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MONADIC APPROACH TO GALOIS DESCENT AND COHOMOLOGY

Dedicated to Dominique Bourn at the occasion of his sixtieth birthday.

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ABSTRACT. We describe a simplified categorical approach to Galois descent theory. It is well known that Galois descent is a special case of Grothendieck descent, and that under mild additional conditions the category of Grothendieck descent data coincides with the Eilenberg-Moore category of algebras over a suitable monad. This also suggests using monads directly, and our monadic approach to Galois descent makes no reference to Grothendieck descent theory at all. In order to make Galois descent constructions perfectly clear, we also describe their connections with some other related constructions of categorical algebra, and make various explicit calculations, especially with 1-cocycles and 1-dimensional non-abelian cohomology, usually omitted in the literature.

Introduction

The so-called Galois descent theory is an old tool used in algebraic geometry, algebraic number theory, and related topics of ring and module theory. It allows to describe certain structures of algebra and geometry defined over a base field B using similar structures defined over a Galois extension E of B. Its results are *descent theorems* that are of the form: "*The category of B-structures is canonically equivalent to the category of descent data of E-structures*", usually supplied by descriptions of the set of isomorphism classes of B-structures with a fixed extension to E in terms of 1-dimensional cohomology of the Galois group of the Galois extension E/B. If one replaces the Galois extension E/B with an arbitrary morphism $p: E \to B$ in a category $\mathbf{C}$ equipped with a Grothendieck fibration $(\mathbf{A}, F)$ of categories over $\mathbf{C}$, there is still a *canonical comparison functor* $K^p: \mathbf{A}^B \to \mathrm{Des}(p)$ from the fibre $\mathbf{A}^B$ of $(\mathbf{A}, F)$ over B to the category $\mathrm{Des}(p)$ of suitably defined *descent data*. In the Galois descent constructions the role of $\mathbf{C}$ is usually played by the dual category of commutative rings, which explains the direction of the arrow p (and this is why we use the opposite direction in our Section 4). According to Grothendieck descent theory, the morphism $p: E \to B$ is said to be an *effective descent morphism* if

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K^p is a category equivalence. Under certain additional conditions on $(\mathbf{A}, F)$ that hold in many known examples, $p: E \to B$ is an effective descent morphism if and only if the change-of-base functor $p^*: \mathbf{A}^B \to \mathbf{A}^E$ is monadic. This follows from the fact that K^p becomes nothing but the comparison functor, in the sense of monad theory, from $\mathbf{A}^B$ to the category of algebras over the monad determined by p^* and its left adjoint.

In this paper we develop a simplified categorical approach to Galois descent by presenting the category of Galois descent data as the category of algebras over a suitable monad directly, i.e. not involving the intermediate context of Grothendieck fibrations and Grothendieck descent. We also show that the monadic approach is helpful in presenting the (1-dimensional) cohomological side of the story by transforming descent data into suitable 1-cocycles - again directly, i.e. not using Grothendieck's cocycle condition. In spite of our goal of simplification, we present some intermediate categorical constructions and various explicit calculations usually omitted in the literature; we believe they are necessary. On the other hand we omit some other important constructions and their natural generalizations; e.g. we do not consider torsors, do not discuss generalizations involving categorical Galois theory whose Galois groupoids are internal groupoids in an abstract category, and do not even consider infinite Galois extensions of fields.

Section 1 begins with 1-dimensional cocycles and cohomology $H^1(G, A)$ of a group G with abelian and then with non-abelian coefficients. We then extend it replacing a G-group A with a G-category $\mathbf{X}$, but also show that in some case the G-category case reduces to the G-group case (Proposition 1.4). Next, we show that involving 2-dimensional category theory makes all constructions more natural. In particular the category $Z^1(G, \mathbf{X})$ of 1-cocycles $G \to \mathbf{X}$ becomes nothing but the category $\mathrm{PsNat}(\mathbf{1}, \mathbf{X})$ of pseudo-natural transformations $\mathbf{1} \to \mathbf{X}$, and the reduction above can be obtained from a canonical factorization of pseudo-natural transformations $\mathbf{1} \to \mathbf{X}$.

Section 2 presents a G-category structure on $\mathbf{X}$ as a special ("split") monoid in the monoidal category $\mathrm{Fam}(\mathbf{X}^{\mathbf{X}})$ of families of endofunctors of $\mathbf{X}$. A 1-cocycle $G \to \mathbf{X}$ then becomes an action of $\mathbf{X}$ on an object in $\mathbf{X}$. The coproduct functor $\mathrm{Fam}(\mathbf{C}) \to \mathbf{C}$, in the case of $\mathbf{C}$ being the monoidal category $\mathbf{X}^{\mathbf{X}}$, transforms the monoid above into a monad on $\mathbf{X}$, provided $\mathbf{X}$ admits G-indexed coproducts. That monad, denoted by $G(-)$, plays a crucial role in Galois descent theory. It is in fact extensively used in Galois theory of commutative rings, where, for a Galois extension E/B, GE is the algebra of all maps from G to E; however, as far as we know, it was never considered there as a monad. Note also, that involving the monoidal category $\mathrm{Fam}(\mathbf{X}^{\mathbf{X}})$ (instead of just $\mathbf{X}^{\mathbf{X}}$) is not absolutely necessary, but makes all constructions more elegant since the monoidal structure of $\mathrm{Fam}(\mathbf{X}^{\mathbf{X}})$ is automatically distributive with respect to coproducts, while in $\mathbf{X}^{\mathbf{X}}$ it only happens for coproducts of very special objects. Section 2 ends with Theorem 2.5, according to which a 1-cocycle $G \to \mathbf{X}$ is nothing but a $G(-)$-algebra.

Section 3 begins by comparing split and connected monoids in $\mathrm{Fam}(\mathbf{C})$, where "split" (as above) means being "a family of unit objects" while "connected" means being "inside $\mathbf{C}$", i.e. being "a one-member family". This allows to construct a canonical morphism from $G(-)$ to any monad T on $\mathbf{X}$ that is invariant (in a suitable sense) under the G-action, and T is said to be a G-Galois monad if that canonical morphism is an isomorphism. A monadic adjunction $(F, U, \eta, \varepsilon)$ from $\mathbf{X}$ to $\mathbf{Y}$ is then called a G-Galois adjunction if it determines a G-Galois monad on $\mathbf{X}$. The main descent theorem (Theorem 3.6) that establishes and describes the equivalence $\mathbf{Y} \sim Z^1(G, \mathbf{X})$ for a Galois adjunction, is a straightforward consequence of monadicity in this context. In particular $Z^1(G, \mathbf{X})$ plays the role of the category of descent data. The second main result (Theorem 3.7) describes, using Theorem 3.6 and abovementioned Proposition 1.4, the full subcategory $\mathbf{Y}[Y_0]$ of $\mathbf{Y}$ with objects all Y in $\mathbf{Y}$ with $U(Y) \cong U(Y_0)$ for a fixed Y_0, as the category of 1-cocycles $G \to \mathrm{End}(U(Y_0))$, where the monoid $\mathrm{End}(U(Y_0))$ is equipped with a G-action induced (in a suitable sense) by the 1-cocycle ξ_0 corresponding to Y_0. Accordingly, Theorem 3.7 also describes the set of isomorphic classes of objects in $\mathbf{Y}[Y_0]$, presented its bijection with $H^1(G, \mathrm{End}(U(Y_0))_{\xi_0}) = H^1(G, \mathrm{Aut}(U(Y_0))_{\xi_0})$.

Section 4 applies our results to the adjunction $(E\text{-Mod})^{\mathrm{op}} \to (B\text{-Mod})^{\mathrm{op}}$ determined by a homomorphism $p: B \to E$ of commutative rings, where E is equipped with a right G-action making $(E\text{-Mod})^{\mathrm{op}}$ a G-category. We show that this adjunction is a G-Galois adjunction if and only if $p: B \to E$ is a G-Galois extension of commutative rings in the sense of [Chase, Harrison, Rosenberg, 1965]; in particular this implies that G must be finite. After that we present the Galois descent/cohomology results obtained as direct translations of Theorems 3.6 and 3.7. The rest of Section 4 is devoted to three classical examples. Much more examples could be presented of course, but we only wanted to explain how the general theory works in practice.

Finally, let us mention that it is not clear how old the idea of Galois descent really is. Apart from Grothendieck's work, relatively recent sources (among many others) are [Serre, 1964] (mentioning "descent for forms"), [Knus, Ojanguren, 1974], [Greither, 1992], [Jahnel, 2000]. As it was pointed out to the third named author during his talk in Budapest by Tamás Szamuely, Galois descent is also mentioned by J.-P. Serre in [Serre, 1959] with a reference to [Weil, 1956], although Serre himself also mentions there an older work of F. Châtelet.

