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BICATEGORIES OF SPANS AS CARTESIAN BICATEGORIES

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Abstract. Bicategories of spans are characterized as cartesian bicategories in which every comonad has an Eilenberg-Moore object and every left adjoint arrow is comonadic.

1. Introduction

Let $\mathcal{E}$ be a category with finite limits. For the bicategory $\mathrm{Span}\,\mathcal{E}$, the locally full subbicategory $\mathrm{MapSpan}\,\mathcal{E}$ determined by the left adjoint arrows is essentially locally discrete, meaning that each hom category $\mathrm{MapSpan}\,\mathcal{E}(X,A)$ is an equivalence relation, and so is equivalent to a discrete category. Indeed, a span $x\colon X \leftarrow S \rightarrow A\colon a$ has a right adjoint if and only if $x\colon S \rightarrow X$ is invertible. The functors

$$\mathrm{MapSpan}\,\mathcal{E}(X,A) \longrightarrow \mathcal{E}(X,A) \quad \text{given by} \quad (x,a) \mapsto ax^{-1}$$

provide equivalences of categories which are the effects on homs for a biequivalence

$$\mathrm{MapSpan}\,\mathcal{E} \rightarrow \mathcal{E}\,.$$

Since $\mathcal{E}$ has finite products, $\mathrm{MapSpan}\,\mathcal{E}$ has finite products *as a bicategory*. We refer the reader to [CKWW] for a thorough treatment of bicategories with finite products. Each hom category $\mathrm{Span}\,\mathcal{E}(X,A)$ is *isomorphic* to the slice category $\mathcal{E}/(X \times A)$ which has binary products given by pullback in $\mathcal{E}$ and terminal object $1\colon X \times A \rightarrow X \times A$. Thus $\mathrm{Span}\,\mathcal{E}$ is a *precartesian bicategory* in the sense of [CKWW]. The canonical lax monoidal structure

$$\mathrm{Span}\,\mathcal{E} \times \mathrm{Span}\,\mathcal{E} \longrightarrow \mathrm{Span}\,\mathcal{E} \longleftarrow \mathbf{1}$$

for this precartesian bicategory is seen to have its binary aspect given on arrows by

$$(X \xleftarrow{x} S \xrightarrow{y} A\,,\; Y \xleftarrow{y} T \xrightarrow{b} B) \mapsto (X \times Y \xleftarrow{x\times y} S \times T \xrightarrow{a\times b} A \times B)\,,$$

and its nullary aspect provided by

$$1 \xleftarrow{1} 1 \xrightarrow{1} 1\,,$$

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the terminal object of $\mathrm{Span}\,\mathcal{E}(1,1)$. Both of these lax functors are readily seen to be pseudofunctors so that $\mathrm{Span}\,\mathcal{E}$ is a *cartesian bicategory* as in [CKWW].

The purpose of this paper is to characterize those cartesian bicategories $\mathbf{B}$ which are biequivalent to $\mathrm{Span}\,\mathcal{E}$, for some category $\mathcal{E}$ with finite limits. Certain aspects of a solution to the problem are immediate. A biequivalence $\mathbf{B} \sim \mathrm{Span}\,\mathcal{E}$ provides

$$\mathrm{Map}\mathbf{B} \sim \mathrm{MapSpan}\,\mathcal{E} \sim \mathcal{E}$$

so that we must ensure firstly that Map$\mathbf{B}$ is essentially locally discrete. From the characterization of bicategories of relations as locally ordered cartesian bicategories in [C&W] one suspects that the following axiom will figure prominently in providing essential local discreteness for Map$\mathbf{B}$.

