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STRONGLY SEPARABLE MORPHISMS IN GENERAL CATEGORIES

Dedicated to Dominique Bourn on the occasion of his sixtieth birthday

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Abstract. We clarify the relationship between separable and covering morphisms in general categories by introducing and studying an intermediate class of morphisms that we call strongly separable.

1. Introduction

The so-called *separable Galois theory* of commutative rings is closely related to Grothendieck's *Galois theory of coverings of schemes* in algebraic geometry. In fact, both of them are special cases of the *purely-categorical Galois theory*, the details of which are explained in [BJ] and [J2], with many references to the relevant literature on Galois theory (see especially [AG], [CHR], [Janusz], [G], [DI], [VZ1], [VZ2], and [M]). We recall here in particular the following fundamental notion:

1.1. Definition.

(a) *Let R be a commutative ring. A commutative R-algebra S is said to be separable if it is projective as an $S \otimes_R S$-module.*

(b) *A commutative separable R-algebra is said to be strongly separable if it is projective as an R-module.*

(All rings and algebras in this paper are supposed to be with 1; many authors also require $1 \neq 0$, but it is better not to do so.)

It is known that a commutative separable R-algebra is strongly separable if, and only if, the structure homomorphism $R \to S$, considered as a morphism $S \to R$ of the corresponding affine schemes, is a covering morphism. In other words, the projectivity condition of Definition 1.1(b), which becomes trivial in the classical case of fields, is exactly what one needs to add to the definition of separable algebra in order to make

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it equivalent to the definition of covering morphism (of affine schemes). Now, Definition 1.1(a) has a straightforward categorical reformulation, first introduced and studied in [CJ] for lextensive categories, and then in [JT1] for arbitrary categories equipped with an *admissible Galois structure*, or simply with an endofunctor satisfying suitable mild conditions. Furthermore, the categorical notion of covering (see e.g. [J2] and references there) plays a central role in categorical Galois theory. Therefore it is natural to search for a purely-categorical condition which, when added to the definition of separable morphism, will again make it equivalent to the definition of covering morphism — which in fact was done in [CJ] in the special case of lextensive categories.

The purpose of the present paper is to present such a condition. More precisely, we introduce a notion of strongly separable morphism and prove the following (see Theorems 4.3 and 4.5):

- A strongly separable morphism is a covering if and only if it satisfies a certain pullback-stability condition (which partly generalizes Theorem 25 of [CJ]);
- In the context of [CJ] and in the presence of a suitable Galois structure, a covering morphism is always strongly separable. This statement uses the corresponding separabililty result from [CJ] and cannot be extended to the general context, since we gave an example of a non-separable covering in [JT1].

Unfortunately we do not know if our categorical notion of strongly separable fully agrees with Definition 1.1(b). We could avoid this uncertainty by adding the above-mentioned pullback-stability condition to our definition of a strongly separable morphism, but that does not seem to be a good idea, particularly because the definition chosen here yields the desirable "equation"

$$\text{strongly separable} = \mathcal{F}\text{-separated} + \mathcal{F}\text{-compact (Proposition 3.2)},$$

in the sense of [T] for a suitable $\mathcal{F}$. Still, the above-mentioned additional condition is itself "Galois-theoretic", which is less visible in the special case already considered in [CJ], but becomes clearly motivated by the results of [CJKP] and [JK] in the general case we consider here.

The conclusion seems to be that, although the algebraic and categorical definitions of separable and covering morphisms agree, the algebraic and categorical ways to compare them may differ. Note, however, that the categorical approach uses effective descent morphisms, and, hence, suggests (see e.g. [JT2]) that "projective as R-module" be replaced by "makes $R \to S$ a pure monomorphism of R-modules" in Definition 1.1(b).

2. Separable = separated

Let $(I, H, \eta, \varepsilon) : \mathbf{C} \to \mathbf{X}$ be a fixed adjunction between categories with finite limits, B an object in $\mathbf{C}$ and $(I^B, H^B, \eta^B, \varepsilon^B) : (\mathbf{C} \downarrow B) \to (\mathbf{X} \downarrow I(B))$ the induced adjunction, in which $I^B(A,\alpha) = (I(A), I(\alpha))$ and $H^B(X,\varphi) = (B \times_{HI(B)} H(X), \mathsf{pr}_1)$. We will always assume that the adjunction $(I, H, \eta, \varepsilon)$ is admissible in the sense of categorical Galois theory (see [J1, BJ, J2], and references there), which calls for the co-units $\varepsilon^B : I^B H^B \to 1_{(\mathbf{X}\downarrow I(B))}$ to be isomorphisms, for every object B; for simplicity, we will also assume that $\varepsilon : IH \to 1_{\mathbf{X}}$ itself is an isomorphism, in fact the identity transformation on a full subcategory $\mathbf{X}$ in $\mathbf{C}$, with $H : \mathbf{X} \to \mathbf{C}$ the inclusion functor. In particular, we will write

$$H^B(X,\varphi) = (B \times_{I(B)} X, \mathsf{pr}_1). \tag{2.1}$$

Accordingly, it is convenient to identify the composite HI with I, and consider (I, η) as a pointed endofunctor of $\mathbf{C}$, but we note that many of the definitions and results below can be formulated for more general pointed endofunctors of $\mathbf{C}$ as in [JT1]. We also note that our assumptions may be expressed equivalently by saying that ($\mathbf{X}$ is a full reflective subcategory in $\mathbf{C}$ and) the reflection $I : \mathbf{C} \to \mathbf{X}$ is *semi-left-exact* in the sense of C. Cassidy, M. Hébert, and G.M. Kelly [CHK].