1. G-categories and cohomology

Ordinary non-abelian cohomology of groups. Let G be a group, and A a G-module - which is the same as a module over the integral group ring $\mathbb{Z}[G]$. Classically, a map $\varphi: G \to A$ is said to be a 1-*cocycle* if

$$\varphi(hg) = \varphi(h) + h\varphi(g) \tag{1.1}$$

for all $g, h \in G$. The set $Z^1(G, A)$ of all 1-cocycles forms an abelian group under the pointwise addition, and the map $A \to Z^1(G, A)$ defined by $a \mapsto (g \mapsto ga - a)$ is a group homomorphism. Denoting the image of that map by $B^1(G, A)$ (=the group of *boundaries*) one then defines the 1-dimensional cohomology group $H^1(G, A)$ as the quotient group

$$H^1(G, A) = Z^1(G, A)/B^1(G, A). \tag{1.2}$$

More generally, this extends to the *non-abelian* context as follows:
• First of all the non-abelian version of the notion of G-module is to be chosen. Since a G-module is the same as a G-object in the category of abelian groups, the obvious candidate is a G-group, i.e. a G-object in the category of groups. Such a G-group A is a group A equipped with a G-action satisfying $g(ab) = g(a)g(b)$, for all $g \in G$ and $a, b \in A$. Equivalently, a G-group is an internal group (=group object) in the category of G-sets.
• We then repeat the definition of a 1-cocycle $\varphi : \ G \to A$ with

$$\varphi(hg) = (\varphi(h))(h\varphi(g)) \tag{1.3}$$

instead of (1.1). However, now the set $Z^1(G, A)$ does not have any natural group structure - it is just a pointed set with the trivial group homomorphism $G \to A$ being the distinguished point.
• Yet, the group A acts on $Z^1(G, A)$ via

$$(a\varphi)(g) = a(\varphi(g))(ga)^{-1}, \tag{1.4}$$

and we can now define the non-abelian $H^1(G, A)$ as the pointed set $H^1(G, A) = Z^1(G, A)/A$ of orbits. Another well-known way to do this is to regard $Z^1(G, A)$ as a groupoid, and then define $H^1(G, A)$ as the set $H^1(G, A) = \pi_0(Z^1(G, A))$ of its connected components. Accordingly the objects of $Z^1(G, A)$ are 1-cocycles $G \to A$, and a morphism $\varphi \to \psi$ is an element a in A such that $a\varphi = \psi$ under the action (1.4).

From G-groups to G-categories. We are now going to replace a G-group A by a G-category $\mathbf{X}$ defined as a G-object in the category **Cat** of all categories. We will hence consider $\mathbf{X}$ as an ordinary category equipped with a G-indexed family of functors $g(-) : \ \mathbf{X} \to \mathbf{X}$, with $1(-) = 1_{\mathbf{X}}$ and $(hg)(-) = h(-)g(-)$. We introduce

1.1. Definition. Let G be a group and $\mathbf{X}$ a G-category. Then:

(a) A 1-cocycle $(X, \xi) : \ G \to \mathbf{X}$ is an object X in $\mathbf{X}$ together with a family $\xi = (\xi_g)_{g \in G}$ of morphisms $\xi_g : \ g(X) \to X$ in $\mathbf{X}$ satisfying $\xi_1 = 1_X$ and $\xi_{hg} = \xi_h h(\xi_g)$ for all $g, h \in G$.

(b) A morphism $f : \ (X, \xi) \to (X', \xi')$ of 1-cocycles is a morphism $f : \ X \to X'$ in $\mathbf{X}$ such that $f\xi_g = \xi'_g g(f)$, for all $g \in G$; the thus obtained category of 1-cocycles $G \to \mathbf{X}$ will be denoted by $Z^1(G, \mathbf{X})$.

(c) The isomorphism class of an object (X, ξ) in $Z^1(G, \mathbf{X})$ will be denoted by $[X, \xi]$, and the set of all such classes by $H^1(G, \mathbf{X})$; we will call it the 1-dimensional cohomology set of the group G with coefficients in the G-category.

This indeed generalizes the definition for groups, but we have to make

1.2. Remark. (a) Let $(X, \xi): G \to \mathbf{X}$ be a 1-cocycle. Then each ξ_g is an isomorphism. Indeed, applying $\xi_1 = 1_X$ and $\xi_{hg} = \xi_h h(\xi_g)$ to the case $h = g^{-1}$, we conclude that each $\xi_{g^{-1}}$ is a split epimorphism and each $g^{-1}(\xi_g)$ is a split monomorphism; since every element of G can be presented as a g^{-1}, and all functors of the form $g(-)$ ($g \in G$) are isomorphisms, this tells us that each ξ_g is a split epimorphism and a split monomorphism at the same time, i.e. each ξ_g is an isomorphism. This yields an alternative definition of a 1-cocycle: if we require each ξ_g to be an isomorphism (or just ξ_1 to be an isomorphism), then $\xi_1 = 1_X$ can be deduced from $\xi_{hg} = \xi_h h(\xi_g)$. Indeed, applying it to $g = h = 1$ we obtain $\xi_1\xi_1 = \xi_1$, and so ξ_1 is an identity whenever it is an isomorphism. In particular, when $\mathbf{X}$ is a groupoid (i.e. every morphism in it is an isomorphism) we can simply omit the condition $\xi_1 = 1_X$ in Definition 1.1(a).

(b) If $\mathbf{X}$ is a groupoid, then obviously $Z^1(G, \mathbf{X})$ also is a groupoid, and in this case $H^1(G, \mathbf{X})$ is the same as $\pi_0(Z^1(G, \mathbf{X}))$.

(c) When $\mathbf{X}$ has only one object, a 1-cocycle $(X, \xi): G \to \mathbf{X}$ becomes just a map ξ from G to the monoid of endomorphisms of the unique object of $\mathbf{X}$ written as $g \mapsto \xi_g$ satisfying $\xi_1 = 1$ and $\xi_{hg} = \xi_h h(\xi_g)$ above. If in addition that monoid is a group, then (by (a)) we can omit the first of these two equalities, and the second one becomes precisely (1.3) if we write $\varphi(g)$ for ξ_g, etc. Thus, when $\mathbf{X}$ is a G-group A regarded as a one object G-category, the category $Z^1(G, \mathbf{X})$ and the set $H^1(G, \mathbf{X})$ coincide with the ordinary (groupoid) $Z^1(G, A)$ and (set) $H^1(G, A)$ respectively.

(d) Unlike the case of a G-group, the general $H^1(G, \mathbf{X})$ has no distinguished point; in order to get such a point we have to take $\mathbf{X}$ itself to be pointed, i.e. to have a distinguished object X_0 such that $g(X_0) = X_0$ for all $g \in G$.

From G-categories to G-groups. Given an object X in a G-category $\mathbf{X}$, let us compare:

- the 1-cocycles $(X, \xi): G \to \mathbf{X}$ with the same fixed X; their full subcategory in $Z^1(G, \mathbf{X})$ will be denoted by $Z^1(G, \mathbf{X})_X$, and the corresponding subset in $H^1(G, \mathbf{X})$ will be denoted by $H^1(G, \mathbf{X})_X$;
- the 1-cocycles $G \to \mathrm{End}(X)_\rho$, where $\mathrm{End}(X)_\rho$ is the endomorphism monoid of X considered as a category (= the full subcategory in X with the unique object X) and equipped with a G-category structure ρ;
- the 1-cocycles $G \to \mathrm{Aut}(X)_\rho$, where $\mathrm{Aut}(X)_\rho$ is the automorphism group $\mathrm{Aut}(X)$ of X in $\mathbf{X}$ equipped with a G-group structure ρ.

1.3. Observation. Clearly the G-category structure of $\mathbf{X}$ restricts to a G-category structure on $\mathrm{End}(X)$ if and only if the object X is G-invariant. However there is another way to make such a structure. For any 1-cocycle $(X, \xi): G \to \mathbf{X}$ (with the same X), $g \in G$, and an endomorphism $a: X \to X$, we define $ga = \xi_g g(a)(\xi_g)^{-1}$ (writing ga for the new G-action in distinction from the old $g(a)$). We will denote this structure by $\rho(\xi)$, and write $Z^1(G, \mathrm{End}(X)_\xi)$ for the corresponding category of 1-cocycles and $H^1(G, \mathrm{End}(X)_\xi)$ for the corresponding 1-dimensional cohomology set respectively. The action $\rho(\xi)$ clearly restricts to a G-group structure on $\mathrm{Aut}(X)$; we will then write $Z^1(G, \mathrm{Aut}(X)_\xi)$ and

$H^1(G, \mathrm{Aut}(X)_\xi)$ respectively. Of course $Z^1(G, \mathrm{Aut}(X)_\xi)$ is just the groupoid whose morphisms are all isomorphisms in $Z^1(G, \mathrm{End}(X)_\xi)$, and therefore $H^1(G, \mathrm{Aut}(X)_\xi)$ is the same as $H^1(G, \mathrm{End}(X)_\xi)$

After this a straightforward calculation proves

1.4. Proposition. *Let $(X, \xi) : G \to \mathbf{X}$ be a fixed 1-cocycle. The assignment $\varphi \mapsto \big(X, (\varphi(g)\xi_g)_{g\in G}\big)$ extends to a category isomorphism $Z^1(G, \mathrm{End}(X)_\xi) \cong Z^1(G, \mathbf{X})_X$, and in particular*

$$H^1(G, \mathrm{Aut}(X)_\xi) = H^1(G, \mathrm{End}(X)_\xi) \cong H^1(G, \mathbf{X})_X.$$

1.5. Remark. Proposition 1.4 *almost* reduces the description of $H^1(G, \mathbf{X})$ to the case where $\mathbf{X}$ is a G-group. Since $H^1(G, \mathbf{X})$ obviously is the disjoint union of $H^1(G, \mathbf{X})_X$ for all objects X from any fixed skeleton of the category $\mathbf{X}$, and each $H^1(G, \mathbf{X})_X$ is either empty, or bijective to $H^1(G, \mathrm{Aut}(X)_\xi)$ for some ξ, it gives a bijection of the form

$$H^1(G, \mathbf{X}) \cong \sqcup_{(X,\xi)\in\Xi} H^1(G, \mathrm{Aut}(X)_\xi), \tag{1.5}$$

where Ξ is any set of 1-cocycles $(X, \xi) : G \to \mathbf{X}$ satisfying the following conditions:
(a) if (X, ξ) and (X', ξ') are in Ξ, then X and X' are not isomorphic in $\mathbf{X}$;
(b) Ξ is a maximal set with the property (a).

