Theory and Applications of Categories, Vol. 23, No. 8, 2010, pp. 150–169.

THE THIRD COHOMOLOGY GROUP CLASSIFIES DOUBLE CENTRAL EXTENSIONS

Dedicated to Dominique Bourn on the occasion of his sixtieth birthday

DIANA RODELO AND TIM VAN DER LINDEN

Abstract. We characterise the double central extensions in a semi-abelian category in terms of commutator conditions. We prove that the third cohomology group $H^3(Z, A)$ of an object Z with coefficients in an abelian object A classifies the double central extensions of Z by A.

Introduction

The second cohomology group $H^2(Z, A)$ of a group Z with coefficients in an abelian group A is well-known to classify the central extensions of Z by A in the following manner (see for instance [29]). A central extension f of Z by A can be described as a short exact sequence

$$0 \longrightarrow A \overset{\ker f}{\rightarrowtriangle} X \overset{f}{\twoheadrightarrow} Z \longrightarrow 0$$

such that $axa^{-1}x^{-1} = 1$ for all $a \in A$ and $x \in X$. Two extensions $f\colon X \to Z$ and $f'\colon X' \to Z$ are equivalent if and only if there exists a group (iso)morphism $x\colon X \to X'$ satisfying $f' \circ x = f$ and $x \circ \ker f = \ker f'$. The induced equivalence classes, together with the classical Baer sum, form an abelian group $\mathsf{Centr}^1(Z, A)$, and this group is isomorphic to $H^2(Z, A)$.

In [19], see also [7] and [10], this construction was extended categorically from the context of groups to semi-abelian categories [24]. This includes familiar results for, say, Lie algebras over a field, commutative algebras, non-unital rings, or (pre)crossed modules.

The aim of the present work is to prove a two-dimensional version of this result, at once in a categorical context: we show that the third cohomology group $H^3(Z, A)$ of an object Z with coefficients in an abelian object A of a semi-abelian category $\mathcal{A}$ classifies the double central extensions in $\mathcal{A}$ of Z by A. Thus the connections between two interpretations of $H^3(Z, A)$ are made explicit.

Research supported by Centro de Matemática da Universidade de Coimbra, by Fundação para a Ciência e a Tecnologia (under grant number SFRH/BPD/38797/2007) and by Vrije Universiteit Brussel.

Received by the editors 2009-01-14 and, in revised form, 2010-01-20.

Published on 2010-02-03 in the Bourn Festschrift.

2000 Mathematics Subject Classification: 18G50, 18G60, 20J, 55N.

Key words and phrases: cohomology, categorical Galois theory, semi-abelian category, higher central extension, Baer sum.

© Diana Rodelo and Tim Van der Linden, 2010. Permission to copy for private use granted.

On one hand, there is the direction approach to cohomology established by Bourn [5, 6, 7] and further investigated in collaboration with Rodelo [11, 31]; here the cohomology groups H^nA of an internal abelian group A are described through direction functors, in such a way that any short exact sequence of internal abelian groups induces a long exact cohomology sequence. This concept of *direction* may be understood as follows. It is well-known that in a Barr exact context, H^1A can be interpreted in terms of A-*torsors*. An A-torsor is a generalised affine space over A, an "abelian group without zero" where any choice of zero gives back A—its direction. Further borrowing intuition from affine geometry, H^1A is described in terms of autonomous Mal'tsev operations with given direction A. On level 2—the level which corresponds to the "third cohomology group" from the title—the direction functor theory is based on that of level 1: now H^2A is described in terms of internal groupoids with given direction A. By means of higher order internal groupoids, the theory is inductively extended to higher levels H^nA.

On the other hand, there is the approach to semi-abelian homology [2, 14] based on categorical Galois theory [3, 20] initiated by Janelidze [21, 22] and further worked out by Everaert, Gran and Van der Linden [12]. Here the basic situation is given by a semi-abelian category $\mathcal{A}$ and a Birkhoff subcategory $\mathcal{B}$ of $\mathcal{A}$; the derived functors of the reflector $I\colon \mathcal{A} \to \mathcal{B}$ are computed in terms of higher Hopf formulae using the induced Galois structures of higher central extensions. In the specific case where $\mathcal{B}$ is the Birkhoff subcategory $\mathsf{Ab}\mathcal{A}$ determined by the abelian objects in $\mathcal{A}$ and $I = \mathsf{ab}$ is the abelianisation functor, one actually begins with the Galois structure

$$\Gamma = (\mathcal{A} \underset{\supset}{\overset{\mathsf{ab}}{\underset{\perp}{\rightleftarrows}}} \mathsf{Ab}\mathcal{A}, |\mathsf{Ext}\mathcal{A}|, |\mathsf{ExtAb}\mathcal{A}|). \qquad (\mathbf{A})$$

The class of extensions $|\mathsf{Ext}\mathcal{A}|$ (respectively $|\mathsf{ExtAb}\mathcal{A}|$) consists of the regular epimorphisms in $\mathcal{A}$ (in $\mathsf{Ab}\mathcal{A}$) and forms the class of objects of the category $\mathsf{Ext}\mathcal{A}$ (or $\mathsf{ExtAb}\mathcal{A}$) whose morphisms are commutative squares between extensions. The coverings with respect to this Galois structure Γ are exactly the central extensions in the sense of commutator theory: an extension $f\colon X \to Z$ is central if and only if $[R[f], \nabla_X] = \Delta_X$, i.e., the commutator of the kernel pair of f with the largest relation ∇_X on X is the smallest relation Δ_X on X. These central extensions, in turn, determine a reflective subcategory $\mathsf{CExt}\mathcal{A}$ of $\mathsf{Ext}\mathcal{A}$; the reflector $\mathsf{centr}\colon \mathsf{Ext}\mathcal{A} \to \mathsf{CExt}\mathcal{A}$ which sends f to the central extension $\mathsf{centr}f\colon X/[R[f], \nabla_X] \to Z$ is the *centralisation functor*. Thus we obtain the Galois structure

$$\Gamma_1 = (\mathsf{Ext}\mathcal{A} \underset{\supset}{\overset{\mathsf{centr}}{\underset{\perp}{\rightleftarrows}}} \mathsf{CExt}\mathcal{A}, |\mathsf{Ext}^2\mathcal{A}|, |\mathsf{ExtCExt}\mathcal{A}|). \qquad (\mathbf{B})$$

The class $|\mathsf{Ext}^2\mathcal{A}|$ consists of *double extensions* in $\mathcal{A}$, which are defined as commutative squares

$$\begin{array}{ccc} X & \xrightarrow{c} & C \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle g} \\ D & \xrightarrow[f]{} & Z \end{array}$$

such that the maps c, d, f, g and the comparison map $(d, c)\colon X \to D \times_Z C$ to the pullback of f with g are regular epimorphisms. The elements of $|\mathsf{ExtCExt}\mathcal{A}|$ are such double extensions, but with the extra condition that both d and g are central. The coverings with respect to the Galois structure Γ_1 are used in the computation of the third homology functor $H_3(-, \mathsf{ab})\colon \mathcal{A} \to \mathsf{Ab}\mathcal{A}$ (see [12]) and form the main subject of the present paper—they are the "double central extensions" from the title.

We start by recalling the main properties of the Galois structure Γ_1 in Section 1. In Section 2 we characterise the Γ_1-coverings in terms of commutators (as Janelidze does in the category of groups [21] and Gran and Rossi do in the context of Mal'tsev varieties [18]) and in terms of internal pregroupoids in the sense of [27]. Section 3 recalls the definition of the third cohomology group in semi-abelian categories from [31]. We obtain a natural notion of direction for double extensions and show in Section 4 that the set $\mathsf{Centr}^2(Z, A)$ of equivalence classes of double central extensions of an object Z by an abelian object A carries a canonical abelian group structure. In Section 5 we conclude the paper with the isomorphism $H^3(Z, A) \cong \mathsf{Centr}^2(Z, A)$ between the third cohomology group of an object Z with coefficients in an abelian object A and the group $\mathsf{Centr}^2(Z, A)$.