Let us recall (see, for example, [J2]):

2.1. Definition. *A morphism $\alpha : A \to B$ (or an object (A,α) in $(\mathbf{C} \downarrow B)$) in $\mathbf{C}$ is said to be*

(a) a trivial covering (of B), if it satisfies the following equivalent conditions:

(i) the morphism $\eta^B_{(A,\alpha)} : (A,\alpha) \to H^B I^B(A,\alpha)$ is an isomorphism,

(ii) $(A,\alpha) \cong H^B(X,\varphi)$ for some $\varphi : X \to I(B)$,

(iii) the diagram

$$\begin{tikzcd} A \arrow[r, "\eta^A"] \arrow[d, "\alpha"'] & I(A) \arrow[d, "I(\alpha)"] \\ B \arrow[r, "\eta^B"'] & I(B) \end{tikzcd} \tag{2.2}$$

is a pullback;

(b) a covering, if there exists an effective descent morphism $p : E \to B$ such that (A,α) is split over (E,p), i.e. the image of (A,α) under the pullback functor $p^* : (\mathbf{C} \downarrow B) \to (\mathbf{C} \downarrow E)$ is a trivial covering of E;

(c) separable, if the diagonal $\Delta_\alpha = \langle 1_A, 1_A \rangle : A \to A \times_B A$ is a trivial covering.

The separable morphisms here are the same as the T-separable morphisms in the sense of [JT1], where $T = HI$; this notion of separability is essentially far more general than the separability in the sense of [CJ] (see Example 2.5). (The preservation of the terminal object by T required in [JT1] is equivalent to the same property of I; we circumvented this condition here by requiring not only the local co-units ε^B but also ε itself to be isomorphisms.) From the results of [JT1] we obtain:

2.2. Proposition. *The class of separable morphisms in* $\mathbf{C}$ *contains all monomorphisms and all trivial coverings; it is closed under composition and all limits in the arrow category of* $\mathbf{C}$*, and it is left cancelable (i.e.* $\beta\alpha$ *separable implies* α *separable), and hence pullback stable.*

The following observation is not mentioned in [JT1] explicitly, but can be deduced from the results there:

2.3. Proposition. *If* $\alpha : A \to B$ *is a split monomorphism, then it is a trivial covering if and only if it can be presented as the equalizer of two parallel morphisms into an object in* $\mathbf{X}$.

Proof. Presenting α as the equalizer of two parallel morphisms into an object X is the same as presenting it as a pullback of the form

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle \Delta_X = \langle 1_X, 1_X\rangle} \\ B & \longrightarrow & X \times X \end{array} \tag{2.3}$$

Since every morphism in $\mathbf{X}$ is a trivial covering, and the class of trivial coverings is obviously pullback stable, the possibility of such a presentation (with X in $\mathbf{X}$) implies that α is a trivial covering. Conversely, if α is a trivial covering, then it is a pullback of $I(\alpha)$, which is a regular monomorphism in $\mathbf{X}$ (because α was supposed to be a split monomorphism, and hence so is $I(\alpha)$). And since every regular monomorphism in $\mathbf{X}$ has a presentation as needed, it now suffices to just compose the two pullbacks in the diagram

$$\begin{array}{ccccc} A & \longrightarrow & I(A) & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ B & \longrightarrow & I(B) & \longrightarrow & X \times X\ . \end{array}$$

∎

2.4. Corollary. *A morphism* $\alpha : A \to B$ *is separable if and only if it is separated, i.e. there exists an equalizer diagram of the form*

$$A \xrightarrow{\Delta_\alpha} A \times_B A \rightrightarrows X \tag{2.4}$$

with X *in* $\mathbf{X}$.