Involving 2-dimensional category theory. Consider the following two questions:
Question A: What is an appropriate notion of a morphism of G-categories?
Question B: Are the ordinary and the generalized notions of 1-cocycle merely technical, or do they occur as special cases of natural categorical constructions?

Considering G-categories as G-objects in **Cat**, or as internal categories in the category $\mathbf{Sets}^G$ of G-sets, one would of course define the morphisms between them as the G-functors, i.e. as the morphisms of G-objects in **Cat**, or, equivalently, as internal functors in $\mathbf{Sets}^G$. However, thinking of a G-category as a functor from G to **Cat**, and having in mind that **Cat** is a 2-dimensional category, we have a choice between the natural transformations which will give us the G-functors again, and the so-called *pseudo-natural transformations*. The second choice suggests the following answer to Question A:

1.6. Definition. Let $\mathbf{C}$ and $\mathbf{D}$ be G-categories. A morphism from $\mathbf{C}$ to $\mathbf{D}$ is a pair (F, ξ), in which F is a functor from $\mathbf{C}$ to $\mathbf{D}$ considered as ordinary categories, and $\xi = (\xi_{g,C})_{g\in G, C\in \mathrm{Ob}(\mathbf{C})}$ a family of isomorphisms $\xi_{g,C} : g(F(C)) \to F(g(C))$ natural in C and satisfying $\xi_{1,C} = \mathbf{1}_{F(C)}$ and $\xi_{hg,C} = \xi_{h,g(C)} h(\xi_{g,C})$ for all objects $C \in \mathbf{C}$ and all $g, h \in G$.

Furthermore, there are morphisms between pseudo-natural transformations called modifications, and for given $\mathbf{C}$ and $\mathbf{D}$, they together form a category $\mathrm{PsNat}(\mathbf{C}, \mathbf{D})$. The objects of $\mathrm{PsNat}(\mathbf{C}, \mathbf{D})$ are thus the morphisms from $\mathbf{C}$ to $\mathbf{D}$ in the sense of Definition 1.6, and a morphism $f : (F, \xi) \to (F', \xi')$ in $\mathrm{PsNat}(\mathbf{C}, \mathbf{D})$ is a natural transformation $f : F \to F'$ such that

$$f_{g(C)}\xi_{g,C} = \xi'_{g,C} g(f_C). \tag{1.6}$$

In particular if we take $\mathbf{C} = \mathbf{1}$ (the category with only one object and one morphism) and write $\mathbf{X}$ instead of $\mathbf{D}$, then $\mathrm{PsNat}(\mathbf{C}, \mathbf{D})$ becomes nothing but what we called $Z^1(G, \mathbf{X})$, yielding a good affirmative answer to our Question B.

2-FUNCTORIALITY OF $Z^1(G, \mathbf{X})$ IN $\mathbf{X}$. In order to take $\mathbf{X} \mapsto Z^1(G, \mathbf{X})$ into a 2-functor we need to make one more step into 2-dimensional category theory:
Given three G-categories $\mathbf{C}, \mathbf{D}, \mathbf{E}$, we construct the composition functor

$$\otimes = \otimes_{\mathbf{C},\mathbf{D},\mathbf{E}} : \ \mathrm{PsNat}(\mathbf{D}, \mathbf{E}) \times \mathrm{PsNat}(\mathbf{C}, \mathbf{D}) \to \mathrm{PsNat}(\mathbf{C}, \mathbf{E})$$

as follows
• $(\underline{F}, \underline{\xi}) \otimes (F, \xi) = (\underline{F}F, \xi \otimes \underline{\xi})$ on objects, where $\otimes$ is the *argumentwise pasting operation*, i.e. $\xi \otimes \underline{\xi}$ is defined by $(\xi \otimes \underline{\xi})_{g,C} = (\underline{F}(\xi_{g,C}))\underline{\xi}_{g,F(C)}$;
• on morphisms the functor $\otimes$ is the ordinary *horizontal composition* of natural transformations, i.e. for $\underline{f} : \ (\underline{F}, \underline{\xi}) \to (\underline{F}', \underline{\xi}')$ in $\mathrm{PsNat}(\mathbf{D}, \mathbf{E})$ and $f : \ (F, \xi) \to (F', \xi')$ in $\mathrm{PsNat}(\mathbf{C}, \mathbf{D})$, the composite $\underline{f} \otimes f : \ (\underline{F}, \underline{\xi}) \otimes (F, \xi) \to (\underline{F}', \underline{\xi}') \otimes (F', \xi')$ is defined by $(\underline{f} \otimes f)_C = (\underline{f}_{F'(C)})(\underline{F}(f_C))$ (which is the same as $(\underline{F}'(f_C))(\underline{f}_{F(C)})$).

The collection of all G-categories, their pseudo-natural transformations, and modifications with the composition $\otimes$ and the obvious *identity pseudo-natural transformations* $1_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$ (for all $\mathbf{C}$) form a 2-dimensional category G-$\mathbf{Cat}$, in which the hom-categories $\hom_{G\text{-}\mathbf{Cat}}(\mathbf{C}, \mathbf{D})$ are the categories $\mathrm{PsNat}(\mathbf{C}, \mathbf{D})$. In particular, in the notation above, we have

$$Z^1(G, \mathbf{X}) = \hom_{G\text{-}\mathbf{Cat}}(\mathbf{1}, \mathbf{X}) = \mathrm{PsNat}(\mathbf{1}, \mathbf{X}), \tag{1.7}$$

yielding a 2-functor $Z^1(G, -) : \ G\text{-}\mathbf{Cat} \to \mathbf{Cat}$.

A REMARK ON $Z^0(G, \mathbf{X})$ AND $H^0(G, \mathbf{X})$. Let us repeat our constructions "decreasing the dimension by one". That is, we take the G-category $\mathbf{X}$ to be a *discrete* category, i.e. a category with no nonidentity morphisms, $\mathbf{X} = \mathrm{Dis}(S)$ - denoting the G-set of its objects by S. Then it is easy to see that the pseudo-natural transformations above become just the natural transformations, the modifications become identities, i.e. $Z^1(G, \mathrm{Dis}(S))$ is discrete, and we have

$$Z^1(G, \mathrm{Dis}(S)) \cong \mathrm{Dis}\{s \in S \mid gs = s \ \text{ for all } \ g \in G\}. \tag{1.8}$$

Since the classical definitions of $Z^0(G, S)$ and $H^0(G, S)$ say

$$H^0(G, S) = Z^0(G, S) = \mathrm{Dis}\{s \in S \mid gs = s \ \text{ for all } \ g \in G\}, \tag{1.9}$$

we conclude that $Z^1(G, \mathbf{X})$ and $H^1(G, \mathbf{X})$ defined for an arbitrary G-category $\mathbf{X}$, generalize not only $Z^1(G, A)$ and $H^1(G, A)$ for a G-group A, but also $Z^0(G, S)$ and $H^0(G, S)$ for a G-set S.

A CATEGORICAL CONSTRUCTION BEHIND PROPOSITION 1.4. Any functor $F: \mathbf{C} \to \mathbf{D}$ canonically factorizes as

$$\mathbf{C} \xrightarrow{F^{(1)}} \mathbf{fact}(F) \xrightarrow{F^{(2)}} \mathbf{D} \tag{1.10}$$

where:
• the objects in $\mathbf{fact}(F)$ are as in $\mathbf{C}$ and the morphisms as in $\mathbf{D}$, that is $\hom_{\mathbf{fact}(F)}(C, C') = \hom_{\mathbf{D}}(F(C), F(C'))$;
• the functor $F^{(1)}$ is defined by $F^{(1)}(C) = C$ on objects and $F^{(1)}(u) = F(u)$ on morphisms;
• the functor $F^{(2)}$ is defined by $F^{(2)}(C) = F(C)$ on objects and $F^{(2)}(u) = u$ on morphisms.