We conjecture that this result may be generalised to higher degrees, so that also for $n > 2$ the $(n + 1)$-st cohomology group $H^{n+1}(Z, A)$ of Z with coefficients in A classifies the n-fold central extensions of Z by A. This will be the subject of future work.

Acknowledgements. Many thanks to Tomas Everaert and George Janelidze for important comments and suggestions on the text. Thanks also to Tomas for the present, more elegant, proof of Theorem 2.8.

1. Preliminaries

1.1. Semi-abelian categories. The basic context where we shall be working is that of *semi-abelian categories* [24]. Some examples are the categories Gp of all groups, Rng of non-unital rings, $\mathsf{Lie}_{\mathbb{K}}$ of Lie algebras over a field $\mathbb{K}$, XMod of crossed modules, and Loop of loops. We briefly recall the main definitions.

A category is *semi-abelian* when it is pointed, Barr exact and Bourn protomodular and has binary coproducts. A category is *pointed* when it has a zero object 0, a terminal object which is also initial. A *Barr exact* category is *regular*—finitely complete with pullback-stable regular epimorphisms and coequalisers of kernel pairs—and such that every internal equivalence relation is a kernel pair [1]. When a category is pointed and regular, *Bourn protomodularity* can be defined via the *regular Short Five Lemma* [4]: given any commutative diagram of regular epimorphisms with their kernels

$$\begin{array}{ccccc} K[f] & \overset{\ker f}{\rightarrowtriangle} & X & \overset{f}{\twoheadrightarrow} & Z \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle z} \\ K[f'] & \underset{\ker f'}{\rightarrowtriangle} & X' & \underset{f'}{\twoheadrightarrow} & Z' \end{array}$$

the morphisms k and z being isomorphisms implies that x is an isomorphism.

Any semi-abelian category $\mathcal{A}$ is a *Mal'tsev category*, i.e., it is finitely complete and in $\mathcal{A}$ every reflexive relation is an equivalence relation.

1.2. Internal pregroupoids. The concept of *internal pregroupoid* due to Kock [27] (called *herdoid* in [26]; see also [25]) generalises internal groupoids in the following manner: in a pregroupoid, the domain and codomain of a map may live in different objects, and no identities need to exist.

1.3. Definition. *Let $\mathcal{A}$ be a finitely complete category. A* pregroupoid (X, d, c, p) *in $\mathcal{A}$ is a span*

$$\begin{array}{ccccc} & & X & & \\ & {\scriptstyle d}\swarrow & & \searrow{\scriptstyle c} & \\ D & & & & C \end{array} \qquad (\mathbf{C})$$

with a partial ternary operation p on X satisfying

1. *$p(\alpha, \beta, \gamma)$ is defined if and only if $c(\alpha) = c(\beta)$ and $d(\gamma) = d(\beta)$;*
2. *$dp(\alpha, \beta, \gamma) = d(\alpha)$ and $cp(\alpha, \beta, \gamma) = c(\gamma)$ if $p(\alpha, \beta, \gamma)$ is defined;*
3. *$p(\alpha, \alpha, \gamma) = \gamma$ if $p(\alpha, \alpha, \gamma)$ is defined, and $p(\alpha, \gamma, \gamma) = \alpha$ if $p(\alpha, \gamma, \gamma)$ is defined;*
4. *$p(\alpha, \beta, p(\gamma, \delta, \epsilon)) = p(p(\alpha, \beta, \gamma), \delta, \epsilon)$ if either side is defined.*

*We denote the category of (pre)groupoids in $\mathcal{A}$ by (*Pre*)*Gd$\mathcal{A}$.

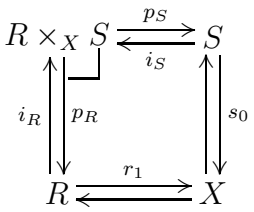
An "element" α of X should be interpreted as a map $\alpha\colon d(\alpha) \to c(\alpha)$; its domain $d(\alpha)$ is an element of D, while its codomain $c(\alpha)$ is an element of C. The operation p sends a composable triple (α, β, γ)

$$\begin{array}{ccc} d(\alpha) & \xrightarrow{\alpha} & c(\alpha) \\ & {\scriptstyle\delta}\searrow \quad \nearrow{\scriptstyle\beta} & \\ d(\gamma) & \xrightarrow[\gamma]{} & c(\gamma) \end{array}$$

to the dotted diagonal $\delta = p(\alpha, \beta, \gamma)\colon d(\alpha) \to c(\gamma)$.

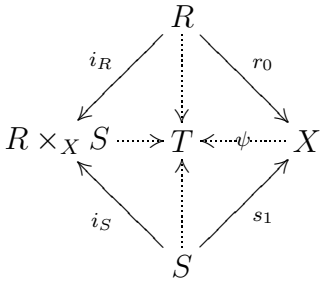
When $\mathcal{A}$ is a Mal'tsev category, conditions (2) and (4) of Definition 1.3 are automatically satisfied, see Proposition 2.6.11 in [2] or Proposition 4.1 in [8]. Moreover, if it exists, a pregroupoid structure p on a span (X, d, c) is necessarily unique. In this case we shall say that the span (X, d, c) *is* a pregroupoid and drop the structure p from the notation. In case (X, d, c) is the underlying span of a reflexive graph, the given splitting $i\colon C = D \to X$ of d and c provides identities for the composition, so that the pregroupoid becomes a groupoid, and the composite $p(\alpha, \beta, \gamma)$ may be viewed as $\gamma \circ \beta^{-1} \circ \alpha$.

1.4. The commutator of equivalence relations. Let $R = (R, r_0, r_1)$ and $S = (S, s_0, s_1)$ be equivalence relations on an object X of an exact Mal'tsev category $\mathcal{A}$. Let $R \times_X S$ denote the pullback of r_1 and s_0.



The object $R \times_X S$ "consists of" triples (α, β, γ) where $\alpha R \beta$ and $\beta S \gamma$. We say that R and S *centralise each other* when there exists a connector between R and S: a morphism $p\colon R \times_X S \to X$ which satisfies $p(\alpha, \alpha, \gamma) = \gamma$ and $p(\alpha, \gamma, \gamma) = \alpha$ [8]; see also [2, Definition 2.6.1]. As explained in the introduction of [8], an internal pregroupoid structure p on a span (X, d, c) is the same thing as a connector between the kernel pairs $R[c]$ and $R[d]$ of c and d; so a connector between equivalence relations R and S is nothing but a pregroupoid structure on the induced span of coequalisers $X/R \leftarrow X \rightarrow X/S$.

When $\mathcal{A}$ is a semi-abelian category, the *commutator* of R and S [30], denoted by $[R, S]$, is the universal equivalence relation on X which, when divided out, makes them centralise each other. More precisely, $[R, S]$ is the kernel pair $R[\psi]$ of the map ψ in the diagram



where the dotted arrows denote the colimit of the outer square [2, Section 2.8]. The direct images ψR and ψS of R and S along the regular epimorphism ψ centralise each other; hence R and S centralise each other if and only if $[R, S] = \Delta_X$ [8, Proposition 4.2].

An equivalence relation R on an object X is *central* when R and ∇_X centralise each other—when $[R, \nabla_X] = \Delta_X$. An object X is *abelian* when $[\nabla_X, \nabla_X] = \Delta_X$; this happens when X admits a (unique) internal abelian group structure.

1.5. Central extensions. The definition of central extensions for groups was extended to varieties of Ω-groups by Fröhlich and Lue [16, 28] and then to the categorical context by Janelidze and Kelly [23]. They have also been defined via commutator theory; as explained in [17], the two approaches are equivalent. We sketch the categorical definition.

Let $\mathcal{A}$ be a semi-abelian category. For any object X of $\mathcal{A}$ we may take the kernel of

the X-component of the unit of the adjunction in $\mathbf{A}$ to obtain a short exact sequence

$$0 \longrightarrow \langle X\rangle \overset{\mu_X}{\rhd\!\!\longrightarrow} X \overset{\eta_X}{\longrightarrow\!\!\rhd} \mathsf{ab}X \longrightarrow 0.$$

Thus we acquire a functor $\langle -\rangle\colon \mathcal{A} \to \mathcal{A}$ and a natural transformation $\mu\colon \langle -\rangle \Rightarrow 1_{\mathcal{A}}$.

