathsf{Ker}\,(\delta_S)"'] & K[\delta_T] \arrow[l, shift right, "\overline{s}"'] \arrow[r] \arrow[d, "\gamma_T \circ \mathsf{Ker}\,(\delta_T)"] & 0 \\
0 \arrow[r] & A_0 \arrow[r, "k_0"] \arrow[d, "q_R"'] & B_0 \arrow[r, shift right, "f_0"'] \arrow[d, "q_S"'] & C_0 \arrow[l, shift right, "s_0"'] \arrow[r] \arrow[d, "q_T"] & 0 \\
 & \pi_0(\mathbf{R}) \arrow[r, "\pi_0(\mathbf{k})"'] \arrow[d] & \pi_0(\mathbf{S}) \arrow[r, shift right, "\pi_0(\mathbf{f})"'] \arrow[d] & \pi_0(\mathbf{T}) \arrow[l, shift right, "\pi_0(\mathbf{s})"'] \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where the top sequence, induced by the "normalisation" of the equivalence relations $\mathbf{R}$, $\mathbf{S}$ and $\mathbf{T}$ (the vertical sequences are then exact by construction), is easily seen to be exact. By the 3×3-Lemma [6] it follows that the lower sequence is exact. ■

As we mentioned in the introduction, the fact that $\mathcal{A}$ is semi-abelian implies that the category $\mathsf{Gpd}^n(\mathcal{A})$ of n-fold groupoids is semi-abelian. When considering the composite adjunction (1) we shall write $\Pi_0^n = \pi_0 \circ \pi_0^2 \circ \ldots \circ \pi_0^n \colon \mathsf{Gpd}^n(\mathcal{A}) \to \mathcal{A}$ for the composite reflector.

1.3. Proposition. *Let $\mathcal{A}$ be a semi-abelian category. Then $\Pi_0^n\colon \mathsf{Gpd}^n(\mathcal{A}) \longrightarrow \mathcal{A}$ is a protoadditive functor for all $n \geq 1$.*

Proof. This follows easily from Theorem 1.2, by taking into account the fact that $\mathsf{Gpd}^n(\mathcal{A}) = \mathsf{Gpd}(\mathsf{Gpd}^{n-1}(\mathcal{A}))$. ■

2. Higher central extensions of n-fold internal groupoids

In this section, we shall characterise the higher central extensions of n-fold groupoids in a semi-abelian category. As an extension of Janelidze and Kelly's categorical notion of central extension [22], *higher central extensions* have been defined in [15] in order to study the Brown-Ellis-Hopf formulae [10] in the context of semi-abelian categories. We shall adopt here the axiomatic approach to (higher) extensions from [14], which we now briefly recall.

Let us write $\mathsf{Arr}\mathcal{A}$ for the category of arrows in a category $\mathcal{A}$. With a class of morphisms $\mathcal{E}$ in a category with pullbacks $\mathcal{A}$, one can associate a class $\mathcal{E}^-$ of objects of $\mathcal{A}$, and a class $\mathcal{E}^1$ of morphisms of $\mathsf{Arr}\mathcal{A}$ (commutative squares in $\mathcal{A}$), as follows: $\mathcal{E}^-$ consists of all objects $A \in \mathcal{A}$ with the property that there exists in $\mathcal{E}$ at least one arrow $A \longrightarrow B$ or one arrow $C \longrightarrow A$; $\mathcal{E}^1$ consists of all morphisms $(f_1, f_0)\colon a \longrightarrow b$ in $\mathsf{Arr}\mathcal{A}$

$$\begin{tikzcd}
A_1 \arrow[rd, dotted, "r"] \arrow[rrd, bend left, "f_1"] \arrow[rdd, bend right, "a"'] & & \\
& P \arrow[r, dotted] \arrow[d, dotted] \arrow[rd, phantom, very near start, "\lrcorner"] & B_1 \arrow[d, "b"] \\
& A_0 \arrow[r, "f_0"'] & B_0,
\end{tikzcd}$$

such that every arrow in the above commutative diagram is in $\mathcal{E}$, where r is the unique factorisation to the pullback $P = A_0 \times_{B_0} B_1$. The full subcategory of $\mathcal{A}$ determined by $\mathcal{E}^-$ will be denoted by $\mathcal{A}_{\mathcal{E}}$.

2.1. Definition. *A class $\mathcal{E}$ of regular epimorphisms in a semi-abelian category $\mathcal{A}$ is a* class of extensions *whenever it satisfies the following properties:*

1. *$\mathcal{E}$ contains all split epimorphisms in $\mathcal{A}_{\mathcal{E}}$, and $0 \in \mathcal{E}^-$;*
2. *$\mathcal{E}$ is closed under pulling back along arrows of $\mathcal{A}_{\mathcal{E}}$, is closed under composition, and if a composite $g \circ f$ of morphisms of $\mathcal{A}_{\mathcal{E}}$ is in $\mathcal{E}$, then also $g \in \mathcal{E}$;*
3. *$\mathcal{E}$ is completely determined by the class of objects $\mathcal{E}^-$ in the following way: a regular epimorphism $f\colon A \longrightarrow B$ is in $\mathcal{E}$ if and only if both its domain A and its kernel $K[f]$ are in $\mathcal{E}^-$;*
4. *For a commutative diagram with short exact rows in $\mathcal{A}$ as below,*

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d, "k"'] & A \arrow[r] \arrow[d, "a"'] & B \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L \arrow[r] & C \arrow[r] & B \arrow[r] & 0,
\end{tikzcd}$$

one has: if $k \in \mathcal{E}$ and a lies in $\mathcal{A}_{\mathcal{E}}$, then $a \in \mathcal{E}$.

2.2. Example. The class of all regular epimorphisms in a semi-abelian category is a class of extensions in the sense of Definition 2.1. In this case $\mathcal{E}^- = \mathrm{Ob}(\mathcal{A})$ and $\mathcal{A}_{\mathcal{E}} = \mathcal{A}$.