If $\mathbf{C}$ and $\mathbf{D}$ are G-categories and F is a G-functor, then, since the factorization (1.10) is functorial, this makes $\mathbf{fact}(F)$ a G-category and $F^{(1)}$ and $F^{(2)}$ G-functors. More generally, if $(F, \xi): \mathbf{C} \to \mathbf{D}$ is a morphism of G-categories in the sense of Definition 1.6, then $\mathbf{fact}(F)$ can still be regarded as a G-category and there are canonical $\xi^{(1)}$ and $\xi^{(2)}$ making $(F^{(1)}, \xi^{(1)})$ and $(F^{(2)}, \xi^{(2)})$ morphisms of G-categories. Explicitly:
• For each $g \in G$, the functor $g(-): \mathbf{fact}(F) \to \mathbf{fact}(F)$ is defined as $g(-): C \to C$ on objects and via

$$\begin{tikzcd} gF(C) \arrow[r, "\xi_{g,C}"] \arrow[d, "g(a)"'] & Fg(C) \arrow[d, "g(a)"] \\ gF(C') \arrow[r, "\xi_{g,C'}"] & Fg(C') \end{tikzcd} \tag{1.11}$$

on morphisms; here the first vertical arrow represents the image under $g(-): D \to D$ of an arbitrary $a: F(C) \to F(C')$ in $\mathbf{D}$, but the second one is the new $g(a)$ to be defined as a morphism from $Fg(C)$ to $Fg(C')$ in $\mathbf{D}$.
• Since $\xi_{g,C}$ is natural in C, it is easy to check that the diagram

$$\begin{tikzcd} \mathbf{C} \arrow[r, "F^{(1)}"] \arrow[d, "g(-)"'] & \mathbf{fact}(F) \arrow[d, "g(-)"] \\ \mathbf{C} \arrow[r, "F^{(1)}"] & \mathbf{fact}(F) \end{tikzcd} \tag{1.12}$$

commutes for every $g \in G$, and so $\xi^{(1)}$ is to be defined as the appropriate family of identity morphisms.
• For a morphism $a: C \to C'$ in $\mathbf{fact}$(F), the diagram (1.11) can now be rewritten as

$$\begin{tikzcd} gF^{(2)}(C) \arrow[r, "\xi_{g,C}"] \arrow[d, "gF^{(2)}(a)"'] & F^{(2)}g(C) \arrow[d, "F^{(2)}g(a)"] \\ gF^{(2)}(C') \arrow[r, "\xi_{g,C'}"] & F^{(2)}g(C') \end{tikzcd} \tag{1.13}$$

and accordingly $\xi^{(2)}$ is defined as the family of isomorphisms

$$\xi_{g,C}: \ gF^{(2)}(C) = gF(C) \to Fg(C) = F^{(2)}g(C).$$

1.7. Definition. Let $(F, \xi) : \mathbf{C} \to \mathbf{D}$ be a morphism of G-categories. With notation as above, we will say that $(F, \xi) = (F^{(2)}, \xi^{(2)}) \otimes (F^{(1)}, \xi^{(1)})$ is the canonical factorization of (F, ξ).

1.8. Example. Let $(X, \xi) : G \to \mathbf{X}$ be a 1-cocycle as in Proposition 1.4 and $(F, \xi) = (F^{(2)}, \xi^{(2)}) \otimes (F^{(1)}, \xi^{(1)})$ the canonical factorization of the corresponding morphism $(F, \xi) : \mathbf{1} \to \mathbf{X}$. Then
(a) the category $\mathbf{fact}(F)$ is the same as the monoid $\mathrm{End}(X)$;
(b) the G-category structure $\rho(\xi)$ on $\mathrm{End}(X)$ described in Observation 1.3 is the same as the G-category structure on $\mathbf{fact}(F)$ described now ((1.11));
(c) the composite of the isomorphism $Z^1(G, \mathrm{End}(X)_\xi) \cong Z^1(G, \mathbf{X})_X$ described in Proposition 1.4 and the embedding $Z^1(G, \mathbf{X})_X \to Z^1(G, \mathbf{X})$ is the same as the functor

$$Z^1(G, (F^{(2)}, \xi^{(2)})) : \; Z^1(G, \mathbf{fact}(F)) \to Z^1(G, \mathbf{X}).$$

2. 1-Cocycles as algebras over a monad

G-Category structures as monoids. Given a category $\mathbf{C}$, let $\mathrm{Fam}(\mathbf{C})$ be the *category of families* of objects in $\mathbf{C}$. Recall that the objects of $\mathrm{Fam}(\mathbf{C})$ are all families $(A_i)_{i \in I}$ with $A_i \in \mathbf{C}$, and a morphism $(A_i)_{i \in I} \to (B_j)_{j \in J}$ is a pair (f, u) in which $f : I \to J$ is a map of sets, and u is a family of morphisms $u_i : A_i \to B_{f(i)}$. If $\mathbf{C}$ is a monoidal category $\mathbf{C} = (\mathbf{C}, \otimes, 1, \alpha, \lambda, \rho)$, then so is $\mathrm{Fam}(\mathbf{C}) = (\mathrm{Fam}(\mathbf{C}), \otimes, 1, \alpha', \lambda', \rho')$ with

$$(A_i)_{i \in I} \otimes (B_j)_{j \in J} = (A_i \otimes B_j)_{(i,j) \in I \times J}, \tag{2.1}$$

with 1 in $\mathrm{Fam}(\mathbf{C})$ being the one-member family corresponding to the 1 in $\mathbf{C}$, and with obvious α', λ', ρ'. According to this description, a *monoid* in $(\mathrm{Fam}(\mathbf{C}), \otimes, 1)$ can be identified with a system $((M_i)_{i \in I}, e, m)$, in which:
• I is an ordinary monoid;
• m is a family of morphisms $m_{(i,j)} : M_i \otimes M_j \to M_{ij}$ in $\mathbf{C}$ satisfying the suitable associativity condition;
• e is a morphism in $\mathbf{C}$ from 1 to M_1 (where the index 1 of M_1 is the identity element of the monoid I) satisfying the suitable identity condition.
In particular, if $\mathbf{C} = (\mathbf{C}, \otimes, 1)$ is a strict monoidal category, and $(M_i)_{i \in I}$ an object in $\mathrm{Fam}(\mathbf{C})$ in which
• I is equipped with (an ordinary) monoid structure;
• $M_i \otimes M_j = M_{ij}$ for all $i, j \in I$;
• $M_1 = 1$
- then the system $((M_i)_{i \in I}, e, m)$, in which all $m_{(i,j)}$ and e are identity morphisms, is a monoid in $\mathrm{Fam}(\mathbf{C})$ (although $\mathrm{Fam}(\mathbf{C})$ is not strict!); let us call such a monoid *split*. Comparing this notion with the notion of G-category considered in Section 1, we immediately observe:

Let G be a group. To make a category $\mathbf{X}$ a G-category is the same thing as to give a split monoid $(M_i)_{i \in I}$ in the monoidal category $\mathrm{Fam}(\mathbf{X}^{\mathbf{X}})$ associated to the monoidal category $\mathbf{X}^{\mathbf{X}} = (\mathbf{X}^{\mathbf{X}}, \circ, 1_{\mathbf{X}})$ of endofunctors of $\mathbf{X}$, with $I = G$. According to the notation of Section 1, (the underlying object of) that monoid is to be written as

$$(g(-))_{g \in G}. \tag{2.2}$$

Split monoid actions as 1-cocycles. Let $M = (M, e, m)$ be a monoid in a monoidal category $\mathbf{C}$. Recall that an M-action in $\mathbf{C}$ is a pair (X, ξ), where X is an object in $\mathbf{C}$ and $\xi : M \otimes X \to X$ a morphism in $\mathbf{C}$ satisfying the associativity and identity properties with respect to the monoid structure of M. We will also say that ξ is an M-action on X. Consider the following

2.1. Situation. We take:
- $\mathbf{C}$ to be a strict monoidal category, but then replace it by the monoidal category $\mathrm{Fam}(\mathbf{C})$ as defined above.
- $M = ((M_i)_{i \in I}, e, m)$ a split monoid in $\mathrm{Fam}(\mathbf{C})$.
- X an object in $\mathbf{C}$ regarded as a one-member family, and therefore an object in $\mathrm{Fam}(\mathbf{C})$.

In this case an M-action on X can be described as a family $\xi = (\xi_i)_{i \in I}$ of morphisms $\xi_i : M_i \otimes X \to X$ with

$$\xi_1 = 1_X \quad \text{and} \quad \xi_{ij} = \xi_i(M_i \otimes \xi_j). \tag{2.3}$$

After that consider a further special case:

2.2. Situation. We take $(\mathbf{C}, \otimes, 1) = (\mathbf{X}^{\mathbf{X}}, \circ, 1_{\mathbf{X}})$ (where $\mathbf{X}$ is a G-category as in Section 1), $(M_i)_{i \in I}$ the monoid (2.2) , and X an object in $\mathbf{X}$ regarded as a constant functor $\mathbf{X} \to \mathbf{X}$. We will say that (X, ξ) is a $(g(-))_{g \in G}$-action on an object in $\mathbf{X}$, if ξ_g is a constant natural transformation for each g in G.