1.6. Lemma. *The functors* ab *and* $\langle -\rangle$ *preserve pullbacks of regular epimorphisms along split epimorphisms.*

Proof. It is shown in [17] that the functor ab has this property. Since kernels commute with pullbacks, it follows that the functor $\langle -\rangle$ has the same property. ∎

An extension $f\colon X \to Z$ is *central* (with respect to the Galois structure Γ in diagram $\mathbf{A}$) if and only if either one of the projections p_0 or p_1 of its kernel pair $(R[f], p_0, p_1)$ is a trivial extension, i.e., a pullback of an extension in $\mathsf{Ab}\mathcal{A}$. It follows that f is central if and only if the right hand side square in the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \langle R[f]\rangle & \overset{\mu_{R[f]}}{\rhd\!\!\longrightarrow} & R[f] & \overset{\eta_{R[f]}}{\longrightarrow\!\!\rhd} & \mathsf{ab}R[f] & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \langle p_0\rangle} & & \downarrow{\scriptstyle p_0} & & \downarrow{\scriptstyle \mathsf{ab}p_0} & & \\
0 & \longrightarrow & \langle X\rangle & \underset{\mu_X}{\rhd\!\!\longrightarrow} & X & \underset{\eta_X}{\longrightarrow\!\!\rhd} & \mathsf{ab}X & \longrightarrow & 0
\end{array}$$

is a pullback or, equivalently, $\langle p_0\rangle$ is an isomorphism. Hence the kernel of $\langle p_0\rangle$, which is denoted by $\langle f\rangle_1$, is zero if and only if f is central. The object $\langle f\rangle_1$ may be considered as a normal subobject of X through the composite $\mu_X \circ \langle p_1\rangle \circ \ker p_0$, see Lemma 4.12 in [13]. Now the *centralisation functor* $\mathsf{centr}\colon \mathsf{Ext}\mathcal{A} \to \mathsf{CExt}\mathcal{A}$ from the Galois structure Γ_1 in diagram $\mathbf{B}$ takes the extension $f\colon X \to Z$ and maps it to the quotient $\mathsf{centr}f\colon X/\langle f\rangle_1 \to Z$ of $f\colon X \to Z$ by the extension $\langle f\rangle_1 \to 0$.

Given an object Z and an abelian object A, a *central extension of* Z *by* A is a central extension $f\colon X \to Z$ with kernel $K[f] = A$. The group of isomorphism classes of central extensions of Z by A is denoted $\mathsf{Centr}^1(Z, A)$. Recall the following result from [19].

1.7. Proposition. *If* $\mathcal{A}$ *is a semi-abelian category and* Z *is an object of* $\mathcal{A}$ *then the functor* $\mathsf{Centr}^1(Z, -)\colon \mathsf{Ab}\mathcal{A} \to \mathsf{Ab}$ *preserves finite products.*

Proof. We shall only repeat the main point of the construction behind [19, Proposition 6.1]. Let $a\colon A \to B$ be a morphism of abelian objects in $\mathcal{A}$ and $f\colon X \to Z$ a central extension of Z by A. Let $A \oplus B$ denote the biproduct of A with B in $\mathsf{Ab}\mathcal{A}$. The functor $\mathsf{Centr}^1(Z, -)$ maps the equivalence class of f to the equivalence class of the central extension f' in the diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A \oplus B & \overset{\ker f \times 1_B}{\rhd\!\!\longrightarrow} & X \times B & \overset{f \circ \mathrm{pr}_X}{\longrightarrow\!\!\rhd} & Z & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle [a, 1_B]} & & \downarrow & & \| & & \\
0 & \longrightarrow & B & \rhd\!\!\longrightarrow & X' & \underset{f'}{\longrightarrow\!\!\rhd} & Z & \longrightarrow & 0.
\end{array} \qquad (\mathbf{D})$$

The extension f' is central as a quotient of the central extension $f \circ \mathrm{pr}_X$. ∎

1.8. Double central extensions. Let $\mathcal{A}$ be a semi-abelian category. Recall that a *double extension (of an object Z)* in $\mathcal{A}$ is a commutative square

$$\begin{array}{ccc} X & \xrightarrow{c} & C \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle g} \\ D & \xrightarrow[f]{} & Z \end{array} \qquad (\mathbf{E})$$

such that all its maps and the comparison map $(d, c)\colon X \to D \times_Z C$ to the pullback of f with g are regular epimorphisms.

By definition, a double extension is *central* when it is a covering with respect to the Galois structure Γ_1. Hence the double extension $\mathbf{E}$, considered as a map $(c, f)\colon d \to g$ in the category $\mathsf{Ext}\mathcal{A}$, is central if and only if the first projection

$$\begin{array}{ccc} R[c] & \xrightarrow{p_0} & X \\ {\scriptstyle R[(c,f)]}\downarrow & & \downarrow{\scriptstyle d} \\ R[f] & \xrightarrow[p_0]{} & D \end{array} \qquad\qquad \begin{array}{ccc} R[c] & \xrightarrow{\quad p_0 \quad} & X \\ \downarrow & & \downarrow \\ R[c]/\langle R[(c,f)]\rangle_1 & \longrightarrow & X/\langle d\rangle_1 \end{array}$$

of its kernel pair—the left hand side square—is a trivial extension with respect to Γ_1. (Alternatively, one could use the square of second projections.) This means that the comparison map to its reflection into $\mathsf{CExt}\mathcal{A}$—the right hand side square—is a pullback. For this to happen, the natural map $\langle R[(c, f)]\rangle_1 \to \langle d\rangle_1$ must be an isomorphism. This, in turn, is equivalent to the square

$$\begin{array}{ccc} \langle R[d]\square R[c]\rangle & \xrightarrow{\langle p_0\rangle} & \langle R[d]\rangle \\ {\scriptstyle \langle p_0\rangle}\downarrow & & \downarrow{\scriptstyle \langle p_0\rangle} \\ \langle R[c]\rangle & \xrightarrow[\langle p_0\rangle]{} & \langle X\rangle \end{array}$$

being a pullback, because $\langle R[(c, f)]\rangle_1$ and $\langle d\rangle_1$ are the kernels of the vertical maps above. Here $(R[d]\square R[c], p_0, p_1)$ denotes the kernel pair of $R[(c, f)]$; it consists of all quadruples $(\alpha, \beta, \gamma, \delta) \in X^4$ in the configuration

$$\begin{pmatrix} \alpha & c & \beta \\ d & & d \\ \delta & c & \gamma \end{pmatrix},$$

called a *diamond* in [25]: $d(\alpha) = d(\delta)$, $c(\alpha) = c(\beta)$, $c(\gamma) = c(\delta)$ and $d(\gamma) = d(\beta)$.

2. Characterisation of double central extensions in terms of commutators

In this section we characterise the coverings with respect to the Galois structure Γ_1 in terms of internal pregroupoids. This characterisation turns out to be equivalent to the conditions given by Janelidze in [21] and Gran and Rossi in [18]—and thus we prove a categorical version of the next result.

2.1. PROPOSITION. [18, 21] *Let $\mathcal{A}$ be a Mal'tsev variety. A double extension $\mathbf{E}$ in $\mathcal{A}$ is central if and only if $[R[d], R[c]] = \Delta_X = [R[d] \cap R[c], \nabla_X]$.*

Double extensions may be characterised in terms of spans in a slice category as follows.