1.1. Axiom. Frobenius: *A cartesian bicategory $\mathbf{B}$ is said to satisfy the* Frobenius *axiom if, for each A in $\mathbf{B}$, A is Frobenius.*

Indeed Frobenius objects in cartesian bicategories were defined and studied in [W&W] where amongst other things it is shown that if A is Frobenius in cartesian $\mathbf{B}$ then, for all X, $\mathrm{Map}\mathbf{B}(X,A)$ is a groupoid. (This theorem was generalized considerably in [LSW] which explained further aspects of the Frobenius concept.) However, essential local discreteness for Map$\mathbf{B}$ requires also that the $\mathrm{Map}\mathbf{B}(X,A)$ be ordered sets (which is automatic for locally ordered $\mathbf{B}$). Here we study also *separable* objects in cartesian bicategories for which we are able to show that if A is separable in cartesian $\mathbf{B}$ then, for all X, $\mathrm{Map}\mathbf{B}(X,A)$ is an ordered set and a candidate axiom is:

1.2. Axiom. Separable: *A cartesian bicategory $\mathbf{B}$ is said to satisfy the* Separable *axiom if, for each A in $\mathbf{B}$, A is separable.*

In addition to essential local discreteness, it is clear that we will need an axiom which provides *tabulation* of each arrow of $\mathbf{B}$ by a span of maps. Since existence of Eilenberg-Moore objects is a basic 2-dimensional limit concept, we will express tabulation in terms of this requirement; we note that existence of pullbacks in Map$\mathbf{B}$ follows easily from tabulation. In the bicategory $\mathrm{Span}\,\mathcal{E}$, the comonads $G : A \nrightarrow A$ are precisely the symmetric spans $g : A \nleftarrow X \nrightarrow A : g$; the map $g : X \nrightarrow A$ together with $g\eta_g : g \nrightarrow gg^*g$ provides an Eilenberg-Moore coalgebra for $g : A \nleftarrow X \nrightarrow A : g$. We will posit:

1.3. Axiom. Eilenberg-Moore for Comonads: *Each comonad (A,G) in $\mathbf{B}$ has an Eilenberg-Moore object.*

Conversely, any map (left adjoint) $g : X \nrightarrow A$ in $\mathrm{Span}\,\mathcal{E}$ provides an Eilenberg-Moore object for the comonad gg^*. We further posit:

1.4. Axiom. Maps are Comonadic: *Each left adjoint $g : X \nrightarrow A$ in $\mathbf{B}$ is comonadic.*

from which, in our context, we can also deduce the Frobenius and Separable axioms.

In fact we shall also give, in Proposition 3.1 below, a straightforward proof that Map$\mathbf{B}$ is locally essentially discrete whenever Axiom 1.4 holds. But due to the importance of the Frobenius and separability conditions in other contexts, we have chosen to analyze them in their own right.

2. Preliminaries

We recall from [CKWW] that a bicategory $\mathbf{B}$ (always, for convenience, assumed to be normal) is said to be *cartesian* if the subbicategory of maps (by which we mean left adjoint arrows), $\mathbf{M} = \mathrm{Map}\mathbf{B}$, has finite products $- \times -$ and 1; each hom-category $\mathbf{B}(B, C)$ has finite products $- \wedge -$ and $\top$; and a certain derived tensor product $- \otimes -$ and I on $\mathbf{B}$, extending the product structure of $\mathbf{M}$, is functorial. As in [CKWW], we write p and r for the first and second projections at the global level, and similarly π and ρ for the projections at the local level. If f is a map of $\mathbf{B}$ — an arrow of $\mathbf{M}$ — we will write $\eta_f, \epsilon_f : f \dashv f^*$ for a chosen adjunction in $\mathbf{B}$ that makes it so. It was shown that the derived tensor product of a cartesian bicategory underlies a symmetric monoidal bicategory structure. We recall too that in [W&W] Frobenius objects in a general cartesian bicategory were defined and studied. We will need the central results of that paper too. Throughout this paper, $\mathbf{B}$ is assumed to be a cartesian bicategory.