Comparing the formulae (2.3) with those in Definition 1.1(a), we obtain:

2.3. Proposition. *A pair $(X, \xi) = (X, (\xi_g)_{g \in G})$ is an object in $Z^1(G, \mathbf{X})$ if and only if it is a $(g(-))_{g \in G}$-action (on X). This yields a canonical isomorphism between the category $Z^1(G, \mathbf{X})$ of 1-cocycles $G \to \mathbf{X}$ and the category of $(g(-))_{g \in G}$-actions on objects in $\mathbf{X}$.*

Involving the coproduct functor $\mathrm{Fam}(\mathbf{C}) \to \mathbf{C}$. In (say) Situation 2.1, if $\mathbf{C}$ had coproducts preserved by the functors $C \otimes -,\ - \otimes C : \mathbf{C} \to \mathbf{C}$ for every object C in $\mathbf{C}$, then the monoid structure of $(M_i)_{i \in I}$ would of course induce a monoid structure on the coproduct $\coprod_{i \in I} M_i$, and there would be a straightforward way to compare the actions of these two monoids. However having in mind Situation 2.2, we cannot assume the preservation of coproducts by $C \otimes -$ for every C; what we really need is the following:

2.4. Proposition. *Let $\mathbf{C}$ be a monoidal category with coproducts preserved by $- \otimes C : \mathbf{C} \to \mathbf{C}$ for each object C in $\mathbf{C}$, and $M = ((M_i)_{i \in I}, e, m)$ a monoid in $\mathrm{Fam}(\mathbf{C})$ such that $M_i \otimes - : \mathbf{C} \to \mathbf{C}$ preserves coproducts for each $i \in I$. Then the monoid structure of $(M_i)_{i \in I}$ induces a monoid structure on the coproduct $\coprod_{i \in I} M_i$ and there is a canonical isomorphism between the category of $(M_i)_{i \in I}$-actions (i.e. M-actions) on one-member families in $\mathrm{Fam}(\mathbf{C})$ and the category of $\coprod_{i \in I} M_i$-actions in $\mathbf{C}$.*

In Situation 2.2, assuming the existence of G-indexed coproducts in $\mathbf{X}$, all the required conditions hold since all the functors $g(-)$ $(g \in G)$ are isomorphisms and therefore preserve coproducts. Here $\coprod_{i\in I} M_i = \coprod_{g\in G} g(-)$ being a monoid in $(\mathbf{X}^{\mathbf{X}}, \circ, 1_{\mathbf{X}})$ is a monad on $\mathbf{X}$; let us denote it by $G(-)$. From Propositions 2.3 and 2.4, we obtain

2.5. Theorem. *Let G be a group and $\mathbf{X}$ a G-category with G-indexed coproducts. Then there is a canonical isomorphism between the category $Z^1(G, \mathbf{X})$ of 1-cocycles $G \to \mathbf{X}$ and the category $\mathbf{X}^{G(-)}$ of $G(-)$-algebras. For a 1-cocycle $(X, \xi) : G \to \mathbf{X}$ with $\xi = (\xi_g)_{g\in G}$, the corresponding $G(-)$-algebra is the pair (X, ξ_*) with the same X and $\xi_* : G(X) = \coprod_{g\in G} g(X) \to X$ induced by all $\xi_g : g(X) \to X$ $(g \in G)$.*

Convention: we will identify ξ and ξ_*, and just write ξ.

3. G-Galois monads and adjunctions

Relating split and connected monoids. Let $M = ((M_i)_{i\in I}, e, m)$ be a split monoid as in Situation 2.1, and $M' = (M', e', m')$ a monoid in the monoidal category $\mathbf{C}$. Regarding $\mathbf{C}$ as a monoidal subcategory of $\mathrm{Fam}(\mathbf{C})$, we can regard M' as a monoid in $\mathrm{Fam}(\mathbf{C})$, and we will call such a monoid in $\mathrm{Fam}(\mathbf{C})$ *connected*. We need the description of monoid homomorphisms from M to M', i.e. morphisms $u : M \to M'$ in $\mathrm{Fam}(\mathbf{C})$ making the diagrams

$$\begin{tikzcd} M \arrow[rr, "u"] & & M' \\ & 1 \arrow[ul, "e"] \arrow[ur, "e'"'] & \end{tikzcd} \qquad \begin{tikzcd} M \otimes M \arrow[r, "u\otimes u"] \arrow[d, "m"'] & M' \otimes M' \arrow[d, "m'"] \\ M \arrow[r, "u"] & M' \end{tikzcd} \tag{3.1}$$

commute. Since M' regarded as an object in $\mathrm{Fam}(\mathbf{C})$ is a one-member family, a morphism $u : M \to M'$ in $\mathrm{Fam}(\mathbf{C})$ is nothing but an I-indexed family of morphisms $u_i : M_i \to M'$ - and the commutativity of (3.1) expressed in the language of u_i's becomes

$$\begin{tikzcd} M_1 \arrow[rr, "u_1"] & & M' \\ & 1 \arrow[ul, "="] \arrow[ur, "e'"'] & \end{tikzcd} \qquad \begin{tikzcd} M_i \otimes M_j \arrow[r, "u_i\otimes u_j"] \arrow[d, "="'] & M' \otimes M' \arrow[d, "m'"] \\ M_{ij} \arrow[r, "u_{ij}"] & M' \end{tikzcd} \tag{3.2}$$

i.e.

$$e' = u_1, \quad m'(u_i \otimes u_j) = u_{ij} \ \ (i, j \in I). \tag{3.3}$$

We will use an example of such a homomorphism provided by

3.1. Lemma. *Let $M = ((M_i)_{i\in I}, e, m)$ and $M' = (M', e', m')$ be as above, and let*

$$M' \otimes M_i = M' \ \text{ and } \ m' \otimes M_i = m' \tag{3.4}$$

for every $i \in I$. Then the family $u = (u_i)_{i\in I}$, where $u_i = e' \otimes M_i$ is the morphism

$$M_i = M_{1i} = M_1 \otimes M_i \xrightarrow{e'\otimes M_i} M' \otimes M_i = M', \tag{3.5}$$

is a monoid homomorphism from M to M'.

Proof. We have that $e' = e' \otimes 1 = e' \otimes M_1 = u_1$, and

$$\begin{aligned} m'(u_i \otimes u_j) &= m'(M' \otimes u_j)(u_i \otimes M_j) \quad \text{(since } \otimes \text{ is a bifunctor)} \\ &\overset{(3.4)}{=} (m' \otimes M_j)(M' \otimes e' \otimes M_j)(e' \otimes M_i \otimes M_j) \\ &= \big((m'(m' \otimes e')) \otimes M_j\big)(e' \otimes M_{ij}) \quad \text{(since } (-) \otimes M_j \text{ is a functor)} \\ &= (1_{M'} \otimes M_j)u_{ij} \quad \text{(since } M' \text{ is a monoid)} \\ &= u_{ij}, \end{aligned}$$

as desired (see (3.3)). ■

Lemma 3.1 easily gives

3.2. Corollary. *Under the assumptions of Proposition 2.4 and Lemma 3.1, the morphism $\coprod_{i \in I} M_i \to M'$ induced by the family of morphisms (3.5) is a monoid homomorphism in the monoidal category* $\mathbf{C}$.

Applying this to Situation 2.2, we obtain

3.3. Corollary. *Let (T, η, μ) be a monad on $\mathbf{X}$ with $T(g(-)) = T$ and $\mu(g(-)) = \mu$ for all $g \in G$. Then the natural transformation $\gamma : G(-) = \coprod_{g \in G} g(-) \to T$ induced by the natural transformations*

$$g(-) = 1_{\mathbf{X}}(g(-)) \xrightarrow{\eta(g(-))} T(g(-)) = T \quad (g \in G), \tag{3.6}$$

is a morphism of monads.

G-Galois monads and adjunctions. Take a monad $T = (T, \eta, \mu)$ on $\mathbf{X}$, and let (F^T, U^T) be the associated free-forgetful adjunction between $\mathbf{X}$ and $\mathbf{X}^T$, with unit $\eta : 1_{\mathbf{X}} \to U^T F^T$ and counit $\varepsilon : F^T U^T \to 1_{\mathbf{X}^T}$. The assumptions $T(g(-)) = T$ and $\mu(g(-)) = \mu$ of Corollary 3.3 can be reformulated as $F^T(g(-)) = F^T$, and $\gamma : G(-) = \coprod_{g \in G} g(-) \to T$ being a morphism of monads induces a functor

$$K^\gamma : \mathbf{X}^T \to \mathbf{X}^{G(-)} \cong Z^1(G, \mathbf{X}). \tag{3.7}$$

Explicitly, for a T-algebra (X, ζ), we have

$$K^\gamma(X, \zeta) = (X, \xi) \;\text{ with }\; \xi_g = \zeta\eta_{g(X)} \quad (g \in G). \tag{3.8}$$

3.4. Definition. Let G be a group and $\mathbf{X}$ a G-category with G-indexed coproducts. A monad $T = (T, \eta, \mu)$ on $\mathbf{X}$ is said to be a G-Galois monad if

(a) $T(g(-)) = T$ and $\mu(g(-)) = \mu$ for all $g \in G$;

(b) the morphism $\gamma : G(-) = \coprod_{g \in G} g(-) \to T$ is an isomorphism.

Equivalently, T is a G-Galois monad if the free functor F^T satisfies $F^T(g(-)) = F^T$ for all $g \in G$ and the functor K^γ is an isomorphism.