2.2. DEFINITION. *A span (X, d, c) in a regular category $\mathcal{A}$*

1. has global support *when $!_D \colon D \to 1$ and $!_C \colon C \to 1$ are regular epimorphisms;*

2. *is* aspherical *when also $(d, c) \colon X \to D \times C$ is a regular epimorphism.*

2.3. PROPOSITION. *Let $\mathcal{A}$ be a semi-abelian category. A commutative square $\mathbf{E}$ in $\mathcal{A}$ is a double extension if and only if (X, d, c) is an aspherical span in $\mathcal{A} \downarrow Z$.*

2.4. DEFINITION. *Suppose that $\mathcal{A}$ is regular. A pregroupoid (X, d, c, p)* has global support *or is* aspherical *whenever the span (X, d, c) has global support or is aspherical. This definition applies in the obvious way to internal groupoids.*

Because of Proposition 2.3, which exhibits the close connection between double extensions in $\mathcal{A}$ and spans in a slice category $\mathcal{A} \downarrow Z$, we are also mostly interested in pregroupoids in slice categories. For $\mathcal{A}$ Mal'tsev, asking that a span (X, d, c) is a pregroupoid in $\mathcal{A} \downarrow Z$ amounts to asking that (X, d, c) is a pregroupoid in $\mathcal{A}$. When $\mathcal{A}$ is semi-abelian, this happens precisely when the first equality $[R[d], R[c]] = \Delta_X$ of Proposition 2.1 holds.

2.5. DEFINITION. *Suppose that $\mathcal{A}$ is semi-abelian and let Z be an object of $\mathcal{A}$. An aspherical (pre)groupoid (X, d, c) in $\mathcal{A} \downarrow Z$ is* central *when $(d, c) \colon X \to D \times_Z C$ is a central extension in $\mathcal{A}$.*

Since $R[d] \cap R[c] = R[(d, c) \colon X \to D \times_Z C]$, this makes the centrality of the aspherical pregroupoid (X, d, c) equivalent to the second equality $[R[d] \cap R[c], \nabla_X] = \Delta_X$ of Proposition 2.1. And thus we proved:

2.6. PROPOSITION. *Let $\mathcal{A}$ be a semi-abelian category. A double extension $\mathbf{E}$ in $\mathcal{A}$ satisfies*

$$[R[d], R[c]] = \Delta_X = [R[d] \cap R[c], \nabla_X] \qquad (\mathbf{F})$$

if and only if the span (X, d, c) is a central pregroupoid in the slice category $\mathcal{A} \downarrow Z$.

2.7. PROPOSITION. *In a semi-abelian category, condition $\mathbf{F}$ is preserved and reflected by pullbacks of double extensions along double extensions.*

PROOF. The proof given in Section 4 of [18] in the context of Mal'tsev varieties is still valid in the present situation. ∎

We obtain the next result, which is further worked out in [15] in the more general context of exact Mal'tsev categories.

2.8. THEOREM. *Consider a double extension* $\mathbf{E}$ *in a semi-abelian category* $\mathcal{A}$. *The following are equivalent:*

1. $\mathbf{E}$ *is a double central extension;*
2. (X, d, c) *is a central pregroupoid in* $\mathcal{A} \downarrow Z$;
3. $[R[d], R[c]] = \Delta_X = [R[d] \cap R[c], \nabla_X]$.

PROOF. By Proposition 2.6 we already know that (2) and (3) are equivalent. To see that (1) implies (3), suppose that $\mathbf{E}$ is a double central extension. Then either one of the projections of its kernel pair is trivial with respect to Γ_1, meaning that it is a pullback of a double extension between central extensions (i.e., a morphism of the category $\mathsf{CExt}\mathcal{A}$). This latter double extension satisfies the condition corresponding to $\mathbf{F}$; hence applying Proposition 2.7 twice shows that (3) holds.

Now we prove that (2) implies (1). The pregroupoid structure of (X, d, c) is a connector $p\colon R[c] \times_X R[d] \to X$. As explained in Subsection 1.8, we are to show that the outer square in the diagram

$$\begin{array}{ccc}
\langle R[d]\square R[c]\rangle & \xrightarrow{\langle p_0\rangle} & \langle R[d]\rangle \\
\downarrow{\scriptstyle \langle p_0\rangle} & \overset{\langle\pi\rangle}{\searrow} \nearrow & \downarrow{\scriptstyle \langle p_0\rangle} \\
\langle R[c]\rangle & \xrightarrow[\langle p_0\rangle]{} & \langle X\rangle
\end{array}
\qquad \langle R[c] \times_X R[d]\rangle$$

is a pullback. Here $\pi\colon R[d]\square R[c] \to R[c] \times_X R[d]$ is defined by

$$\begin{pmatrix} \alpha & c & \beta \\ d & & d \\ \delta & c & \gamma \end{pmatrix} \mapsto (\alpha, \beta, \gamma).$$

By Lemma 1.6 we know that the inner quadrangle is a pullback, hence it suffices that $\langle\pi\rangle$ is an isomorphism. The left hand side square

$$\begin{array}{ccc}
R[d]\square R[c] & \xrightarrow{\pi} & R[c] \times_X R[d] \\
\downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\
R[d] \cap R[c] & \xrightarrow[p_0]{} & X
\end{array}
\qquad
\begin{array}{ccc}
\langle R[d]\square R[c]\rangle & \xrightarrow{\langle\pi\rangle} & \langle R[c] \times_X R[d]\rangle \\
\downarrow{\scriptstyle \langle q\rangle} & & \downarrow{\scriptstyle \langle p\rangle} \\
\langle R[d] \cap R[c]\rangle & \xrightarrow[\langle p_0\rangle]{} & \langle X\rangle,
\end{array}$$

where q is defined by

$$\begin{pmatrix} \alpha & c & \beta \\ d & & d \\ \delta & c & \gamma \end{pmatrix} \mapsto (p(\alpha, \beta, \gamma), \delta),$$

is a pullback. Since p_0 is a split epimorphism we may again use Lemma 1.6 to show that also the right hand side square above is a pullback. It follows that $\langle \pi \rangle$ is an isomorphism if and only if $\langle p_0 \rangle$ is an isomorphism, so that the internal pregroupoid (X, d, c) is central if and only if $\mathbf{E}$ is a double central extension. ∎

3. The third cohomology group

In this section we translate the description of the *second order direction functor* and its associated cohomology groups, developed in [31] for Barr exact categories, to the context of semi-abelian categories. A similar translation was made in [31] for *Moore categories* (i.e., strongly protomodular semi-abelian categories) where the connection with n-fold crossed extensions is explored. In that context, Z-modules are simpler since, due to strong protomodularity, they are just split exact sequences whose kernel is an abelian object (compare with the definition of Z-module given below). We shall, however, not need this simplification; in fact it turns out that we may focus on trivial modules. Note that what we call the *third* cohomology group here is actually the *second* cohomology group in [31]; the dimension shift is there for historical reasons, in order to comply with the "non-abelian" numbering used in classical cohomology of groups. From now on, $\mathcal{A}$ will denote a semi-abelian category and Z a fixed object of $\mathcal{A}$.

An aspherical (abelian) groupoid in $\mathcal{A} \downarrow Z$ consists of a commutative diagram

$$\begin{tikzcd} X \arrow[rr, shift left, "d"] \arrow[rr, shift right, "c"'] \arrow[dr, "f\circ d = f\circ c"'] & & Y \arrow[ll, "i"'] \arrow[dl, "f"] \\ & Z & \end{tikzcd} \tag{G}$$

such that the top line is a groupoid in $\mathcal{A}$, and both f and $(d, c)\colon X \to R[f]$ are regular epimorphisms. Such an internal groupoid has an underlying double extension

$$\begin{tikzcd} X \arrow[r, "c"] \arrow[d, "d"'] & Y \arrow[d, "f"] \\ Y \arrow[r, "f"'] & Z. \end{tikzcd} \tag{H}$$

We denote by $\mathsf{AsphGd}(\mathcal{A} \downarrow Z)$ the category of aspherical groupoids in $\mathcal{A} \downarrow Z$.

The category $\mathsf{Mod}_Z\mathcal{A}$ of *Z-modules* is the category $\mathsf{Ab}(\mathcal{A} \downarrow Z)$ of abelian groups in $\mathcal{A} \downarrow Z$. So, a Z-module gives us a split exact sequence

$$0 \longrightarrow A \overset{\ker p}{\rightarrowtail} P \underset{s}{\overset{p}{\rightleftarrows}} Z \longrightarrow 0$$

where A is an abelian object and p is a split epimorphism (equipped with an additional structure making it an abelian group in $\mathcal{A} \downarrow Z$). Using the equivalence between split epimorphisms and internal actions [9], we can replace P with a semi-direct product $Z \ltimes (A, \xi)$. For simplicity, we denote a Z-module just by its induced Z-algebra (A, ξ).