Let us also recall some important examples of separable morphisms:

2.5. Example. A morphism $\alpha : A \to B$ in a lextensive category $\mathbf{C}$ is called separable (=*decidable*) in [CJ] if the diagonal $\Delta_\alpha : A \to A \times_B A$ is a coproduct injection. This definition does not involve $\mathbf{X}$ and hence is not exactly a special case of the present one. However there are (at least) two reasonable contexts where they agree and apply for many *geometric* applications :

(a) Let $\mathbf{C} = \mathsf{Fam}(\mathbf{A})$ be the category of families (="free coproduct completion") of objects in a category $\mathbf{A}$ such that $\mathbf{C}$ has pullbacks, and $\mathbf{A}$ has a terminal object. Since the category of sets can be identified with $\mathsf{Fam}(\mathbf{1})$, and $\mathsf{Fam}(-)$ is a 2-functor, the adjunction $\mathbf{A} \to \mathbf{1}$ induces an admissible adjunction $\mathbf{C} \to \mathbf{Sets}$ (see [BJ] or [J2] for details). It is then easy to see that the separable morphisms in the sense of our definition above are the same as the separable morphisms in $\mathbf{C}$ in the sense of [CJ]. In particular $\mathbf{C}$ could be any locally connected (=molecular) topos; then the separable morphisms are precisely the decidable ones in the topos-theoretic sense.

(b) The same can be said about finite families and finite sets. The notion of locally connected topos is then also to be replaced with its appropriate finite version, which for instance will include all toposes with finite hom-sets.

2.6. Example. Let R be a commutative ring and S a commutative R-algebra (both with 1 preserved by the structure homomorphism $R \to S$). The following conditions are well-known to be equivalent (see e.g. [DI]):

(a) S is a separable R-algebra (see Definition 1.1);

(b) the multiplication map $\mu_S : S \otimes_R S \to S$ is a split epimorphism of $S \otimes_R S$-modules;

(c) there exists an idempotent e in $S \otimes_R S$ with $\mu_S(e) = 1$;

(d) $\mu_S : S \otimes_R S \to S$ is a product projection in the category of (commutative) rings;

(e) the structure homomorphism $R \to S$ considered as a morphism $S \to R$ in the opposite category of commutative rings, is separable in the sense of [CJ].

Let $\mathbf{C}$ be a full subcategory of the opposite category $(\mathbf{Comm}\text{-}R\text{-}\mathbf{alg})^{\mathrm{op}}$ of commutative R-algebras (with 1) satisfying the following conditions:

(i) it is closed under pullbacks in $(\mathbf{Comm}\text{-}R\text{-}\mathbf{alg})^{\mathrm{op}}$;

(ii) the set of idempotents in any S in $\mathbf{C}$ is finite;

(iii) if $S \cong S_1 \times S_2$ in $\mathbf{Comm}\text{-}R\text{-}\mathbf{alg}$, then S is in $\mathbf{C}$ if and only if so are S_1 and S_2.

When R is *connected*, i.e. has no non-trivial idempotents, we can use Example 2.5(b) to construct an adjunction $\mathbf{C} \to \mathbf{Finite\ Sets}$, and a morphism $\alpha : A \to B$ in $\mathbf{C}$ will be separable if and only if A is a separable B-algebra. In particular we could take R to be a field and $\mathbf{C}$ the opposite category of commutative R-algebras (with 1 again) that are finite-dimensional as vector spaces over R. Recall that an R-algebra S would then be separable if and only if it is a finite product of finite separable field extensions of R.

2.7. Example. Let $\mathbf{C}$ be an elementary topos, j a Lawvere-Tierney topology on $\mathbf{C}$, $\mathbf{X} = \mathsf{shv}_j(\mathbf{C})$ the corresponding category of internal sheaves, $H : \mathbf{X} \to \mathbf{C}$ the inclusion functor, $I : \mathbf{C} \to \mathbf{X}$ the sheafification functor, and A an object in $\mathbf{C}$. Then the following conditions are equivalent:

(a) $A \to 1$ is a separable morphism;

(b) A is an internal separated presheaf in the topos-theoretic sense.

3. Strongly separable morphisms

Let $(I, H, \eta, \varepsilon) : \mathbf{C} \to \mathbf{X}$ be as in Section 2. Since it is a semi-left-exact reflection in the sense of [CHK], it produces a corresponding *reflective factorization system* $(\mathcal{E}, \mathcal{M})$ in $\mathbf{C}$. As it was later observed in [CJKP] and in [JK], this construction from [CHK] plays an important role in Galois theory; in particular, $\mathcal{M}$ coincides with the class of trivial covering morphisms. At the moment we just need to recall that for a morphism $\alpha : A \to B$ in $\mathbf{C}$, the canonical factorization $\alpha = m_\alpha e_\alpha$ is given by the commutative diagram

$$\begin{tikzcd}
A \arrow[dr, "e_\alpha"] \arrow[drr, "\eta_A"] \arrow[ddr, "\alpha"'] & & \\
& B \times_{I(B)} I(A) \arrow[r, "\mathsf{pr}_2"'] \arrow[d, "m_\alpha = \mathsf{pr}_1"] & I(A) \arrow[d, "I(\alpha)"] \\
& B \arrow[r, "\eta_B"'] & I(B)
\end{tikzcd} \tag{3.1}$$

We can now introduce the following definition:

3.1. Definition. *A morphism $\alpha : A \to B$ in $\mathbf{C}$ is said to be strongly separable if it is separable and, for every pullback α' of α, the morphism $e_{\alpha'}$ is an effective descent morphism.*

In order to describe the basic properties of the class of strongly separable morphisms, it is convenient to introduce operators Sep, St, Loc, $-\#\mathbf{X}$ as follows: if $\mathcal{F}$ is a class of morphisms in $\mathbf{C}$, then

- $\mathsf{St}(\mathcal{F})$ is the stabilization of $\mathcal{F}$, i.e. the class of all morphisms in $\mathbf{C}$ each pullback of which is in $\mathcal{F}$; in [T] this class is denoted by $\mathrm{c}(\mathcal{F})$ and its elements are called $\mathcal{F}$-compact morphisms.

- $\mathsf{Sep}(\mathcal{F})$ is the class of $\mathcal{F}$-separable (=$\mathcal{F}$-separated) morphisms, i.e. those morphisms $\alpha : A \to B$ in $\mathbf{C}$ for which $\Delta_\alpha : A \to A \times_B A$ is in $\mathsf{St}(\mathcal{F})$; in [T] this class is denoted by $\mathrm{d}(\mathcal{F})$ and its elements are also called $\mathcal{F}$-separated morphisms.

- $\mathsf{Loc}(\mathcal{F})$ is the localization of $\mathcal{F}$, i.e. the class of all morphisms in $\mathbf{C}$ some pullback of which along an effective descent morphism is in $\mathcal{F}$.

- $\mathcal{F}\#\mathbf{X}$ is the class of morphisms in $\mathbf{C}$ which are in $\mathcal{F}$ up to $\mathbf{X}$, i.e. the class of those α's which have e_α in $\mathcal{F}$.

We will apply these operators to the following two classes, each of which contains all isomorphisms, is closed under composition, and is pullback stable:

- $\mathsf{TrivCov}(\mathbf{C})$, the class of trivial coverings in $\mathbf{C}$;
- $\mathsf{EffDes}(\mathbf{C})$, the class of effective descent morphisms in $\mathbf{C}$.

According to this notation we have

- $\mathsf{St}(\mathsf{TrivCov}(\mathbf{C})) = \mathsf{TrivCov}(\mathbf{C})$ and $\mathsf{St}(\mathsf{EffDes}(\mathbf{C})) = \mathsf{EffDes}(\mathbf{C})$;
- $\mathsf{Sep}(\mathsf{TrivCov}(\mathbf{C}))$ is the class of separable morphisms in $\mathbf{C}$;
- $\mathsf{Sep}(\mathsf{TrivCov}(\mathbf{C})) \cap \mathsf{St}(\mathsf{EffDes}(\mathbf{C})\#\mathbf{X})$ is the class of strongly separable morphisms in $\mathbf{C}$;
- $\mathsf{Loc}(\mathsf{TrivCov}(\mathbf{C}))$ is the class of covering morphisms in $\mathbf{C}$.

Note also that, since a monomorphism is an effective descent morphism if and only if it is an isomorphism, we have $\mathsf{Sep}(\mathsf{TrivCov}(\mathbf{C})) = \mathsf{Sep}(\mathsf{EffDes}(\mathbf{C})\#\mathbf{X})$, and we deduce:

3.2. PROPOSITION. *A morphism in $\mathbf{C}$ is strongly separable if and only if it is $\mathsf{EffDes}(\mathbf{C})\#\mathbf{X}$-separated and $\mathsf{EffDes}(\mathbf{C})\#\mathbf{X}$-compact in the sense of [T].*

It is easy to show that the class $\mathsf{EffDes}(\mathbf{C})\#\mathbf{X}$ is closed under composition (and of course contains all isomorphisms), and so we can use various results of section 3 of [T] on $\mathcal{F}$-separated and $\mathcal{F}$-compact morphisms for $\mathcal{F} = \mathsf{EffDes}(\mathbf{C})\#\mathbf{X}$. In particular, we obtain:

3.3. PROPOSITION. *The class of strongly separable morphisms in $\mathbf{C}$ contains all isomorphisms and all trivial coverings, is closed under composition and pullback stable, and has the following cancellation property: if $\beta\alpha$ is strongly separable and β separable then α is separable.*

It can also be proved that if $\beta\alpha$ is strongly separable, β separable, and $I(\alpha)$ an effective descent morphism (in $\mathbf{C}$), then β is strongly separable.

4. Locally stable factorization and main theorems

Let $(\mathcal{E}, \mathcal{M})$ be the factorization system on $\mathbf{C}$ used above; that is

$$\mathcal{E} = \{e_\alpha \mid \alpha : A \to B \text{ in } \mathbf{C}\} = \{\alpha \mid I(\alpha) \text{ is an isomorphism}\},$$

$$\mathcal{M} = \{m_\alpha \mid \alpha : A \to B \text{ in } \mathbf{C}\} = \mathsf{TrivCov}(\mathbf{C}).$$

Note that according to the notation above, $\mathcal{E}\#\mathbf{X}$ is the class of all morphisms in $\mathbf{C}$, and $\mathcal{M}\#\mathbf{X} = \mathcal{M}$.