If instead of a monad T we begin with an adjunction

$$\mathbf{X} \underset{U}{\overset{F}{\rightleftarrows}} \mathbf{Y}, \quad \eta : 1_{\mathbf{X}} \to UF, \; \varepsilon : FU \to 1_{\mathbf{Y}} \tag{3.9}$$

where $\mathbf{X}$ is a G-category again, Definition 3.4 should be reformulated as

3.5. Definition. An adjunction (3.9) is said to be a G-Galois adjunction if

(a) $F(g(-)) = F$, for all $g \in G$;

(b) the category $\mathbf{X}$ admits G-indexed coproducts and the morphism $\gamma_X : \coprod_{g\in G} g(X) \to UF(X)$ induced by the collection of morphisms $\eta_{g(X)} : g(X) \to UF(g(X)) = UF(X)$ $(g \in G)$ is an isomorphism for each object $X \in \mathbf{X}$;

(c) the functor U is monadic.

Recall that for a general adjunction (3.9) with the corresponding monad $T = (T, \eta, \mu) = (UF, \eta, U\varepsilon F)$, the comparison functor $K : \mathbf{Y} \to \mathbf{X}^T$ is defined by $K(Y) = (U(Y), U(\varepsilon_Y))$, and when U is monadic the quasi-inverse $L : \mathbf{X}^T \to \mathbf{Y}$ of K can be described via the coequalizer diagram

$$FUF(X) \underset{F(\zeta)}{\overset{\varepsilon_{F(X)}}{\rightrightarrows}} F(X) \longrightarrow L(X,\zeta) \tag{3.10}$$

where (X, ζ) is any T-algebra. Moreover, if $\mathbf{Y}$ admits coequalizers, then K always (i.e. even under no monadicity assumption) has a left adjoint L defined in the same way.

Let us provide another construction for the functor L in the situation of Definition 3.5, using the fact that the monad involving there is isomorphic (by 3.5(b)) to the monad $G(-)$. At the same time this will describe the composite of L with the isomorphism between the categories $Z^1(G, \mathbf{X})$ and $\mathbf{X}^T$, i.e. show how to calculate the object in $\mathbf{Y}$ corresponding to a given 1-cocycle $G \to \mathbf{X}$. For, we observe:

• Since $\gamma_X : \coprod_{g\in G} g(X) \to UF(X)$ in Definition 3.5 is an isomorphism, we can replace the two parallel arrows in (3.10) by their composites with $F(\gamma_X)$. Moreover, since the functor F being a left adjoint sends coproducts to coproducts, we can make a further replacement by composing with the canonical isomorphism $\coprod_{g\in G} F(g(X)) \to F(\coprod_{g\in G} g(X))$. After that, instead of defining $L(X,\zeta)$ via the coequalizer diagram (3.10), we can define it via the morphism $F(X) \to L(X,\zeta)$ that composed with

$$F(g(X)) \xrightarrow{F(\iota_g)} F(G(X)) \xrightarrow{F(\gamma_X)} FUF(X) \xrightarrow{\varepsilon_{F(X)}} F(X) \tag{3.11}$$

and with

$$F(g(X)) \xrightarrow{F(\iota_g)} F(G(X)) \xrightarrow{F(\gamma_X)} FUF(X) \xrightarrow{F(\zeta)} F(X) \tag{3.12}$$

makes them equal for every $g \in G$, and is universal with this property (where ι_g is the coproduct injection corresponding to g).

• For the composite (3.11), we have, using the definition of γ for the second equality,

$$\varepsilon_{F(X)}F(\gamma_X)F(\iota_g) = \varepsilon_{F(X)}F(\gamma_X\iota_g) = \varepsilon_{F(X)}F(\eta_{g(X)}) \overset{3.5(a)}{=} \varepsilon_{F(g(X))}F(\eta_{g(X)}),$$

which is the identity morphism of $F(g(X)) = F(X)$ by one of the triangular identities for adjoint functors.

• And for (3.12): $F(\zeta)F(\gamma_X)F(\iota_g) = F(\zeta\gamma_X\iota_g) = f(\zeta\eta_{g(X)}) = f(\xi_g)$, with ξ as in (3.8).
• It follows easily from 1.1(a) and 3.5(a) that the assignment $g \mapsto F(\xi_g)$ determines a group homomorphism $G \to \mathrm{Aut}(F(X))$, i.e. it makes $F(X)$ a G-object in $\mathbf{Y}$. Moreover, if we regard $\mathbf{Y}$ as a G-category with the trivial action of G then the category $Z^1(G, \mathbf{Y})$ of 1-cocycles $G \to \mathbf{Y}$ is of course nothing but the category $\mathbf{Y}^G$ of G-objects in $\mathbf{Y}$, and then F, obviously being a G-functor, induces a functor $Z^1(G, F) : \ Z^1(G, \mathbf{X}) \to Z^1(G, \mathbf{Y}) = \mathbf{Y}^G$. Of course the G-object in $\mathbf{Y}$ obtained above is nothing but $Z^1(G, F)(X, \xi)$.
• If $\mathbf{Y}$ admits "all" colimits (say, as large as its hom-sets), then there is the colimit functor $\mathrm{colim} : \ \mathbf{Y}^G \to \mathbf{Y}$, sending a G-object Y to its *universal quotient with the trivial G-action*. We will briefly write $\mathrm{colim}(Y) = Y/G$, also when that colimit exists just for a given object Y. In particular $L(X, \zeta) = F(X)/G$.

Summarizing, and using the isomorphism (3.7) defined by (3.8) and the definition of the comparison functor $K : \mathbf{Y} \to \mathbf{X}^T$, we obtain

3.6. Theorem. *(a) For an adjunction (3.9) satisfying 3.5(a) and (b) the left adjoint $L : \ \mathbf{X}^T \to \mathbf{Y}$ of the comparison functor $K : \ \mathbf{Y} \to \mathbf{X}^T$ can be described as*

$$L(X, \zeta) = F(X)/G, \tag{3.13}$$

where $F(X)$ is regarded as a G-object in $\mathbf{Y}$ via $g \mapsto F(\xi_g) = F(\zeta\eta_{g(X)}) : \ F(X) \to F(X)$.
(b) If (3.9) is G-Galois adjunction then there is a category equivalence

$$\mathbf{Y} \sim Z^1(G, \mathbf{X}), \tag{3.14}$$

under which an object Y in $\mathbf{Y}$ corresponds to the 1-cocycle $\big(U(Y), (U(\varepsilon_Y)\eta_{g(U(Y))})_{g\in G}\big) :$ $G \to \mathbf{X}$; conversely, a 1-cocycle $(X, \xi) : \ G \to \mathbf{X}$ corresponds to the object $F(X)/G$, where $F(X)$ is regarded as a G-object in $\mathbf{Y}$ via $g \mapsto F(\xi_g) : \ F(X) \to F(X)$.

Furthermore, applying Proposition 1.4, we obtain:

3.7. Theorem. *If (3.9) is a G-Galois adjunction, then every object Y_0 in $\mathbf{Y}$ admits a category equivalence*

$$\mathbf{Y}[Y_0] \sim Z^1(G, \mathrm{End}(U(Y_0))_{\xi_0}), \tag{3.15}$$

in which $\mathbf{Y}[Y_0]$ is the full subcategory in $\mathbf{Y}$ with objects all Y in $\mathbf{Y}$ with $U(Y)$ isomorphic to $U(Y_0)$, and $(U(Y_0), \xi_0) = (U(Y_0), (U(\varepsilon_{Y_0})\eta_{g(U(Y_0))})_{g\in G})$ is the 1-cocycle $G \to \mathbf{X}$ corresponding to Y_0 under the equivalence (3.14). In particular the set of isomorphism classes of objects in $\mathbf{Y}[Y_0]$ is bijective to $H^1(G, \mathrm{End}(U(Y_0))_{\xi_0}) = H^1(G, \mathrm{Aut}(U(Y_0))_{\xi_0})$.

3.8. Remark. The equivalence (3.15) is "less canonical" than the equivalence (3.14) in the following sense: let $\mathbf{Y}[Y_0]$ be the full subcategory in $\mathbf{Y}$ with objects all Y in $\mathbf{Y}$ with $U(Y) = U(Y_0)$, and let

$$\mathbf{Y}(Y_0) \to Z^1(G, \mathbf{X})_{U(Y_0)} \ \text{ and } \ Z^1(G, \mathbf{X})_{U(Y_0)} \to \mathbf{Y}[Y_0] \tag{3.16}$$

be the functors induced by the equivalence (3.14). Since both of them obviously are category equivalences, so are the composites

$$\begin{gathered}\mathbf{Y}(Y_0) \to Z^1(G,\mathbf{X})_{U(Y_0)} \to Z^1(G,\mathrm{End}(U(Y_0))_{\xi_0}) \ \text{ and} \\ Z^1(G,\mathrm{End}(U(Y_0))_{\xi_0}) \to Z^1(G,\mathbf{X})_{U(Y_0)} \to \mathbf{Y}[Y_0],\end{gathered} \tag{3.17}$$

where ξ_0 is as in Theorem 3.7, and the isomorphism $Z^1(G,\mathbf{X})_{U(Y_0)} \cong Z^1(G,\mathrm{End}(U(Y_0))_{\xi_0})$ is obtained from Proposition 1.4. However, since the category $\mathbf{Y}(Y_0)$ is only equivalent, but not equal to $\mathbf{Y}[Y_0]$, the functors (3.17) are not exactly quasi-inverse to each other. One can say that the equivalence (3.15) is only canonical up to a choice of the quasi-inverse of the inclusion functor $\mathbf{Y}(Y_0) \to \mathbf{Y}[Y_0]$. Yet, since the second functor in (3.17) is uniquely determined only up to the choice of colimits in $\mathbf{Y}$, it is useful to describe it explicitly. And, since it is a composite of explicitly given functors, the description is straightforward:

$$(\varphi \in Z^1(G,\mathrm{End}(U(Y_0))_{\xi_0})) \mapsto F(Y_0)/G, \tag{3.18}$$

where G acts on $F(Y_0)$ by

$$g \mapsto F(\varphi(g)U(\varepsilon_{Y_0}\eta_{g(U(Y_0))})), \tag{3.19}$$

since $g \mapsto F(\xi_g)$ in (3.13), the role of ξ_g is played by $\varphi(g)\xi_{0g}$ here according to Proposition 1.4, and $\xi_{0g} = U(\varepsilon_{Y_0}\eta_{g(U(Y_0))})$ as in Theorem 3.7.