If $\mathcal{E}$ is a class of extensions, then a morphism $f \in \mathcal{E}$ is called an ($\mathcal{E}$-)*extension*, and the full subcategory of $\mathsf{Arr}\mathcal{A}$ determined by $\mathcal{E}$ is denoted by $\mathsf{Ext}_{\mathcal{E}}\mathcal{A}$ or, simply, $\mathsf{Ext}\mathcal{A}$. An element of $\mathcal{E}^1$ is called a *double ($\mathcal{E}$-)extension* and the full subcategory of $\mathsf{Arr}^2\mathcal{A} = \mathsf{Arr}(\mathsf{Arr}\mathcal{A})$ determined by $\mathcal{E}^1$ is denoted by $\mathsf{Ext}^2\mathcal{A}$. The class of double $\mathcal{E}$-extensions $\mathcal{E}^1$ is itself a class of extensions, in the (semi-abelian) category $\mathsf{Arr}\mathcal{A}$ (see [14], Proposition 1.8). Therefore, inductively, any class of extensions $\mathcal{E}$ determines, for any $k \geq 1$, a class of extensions $\mathcal{E}^k$ in $\mathsf{Arr}^k\mathcal{A}$, called $(k+1)$*-fold extensions*. The corresponding full subcategory of $\mathsf{Arr}^{k+1}\mathcal{A}$ is denoted by $\mathsf{Ext}^{k+1}\mathcal{A}$. Note that, for any $k \geq 1$, we have that $(\mathcal{E}^k)^- = \mathcal{E}^{k-1}$ and $(\mathsf{Arr}^k\mathcal{A})_{\mathcal{E}^k} = \mathsf{Ext}^k\mathcal{A}$ (where $\mathcal{E}^0 = \mathcal{E}$, $\mathsf{Ext}^1\mathcal{A} = \mathsf{Ext}\mathcal{A}$ and $\mathsf{Arr}^1\mathcal{A} = \mathsf{Arr}\mathcal{A}$).

(Higher) central extensions in a semi-abelian category $\mathcal{A}$ are defined with respect to a strongly $\mathcal{E}$-Birkhoff subcategory $\mathcal{B}$ of $\mathcal{A}$, by which we mean the following:

2.3. Definition. *Let $\mathcal{E}$ be a class of extensions in a semi-abelian category $\mathcal{A}$, and $\mathcal{B}$ a full and replete reflective subcategory of $\mathcal{A}_{\mathcal{E}}$ with reflector $I\colon \mathcal{A}_{\mathcal{E}} \longrightarrow \mathcal{B}$. $\mathcal{B}$ is a* strongly $\mathcal{E}$-Birkhoff *subcategory of $\mathcal{A}$ if for every $\mathcal{E}$-extension $f : A \longrightarrow B$ the canonical commutative square induced by the adjunction*

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] & I(A) \arrow[d, "I(f)"] \\ B \arrow[r, "\eta_B"'] & I(B) \end{tikzcd} \tag{6}$$

is a double $\mathcal{E}$-extension.

Remark that, under the assumptions of Definition 2.3, all $\eta_A\colon A \to I(A)$ are regular epimorphisms (since they are $\mathcal{E}$-extensions), thus $\mathcal{B}$ is a (regular epi)-reflective subcategory of $\mathcal{A}_{\mathcal{E}}$.

2.4. Example. Any Birkhoff subcategory of a semi-abelian category is a strongly $\mathcal{E}$-Birkhoff subcategory of $\mathcal{A}$, with $\mathcal{E}$ the class of all regular epimorphisms [22]. In particular, this accounts for the subcategory $\mathsf{Dis}^{\mathsf{n}}(\mathcal{A})$ of $\mathsf{Gpd}^n(\mathcal{A})$ of n-fold discrete equivalence relations, where $\mathcal{A}$ is a semi-abelian category and $n \geq 1$.

Now, assume that $\mathcal{B}$ is a strongly $\mathcal{E}$-Birkhoff subcategory of a semi-abelian category $\mathcal{A}$, where $\mathcal{E}$ is a class of extensions in $\mathcal{A}$. For an object B of $\mathcal{A}$, we write $\mathsf{Ext}(B)$ for the full subcategory of the comma category $\mathcal{A} \downarrow B$ determined by the arrows $f\colon A \to B$ in $\mathcal{E}$. When $g\colon E \to B$ is an arrow with $E \in \mathcal{E}^-$, one defines the functor $g^*\colon \mathsf{Ext}(B) \to \mathsf{Ext}(E)$ by associating, with an extension $f\colon A \to B$, its pullback $g^*(f)\colon E \times_B A \to A$ along g. We write $\pi_1, \pi_2\colon R[f] \to A$ for the kernel pair projections.

2.5. Definition. *Let $\mathcal{E}$ be a class of extensions in a semi-abelian category $\mathcal{A}$, $\mathcal{B}$ a strongly $\mathcal{E}$-Birkhoff subcategory of $\mathcal{A}$, $f\colon A \to B$ an extension in $\mathcal{E}$.*

1. *$f\colon A \to B$ is a* trivial extension *when the square (6) is a pullback ;*
2. *$f\colon A \to B$ is a* normal extension *when the first projection $\pi_1\colon R[f] \to A$ (equivalently, the second projection π_2) is a trivial extension;*
3. *$f\colon A \to B$ is a* central extension *when there exists a $g\colon E \to B$ in $\mathsf{Ext}(B)$ such that $g^*(f)\colon E \times_B A \to E$ is a trivial extension.*

2.6. REMARK. By definition, every normal extension is central. Under our assumptions one also has the converse (see Proposition 4.5 in [15]).

We shall write $\mathsf{CExt}_{\mathcal{B}}\mathcal{A}$ for the full subcategory of $\mathsf{Ext}\mathcal{A}$ determined by the central extensions with respect to $\mathcal{B}$.