4.1. DEFINITION. *We say that a morphism $\alpha : A \to B$ in $\mathbf{C}$ has*

(a) a stable factorization if it belongs to $\mathsf{St}(\mathcal{E})\#\mathbf{X}$, i.e. if not just e_α itself but every pullback of it is in $\mathcal{E}$;

(b) a locally stable factorization if it belongs to $\mathsf{Loc}(\mathsf{St}(\mathcal{E})\#\mathbf{X})$, i.e. there exists an effective descent morphism $p : E \to B$ in $\mathbf{C}$ such that the pullback $\mathsf{pr}_1 : E \times_B A \to E$ of α along p has a stable factorization.

As shown in [CJKP], the (*Purely inseparable*, *Separable*) factorization for finite-dimensional field extensions and the (*Monotone*, *Light*) factorization for continuous maps of compact Hausdorff spaces are instances of factorization systems of the form $(\mathsf{St}(\mathcal{E}), \mathsf{Loc}(\mathcal{M}))$; the existence theorem for those factorization systems proved in [CJKP] can be formulated as follows:

4.2. THEOREM. *The following conditions are equivalent:*

(a) $(\mathsf{St}(\mathcal{E}), \mathsf{Loc}(\mathcal{M}))$ is a factorization system;

(b) every morphism in $\mathbf{C}$ has a locally stable factorization.

In the next section we will prove a number of technical results in order to deduce:

4.3. THEOREM. *A strongly separable morphism is*

(a) a trivial covering if and only if it has a stable factorization;

(b) a covering if and only if it has a locally stable factorization.

In particular, we obtain:

4.4. COROLLARY. *If the equivalent conditions of Theorem 4.2 hold, then every strongly separable morphism in $\mathbf{C}$ is a covering.*

An example of a non-separable covering morphism is given in [JT1]; however, in Section 6 we will prove:

4.5. THEOREM. *In the situation considered in Example 2.5, every covering morphism is strongly separable.*

5. Proof of Theorem 4.3

5.1. LEMMA. *In a commutative diagram of the form*

$$\begin{tikzcd}
D \arrow[rrr, "q"] \arrow[ddd, "\delta"'] \arrow[dr, "e_\delta"] & & & A \arrow[dl, "e_\alpha"'] \arrow[ddd, "\alpha"] \\
& D' \arrow[r, "q'"] \arrow[dl, "m_\delta"] & A' \arrow[dr, "m_\alpha"'] & \\
& & & \\
E \arrow[rrr, "p"'] & & & B
\end{tikzcd} \tag{5.1}$$

with m_δ and q' jointly monic, the following implication holds for any pair d_1, d_2 of parallel morphisms with codomain D:

$$(\delta d_1 = \delta d_2 \ \textit{ and } \ \eta_A q d_1 = \eta_A q d_2) \Rightarrow (\eta_D d_1 = \eta_D d_2).$$

Proof. As we see from diagram (3.1), for every pair v_1, v_2 of parallel morphisms with codomain A we have $(\alpha v_1 = \alpha v_2$ and $\eta_A v_1 = \eta_A v_2) \Leftrightarrow (e_\alpha v_1 = e_\alpha v_2)$; similarly we have $(\delta d_1 = \delta d_2$ and $\eta_D d_1 = \eta_D d_2) \Leftrightarrow (e_\delta d_1 = e_\delta d_2)$. After that we observe:

(i) since $\alpha q = p\delta$, the equality $\delta d_1 = \delta d_2$ implies the equality $\alpha q d_1 = \alpha q d_2$;

(ii) therefore assuming $\delta d_1 = \delta d_2$ and $\eta_A q d_1 = \eta_A q d_2$, we obtain $e_\alpha q d_1 = e_\alpha q d_2$;

(iii) since $q' e_\delta = e_\alpha q$, this also gives $q' e_\delta d_1 = q' e_\delta d_2$;

(iv) moreover, since $\delta = m_\delta e_\delta$ and m_δ and q' are jointly monic, (iii) tells us that our assumption implies $e_\delta d_1 = e_\delta d_2$;

(v) since $e_\delta d_1 = e_\delta d_2$ implies $\eta_D d_1 = \eta_D d_2$, this completes the proof.