As in [CKWW] we write

$$\begin{array}{ccccc} & & \mathbf{G} = \mathrm{Gro}\mathbf{B} & & \\ & \overset{\partial_0}{\swarrow} & & \overset{\partial_1}{\searrow} & \\ \mathbf{M} & & & & \mathbf{M} \end{array}$$

for the Grothendieck span corresponding to

$$\mathbf{M}^{\mathrm{op}} \times \mathbf{M} \xrightarrow{i^{\mathrm{op}} \times i} \mathbf{B}^{\mathrm{op}} \times \mathbf{B} \xrightarrow{\mathbf{B}(-,-)} \mathbf{CAT}$$

where $i : \mathbf{M} \to \mathbf{B}$ is the inclusion. A typical arrow of $\mathbf{G}$, $(f, \alpha, u) : (X, R, A) \to (Y, S, B)$ can be depicted by a square

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ {\scriptstyle R}\downarrow & \xrightarrow{\ \alpha\ } & \downarrow{\scriptstyle S} \\ A & \xrightarrow[\ u\ ]{} & B \end{array} \tag{1}$$

and such arrows are composed by pasting. A 2-cell $(\phi, \psi) : (f, \alpha, u) \to (g, \beta, v)$ in $\mathbf{G}$ is a pair of 2-cells $\phi : f \to g$, $\psi : u \to v$ in $\mathbf{M}$ which satisfy the obvious equation. The (strict) pseudofunctors ∂_0 and ∂_1 should be regarded as *domain* and *codomain* respectively. Thus, applied to (1), ∂_0 gives f and ∂_1 gives u. The bicategory $\mathbf{G}$ also has finite products, which are given on objects by $- \otimes -$ and I; these are preserved by ∂_0 and ∂_1.

The Grothendieck span can also be thought of as giving a double category (of a suitably weak flavour), although we shall not emphasize that point of view.

2.1. The arrows of $\mathbf{G}$ are particularly well suited to relating the various product structures in a cartesian bicategory. In 3.31 of [CKWW] it was shown that the local binary

product, for $R, S{:}X \rightrightarrows A$, can be recovered to within isomorphism from the defined tensor product by

$$R \wedge S \cong d_A^*(R \otimes S)d_X$$

A slightly more precise version of this is that the mate of the isomorphism above, with respect to the single adjunction $d_A \dashv d_A^*$, defines an arrow in $\mathbf{G}$

$$\begin{array}{ccc} X & \xrightarrow{d_X} & X \otimes X \\ {\scriptstyle R\wedge S}\downarrow & \longrightarrow & \downarrow{\scriptstyle R\otimes S} \\ A & \xrightarrow[d_A]{} & A \otimes A \end{array}$$

which when composed with the projections of $\mathbf{G}$, recovers the local projections as in

$$\begin{array}{ccccc} X & \xrightarrow{d_X} & X \otimes X & \xrightarrow{p_{X,X}} & X \\ {\scriptstyle R\wedge S}\downarrow & \longrightarrow & {\scriptstyle R\otimes S}\downarrow & \multimap_{\tilde{p}_{R,S}} & \downarrow{\scriptstyle R} \\ A & \xrightarrow[d_A]{} & A \otimes A & \xrightarrow[p_{A,A}]{} & A \end{array} \quad \cong \quad \begin{array}{ccc} X & \xrightarrow{1_X} & X \\ {\scriptstyle R\wedge S}\downarrow & \multimap_{\pi} & \downarrow{\scriptstyle R} \\ A & \xrightarrow[1_A]{} & A \end{array}$$

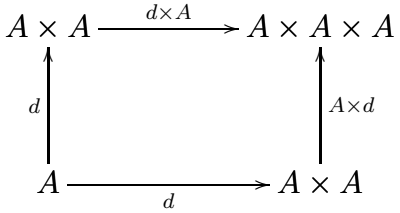
for the first projection, and similarly for the second. The unspecified $\cong$ in $\mathbf{G}$ is given by a pair of convenient isomorphisms $p_{X,X}d_X \cong 1_X$ and $p_{A,A}d_A \cong 1_A$ in $\mathbf{M}$. Similarly, when $R \wedge S \nrightarrow R \otimes S$ is composed with $(r_{X,X}, \tilde{r}_{R,S}, r_{A,A})$ the result is $(1_X, \rho, 1_A){:}R \wedge S \nrightarrow S$.