Note that it follows from Definition 2.1 that the category $\mathcal{A}_{\mathcal{E}}$ has pullbacks of arbitrary morphisms along extensions, which are pullbacks in $\mathcal{A}$. Since, moreover, any split epimorphism in $\mathcal{A}_{\mathcal{E}}$ is an extension, it follows that $\mathcal{A}_{\mathcal{E}}$ is protomodular because $\mathcal{A}$ is so. Furthermore, since $\mathcal{B}$ is a replete (regular epi)-reflective subcategory of $\mathcal{A}_{\mathcal{E}}$, $\mathcal{B}$ is closed in $\mathcal{A}_{\mathcal{E}}$ under products and subobjects, so that $\mathcal{B}$ is closed in $\mathcal{A}_{\mathcal{E}}$ under limits. Consequently, $\mathcal{B}$ is protomodular as well.

The case where the reflector $I\colon \mathcal{A}_{\mathcal{E}} \longrightarrow \mathcal{B}$ is protoadditive, is of interest:

2.7. PROPOSITION. *Let $\mathcal{B}$ be a strongly $\mathcal{E}$-Birkhoff subcategory of $\mathcal{A}$ such that the reflector $I\colon \mathcal{A}_{\mathcal{E}} \longrightarrow \mathcal{B}$ is a protoadditive functor. Then an extension $f\colon A \longrightarrow B$ is central if and only if $K[f] \in \mathcal{B}$.*

PROOF. Let $f\colon A \longrightarrow B$ be an extension. Using Remark 2.6, f is a central extension if and only if the first projection π_1 of its kernel pair is trivial, i. e. if the right hand square below is a pullback:

$$\begin{tikzcd}
K[\pi_1] \arrow[r] \arrow[d] & R[f] \arrow[r, "\pi_1"] \arrow[d] & A \arrow[d] \\
I(K[\pi_1]) \arrow[r] & I(R[f]) \arrow[r, "I(\pi_1)"'] & I(A).
\end{tikzcd}$$

Using that I preserves split short exact sequences, we see that both rows of the diagram are short exact sequences, so that the right hand square is a pullback if and only if the left hand vertical morphism is an isomorphism, i. e. if and only if $K[\pi_1] \in \mathcal{B}$. It suffices now to note that $K[\pi_1] = K[f]$. ■

Applying this proposition to the particular case of the adjunction (3), we find:

2.8. COROLLARY. *[17] If $\mathcal{A}$ is a semi-abelian category, then the central extensions in $\mathsf{Gpd}(\mathcal{A})$, with respect to $\mathsf{Dis}(\mathcal{A})$, are precisely the extensions $\mathbf{f}\colon \mathbf{A} \longrightarrow \mathbf{B}$ with $K[d_A] \cap K[f] = 0$ (or, equivalently, $K[c_A] \cap K[f] = 0$) i.e. the discrete fibrations.*

PROOF. Consider an extension $\mathbf{f}\colon \mathbf{A} \longrightarrow \mathbf{B}$ of groupoids and its kernel $K[\mathbf{f}]$, as in the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & K[f] \arrow[r] \arrow[d, shift right, "d_{K[f]}"'] \arrow[d, shift left, "c_{K[f]}"] & A \arrow[r, "f"] \arrow[d, shift right, "d_A"'] \arrow[d, shift left, "c_A"] & B \arrow[r] \arrow[d, shift right, "d_B"'] \arrow[d, shift left, "c_B"] & 0 \\
0 \arrow[r] & K[f_0] \arrow[r] & A_0 \arrow[r, "f_0"'] & B_0 \arrow[r] & 0.
\end{tikzcd}$$

By Proposition 2.7 and Theorem 1.2, we know that $\mathbf{f}$ is central if and only if $K[\mathbf{f}]$ is a discrete equivalence relation. By protomodularity of $\mathcal{A}$ this is equivalent to $\mathbf{f}$ being a discrete fibration. Since the right hand squares in the diagram are double extensions as split epimorphisms between extensions, this is also equivalent to any of the two factorisations $A \longrightarrow A_0 \times_{B_0} B$ being a monomorphism. Again, by protomodularity, this amounts to any of the conditions $K[d_A] \cap K[f] = 0$ or $K[c_A] \cap K[f] = 0$. ∎

Let us now consider the case of n-fold groupoids, for $n \geq 1$. As in the case $n = 1$, we write A for the "object of n-fold arrows" of an n-fold groupoid $\mathbf{A}$. If $\mathbf{f}\colon \mathbf{A} \longrightarrow \mathbf{B}$ is a morphism in $\mathsf{Gpd}^n(\mathcal{A})$, we write f for the morphism $A \longrightarrow B$ "at the level of n-fold arrows". Furthermore, we write d^i_A $(1 \leq i \leq n)$ for the domain arrows starting from the "object of n-fold arrows"; similarly, c^i_A will denote the codomain arrows.

Corollary 2.8 extends to the following:

2.9. Proposition. *The central extensions with respect to the adjunction*

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift right, "D^n\circ\ldots\circ D"'] \arrow[r, phantom, "\perp"] & \mathsf{Gpd}^n(\mathcal{A}) \arrow[l, shift right, "\pi_0\circ\ldots\circ\pi_0^n"'] \end{tikzcd}$$

are precisely the extensions $\mathbf{f}\colon \mathbf{A} \longrightarrow \mathbf{B}$ *with* $K[d^i_A] \cap K[f] = 0$ *for all* $1 \leq i \leq n$.