∎

5.2. Lemma. *If $\alpha : A \to B$ is separable and has a stable factorization, then it is* dissonant, *i.e. e_α is a monomorphism.*

Proof. Given $a_1, a_2 : T \to A$ with $e_\alpha a_1 = e_\alpha a_2$, consider the diagrams

$$\begin{tikzcd} T \arrow[r, "a_i"] \arrow[d, "\langle a_i, 1_T \rangle"'] & A \arrow[r, "\eta_A"] \arrow[d, "\Delta_\alpha"'] & I(A) \arrow[d, "I(\Delta_\alpha)"] \\ A \times_B T \arrow[r, "1_A \times a_i"'] & A \times_B A \arrow[r, "\eta_{A \times_B A}"'] & I(A \times_B A) \end{tikzcd} , \tag{5.2}$$

where $A \times_B T$ is the pullback of α and $\alpha a_i (i = 1, 2)$. If we knew that

$$\eta_{A \times_B A}(1_A \times a_1) = \eta_{A \times_B A}(1_A \times a_2), \tag{5.3}$$

then, since both squares in (5.2) are pullbacks, we could conclude that $\langle a_1, 1_T \rangle$ and $\langle a_2, 1_T \rangle$ represent the same subobject of $A \times_B T$ and so $a_1 = a_2$ as desired. Therefore we only need to prove (5.3). But this follows from Lemma 5.1 applied to

- $E = A$;
- $p = \alpha$;
- $D = A \times_B A$ with δ and q the pullback projections;
- $d_i = 1_A \times a_i : A \times_B T \to A \times_B A \ (i = 1, 2)$,

where $\delta d_1 = \delta d_2$ becomes trivial and $\eta_A q d_1 = \eta_A q d_2$ follows from our assumption $e_\alpha a_1 = e_\alpha a_2$; the fact that m_δ and q' are jointly monic follows from the stability of the factorization $\alpha = m_\alpha e_\alpha$. ∎

Now we are ready to complete the proof of Theorem 4.3:

Since $\mathsf{St}(\mathcal{E})$ contains all isomorphisms, every trivial covering has a stable factorization and every covering has a locally stable factorization. That is, the "only if" parts of 4.3(a) and 4.3(b) are trivial.

If α is strongly separable, then e_α is an effective descent morphism, and so it is an isomorphism if (and only if) it is a monomorphism. Therefore 4.3(a) follows from Lemma 5.2. On the other hand since the class of strongly separable morphisms is pullback stable, 4.3(b) immediately follows from 4.3(a).

5.3. Remark. Let us say that α *has a weakly stable factorization* , if for every morphism α' obtained from α by pulling back along any morphism p (with the same codomain), the canonical morphism from the domain of $m_{\alpha'}$ to the domain of the pullback of m_α along p is a monomorphism. As we see from the proof above, whose main ingredient was the assumption that m_δ and q' are jointly monic in Lemma 5.1, we could extend the statement of 4.3(a) by: "and if and only if it has a weakly stable factorization". The same is of course true for 4.3(b) with the obvious notion of locally weakly stable factorization.

6. Four lemmas on effective descent morphisms in lextensive categories

In this section we assume that $\mathbf{C}$ is (infinitary) lextensive, i.e. it admits finite limits and arbitrary (small) coproducts, and for every family $(C_\lambda)_{\lambda\in\Lambda}$ of objects in $\mathbf{C}$ the coproduct functor

$$\sum : \prod_{\lambda\in\Lambda}(\mathbf{C}\downarrow C_\lambda) \to (\mathbf{C}\downarrow \sum_{\lambda\in\Lambda} C_\lambda) \tag{6.1}$$

is a category equivalence.

6.1. Lemma. *Let $p_\lambda : E_\lambda \to B_\lambda$ $(\lambda\in\Lambda)$ be a family of morphisms in $\mathbf{C}$. The morphism $\sum_{\lambda\in\Lambda} p_\lambda : \sum_{\lambda\in\Lambda} E_\lambda \to \sum_{\lambda\in\Lambda} B_\lambda$ is an effective descent morphism if and only if so are all the p_λ.*

Proof. Just use the equivalence (6.1). ∎

6.2. Lemma. *Let $p_\lambda : E_\lambda \to B_\lambda$ $(\lambda\in\Lambda)$ be a family of morphisms in $\mathbf{C}$. If at least one p_λ is an effective descent morphism then the induced morphism $\sum_{\lambda\in\Lambda} E_\lambda \to B$ is also an effective descent morphism.*

Proof. Choose an index $\mu\in\Lambda$ for which p_μ is an effective descent morphism, and let $q_\lambda : E_\lambda \to B_\lambda$ $(\lambda\in\Lambda)$ be the family of morphisms in which $q_\mu = p_\mu$ and $q_\lambda : E_\lambda \to B_\lambda$ is the identity morphism of E_λ for all $\lambda \neq \mu$. Then $\sum_{\lambda\in\Lambda} E_\lambda \to B$ is the composite of $\sum_{\lambda\in\Lambda} q_\lambda : \sum_{\lambda\in\Lambda} E_\lambda \to \sum_{\lambda\in\Lambda} B_\lambda$, which is an effective descent morphism by Lemma 6.1, and the split epimorphism $\sum_{\lambda\in\Lambda} B_\lambda \to B$ induced by the identity morphism of $B_\mu = B$ and all $p_\lambda : B_\lambda = E_\lambda \to B$ $(\lambda\neq\mu)$. Therefore $\sum_{\lambda\in\Lambda} E_\lambda \to B$ itself is an effective descent morphism. ∎

6.3. LEMMA. *Let* $p_\lambda : E_\lambda \to B$ $(\lambda = 1, 2)$ *be two morphisms making the induced morphism* $p : E_1 + E_2 \to B$ *an effective descent morphism. If* B *is connected (=indecomposable into a non-trivial coproduct) and* $E_1 \neq 0 \neq E_2$*, then* $E_1 \times_B E_2 \neq 0$.