2.2. Quite generally, an arrow of $\mathbf{G}$ as given by the square (1) will be called a *commutative* square if α is invertible. An arrow of $\mathbf{G}$ will be said to satisfy the *Beck condition* if the mate of α under the adjunctions $f \dashv f^*$ and $u \dashv u^*$, as given in the square below (no longer an arrow of $\mathbf{G}$), is invertible.

$$\begin{array}{ccc} X & \xleftarrow{f^*} & Y \\ {\scriptstyle R}\downarrow & \multimap_{\alpha^*} & \downarrow{\scriptstyle S} \\ A & \xleftarrow[u^*]{} & B \end{array}$$

Thus Proposition 4.7 of [CKWW] says that projection squares of the form $\tilde{p}_{R,1_Y}$ and $\tilde{r}_{1_X,S}$ are commutative while Proposition 4.8 of [CKWW] says that these same squares satisfy the Beck condition. If R and S are also maps and α is invertible then α^{-1} gives rise to another arrow of $\mathbf{G}$, from f to u with reference to the square above, which may or may not satisfy the Beck condition. The point here is that a commutative square of maps gives rise to two, generally distinct, Beck conditions. It is well known that, for bicategories of the form $\mathrm{Span}\,\mathcal{E}$ and $\mathrm{Rel}\,\mathcal{E}$, all pullback squares of maps satisfy both Beck conditions. A

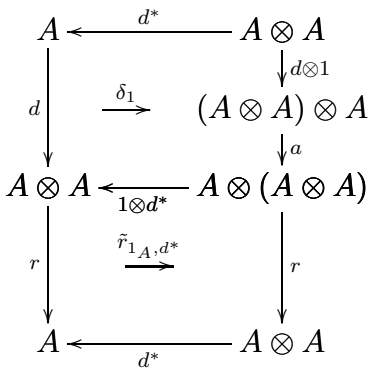
category with finite products has automatically a number of pullbacks which we might call *product-absolute* pullbacks because they are preserved by all functors which preserve products. In [W&W] the Beck conditions for the product-absolute pullback squares of the form



were investigated. (In fact, in this case it was shown that either Beck condition implies the other.) The objects for which these conditions are met are called *Frobenius* objects.

2.3. Proposition. *For a cartesian bicategory, the axiom* Maps are Comonadic *implies the axiom* Frobenius.

Proof. It suffices to show that the 2-cell δ_1 below is invertible:



The paste composite of the squares is invertible (being essentially the identity 2-cell on d^*). The lower 2-cell is invertible by Proposition 4.7 of [CKWW] so that the whisker composite $r\delta_1$ is invertible. Since r is a map it reflects isomorphisms, by Maps are Comonadic, and hence δ_1 is invertible. ∎

2.4. Remark. It was shown in [W&W] that, in a cartesian bicategory, the Frobenius objects are closed under finite products. It follows that the full subbicategory of a cartesian bicategory determined by the Frobenius objects is a cartesian bicategory which satisfies the Frobenius axiom.

3. Separable Objects and Discrete Objects in Cartesian Bicategories

In this section we look at separability for objects of cartesian bicategories. Since for an object A which is both separable and Frobenius, the hom-category Map$\mathbf{B}(X, A)$ is essentially discrete, for all X, we shall then be able to show that Map$\mathbf{B}$ is essentially discrete by showing that all objects in $\mathbf{B}$ are separable and Frobenius. But first we record the following direct argument:

3.1. PROPOSITION. *If* $\mathbf{B}$ *is a bicategory in which all maps are comonadic and* Map$\mathbf{B}$ *has a terminal object, then* Map$\mathbf{B}$ *is locally essentially discrete.*