Proof. Since $\Pi^n_0 = \pi_0 \circ \ldots \circ \pi^n_0$ is a protoadditive functor by Proposition 1.3, an extension $\mathbf{f}\colon \mathbf{A} \longrightarrow \mathbf{B}$ is central if and only if $K[\mathbf{f}] \in \mathsf{Dis}^{\mathsf{n}}(\mathcal{A})$, which means that all the arrows in the diagram of $K[\mathbf{f}]$ are isomorphisms. This is easily seen to be equivalent to all $d^i_{K[f]}$ being isomorphisms $(1 \leq i \leq n)$. By protomodularity of $\mathcal{A}$, and because the $d^i_{K[f]}$ are restrictions of the d^i_A to $K[f]$, this is equivalent to the conditions $K[d^i_A] \cap K[f] = 0$ $(1 \leq i \leq n)$. ∎

Recall from [14] that the category $\mathsf{CExt}_{\mathcal{B}}\mathcal{A}$ is a strongly $\mathcal{E}^1$-Birkhoff subcategory of $\mathsf{Arr}\mathcal{A}$. This allows one to define *double central extensions* as those double extensions that are central with respect to $\mathsf{CExt}_{\mathcal{B}}\mathcal{A}$ and then, inductively, for $k \geq 2$, *k-fold central extensions* as those k-fold extensions that are central with respect to $\mathsf{CExt}^{k-1}_{\mathcal{B}}\mathcal{A}$ (where we have put $\mathsf{CExt}^1_{\mathcal{B}}\mathcal{A} = \mathsf{CExt}_{\mathcal{B}}\mathcal{A}$, and $\mathsf{CExt}^{k-1}_{\mathcal{B}}\mathcal{A}$ for the category of $(k-1)$-fold central extensions). If I is the reflector $\mathcal{A}_{\mathcal{E}} \longrightarrow \mathcal{B}$, with unit η, then we shall write I_k for the reflector $\mathsf{Ext}^k\mathcal{A} \longrightarrow \mathsf{CExt}^k_{\mathcal{B}}\mathcal{A}$ and η^k for the corresponding unit. If I is a protoadditive functor, then so is I_1, hence I_k, for *any* $k \geq 1$, as we shall prove in the next proposition.

We denote the kernel of the unit $\eta_A\colon A \longrightarrow I(A)$ by $\kappa_A\colon [A] \longrightarrow A$, for $A \in \mathcal{A}$, so that $I(A) = A/[A]$. This defines a functor $[\cdot]\colon \mathcal{A}_{\mathcal{E}} \longrightarrow \mathcal{A}_{\mathcal{E}}$ which is easily seen to be protoadditive whenever I is a protoadditive functor.

2.10. Proposition. *Let $\mathcal{B}$ be a strongly $\mathcal{E}$-Birkhoff subcategory of a semi-abelian category $\mathcal{A}$ with reflector $I\colon \mathcal{A}_{\mathcal{E}} \longrightarrow \mathcal{B}$. If I is a protoadditive functor, then so is the reflector $I_1\colon \mathsf{Ext}\mathcal{A} \longrightarrow \mathsf{CExt}_{\mathcal{B}}\mathcal{A}$. Moreover, I_1 sends an extension $f\colon A \longrightarrow B$ to the induced extension $A/[K[f]] \longrightarrow B$.*

Proof. It suffices to prove that the reflection into $\mathsf{CExt}_{\mathcal{B}}\mathcal{A}$ of an extension $f\colon A \longrightarrow B$ is given by the induced extension $A/[K[f]] \longrightarrow B$. From this and from the fact that $[\cdot]$ is protoadditive it will then easily follow, by using the 3×3-Lemma, that I_1 is a protoadditive functor, as desired.

First observe that $[K[f]]$ is a normal subobject of A: to see this, consider the following diagram

$$\begin{tikzcd} {[K[\pi_1]]} \arrow[r, "{[\mathsf{Ker}\,(\pi_1)]}"] \arrow[d, "\kappa_{K[\pi_1]}"'] & {[R[f]]} \arrow[r, "{[\pi_1]}"] \arrow[d, "\kappa_{R[f]}"'] & {[A]} \arrow[d, "\kappa_A"] \\ K[\pi_1] \arrow[r, "\mathsf{Ker}\,(\pi_1)"'] & R[f] \arrow[r, "\pi_1"'] & A \end{tikzcd}$$

where $K[\pi_1] = K[f]$. Since $[\cdot]\colon \mathcal{A}_{\mathcal{E}} \to \mathcal{A}_{\mathcal{E}}$ is a protoadditive functor, both rows are short exact sequences. It follows that the left hand square is a pullback, because κ_A is a monomorphism. Consequently, $[K[\pi_1]]$ is a normal subobject of $R[f]$, as an intersection of normal subobjects. The regular image of $\mathsf{Ker}\,(\pi_1) \circ \kappa_{K[\pi_1]}\colon [K[\pi_1]] \to R[f]$ along π_2 is then also a normal monomorphism: but this is just $\pi_2 \circ \mathsf{Ker}\,(\pi_1) \circ \kappa_{K[\pi_1]} = \mathsf{Ker}\,(f) \circ \kappa_{K[f]}$, as desired.

Next remark that $A/[K[f]] \longrightarrow B$ is a central extension: indeed, by the "double quotient" isomorphism theorem (see Theorem 4.3.10 in [1]) its kernel is $K[f]/[K[f]]$, so that it belongs to $\mathcal{B}$, as $K[f]/[K[f]] = I(K[f])$. Then, by Proposition 2.7, $A/[K[f]] \longrightarrow B$ is a central extension.

Now, let $g\colon C \longrightarrow D$ be a central extension as well, and $(a, b)\colon f \longrightarrow g$ a morphism of extensions. We need to show that there is a morphism $\overline{a}$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left, "a"] & \frac{A}{[K[f]]} \arrow[r, dotted, "\overline{a}"] \arrow[d, "I_1(f)"] & C \arrow[d, "g"] \\ B \arrow[r, equal] & B \arrow[r, "b"'] & D. \end{tikzcd}$$

For this, it suffices to note that there is a commutative square

$$\begin{tikzcd} {[K[f]]} \arrow[r] \arrow[d, "\mathsf{Ker}\,(f)\circ\kappa_{K[f]}"'] & {[K[g]]} \arrow[d, "\mathsf{Ker}\,(g)\circ\kappa_{K[g]}"] \\ A \arrow[r, "a"'] & C, \end{tikzcd}$$

and that $[K[g]] = 0$ because g is central, so that $a \circ \mathsf{Ker}\,(f) \circ \kappa_{K[f]} = 0$. ■

The above proposition will allow us to characterise the k-fold central extensions of n-fold groupoids with respect to the Birkhoff subcategory $\mathsf{Dis}^{\mathsf{n}}(\mathcal{A})$ in Theorem 2.12. First, we continue to consider the general case.