4. Galois descent for modules and algebras

Galois adjunctions correspond to finite Galois extensions. Let us take the adjunction (3.9) to be

$$(E\text{-}\mathbf{Mod})^{\mathrm{op}} \underset{U=E\otimes_B(-)}{\overset{F=(-)_p}{\rightleftarrows}} (B\text{-}\mathbf{Mod})^{\mathrm{op}}\ , \tag{4.1}$$

$$\eta:\ 1_{(E\text{-}\mathbf{Mod})^{\mathrm{op}}} \to UF,\ \varepsilon:\ FU \to 1_{(B\text{-}\mathbf{Mod})^{\mathrm{op}}},$$

where

• $p:\ B \to E$ is a homomorphism of non-trivial commutative rings with 1;
• $F:\ (E\text{-}\mathbf{Mod})^{\mathrm{op}} \to (B\text{-}\mathbf{Mod})^{\mathrm{op}}$ is the (dual of) the restriction-of-scalars functor from the dual (=opposite) category of E-modules to the dual category of B-modules, for which we shall write $F(D) = D_p$;
• $U:\ (B\text{-}\mathbf{Mod})^{\mathrm{op}} \to (E\text{-}\mathbf{Mod})^{\mathrm{op}}$ is the (dual of) the induction functor, for which we shall write $U(A) = E \otimes_B A$;
• $\eta = (\eta_D:\ E \otimes_B D_p \to D_p)_{D\in(E\text{-}\mathbf{Mod})^{\mathrm{op}}}$ is defined by $\eta_D(e \otimes d) = ed$ (here and below, whenever it makes sense, we keep the direction of arrows as in the categories of modules, not as in their duals);
• $\varepsilon = \big(\varepsilon_A:\ A \to (E \otimes_B A)_p\big)_{A\in(B\text{-}\mathbf{Mod})^{\mathrm{op}}}$ is defined by $\varepsilon_A(a) = 1 \otimes a$.

We will also assume that (E,p) is equipped with a right action of a group G, i.e. G acts on E on the right (via ring automorphisms of E) with $p(b)g = p(b)$ for each $b \in B$

and $g \in G$. This makes $(E\text{-}\mathbf{Mod})^{\mathrm{op}}$ a G-category, in which $g(D) = D_g$, i.e. $g(D) = D$ as abelian groups and, for $e \in E$ and $d \in D$, we have:

$$ed \in g(D) \text{ is the same as } (eg)d \in D. \tag{4.2}$$

4.1. Theorem. *The adjunction (4.1) equipped with the G-category structure on $(E\text{-}\mathbf{Mod})^{\mathrm{op}}$ defined by (4.2) is a G-Galois adjunction if and only if the following conditions hold:*

(a) $p : B \to E$ is an effective (co)descent morphism, i.e., considered as a homomorphism of B-modules, it is a pure monomorphism;

(b) the map $h : E \otimes_B E \to GE$, where GE denotes the E-algebra of all maps from G to E, defined by $h(e \otimes e')(g) = (eg)e'$, is bijective;

(c) the group G is finite.

Proof. (a) simply means that the functor $U : (B\text{-}\mathbf{Mod})^{\mathrm{op}} \to (E\text{-}\mathbf{Mod})^{\mathrm{op}}$ is monadic (see [Janelidze, Tholen, 2004] for details). On the other hand condition 3.5 (b) now becomes: the map

$$\gamma_D : E \otimes_B D_p \to GD, \quad \gamma_D(e \otimes d)(g) = (eg)d, \tag{4.3}$$

is bijective for each D in E-$\mathbf{Mod}$.
Since this condition implies 4.1(b), it suffices to prove that, under conditions 4.1 (a) and 4.1(b), condition 3.5(b) is equivalent to condition 4.1(c). For, note that we can identify h with γ_E, and consider the commutative diagram

$$\begin{tikzcd}
(E \otimes_B E) \otimes_E D_p \arrow[r, "h \otimes 1_D"] \arrow[d] & (GE) \otimes_E D_p \arrow[d] \\
E \otimes_B D_p \arrow[r, "\gamma_D"] & GD,
\end{tikzcd} \tag{4.4}$$

whose vertical arrows are obvious canonical maps. Since its top arrow is bijective by 4.1 (b), and its left-hand vertical arrow is always bijective, the bijectivity of its bottom arrow is equivalent to the bijectivity of its right-hand vertical arrow. Therefore all we need to observe is that the canonical map $(GE) \otimes_E D_p \to GD$ is bijective for every E-module D if and only if G is finite. ∎

4.2. Remark. One can replace ordinary modules with modules equipped with various kinds of additional structure, and, in particular, with various kinds of algebras. To be more precise, Theorem 4.1 still holds if instead of E-$\mathbf{Mod}$ and B-$\mathbf{Mod}$ we have E-$\mathbf{Alg}$ and B-$\mathbf{Alg}$ respectively, where "-$\mathbf{Alg}$" refers to, for example, (a) arbitrary (not necessarily associative or commutative) algebras, with or without 1; (b) associative algebras, with or without 1; (c) (associative and) commutative algebras, with or without 1; (d) Lie algebras; (e) Jordan algebras; (f) differential algebras.
All we need here is to know that the functors F and U of (4.1) induce similar functors for these kinds of algebras, and that replacing modules with algebras does not affect monadicity. For this, see [Janelidze, Tholen, 2004, Sec. 6].

Translation of Theorems 3.6 and 3.7 for modules and algebras. Theorem 4.1 allows us to apply Theorems 3.6 and 3.7 to modules and algebras. This gives:

4.3. Theorem. *Suppose the equivalent conditions of Theorem 4.1 hold. Then the category of B-modules is equivalent to the category $Z^1(G,(E\text{-}\mathbf{Mod})^{\mathrm{op}})^{\mathrm{op}}$, which can be identified with the category of E-modules D on which G acts on the right via B-module automorphisms with $(ed)g=(eg)(dg)$ for all $e\in E$, $d\in D$, and $g\in G$. Under this equivalence:*
(a) a B-module A corresponds to the E-module $E\otimes_B A$, on which G acts on the right via B-module automorphisms with $(e\otimes a)g=eg\otimes a$ for all $e\in E$, $a\in A$, and $g\in G$;
(b) an E-module D equipped with a right G-action as above corresponds to the B-module $D^G=\{d\in D\mid dg=d \text{ for all } g\in G\}$, where the scalar multiplication $B\times D^G\to D^G$ is induced by the scalar multiplication $B\times D_p\to D_p$.

Proof. A 1-cocycle $(D,\xi)\ :G\to(E\text{-}\mathbf{Mod})^{\mathrm{op}}$ is an E-module D together with a family $\xi=(\xi_g)_{g\in G}$ of E-module isomorphisms $\xi_g:\ D\to D_g$ in $\mathbf{X}$ satisfying $\xi_1=1_X$ and $\xi_{hg}=\xi_g\xi_h$ (having in mind that $h(\xi_g)=\xi_g$, since $h(-):\ (E\text{-}\mathbf{Mod})^{\mathrm{op}}\to(E\text{-}\mathbf{Mod})^{\mathrm{op}}$ is the restriction-of-scalars functor corresponding to h). Since the E-module structure on D_g is given by (4.2), it is easy to see that to say that $\xi_g:\ D\to D_g$ is an E-module isomorphism is to say that $\xi_g(ed)=(eg)\xi_g(d)$ for all $e\in E$, $d\in D$, and $g\in G$. After that we put $dg=\xi_g(d)$, and this gives the desired description of the category $Z^1(G,(E\text{-}\mathbf{Mod})^{\mathrm{op}})^{\mathrm{op}}$. It remains to prove (a) and (b).

(a) By Theorem 3.6(b), the 1-cocycle $G\to(E\text{-}\mathbf{Mod})^{\mathrm{op}}$ corresponding to a B-module A, is the pair $(E\otimes_B A,(\eta_{g(E\otimes_B A))}(E\otimes_B\varepsilon_A))_{g\in G})$. Using the explicit formulae for ε and η given in the description of the adjunction (4.1), we have

$$(\eta_{g(E\otimes_B A))}(E\otimes_B\varepsilon_A))(e\otimes a)=\eta_{g(E\otimes_B A))}(e\otimes(1\otimes a))=e(1\otimes a)=eg\otimes a,$$

where the last equality follows from the fact that $e(1\otimes a)$ is calculated in $g(E\otimes_B A)$. According to our way of defining a right G-action out of a 1-cocycle $G\to(E\text{-}\mathbf{Mod})^{\mathrm{op}}$ this gives

$$(e\otimes a)g=(\eta_{g(E\otimes_B A))}(E\otimes_B\varepsilon_A))(e\otimes a)=eg\otimes a,$$

as desired.