PROOF. It is easy to see that if $E_1 \times_B E_2 = 0$, then each of the coproduct injections $E_\lambda \to E_1 + E_2$ determines a descent datum over p. Therefore, in that case, descending along p would give a coproduct decomposition for B which is non-trivial since $E_1 \neq 0 \neq E_2$. ■

6.4. LEMMA. *Let*

$$\begin{tikzcd} D_1 + D_2 \arrow[r, "q"] \arrow[d, "\delta_1+\delta_2"'] & A \arrow[d, "\alpha"] \\ E_1 + E_2 \arrow[r, "p"'] & B \end{tikzcd} \tag{6.2}$$

be a pullback diagram, in which p *and* δ_1 *are effective descent morphisms,* B *is connected, and* $E_1 \neq 0 \neq E_2$*. Then* $D_2 \neq 0$.

PROOF. Consider the composite $p(\mathsf{in}_1)(\mathsf{pr}_1) = p(\mathsf{in}_2)(\mathsf{pr}_2)$ of the pullback projection $\mathsf{pr}_\lambda : E_1 \times_B E_2 \to E_\lambda$, coproduct injection $\mathsf{in}_\lambda : E_\lambda \to E_1 + E_2$, and p $(\lambda = 1, 2)$, and let α' be the pullback of α along that composite. Then:

(i) since α' can be considered as the pullback of δ_1, it is an effective descent morphism;

(ii) since $E_1 \times_B E_2 \neq 0$ by Lemma 6.3, we conclude that α' has a non-zero domain;

(iii) since α' can also be considered as the pullback of δ_2, this gives $D_2 \neq 0$.

■

7. Proof of Theorem 4.5

In this section $\mathbf{C} = \mathsf{Fam}(\mathbf{A})$, $\mathbf{X} = \mathbf{Sets}$, and I and H are as in Example 2.5(a). In particular $\mathbf{C}$ is lextensive.

It is convenient to display the objects of $\mathbf{C}$ as $A = (A_i)_{i \in I(A)}$ (as e.g. in [BJ]); note that a morphism $\alpha : A \to B$ in $\mathbf{C}$ is uniquely determined by the map $I(\alpha)$ and the family $(\alpha_i : A_i \to B_{I(\alpha)(i)})_{i \in I(A)}$ (which can be any map from $I(A)$ to $I(B)$ and any family of morphisms $A_i \to B_{I(\alpha)(i)}$ defined for all $i \in I(A)$). In this notation the factorization $\alpha = m_\alpha e_\alpha$ displays as

$$(A_i)_{i \in I(A)} \xrightarrow{e_\alpha} (B_{I(\alpha)(i)})_{i \in I(A)} \xrightarrow{m_\alpha} (B_i)_{i \in I(B)}, \tag{7.1}$$

$$I(e_\alpha) = 1_{I(A)}, (e_\alpha)_i = \alpha_i, I(m_\alpha) = I(\alpha), (m_\alpha)_i = 1_{B_{I(\alpha)(i)}},$$

and we have

$$\mathcal{E} = \{\alpha \mid I(\alpha) \text{ is a bijection}\}, \mathcal{M} = \{\alpha \mid \text{each } \alpha_i \text{ is an isomorphism}\}. \tag{7.2}$$

Lemma 6.1 gives:

7.1. Corollary. *For any morphism α in $\mathbf{C}$ the following conditions are equivalent:*

(a) e_α is an effective descent morphism;

(b) each α_i ($i \in I(A)$) considered as a morphism in $\mathbf{C}$ is an effective descent morphism.

7.2. Proposition. *Let*

$$\begin{tikzcd} D \arrow[r, "q"] \arrow[d, "\delta"'] & A \arrow[d, "\alpha"] \\ E \arrow[r, "p"'] & B \end{tikzcd} \tag{7.3}$$

be a pullback diagram with p an effective descent morphism. Then:

(a) if e_δ is an effective descent morphism then so is e_α;

(b) if δ is separable then so is α;

(c) if δ is strongly separable then so is α.