In order to fix notations, note that a k-fold arrow in a category $\mathcal{A}$ ($k \geq 1$) can be seen as a contravariant functor $A\colon \mathcal{P}(k)^{\mathrm{op}} \longrightarrow \mathcal{A}$, where $\mathcal{P}(k)$ is the powerset of $k = \{0, 1, \ldots, k-1\}$ (here we consider the natural numbers by their von Neumann construction). If $S \subseteq T$ are subsets of k, let us write A_S for the image of S by the functor A and $a_S^T\colon A_T \longrightarrow A_S$ for the image of $S \subseteq T$. If $f\colon A \longrightarrow B$ is a morphism of k-fold arrows—a natural transformation—we write $f_S\colon A_S \longrightarrow B_S$ for the S-component of f. Furthermore, we also write $(A_S)_{S\subseteq k}$ instead of A and $(f_S)_{S\subseteq k}$ instead of f. Moreover, in order to simplify our notations, we write a_i instead of $a^k_{k\setminus\{i\}}$ for the "initial" arrows $A_k \longrightarrow A_{k\setminus\{i\}}$ in the diagram of A ($0 \leq i \leq k-1$). Note that $\mathsf{Arr}^k\mathcal{A}$ can be identified with $\mathsf{Arr}(\mathsf{Arr}^{k-1}\mathcal{A})$ via the isomorphism $\delta\colon \mathsf{Arr}^k\mathcal{A} \longrightarrow \mathsf{Arr}(\mathsf{Arr}^{k-1}\mathcal{A})$ given by sending a k-fold arrow A in $\mathcal{A}$ to the morphism

$$\left(a_S^{S\cup\{k-1\}}\right)_{S\subseteq k-1}\colon (A_{S\cup\{k-1\}})_{S\subseteq k-1} \longrightarrow (A_S)_{S\subseteq k-1}$$

of $(k-1)$-fold arrows in $\mathcal{A}$.

For a k-fold extension A, $[A]_k$ denotes the object $K[\eta^k_A]_k$, the "initial" (and only non-zero [13]) object in the diagram of $K[\eta^k_A]_k$. Note that one has that $(I_kA)_k = A_k/[A]_k$. If the reflector $I\colon \mathcal{A}_{\mathcal{E}} \longrightarrow \mathcal{B}$ is protoadditive, $[A]_k$ can be characterised as follows:

2.11. Proposition. *Let $\mathcal{B}$ be a strongly $\mathcal{E}$-Birkhoff subcategory of a semi-abelian category $\mathcal{A}$, where $\mathcal{E}$ is a class of extensions in $\mathcal{A}$. If the reflector $I\colon \mathcal{A}_{\mathcal{E}} \longrightarrow \mathcal{B}$ is protoadditive, then, for any $k \geq 1$ and any k-fold extension A, one has that*

$$[A]_k = [\bigcap_{0\leq i\leq k-1} K[a_i]].$$

It follows that the k-fold central extensions with respect to $\mathcal{B}$ are precisely the k-fold extensions A with

$$\bigcap_{0\leq i\leq k-1} K[a_i] \in \mathcal{B}.$$

Proof. The proof is by induction on k. Recall from Proposition 2.10 that, as soon as $I\colon \mathcal{A}_{\mathcal{E}} \longrightarrow \mathcal{B}$ is protoadditive, the reflector $I_1\colon \mathsf{Ext}\mathcal{A} \longrightarrow \mathsf{CExt}_{\mathcal{B}}\mathcal{A}$, which is protoadditive as well, is obtained as follows: for any extension $f\colon A \longrightarrow B$, $I_1(f)$ is the induced extension $A/[K[f]] \longrightarrow B$. This provides the case $k = 1$. Now suppose that, for some $k-1 \geq 1$, the reflection $I_{k-1}(A)$ of any $(k-1)$-fold extension A is the induced $(k-1)$-fold extension determined by

$$(I_{k-1}(A))_{k-1} = A_{k-1}/[\cap_{0\leq i\leq k-2} K[a_i]]$$

and $(I_{k-1}(A))_S = A_S$, for $S \subsetneq k-1$. Then, again taking into account Proposition 2.10, as well as the induction hypothesis, the reflection $I_k(A)$ of a k-fold extension A is the induced k-fold extension determined by

$$\begin{aligned}(I_k(A))_k &= A_k/[K[\delta(A)]]_{k-1}\\ &= A_k/[\bigcap_{0\leq i\leq k-2} K[(K[\delta(A)])_i]]\end{aligned}$$

$$\begin{aligned} &= A_k/[\bigcap_{0\le i\le k-2} K[a_i] \cap K[a_{k-1}]] \\ &= A_k/[\bigcap_{0\le i\le k-1} K[a_i]], \end{aligned}$$

and $(I_k(A))_S = A_S$, for $S \subsetneq k$, as desired. ∎

Let us then go back to the special case of n-fold groupoids. As above, we write d^i_A $(1 \le i \le n)$ and c^i_A for the domain and codomain arrows starting from the "object of n-fold arrows", respectively.

2.12. THEOREM. *The k-fold central extensions with respect to the adjunction*

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift right, "D^n\circ\ldots\circ D"'] \arrow[r, phantom, "\perp"] & \mathsf{Gpd}^n(\mathcal{A}) \arrow[l, shift right, "\pi_0\circ\ldots\circ\pi_0^n"'] \end{tikzcd}$$

are precisely the k-fold extensions $\mathbf{A}$ of n-fold groupoids, with

$$K[d^i_A] \cap \bigcap_{0\le j\le k-1} K[a_j] = 0$$

for all $1 \le i \le n$.