In the context of semi-abelian categories, the direction functor from [31, Definition 3.7] determines a functor $\mathsf{d}_Z \colon \mathsf{AsphGd}(\mathcal{A} \downarrow Z) \to \mathsf{Mod}_Z\mathcal{A}$ which maps an aspherical internal groupoid $\mathbf{G}$ to the Z-module $\mathsf{d}_Z(\mathbf{G}) = (A, \xi)$ defined by the downward pullback/upward pushout

$$\begin{array}{ccc} R[(d,c)] & \longrightarrow\!\!\triangleright & Z \ltimes (A, \xi) \\ {\scriptstyle (1_X, 1_X)}\uparrow\!\downarrow{\scriptstyle p_0} & & {\scriptstyle s}\uparrow\!\downarrow{\scriptstyle p} \\ X & \xrightarrow[f \circ d]{}\!\!\triangleright & Z. \end{array} \qquad (\mathbf{I})$$

More precisely, the pair $(p, s) \colon Z \ltimes (A, \xi) \rightleftarrows Z$ arises as a pushout of $(1_X, 1_X)$ along $f \circ d$ but, using the properties of $\mathbf{G}$, one may show that the square of downward arrows in $\mathbf{I}$ is a pullback [6]. Thus we see that $A = K[p] = K[p_0] = K[(d, c)] = K[d] \cap K[c]$.

3.1. Remark. Suppose we have a symmetric monoidal category $(\mathcal{C}, \otimes, E)$ such that the property

$$\forall C \in \mathcal{C}, \exists \overline{C} \in \mathcal{C} \colon C \otimes \overline{C} \sim E \qquad (\mathbf{J})$$

holds, where $\sim$ means "is connected to (by a zigzag)". Then it is easy to check that the monoidal structure of $\mathcal{C}$ induces an abelian group structure on the set $\pi_0\mathcal{C}$ of its connected components (equivalence classes with respect to $\sim$). The addition is defined by $\{C_1\} + \{C_2\} = \{C_1 \otimes C_2\}$, the zero is $\{E\}$ and $-\{C\} = \{\overline{C}\}$.

It is shown in [6] that the fibres of d_Z are symmetric monoidal categories with property $\mathbf{J}$. The tensor product is called the *Baer sum* since it gives the Baer sum of (2-fold) extensions in the classical examples. So, for any Z-module (A, ξ), $\pi_0\mathsf{d}_Z^{-1}(A, \xi)$ is an abelian group.

3.2. Definition. *[31] Let (A, ξ) be a Z-module. The* third cohomology group

$$H^3(Z, (A, \xi))$$

of Z with coefficients in (A, ξ) *is the abelian group $\pi_0\mathsf{d}_Z^{-1}(A, \xi)$ of equivalence classes of aspherical internal groupoids in $\mathcal{A} \downarrow Z$ with direction (A, ξ). This defines an additive functor*

$$H^3(Z, -) \colon \mathsf{Mod}_Z\mathcal{A} \to \mathsf{Ab}.$$

We are especially interested in the case of trivial Z-modules (A, τ), i.e., abelian objects A with the trivial Z-action τ. In this situation we write $H^3(Z, A)$ for $H^3(Z, (A, \tau))$. The functor $H^3(Z, -)$ restricts to an additive functor $\mathsf{Ab}\mathcal{A} \to \mathsf{Ab}$.

3.3. Proposition. *The direction of an aspherical groupoid $\mathbf{G}$ in $\mathcal{A} \downarrow Z$ is a trivial Z-module (A, τ) in $\mathcal{A}$ if and only if $\mathbf{G}$ is a central groupoid.*

PROOF. Let us first suppose that $\mathsf{d}_Z(\mathbf{G}) = (A, \tau)$. Then, $\mathsf{d}_Z(\mathbf{G})$, defined by

$$(p, s)\colon Z \ltimes (A, \tau) \rightleftarrows Z$$

in diagram **I**, is the product projection with its canonical inclusion

$$(\mathrm{pr}_Z, (1_Z, 0))\colon Z \times A \rightleftarrows Z.$$

It follows that the pullback $(p_0, (1_X, 1_X))\colon R[(d, c)] \rightleftarrows X$ is also a product projection with its canonical inclusion, namely $(\mathrm{pr}_X, (1_X, 0))\colon X \times A \rightleftarrows X$. In particular, the splitting $(1_X, 1_X)$ is a normal monomorphism in $\mathcal{A}$, which by Theorem 5.2 in [8] (see also Corollary 6.1.8 in [2]) means that $R[(d, c)] = R[d] \cap R[c]$ is central. Hence $[R[d] \cap R[c], \nabla_X] = \Delta_X$ and the groupoid is central.

Conversely, suppose that $\mathbf{G}$ is a central groupoid in $\mathcal{A} \downarrow Z$. By the same arguments as above we see that $(p_0, (1_X, 1_X))$ and hence (p, s) are product projections with their canonical inclusions. It follows that A has a trivial Z-action τ. ∎

3.4. COROLLARY. *Let* $\mathbf{G}$ *be an aspherical groupoid in* $\mathcal{A} \downarrow Z$ *and let* $\mathbf{H}$ *be the corresponding double extension. Then* $\mathbf{H}$ *is a double central extension if and only if* $\mathsf{d}_Z(\mathbf{G})$ *is a trivial* Z*-module* (A, τ) *in* $\mathcal{A}$.

Thus we see that the direction of a central internal groupoid $\mathbf{G}$ is just the intersection $A = K[d] \cap K[c]$ of the kernels of d and c; indeed, this object A is always abelian as the kernel of the central extension (d, c). In view of this fact we can extend the concept of direction to double central extensions.

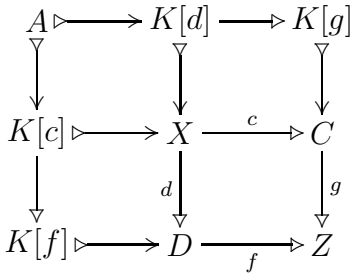
4. The group of equivalence classes of double central extensions

4.1. DEFINITION. *The* direction *of a double central extension* $\mathbf{E}$ *is the abelian object* $K[d] \cap K[c]$. *This defines a functor* $\mathsf{D}_Z\colon \mathsf{CExt}^2_Z\mathcal{A} \to \mathsf{Ab}\mathcal{A}$, *where* $\mathsf{CExt}^2_Z\mathcal{A}$ *denotes the category of double central extensions of the object* Z *of* $\mathcal{A}$.

The fibre $\mathsf{D}_Z^{-1}A$ *of this functor over an abelian object* A *is the category of* double central extensions of Z by A. *Two double central extensions of* Z *by* A *which are connected by a zigzag in* $\mathsf{D}_Z^{-1}A$ *are called* equivalent. *The equivalence classes form the set* $\mathsf{Centr}^2(Z, A) = \pi_0\mathsf{D}_Z^{-1}A$ *of connected components of this category.*

4.2. REMARK. Depending on the context it might not be clear whether $\mathsf{Centr}^2(Z, A)$ is indeed a set (rather than a proper class) but in any case Theorem 5.3 implies that $\mathsf{Centr}^2(Z, A)$ is only as large as is $H^3(Z, A)$.

4.3. Remark. The double central extension $\mathbf{E}$ induces a 3×3 diagram



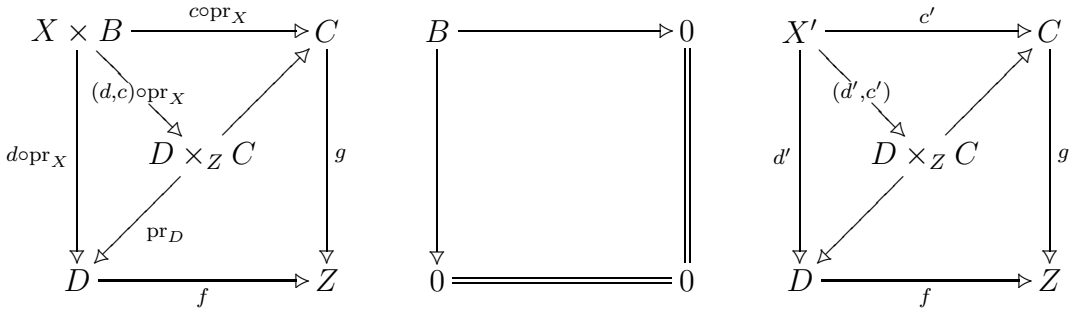
and the object A in this diagram is the direction of $\mathbf{E}$.