(b) now follows immediately from Theorem 3.6(b). ∎

4.4. Theorem. *Suppose the equivalent conditions of Theorem 4.1 hold, and A_0 is a fixed B-module. Then the full subcategory of the category of B-modules*

$$B\text{-}\mathbf{Mod}[A_0]=\{A\in B\text{-}\mathbf{Mod}\mid E\otimes_B A\cong E\otimes_B A_{[0]}\text{ in } E\text{-}\mathbf{Mod}\}$$

is equivalent to the category $Z^1(G,\mathrm{End}(E\otimes_B A_0)^{\mathrm{op}}{}_{\xi_0})^{\mathrm{op}}$, where $(E\otimes_B A_0,\xi_0):\ G\to(E\text{-}\mathbf{Mod})^{\mathrm{op}}$ is the 1-cocycle, corresponding to A_0, which makes $\mathrm{End}(E\otimes_B A_0)^{\mathrm{op}}$ a one-object G-category, as in Observation 1.3. Moreover:

(a) The G-category constructed above on $\mathrm{End}(E\otimes_B A_0)^{\mathrm{op}}$, written as

$$G\times\mathrm{End}(E\otimes_B A_0)^{\mathrm{op}}\to\mathrm{End}(E\otimes_B A_0)^{\mathrm{op}},\quad (g,u)\mapsto gu,$$

has, for each $g \in G$, the E-module isomorphism $g(u):\ E \otimes_B A_0 \to E \otimes_B A_0$ defined by

$$(gu)(t) = (u(tg))g^{-1} \qquad (t \in E \otimes_B A_0), \tag{4.5}$$

using the right action of G on $E \otimes_B A_0$ described in 4.3 (a).

(b) In particular, we have a bijection between B-$\mathbf{Mod}[A_0]/\cong$ and $H^1(G, \mathrm{Aut}(E \otimes_B A_0)^{\mathrm{op}}$, where B-$\mathbf{Mod}[A_0]/\cong$ is the set of isomorphism classes of objects of the category B-$\mathbf{Mod}[A_0]$ and the action of G on $\mathrm{Aut}(E \otimes_B A_0)^{\mathrm{op}}$ is defined by (4.5).

Proof. The first assertion of the theorem is really a direct translation of Theorem 3.7 and the same is true for the assertion 4.4(b) provided we can use 4.4(a). Therefore we only need to prove 4.4(a). For, we observe:

(i) As we see from the proof of 4.3 (a), $\xi_0 = (\xi_{0g})_{g \in G}$, where $\xi_{0g}:\ E \otimes_B A_0 \to (E \otimes_B A_0)_g$ is defined by $\xi_{0g}(e \otimes a) = eg \otimes a$.

(ii) Recall the formula $ga = \xi_g g(a)(\xi_g)^{-1}$ from Observation 1.3. It tells us that gu in 4.4 (a) is defined by $gu = \xi_{0g} u (\xi_{0g})^{-1}$ (since $g(-):\ (E\text{-}\mathbf{Mod})^{\mathrm{op}} \to (E\text{-}\mathbf{Mod})^{\mathrm{op}}$ is the dual restriction-of-scalars functor, we write u instead of $g(u)$).

(iii) $eg \otimes a)$ above is the same as $(e \otimes a)g$ in $E \otimes_B A_0$ (on which G acts via B-module automorphisms as described in 4.3(a)).

(4.5) follows from these three observations. ■

4.5. Remark. The following obvious reformulations are useful:

(a) Formula (4.5) implies that $(gu)(1 \otimes a) = (u(1 \otimes a))g^{-1}$, for each $a \in A_0$ - which determines gu since it is an E-module homomorphism and $E \otimes_B A_0$ is generated by the elements of the form $1 \otimes a$ ($a \in A_0$) as an E-module.

(b) Since $\mathrm{Aut}(E \otimes_B A_0)^{\mathrm{op}}$ and $\mathrm{Aut}(E \otimes_B A_0)$ are isomorphic (by sending automorphisms to their inverses), 4.4(b) can be restated as a bijective correspondence between B-$\mathbf{Mod}[A_0]/\cong$ and $H^1(G, \mathrm{Aut}(E \otimes_B A_0)$, where the action of G on $\mathrm{Aut}(E \otimes_B A_0)$ is defined by (4.5).

4.6. Remark. If we are not interested in the cohomological side of the story, the first assertion of Theorem 4.3 reduces to:

(a) Under the equivalent conditions of Theorem 4.1, the category B-$\mathbf{Mod}$ is equivalent to the category of E-modules D on which G acts on the right via B-module automorphisms with $(ed)g = (eg)(dg)$ for all $e \in E$, $d \in D$, and $g \in G$.

After that 4.3(a) implies that:

(b) Under the equivalent conditions of Theorem 4.1, the category B-$\mathbf{Mod}[A_0]$ is equivalent to the category defined as follows:

- the objects are right actions of G on $E \otimes_B A_0$ via B-module automorphisms with

$$(et)g = (eg)(tg),\ \text{for all } e \in E,\ t \in E \otimes_B A_0,\ \text{and } g \in G\}; \tag{4.6}$$

- the morphisms are E-module homomorphisms that preserve G-action.

This suggests to ask if Theorem 4.4 provides a better (non-cohomological) description of the category B-$\mathbf{Mod}[A_0]$? It is easy to check that the direct translation of the first two assertions of Theorem 4.4 gives:

(c) Under the equivalent conditions of Theorem 4.1, the category B-$\mathbf{Mod}[A_0]$ is equivalent to the category defined as follows:
• the objects are right actions $*$ of G on $E \otimes_B A_0$ via E-module automorphisms, which together with the fixed right G-action defined as in 4.3(a), satisfy

$$t * hg = ((t * h)h * g)h^{-1} \text{ forall } t \in E \otimes_B A_0, \text{ and } g, h \in G; \tag{4.7}$$

• the morphisms are the E-module homomorphisms that preserve the G-action $*$.

Indeed, using (1.3) with (1.4), and denoting $\varphi(g)(t)$ by $t * g$, we have

$$\begin{aligned} t * hg = \varphi(hg)(t) &= ((h\varphi(g))(\varphi(h)))(t) = (h\varphi(g))((\varphi(h)(t)) \\ &= (h\varphi(g))(t * h) = ((\varphi(g)((t * h)h)))h^{-1} = ((t * h)h * g)h^{-1}, \end{aligned}$$

where the reason of writing $(h\varphi(g))(\varphi(h))$ instead of $(\varphi(h))(h\varphi(g))$ is that we are applying (1.3) to 1-cocycles $G \to \mathrm{End}(E \otimes_B A_0)^{\mathrm{op}}$.
In comparison with (b), the advantage of (c) is that the variable G-actions are actions via E-module automorphisms, although formula (4.7) looks complicated unless it is interpreted cohomologically.

Again, all these calculations and results can be copied for various kinds of algebras (see Remark 4.2).

Galois cohomology of commutative rings.

4.7. Theorem. *With notation as above, suppose conditions 4.1(b) and 4.1(c) hold. Then the following conditions are equivalent:*

(a) condition 4.1(a);

(b) p is injective and makes E a finitely generated projective B-module;

(c) p induces an isomorphism $B \cong E^G$.

Proof. $(a) \Rightarrow (c)$: Under the category equivalence described in Theorem 4.3, the B-module E corresponds to the E-module $E \otimes_B B \cong E$, on which G acts as originally on E (see 4.3(a)), and then 4.7(c) follows from 4.3(b). $(c) \Rightarrow (b)$ follows from the results of [Chase, Harrison, Rosenberg, 1965] (see also [Chase, Sweedler, 1969] and [DeMeyer, Ingraham, 1971]). $(b) \Rightarrow (a)$ is well-known, and can be proved easily as follows. E is finitely generated and projective and faithful as a B-module, hence it is a B-progenerator (see for example [DeMeyer, Ingraham, 1971, Corollary I.1.10]), and a fortiori faithfully flat as a B-module. We need to show that, for any B-module N, the map $i_N : N \to N \otimes_B E$, $i_N(n) = n \otimes_B 1_E$ is injective. It suffices to show that $i_N \otimes_B E$ is injective. If $x = \sum_i n_i \otimes_B e_i \in \mathrm{Ker}(i_N \otimes_B E)$, then $\sum_i n_i \otimes_B 1_E \otimes_B e_i = 0$, and, multiplying the second and third tensor factor, we find that $x = 0$. ∎

From the results of [Chase, Harrison, Rosenberg, 1965] we also conclude:

4.8. Corollary. *The equivalent conditions of Theorem 4.1 hold if and only if $p : B \to E$ is a Galois extension with Galois group G in the sense of [Chase, Harrison, Rosenberg, 1965].*

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