PROOF. By Proposition 2.11 a k-fold extension $\mathbf{A}$ of n-fold groupoids is central with respect to $\mathcal{A}$ if and only if

$$\bigcap_{0\le j\le k-1} K[a_j] \in \mathsf{Dis}^{\mathsf{n}}(\mathcal{A}).$$

This can be seen to be equivalent to all

$$d^i_{\bigcap_{0\le j\le k-1} K[a_j]}$$

being isomorphisms and, by protomodularity of $\mathcal{A}$, this amounts to their kernel being 0. It remains to observe that

$$K[d^i_{\bigcap_{0\le j\le k-1} K[a_j]}] = K[d^i_A] \cap \bigcap_{0\le j\le k-1} K[a_j] = 0,$$

for all $1 \le i \le n$. ∎

3. Homology of n-fold internal groupoids

In this section, we shall characterise the generalised higher Hopf formulae introduced in [15, 14] associated with the composite adjunction (1).

Recall that, in general, the Hopf formulae are defined with respect to the following data: a semi-abelian category $\mathcal{A}$, a class of extensions $\mathcal{E}$ in $\mathcal{A}$, and a strongly $\mathcal{E}$-Birkhoff subcategory $\mathcal{B}$ of $\mathcal{A}$ with corresponding unit η. Recall from the previous section that those data induce, for each $k \geq 1$, a class of extensions $\mathcal{E}^k$ of the semi-abelian category $\mathsf{Arr}^k\mathcal{A}$, and a strongly $\mathcal{E}^k$-Birkhoff subcategory $\mathcal{B}_k = \mathsf{CExt}^k_{\mathcal{B}}\mathcal{A}$ with unit η^k. Furthermore, the assumption that $\mathcal{A}_{\mathcal{E}}$ has enough $\mathcal{E}$-projective objects is made. Here an object $P \in \mathcal{A}_{\mathcal{E}}$ is called *$\mathcal{E}$-projective* if, for any extension $f\colon A \longrightarrow B \in \mathcal{E}$ and any morphism $b\colon P \longrightarrow B$ in $\mathcal{A}$, there exists at least one morphism $a\colon P \longrightarrow A$ such that $f \circ a = b$:

$$\begin{tikzcd} & P \arrow[dl, dotted, "a"'] \arrow[d, "b"] \\ A \arrow[r, "f"'] & B. \end{tikzcd}$$

$\mathcal{A}_{\mathcal{E}}$ is said to have *enough $\mathcal{E}$-projective objects* if, for any object $A \in \mathcal{A}_{\mathcal{E}}$, there exists an $\mathcal{E}$-projective object P and an extension $P \longrightarrow A$. Thus, in particular, the Hopf formulae can be considered in the following situation: $\mathcal{A}$ is a semi-abelian category with enough regular projective objects and $\mathcal{B}$ is a Birkhoff subcategory of $\mathcal{A}$.

Let us recall the definition. Consider $\mathcal{A}$, $\mathcal{E}$ and $\mathcal{B}$ with the assumptions mentioned above. First recall that, for $k \geq 1$, a *k-fold presentation* of an object $A \in \mathcal{A}_{\mathcal{E}}$ is a k-fold extension P such that P_S is an $\mathcal{E}$-projective object for any $0 \neq S \subseteq k$, and $P_0 = A$. Since $\mathcal{A}_{\mathcal{E}}$ has enough $\mathcal{E}$-projective objects, it follows that for any object $A \in \mathcal{A}_{\mathcal{E}}$ and $k \geq 1$, there exists at least one k-fold presentation P of the object A. Then the *Hopf formula* $H_{k+1}(A, \mathcal{B})_{\mathcal{E}} = H_{k+1}(A, \mathcal{B})$ *for the* $(k+1)$*st homology of* A*, with respect to* $\mathcal{B}$, is defined as

$$H_{k+1}(A, \mathcal{B}) = \frac{[P_k] \cap \bigcap_{0 \leq i \leq k-1} K[p_i]}{[P]_k}.$$

It turns out that this definition does not depend on the particular choice of k-fold presentation of the object A.

It was shown in [15] that the functors $H_{k+1}(-, \mathcal{B})\colon \mathcal{A}_{\mathcal{E}} \longrightarrow \mathcal{B}$ coincide with the Barr-Beck left derived functors of the reflector $I\colon \mathcal{A}_{\mathcal{E}} \longrightarrow \mathcal{B}$, whenever the category $\mathcal{A}$ is monadic over Set, the category of sets and maps, and $\mathcal{E}$ is the class of all regular epimorphisms in $\mathcal{A}$ (so that $\mathcal{B}$ is a Birkhoff subcategory of $\mathcal{A}$). Now, if $\mathcal{A}_{\mathcal{E}}$ has enough $\mathcal{E}$-projective objects, then $\mathsf{Ext}\mathcal{A} = (\mathsf{Arr}\mathcal{A})_{\mathcal{E}^1}$ has enough $\mathcal{E}^1$-projective objects as well, so that the higher Hopf formulae can be considered with respect to $\mathsf{Arr}\mathcal{A}$, $\mathcal{E}^1$ and $\mathcal{B}_1 = \mathsf{CExt}_{\mathcal{B}}\mathcal{A}$. It was shown in [13] that any extension $f\colon A \longrightarrow B \in \mathcal{E}$ induces a long exact sequence $(\mathbf{G})$

$$\ldots \longrightarrow H_{k+1}(A, \mathcal{B}) \longrightarrow H_{k+1}(B, \mathcal{B}) \longrightarrow K[H_k(f, \mathcal{B}_1)] \longrightarrow H_k(A, \mathcal{B}) \longrightarrow \ldots$$

$$\ldots \longrightarrow H_2(A, \mathcal{B}) \longrightarrow H_2(B, \mathcal{B}) \longrightarrow K[H_1(f, \mathcal{B}_1)] \longrightarrow H_1(A, \mathcal{B}) \longrightarrow H_1(B, \mathcal{B}) \longrightarrow 0$$

in $\mathcal{A}$. Here we recall that, for any $\mathcal{A}$, $\mathcal{E}$, $\mathcal{B}$ and $A \in \mathcal{A}_{\mathcal{E}}$, by definition $H_1(A, \mathcal{B}) = I(A)$. In the case $\mathcal{A} = \mathsf{Gp}$ is the variety of groups, and $\mathcal{B} = \mathsf{Ab}$ the variety of abelian groups, the final part of the above sequence is the classical Stallings Stammbach sequence of low dimensional group homology. As explained in [16], the universality of the above long exact sequence completely determines the Hopf formulae.