We now show that $\mathsf{Centr}^2(Z, A)$ carries a canonical abelian group structure.

4.4. Proposition. *Let $\mathcal{A}$ be a semi-abelian category and let Z be an object of $\mathcal{A}$. Mapping an abelian object A of $\mathcal{A}$ to the set $\mathsf{Centr}^2(Z, A)$ of equivalence classes of double central extensions of Z by A gives a finite product-preserving functor*

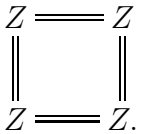
$$\mathsf{Centr}^2(Z, -)\colon \mathsf{Ab}\mathcal{A} \to \mathsf{Set}.$$

Proof. Let $a\colon A \to B$ be a morphism of abelian objects in $\mathcal{A}$ and $\mathbf{E}$ a double central extension of Z by A. Then $(d, c)\colon X \to D \times_Z C$ is a central extension of $D \times_Z C$ by A, and the construction of Proposition 1.7 yields a central extension (d', c') of $D \times_Z C$ by B. The morphism $\mathsf{Centr}^2(Z, a)$ now maps the equivalence class of $\mathbf{E}$ to the class of the right hand side square below. Indeed, since the left hand side square



—which arises from the regular epimorphism in the top sequence in $\mathbf{D}$—is a double central extension as the product of $\mathbf{E}$ with the middle double central extension, so is its right hand side quotient. The functoriality of $\mathsf{Centr}^2(Z, -)$ now follows from the functoriality of $\mathsf{Centr}^1(Z, -)$.

It is clear that $\mathsf{Centr}^2(Z, -)$ preserves the terminal object: any double central extension with direction 0 is connected to



To show that $\mathsf{Centr}^2(Z,-)$ also preserves binary products, we must provide an inverse to the map

$$(\mathsf{Centr}^2(Z,\mathrm{pr}_A),\mathsf{Centr}^2(Z,\mathrm{pr}_B))\colon \mathsf{Centr}^2(Z,A\times B)\to \mathsf{Centr}^2(Z,A)\times\mathsf{Centr}^2(Z,B).$$

This inverse is given by the product in the category $\mathsf{CExt}^2_Z\mathcal{A}$ of double central extensions of Z. Let indeed the two squares

$$\begin{array}{ccc} X & \xrightarrow{c} & C \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle g} \\ D & \xrightarrow[f]{} & Z \end{array} \qquad \text{and} \qquad \begin{array}{ccc} X' & \xrightarrow{c'} & C' \\ {\scriptstyle d'}\downarrow & & \downarrow{\scriptstyle g'} \\ D' & \xrightarrow[f']{} & Z \end{array}$$

be double central extensions of Z by A and B, respectively. Then their product in $\mathsf{CExt}^2_Z\mathcal{A}$ is the square

$$\begin{array}{ccc} X\times_Z X' & \xrightarrow{c\times_Z c'} & C\times_Z C' \\ {\scriptstyle d\times_Z d'}\downarrow & & \downarrow{\scriptstyle g\circ\mathrm{pr}_C} \\ D\times_Z D' & \xrightarrow[f\circ\mathrm{pr}_D]{} & Z. \end{array}$$

In fact, this square represents a pregroupoid in $\mathcal{A}\downarrow Z$ as a product of two such pregroupoids, and the comparison map $(d\times_Z d', c\times_Z c')$ to the pullback is a central extension as a pullback of the central extension $(d,c)\times(d',c')$. Finally, the direction of this double central extension is the kernel of $(d\times_Z d', c\times_Z c')$, which is nothing but $A\times B$. ∎

4.5. Corollary. *The functor* $\mathsf{Centr}^2(Z,-)$ *uniquely factors through the forgetful functor* $\mathsf{Ab}\to\mathsf{Set}$ *to yield a functor* $\mathsf{Centr}^2(Z,-)\colon \mathsf{Ab}\mathcal{A}\to\mathsf{Ab}$.

Proof. Any abelian object of $\mathcal{A}$ carries a canonical internal abelian group structure; we just showed that the functor $\mathsf{Centr}^2(Z,-)$ preserves such structures. See also Remark 5.5. ∎

5. $H^3(Z,A)$ and $\mathsf{Centr}^2(Z,A)$ are isomorphic

5.1. Proposition. *(Cf. Section 3 in [27]) Let* $\mathcal{A}$ *be a finitely complete category. The forgetful embedding* $\mathsf{Gd}\mathcal{A}\hookrightarrow\mathsf{PreGd}\mathcal{A}$ *has a right adjoint* $\mathsf{gd}\colon\mathsf{PreGd}\mathcal{A}\to\mathsf{Gd}\mathcal{A}$. *Moreover, when* $\mathcal{A}$ *is semi-abelian,* Z *is an object of* $\mathcal{A}$ *and* A *is an abelian object of* $\mathcal{A}$*, this adjunction restricts to the fibres of the direction functors* d_Z *and* D_Z

$$\mathsf{d}_Z^{-1}(A,\tau)\underset{\mathsf{gd}}{\overset{\subset}{\underset{\bot}{\rightleftarrows}}}\mathsf{D}_Z^{-1}A. \qquad (\mathbf{K})$$

PROOF. Given an internal pregroupoid (X, d, c), the induced internal groupoid $\mathsf{gd}(X, d, c)$ has as underlying reflexive graph

$$R[c] \times_X R[d] \underset{\text{cod}}{\overset{\text{dom}}{\underset{\text{id}}{\rightleftarrows}}} X,$$

where dom and cod are the first and third projections and id is the diagonal. This reflexive graph is a groupoid, in which the composition maps a pair $(\alpha R[c]\beta R[d]\gamma, \gamma R[c]\delta R[d]\epsilon)$ to the triple $(\alpha, p(\delta, \gamma, \beta), \epsilon)$, where p is the pregroupoid structure of (X, d, c). The (X, d, c)-component of the counit of the adjunction is defined by the map

$$(p, d, c) \colon (R[c] \times_X R[d], \text{dom}, \text{cod}) \to (X, d, c)$$

in $\mathsf{PreGd}\mathcal{A}$; and given an internal groupoid

$$v \circlearrowright X \underset{c}{\overset{d}{\underset{i}{\rightleftarrows}}} Y$$

with inversion map v, the associated unit component is

$$\begin{array}{ccc} X & \underset{c}{\overset{d}{\underset{i}{\rightleftarrows}}} & Y, \\ {\scriptstyle (i\circ d, v, i\circ c)}\downarrow & & \downarrow{\scriptstyle i} \\ R[c] \times_X R[d] & \underset{\text{cod}}{\overset{\text{dom}}{\underset{\text{id}}{\rightleftarrows}}} & X. \end{array} \tag{L}$$

One easily checks that the triangular identities hold.

Corollary 3.4 implies that the embedding $\mathsf{Gd}\mathcal{A} \hookrightarrow \mathsf{PreGd}\mathcal{A}$ restricts to the fibres of d_Z and D_Z. Now suppose that $\mathbf{E} \in \mathsf{D}_Z^{-1}A$; then (X, d, c) is a central pregroupoid in $\mathcal{A} \downarrow Z$ by Theorem 2.8, and $A = K[(d, c)]$. Using that the square

$$\begin{array}{ccc} R[c] \times_X R[d] & \xrightarrow{\ p\ } & X \\ {\scriptstyle (\text{dom},\text{cod})}\downarrow & & \downarrow{\scriptstyle (d,c)} \\ X \times_Z X & \xrightarrow[d\times_Z c]{} & D \times_Z C \end{array} \tag{M}$$

is a pullback, we see that (dom, cod) is a central extension and $A = K[(\text{dom}, \text{cod})]$. Hence the groupoid $\mathsf{gd}(X, d, c)$ in $\mathcal{A} \downarrow Z$ is central, which by Proposition 3.3 means that it has direction (A, τ), that is, it is in the fibre $\mathsf{d}_Z^{-1}(A, \tau)$—so the functor gd also restricts to the fibres of the direction functors d_Z and D_Z.