Proof. (a): The equivalence (6.1) and Corollary 7.1 tell us that without loss of generality we can assume that A and B are connected (i.e. $I(A)$ and $I(B)$ are one-element sets), and then we have to prove that α itself is an effective descent morphism. As shown in [JST] (originally in [ST], but under an additional assumption), the class of effective descent morphisms is closed under composition, is pullback stable, and has the strong right cancellation property. So, since p and (therefore also) q are effective descent morphisms, it suffices to prove that so is δ. On the other hand, since e_δ is an effective descent morphism, so is each δ_i ($i \in I(D)$), and then, as easily follows from Lemmas 6.1 and 6.2, it suffices to show that $I(\delta)$ is surjective.

Let X be the image of $I(\delta)$ and Y its complement in $I(E)$. We rewrite the diagram (7.3) as the diagram (6.2) with:

- $D_1 = D$,
- $D_2 = 0$,
- $E_1 = (E_x)_{x \in X}$,
- $E_2 = (E_y)_{y \in Y}$,
- δ_1 the same as δ but considered as a morphism from $D_1 = D$ to E_1,
- $\delta_2 =$ the unique morphism from 0 to E_2.

Since $D_2 = 0$ together with $E_1 \neq 0 \neq E_2$ would contradict Lemma 6.4, we conclude that either E_1 or E_2 is 0, i.e. is the empty family. However, if E_1 were 0, then so would be also D, which is not the case since q is an effective descent morphism and A is connected. Thus $E_2 = 0$, and therefore $I(\delta)$ is surjective, as desired.

(b) follows from the results of [CJ], and (c) follows from (a) and (b). ∎

Since every trivial covering morphism is strongly separable, Theorem 4.5 immediately follows from the assertion (c) of Proposition 7.2.

References

[AG] M. Auslander and O. Goldman, The Brauer group of a commutative ring, *Trans. AMS* **97**, 1960, 367–409

[BJ] F. Borceux and G. Janelidze, *Galois Theories*, Cambridge Studies in Advanced Mathematics 72, Cambridge University Press, 2001

[CJ] A. Carboni and G. Janelidze, Decidable (=separable) objects and morphisms in lextensive categories, *Journal of Pure and Applied Algebra* **110**, 1996, 219–240

[CJKP] A. Carboni, G. Janelidze, G. M. Kelly, and R. Paré, On localization and stabilization of factorization systems, *Applied Categorical Structures* **5**, 1997, 1–58

[CHK] C. Cassidy, M. Hébert, and G. M. Kelly, Reflective subcategories, localizations, and factorization systems, *Journal of Australian Mathematical Society (Series A)*, 1985, 287–329

[CHR] S. U. Chase, D. K. Harrison, and A. Rosenberg, Galois theory and cohomology of commutative rings, *Mem. AMS* **52**, 1965, 15–33

[DI] F. R. DeMeyer and E. Ingraham, Separable algebras over a commutative ring, *Lecture Notes in Math.* **181**, Springer 1971

[G] A. Grothendieck, Revêtements étales et groupe fondamental, SGA 1, exposé V, *Lecture Notes in Math.* **224**, Springer 1971

[J1] G. Janelidze, Precategories and Galois theory, *Lecture Notes in Math.* **1488**, Springer, 1991, 157–173

[J2] G. Janelidze, Categorical Galois theory: revision and some recent developments, in: *Galois Connections and Applications*, Mathematics and Its Applications 565, Kluwer Academic Publishers B.V., 2004, 139–171

[JK] G. Janelidze and G. M. Kelly, The reflectiveness of covering morphisms in algebra and geometry, *Theory and Applications of Categories* **3**, 1997, 132–159

[JST] G. Janelidze, M. Sobral, and W. Tholen, Beyond Barr exactness: effective descent morphisms, in: *Categorical Foundations; Special Topics in Order, Topology, Algebra, and Sheaf Theory*, Edited by M. C. Pedicchio and W. Tholen, Cambridge University Press, 2004, 359–405

[JT1] G. Janelidze and W. Tholen, Functorial factorization, well-pointedness and separability, *Journal of Pure and Applied Algebra* **142**, 1999, 99–130

[JT2] G. Janelidze and W. Tholen, Facets of Descent III: Monadic descent for rings and algebras, *Applied Categorical Structures* **12**, Nos 5–6, 2004, 461–467

[Janusz] G. J. Janusz, Separable algebras over commutative rings, *Trans. AMS* **122**, 1966, 461–479

[M] A. R. Magid, *The separable Galois theory of commutative rings*, Marcel Dekker, 1974

[ST] M. Sobral and W. Tholen, Effective descent morphisms and effective equivalence relations, *Canadian Math. Soc. Conference Proceedings* **13**, 1992, 421–433

[T] W. Tholen, A categorical guide to separation, compactness and perfectness, *Homology, Homotopy, Appl.* **1**, 1999, 147–161

[VZ1] O. Villamayor and D. Zelinsky, Galois theory for rings with finitely many idempotents, *Nagoya Math. Journal* **27**, 1966, 721–731

[VZ2] O. Villamayor and D. Zelinsky, Galois theory with infinitely many idempotents, *Nagoya Math. Journal* **35**, 1969, 83–98

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