In order to consider the Hopf formulae associated with the Birkhoff subcategory $\mathsf{Dis}^n(\mathcal{A})$ of $\mathsf{Gpd}^n(\mathcal{A})$ ($n \geq 1$), we need that the category $\mathsf{Gpd}^n(\mathcal{A})$ has enough regular projectives. For this, it suffices that the category $\mathcal{A}$ has this property, as we are now going to show. In fact, it suffices for this that the category $\mathcal{A}$ is, more generally, regular and Mal'tsev, with finite colimits, rather than semi-abelian. Recall that a category $\mathcal{A}$ is *Mal'tsev* if any (internal) reflexive relation in $\mathcal{A}$ is an equivalence relation.

Let us denote by U^n the forgetful functor $\mathsf{Gpd}^n(\mathcal{A}) \longrightarrow \mathcal{A}$ that maps an n-fold groupoid $\mathbf{A}$ to its "object of n-fold arrows" A.

3.1. PROPOSITION. *When $\mathcal{A}$ is a regular Mal'tsev category with finite colimits, then the functor $U^n \colon \mathsf{Gpd}^n(\mathcal{A}) \longrightarrow \mathcal{A}$ is monadic, for any $n \geq 1$.*

PROOF. Since $\mathsf{Gpd}^n(\mathcal{A})$ has all coequalisers, which are level-wise in $\mathcal{A}$ [17], U^n preserves and reflects coequalisers, hence it suffices by Beck's Weak Monadicity Theorem (see, exercise $VI.7.2$ [25]) to prove that U^n has a left adjoint. Of course, it will suffice to prove this in the case $n = 1$.

But $U = U^1$ factors as $U_2 \circ U_1$, where U_1 is the inclusion functor and U_2 the functor that sends a reflexive graph (A, A_0, d_A, c_A, i_A) to the object A:

$$\mathsf{Gpd}(\mathcal{A}) \xrightarrow{U_1} \mathsf{RG}(\mathcal{A}) \xrightarrow{U_2} \mathcal{A}.$$

The functor U_1 is known to have a left adjoint (see Theorem 2.8.13 in [1]). We claim that also U_2 has a left adjoint, F_2. Indeed, for an object $A \in \mathcal{A}$, let $F_2(A)$ be the reflexive graph

$$\begin{tikzcd} A + A + A \arrow[r, shift left=2, "{[i_1,i_1,i_2]}"] \arrow[r, shift right=2, "{[i_1,i_2,i_2]}"'] & A + A, \arrow[l, "{[i_1,i_3]}"' description] \end{tikzcd}$$

where the i_j's are the injections of A in the considered coproducts. This defines a functor $F_2 \colon \mathcal{A} \longrightarrow \mathsf{RG}(\mathcal{A})$ (with the obvious definition on morphisms). To see that F_2 is left adjoint to U_2, it suffices to note that, for an object A and a reflexive graph $\mathbf{B} = (B, B_0, d_B, c_B, i_B)$, sending a morphism $f \colon A \longrightarrow B$ to the morphism of reflexive graphs

$$\begin{tikzcd} A + A + A \arrow[rr, "{[i_B \circ d_B \circ f, f, i_B \circ c_B \circ f]}"] \arrow[d, shift right=3, "{[i_1,i_1,i_2]}"'] \arrow[d, shift left=3, "{[i_1,i_2,i_2]}"] & & B \arrow[d, shift right=3, "d_B"'] \arrow[d, shift left=3, "c_B"] \\ A + A \arrow[u] \arrow[rr, "{[d_B \circ f, c_B \circ f]}"'] & & B_0 \arrow[u] \end{tikzcd}$$

defines a bijection

$$\mathrm{Hom}_{\mathcal{A}}(A, B) \cong \mathrm{Hom}_{\mathsf{RG}(\mathcal{A})}(F_2(A), \mathbf{B}),$$

which is natural both in A and $\mathbf{B}$. ∎

3.2. Proposition. *Let $\mathcal{A}$ be a regular Mal'tsev category with finite colimits and $n \geq 1$. Then $\mathsf{Gpd}^n(\mathcal{A})$ has enough regular projective objects as soon as $\mathcal{A}$ has enough regular projective objects.*

Proof. As for the previous proposition, it suffices to prove the case $n = 1$. This follows from the fact that the forgetful functor $U\colon \mathsf{Gpd}(\mathcal{A}) \longrightarrow \mathcal{A}$ is monadic and preserves regular epimorphisms. In order to explain why this is the case, let us write F for the left adjoint to $U\colon \mathsf{Gpd}(\mathcal{A}) \longrightarrow \mathcal{A}$ and μ and φ for the unit and counit of this adjunction.

Now, for any groupoid $\mathbf{A}$ in $\mathcal{A}$, and any projective presentation $p\colon P \longrightarrow U(\mathbf{A})$ of $U(\mathbf{A})$, we claim that $\varphi_{\mathbf{A}} \circ F(p)\colon F(P) \longrightarrow \mathbf{A}$ is a projective presentation of $\mathbf{A}$. Indeed, on the one hand, we have that $\varphi_{\mathbf{A}} \circ F(p)$ is a regular epimorphism because both $F(p)$ and $\varphi_{\mathbf{A}}$ are: it is easily verified in this case that the counit $\varphi_{\mathbf{A}}$ is a regular epimorphism (but this is in fact true for *any* monadic functor U with left adjoint F [25]); and $F(p)$ is a regular epimorphism as the image of the regular epimorphism p by the left adjoint functor F.