To see that these restrictions are still adjoint to each other, it suffices to prove that the components of the unit and the counit are in the fibre of $1_{(A,\tau)}$ (respectively 1_A). This is the case, because both the square $\mathbf{M}$ and the similar square corresponding to $\mathbf{L}$ are pullbacks. ∎

5.2. Remark. Consider an adjunction

$$\mathcal{C} \underset{G}{\overset{F}{\underset{\perp}{\rightleftarrows}}} \mathcal{D}.$$

1. The functors F and G induce functions $\varphi\colon \pi_0\mathcal{C} \to \pi_0\mathcal{D}$, defined by $\varphi\{C\} = \{FC\}$, and $\gamma\colon \pi_0\mathcal{D} \to \pi_0\mathcal{C}$, defined by $\gamma\{D\} = \{GD\}$, respectively.

2. F being left adjoint to G implies that $\varphi^{-1} = \gamma$, i.e., $\pi_0\mathcal{C} \cong \pi_0\mathcal{D}$. In fact, $(\varphi\circ\gamma)\{D\} = \{FGD\} = \{D\}$, for any object D of $\mathcal{D}$, since FGD is connected to D by the D-component of the counit of the adjunction; thus $\varphi\circ\gamma = 1_{\pi_0\mathcal{D}}$. Similarly $\gamma\circ\varphi = 1_{\pi_0\mathcal{C}}$, using the unit of the adjunction instead.

Now suppose that the category $\mathcal{C}$ carries a symmetric monoidal structure $(\mathcal{C}, \otimes, E)$ as in Remark 3.1.

3. $\pi_0\mathcal{C}$ is an abelian group.

4. $\pi_0\mathcal{D}$ is an abelian group with addition given by $\{D_1\} + \{D_2\} = \{F(\,GD_1 \otimes GD_2\,)\}$, zero $\{FE\}$ and $-\{D\} = \{F(\,\overline{GD}\,)\}$.

5. The function φ is a group isomorphism with inverse γ.

5.3. Theorem. *In a semi-abelian category, the third cohomology group $H^3(Z, A)$ of an object Z with coefficients in an abelian object A is isomorphic to the group $\mathsf{Centr}^2(Z, A)$ of equivalence classes of double central extensions of Z by A.*

Proof. By the unicity in Corollary 4.5, to show that $H^3(Z, -)$ and $\mathsf{Centr}^2(Z, -)$ are isomorphic as functors $\mathsf{Ab}\mathcal{A} \to \mathsf{Ab}$, it suffices to give a bijection between the underlying sets $H^3(Z, A)$ and $\mathsf{Centr}^2(Z, A)$, natural in A. Through Remark 5.2, the adjunction $\mathbf{K}$ from Proposition 5.1 induces the needed isomorphisms $\varphi\colon H^3(Z, A) \to \mathsf{Centr}^2(Z, A)$ and $\gamma\colon \mathsf{Centr}^2(Z, A) \to H^3(Z, A)$. ∎

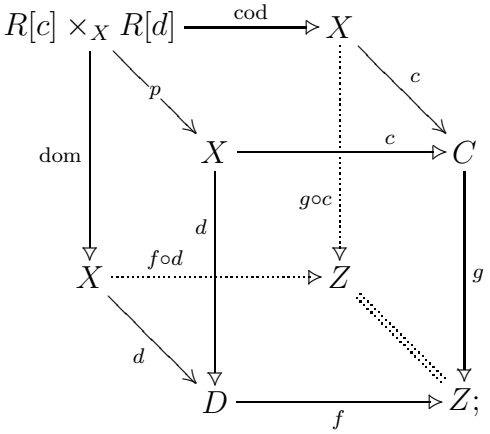
5.4. Remark. The isomorphisms of Theorem 5.3 may also be obtained as follows. We have $\varphi\colon H^3(Z, A) \to \mathsf{Centr}^2(Z, A)\colon \{\mathbf{G}\} \mapsto \{\mathbf{H}\}$ and

$$\gamma\colon \mathsf{Centr}^2(Z, A) \to H^3(Z, A)\colon \{\mathbf{E}\} \mapsto \{\mathsf{gd}(\mathbf{E})\}\,,$$

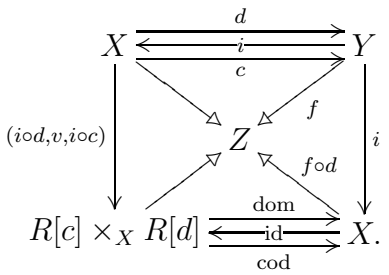
where

$$\mathsf{gd}(\mathbf{E}) = \begin{tikzcd} R[c] \times_X R[d] \arrow[r, shift left=2, "\mathrm{dom}"] \arrow[r, shift right=2, "\mathrm{cod}"'] \arrow[r, phantom, "\overset{\mathrm{id}}{\leftleftarrows}"] \arrow[dr] & X \arrow[d, "f\circ d = g\circ c"] \\ & Z \end{tikzcd}$$

such that $\varphi\circ\gamma = 1_{\mathsf{Centr}^2(Z,A)}$, because for any double central extension $\mathbf{E}$ of Z by A, $(\varphi\circ\gamma)\{\mathbf{E}\}$ is equal to $\{\mathbf{E}\}$ through (p, d, c), the $\mathbf{E}$-component of the counit of the adjunction $\mathbf{K}$



and $\gamma\circ\varphi = 1_{H^3(Z,A)}$, since for any central internal groupoid $\mathbf{G}$, with inversion map v and direction (A, τ), $(\gamma\circ\varphi)\{\mathbf{G}\}$ is equal to $\{\mathbf{G}\}$ through $((i\circ d, v, i\circ c), i)$, the $\mathbf{G}$-component of the unit of the adjunction $\mathbf{K}$



5.5. Remark. We know that $\mathsf{d}_Z^{-1}(A, \tau)$ is a symmetric monoidal category with property $\mathbf{J}$ by Remark 3.1. The arguments in Remark 5.2 show how the addition on $H^3(Z, A)$ is transported to an abelian group structure on $\mathsf{Centr}^2(Z, A)$ as described in Remark 5.2, (4). This makes the connection between the canonical abelian group structure from Proposition 4.4 and Corollary 4.5 and the Baer sum on $\mathsf{d}_Z^{-1}(A, \tau)$ explicit.

References

[1] M. Barr, *Exact categories*, Exact categories and categories of sheaves, Lecture Notes in Mathematics, vol. 236, Springer, 1971, pp. 1–120.

[2] F. Borceux and D. Bourn, *Mal'cev, protomodular, homological and semi-abelian categories*, Mathematics and its Applications, vol. 566, Kluwer Academic Publishers, 2004.

[3] F. Borceux and G. Janelidze, *Galois theories*, Cambridge Studies in Advanced Mathematics, vol. 72, Cambridge University Press, 2001.

[4] D. Bourn, *Normalization equivalence, kernel equivalence and affine categories*, Category Theory, Proceedings Como 1990 (A. Carboni, M. C. Pedicchio, and G. Rosolini, eds.), Lecture Notes in Mathematics, vol. 1488, Springer, 1991, pp. 43–62.

[5] D. Bourn, *Baer sums and fibered aspects of Mal'cev operations*, Cah. Topol. Géom. Differ. Catég. **XL** (1999), 297–316.

[6] D. Bourn, *Aspherical abelian groupoids and their directions*, J. Pure Appl. Algebra **168** (2002), 133–146.

[7] D. Bourn, *Baer sums in homological categories*, J. Algebra **308** (2007), 414–443.

[8] D. Bourn and M. Gran, *Centrality and normality in protomodular categories*, Theory Appl. Categ. **9** (2002), no. 8, 151–165.

[9] D. Bourn and G. Janelidze, *Protomodularity, descent, and semidirect products*, Theory Appl. Categ. **4** (1998), no. 2, 37–46.