On the other hand, we have that $F(P)$ is a projective object of $\mathsf{Gpd}(\mathcal{A})$. Indeed, let $g\colon \mathbf{B} \longrightarrow \mathbf{C}$ be a regular epimorphism and $c\colon F(P) \longrightarrow \mathbf{C}$ be any morphism in $\mathsf{Gpd}(\mathcal{A})$. Since U preserves regular epimorphisms and P is projective, there exists a morphism b in $\mathcal{A}$ such that the right hand triangle here below commutes :

$$\begin{tikzcd} & F(P) \arrow[dl, dotted, "\varphi_{\mathbf{B}} \circ F(b)"'] \arrow[d, "c"] \\ \mathbf{B} \arrow[r, "g"'] & \mathbf{C} \end{tikzcd} \qquad \begin{tikzcd} & P \arrow[dl, dotted, "b"'] \arrow[d, "U(c)\circ\mu_P"] \\ U(\mathbf{B}) \arrow[r, "U(g)"'] & U(\mathbf{C}). \end{tikzcd}$$

Note that the morphism $U(c) \circ \mu_P$ corresponds to c via the bijection

$$\mathrm{Hom}_{\mathcal{A}}(P, U(\mathbf{C})) \cong \mathrm{Hom}_{\mathsf{Gpd}(\mathcal{A})}(F(P), \mathbf{C})$$

and that b corresponds to $\varphi_{\mathbf{B}} \circ F(b)$ via the bijection

$$\mathrm{Hom}_{\mathcal{A}}(P, U(\mathbf{B})) \cong \mathrm{Hom}_{\mathsf{Gpd}(\mathcal{A})}(F(P), \mathbf{B}).$$

Using the naturality of the above bijections, we can now conclude that $\varphi_{\mathbf{B}} \circ F(b)$ makes the left hand triangle commute. ∎

Let us now characterise the higher Hopf formulae associated with the adjunction (1) in the case that $\mathcal{A}$ has enough regular projective objects.

For an n-fold groupoid $\mathbf{A}$ ($n \geq 1$) in a semi-abelian category $\mathcal{A}$, $\Gamma_{\mathbf{A}}(0)$ denotes the full n-fold subgroupoid determined by the connected component of 0: $\Gamma_{\mathbf{A}}(0)$ is the kernel $K[\eta^n_{\mathbf{A}}]$ of the unit $\eta^n\colon 1_{\mathsf{Gpd}^n(\mathcal{A})} \Rightarrow D_n \circ \Pi^n_0$ in $\mathbf{A}$, i. e. $\Gamma_{\mathbf{A}}(0) = [A]_{\mathsf{Dis}^n(\mathcal{A})}$. Thanks to Proposition 2.11, we find that the Hopf formulae take the following shape:

3.3. Theorem. *Let $\mathcal{A}$ be a semi-abelian category with enough regular projective objects, and $\mathbf{A}$ an n-fold groupoid in $\mathcal{A}$. For $k \geq 1$ and $\mathbf{P}$ an arbitrary k-fold presentation of $\mathbf{A}$, one has that*

$$H_{k+1}(\mathbf{A}, \mathsf{Dis^n}(\mathcal{A})) = \frac{\Gamma_{\mathbf{P}_k}(0) \cap \bigcap_{0 \leq i \leq k-1} K[\mathbf{p_i}]}{\Gamma_{\bigcap_{0 \leq i \leq k-1} K[\mathbf{p_i}]}(0)}.$$

In particular, the right hand expression is independent of the choice of k-fold presentation of $\mathbf{A}$.

3.4. Remark. By Proposition 1.5.11 in [13], $H_{k+1}(\mathbf{A}, \mathsf{Dis^n}(\mathcal{A}))$ belongs to the Birkhoff subcategory $\mathcal{A} \cong \mathsf{Dis^n}(\mathcal{A})$ of $\mathsf{Gpd}^n(\mathcal{A})$.

Let us finally mention that the long exact homology sequence $(\mathbf{G})$ particularises to the following:

3.5. Proposition. *Any short exact sequence of n-fold groupoids*

$$0 \longrightarrow \mathbf{K} \longrightarrow \mathbf{A} \xrightarrow{\mathbf{f}} \mathbf{B} \longrightarrow 0$$

in a semi-abelian category $\mathcal{A}$ with enough regular projective objects induces a long exact homology sequence

$$\ldots \longrightarrow H_{k+1}(\mathbf{A}, \mathsf{Dis^n}(\mathcal{A})) \longrightarrow H_{k+1}(\mathbf{B}, \mathsf{Dis^n}(\mathcal{A})) \longrightarrow K[H_k(\mathbf{f}, \mathsf{Dis^n}(\mathcal{A})_1)]$$
$$\longrightarrow H_k(\mathbf{A}, \mathsf{Dis^n}(\mathcal{A})) \longrightarrow H_k(\mathbf{B}, \mathsf{Dis^n}(\mathcal{A})) \longrightarrow \ldots$$
$$\ldots \longrightarrow H_1(\mathbf{K}, \mathsf{Dis^n}(\mathcal{A})) \longrightarrow H_1(\mathbf{A}, \mathsf{Dis^n}(\mathcal{A})) \longrightarrow H_1(\mathbf{B}, \mathsf{Dis^n}(\mathcal{A})) \longrightarrow 0$$

Proof. It suffices to prove that $K[H_1(\mathbf{f}, \mathsf{Dis^n}(\mathcal{A})_1)] = H_1(\mathbf{K}, \mathsf{Dis^n}(\mathcal{A}))$. From Proposition 2.10 it follows that $(\Pi_0^n)_1(\mathbf{f})$ is the morphism $\mathbf{A}/[\mathbf{K}]_{\mathsf{Dis^n}(\mathcal{A})} \longrightarrow \mathbf{B}$ induced by $\mathbf{f}$, so that

$$K[(\Pi_0^n)_1(\mathbf{f})] = \mathbf{K}/[\mathbf{K}]_{\mathsf{Dis^n}(\mathcal{A})} = \Pi_0^n(\mathbf{K}),$$

hence

$$K[H_1(\mathbf{f}, \mathsf{Dis^n}(\mathcal{A})_1)] = K[(\Pi_0^n)_1(\mathbf{f})] = \Pi_0^n(\mathbf{K}) = H_1(\mathbf{K}, \mathsf{Dis^n}(\mathcal{A})).$$

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