[10] D. Bourn and G. Janelidze, *Extensions with abelian kernels in protomodular categories*, Georgian Math. J. **11** (2004), no. 4, 645–654.

[11] D. Bourn and D. Rodelo, *Cohomology without projectives*, Cah. Topol. Géom. Differ. Catég. **XLVIII** (2007), no. 2, 104–153.

[12] T. Everaert, M. Gran, and T. Van der Linden, *Higher Hopf formulae for homology via Galois Theory*, Adv. Math. **217** (2008), 2231–2267.

[13] T. Everaert and T. Van der Linden, *Baer invariants in semi-abelian categories I: General theory*, Theory Appl. Categ. **12** (2004), no. 1, 1–33.

[14] T. Everaert and T. Van der Linden, *Baer invariants in semi-abelian categories II: Homology*, Theory Appl. Categ. **12** (2004), no. 4, 195–224.

[15] T. Everaert and T. Van der Linden, *A note on double central extensions in exact Mal'tsev categories*, Cah. Topol. Géom. Differ. Catég., to appear, 2010.

[16] A. Fröhlich, *Baer-invariants of algebras*, Trans. Amer. Math. Soc. **109** (1963), 221–244.

[17] M. Gran, *Applications of categorical Galois theory in universal algebra*, Galois Theory, Hopf Algebras, and Semiabelian Categories (G. Janelidze, B. Pareigis, and W. Tholen, eds.), Fields Institute Communications Series, vol. 43, American Mathematical Society, 2004, pp. 243–280.

[18] M. Gran and V. Rossi, *Galois theory and double central extensions*, Homology, Homotopy and Appl. **6** (2004), no. 1, 283–298.

[19] M. Gran and T. Van der Linden, *On the second cohomology group in semi-abelian categories*, J. Pure Appl. Algebra **212** (2008), 636–651.

[20] G. Janelidze, *Pure Galois theory in categories*, J. Algebra **132** (1990), 270–286.

[21] G. Janelidze, *What is a double central extension? (The question was asked by Ronald Brown)*, Cah. Topol. Géom. Differ. Catég. **XXXII** (1991), no. 3, 191–201.

[22] G. Janelidze, *Higher dimensional central extensions: A categorical approach to homology theory of groups*, Lecture at the International Category Theory Meeting CT95, Halifax, 1995.

[23] G. Janelidze and G. M. Kelly, *Galois theory and a general notion of central extension*, J. Pure Appl. Algebra **97** (1994), 135–161.

[24] G. Janelidze, L. Márki, and W. Tholen, *Semi-abelian categories*, J. Pure Appl. Algebra **168** (2002), 367–386.

[25] G. Janelidze and M. C. Pedicchio, *Pseudogroupoids and commutators*, Theory Appl. Categ. **8** (2001), no. 15, 408–456.

[26] P. T. Johnstone, *The 'closed subgroup theorem' for localic herds and pregroupoids*, J. Pure Appl. Algebra **70** (1991), 97–106.

[27] A. Kock, *Fibre bundles in general categories*, J. Pure Appl. Algebra **56** (1989), 233–245.

[28] A. S.-T. Lue, *Baer-invariants and extensions relative to a variety*, Math. Proc. Cambridge Philos. Soc. **63** (1967), 569–578.

[29] S. Mac Lane, *Homology*, Die Grundlehren der mathematischen Wissenschaften, vol. 144, Springer, 1967.

[30] M. C. Pedicchio, *A categorical approach to commutator theory*, J. Algebra **177** (1995), 647–657.

[31] D. Rodelo, *Directions for the long exact cohomology sequence in Moore categories*, Appl. Categ. Structures **17** (2009), no. 4, 387–418.

Departamento de Matemática, Faculdade de Ciências e Tecnologia,
Universidade do Algarve, Campus de Gambelas,
8005-139 Faro, Portugal

Centro de Matemática, Universidade de Coimbra,
3001-454 Coimbra, Portugal

Vakgroep Wiskunde, Vrije Universiteit Brussel,
Pleinlaan 2, 1050 Brussel, Belgium

Email: drodelo@ualg.pt
tvdlinde@vub.ac.be

This article may be accessed at http://www.tac.mta.ca/tac/ or by anonymous ftp at ftp://ftp.tac.mta.ca/pub/tac/html/volumes/23/8/23-08.{dvi,ps,pdf}

THEORY AND APPLICATIONS OF CATEGORIES (ISSN 1201-561X) will disseminate articles that significantly advance the study of categorical algebra or methods, or that make significant new contributions to mathematical science using categorical methods. The scope of the journal includes: all areas of pure category theory, including higher dimensional categories; applications of category theory to algebra, geometry and topology and other areas of mathematics; applications of category theory to computer science, physics and other mathematical sciences; contributions to scientific knowledge that make use of categorical methods.
Articles appearing in the journal have been carefully and critically refereed under the responsibility of members of the Editorial Board. Only papers judged to be both significant and excellent are accepted for publication.
Full text of the journal is freely available in .dvi, Postscript and PDF from the journal's server at `http://www.tac.mta.ca/tac/` and by ftp. It is archived electronically and in printed paper format.

SUBSCRIPTION INFORMATION. Individual subscribers receive abstracts of articles by e-mail as they are published. To subscribe, send e-mail to `tac@mta.ca` including a full name and postal address. For institutional subscription, send enquiries to the Managing Editor, Robert Rosebrugh, `rrosebrugh@mta.ca`.

INFORMATION FOR AUTHORS. The typesetting language of the journal is TEX, and LATEX2e strongly encouraged. Articles should be submitted by e-mail directly to a Transmitting Editor. Please obtain detailed information on submission format and style files at `http://www.tac.mta.ca/tac/`.

MANAGING EDITOR. Robert Rosebrugh, Mount Allison University: `rrosebrugh@mta.ca`

TEXNICAL EDITOR. Michael Barr, McGill University: `barr@math.mcgill.ca`

ASSISTANT TEX EDITOR. Gavin Seal, Ecole Polytechnique Fédérale de Lausanne: `gavin_seal@fastmail.fm`

TRANSMITTING EDITORS.
Clemens Berger, Université de Nice-Sophia Antipolis, `cberger@math.unice.fr`
Richard Blute, Université d' Ottawa: `rblute@uottawa.ca`
Lawrence Breen, Université de Paris 13: `breen@math.univ-paris13.fr`
Ronald Brown, University of North Wales: `ronnie.profbrown (at) btinternet.com`
Aurelio Carboni, Università dell Insubria: `aurelio.carboni@uninsubria.it`
Valeria de Paiva, Cuill Inc.: `valeria@cuill.com`
Ezra Getzler, Northwestern University: `getzler(at)northwestern(dot)edu`
Martin Hyland, University of Cambridge: `M.Hyland@dpmms.cam.ac.uk`
P. T. Johnstone, University of Cambridge: `ptj@dpmms.cam.ac.uk`
Anders Kock, University of Aarhus: `kock@imf.au.dk`
Stephen Lack, University of Western Sydney: `s.lack@uws.edu.au`
F. William Lawvere, State University of New York at Buffalo: `wlawvere@acsu.buffalo.edu`
Tom Leinster, University of Glasgow, `T.Leinster@maths.gla.ac.uk`
Jean-Louis Loday, Université de Strasbourg: `loday@math.u-strasbg.fr`
Ieke Moerdijk, University of Utrecht: `moerdijk@math.uu.nl`
Susan Niefield, Union College: `niefiels@union.edu`
Robert Paré, Dalhousie University: `pare@mathstat.dal.ca`
Jiri Rosicky, Masaryk University: `rosicky@math.muni.cz`
Brooke Shipley, University of Illinois at Chicago: `bshipley@math.uic.edu`
James Stasheff, University of North Carolina: `jds@math.unc.edu`
Ross Street, Macquarie University: `street@math.mq.edu.au`
Walter Tholen, York University: `tholen@mathstat.yorku.ca`
Myles Tierney, Rutgers University: `tierney@math.rutgers.edu`
Robert F. C. Walters, University of Insubria: `robert.walters@uninsubria.it`
R. J. Wood, Dalhousie University: `rjwood@mathstat.dal.ca`