PROOF. We must show that for all objects X and A, the hom-category Map$\mathbf{B}(X, A)$ is essentially discrete. As usual, we write 1 for the terminal object of Map$\mathbf{B}$ and $t_A : A \longrightarrow 1$ for the essentially unique map, which by assumption is comonadic. Let $f, g : X \longrightarrow A$ be maps from X to A. If $\alpha : f \longrightarrow g$ is any 2-cell, then $t_A\alpha$ is invertible, since 1 is terminal in Map$\mathbf{B}$. But since t_A is comonadic, it reflects isomorphisms, and so α is invertible. Furthermore, if $\beta : f \longrightarrow g$ is another 2-cell, then $t_A\alpha = t_A\beta$ by the universal property of 1 once again, and now $\alpha = \beta$ since t_A is faithful. Thus there is at most one 2-cell from f to g, and any such 2-cell is invertible. ∎

In any (bi)category with finite products the diagonal arrows $d_A : A \nrightarrow A \times A$ are (split) monomorphisms so that in the bicategory $\mathbf{M}$ the following square is a product-absolute pullback

$$\begin{array}{ccc} A & \xrightarrow{1_A} & A \\ {\scriptstyle 1_A}\downarrow & & \downarrow{\scriptstyle d_A} \\ A & \xrightarrow[d_A]{} & A \otimes A \end{array}$$

that gives rise to a single $\mathbf{G}$ arrow.

3.2. DEFINITION. *An object* A *in a cartesian bicategory is said to be* separable *if the* $\mathbf{G}$ *arrow above satisfies the Beck condition.*

Of course the invertible mate condition here says precisely that the unit $\eta_{d_A} : 1_A \nrightarrow d_A^* d_A$ for the adjunction $d_A \dashv d_A^*$ is invertible. Thus Axiom 1.2, as stated in the Introduction, says that, for all A in $\mathbf{B}$, η_{d_A} is invertible.

3.3. REMARK. For a map f it makes sense to define f *is fully faithful* to mean that η_f is invertible. For a *category* A the diagonal d_A is fully faithful if and only if A is an ordered set.

3.4. PROPOSITION. *For an object* A *in a cartesian bicategory, the following are equivalent:*

i) A *is separable;*

ii) *for all* $f : X \nrightarrow A$ *in* $\mathbf{M}$*, the diagram* $f \nleftarrow f \nrightarrow f$ *is a product in* $\mathbf{B}(X, A)$*;*

iii) $1_A \nleftarrow 1_A \nrightarrow 1_A$ *is a product in* $\mathbf{B}(A, A)$*;*

iv) $1_A \nrightarrow \top_{A,A}$ *is a monomorphism in* $\mathbf{B}(A, A)$*;*

v) *for all* $G \nrightarrow 1_A$ *in* $\mathbf{B}(A, A)$*, the diagram* $G \nleftarrow G \nrightarrow 1_A$ *is a product in* $\mathbf{B}(A, A)$.

PROOF. $[i) \Longrightarrow ii)]$ A local product of maps is not generally a map but here we have:

$$f \wedge f \cong d_A^*(f \otimes f)d_X \cong d_A^*(f \times f)d_X \cong d_A^* d_A f \cong f$$

$[ii) \Longrightarrow iii)]$ is trivial.

$[iii) \Longrightarrow i)]$ Note the use of pseudo-functoriality of $\otimes$:

$$d_A^* d_A \cong d_A^* 1_{A\otimes A} d_A \cong d_A^*(1_A \otimes 1_A)d_A \cong 1_A \wedge 1_A \cong 1_A$$

$[iii) \Longrightarrow iv)]$ To say that $1_A \leftarrowtail 1_A \rightarrowtail 1_A$ is a product in $\mathbf{B}(A, A)$ is precisely to say that

$$\begin{array}{ccc} 1_A & \xrightarrow{1_{1_A}} & 1_A \\ {\scriptstyle 1_{1_A}}\downarrow & & \downarrow \\ 1_A & \longrightarrow & \top_{A,A} \end{array}$$

is a pullback in $\mathbf{B}(A, A)$ which in turn is precisely to say that $1_A \rightarrowtail \top_{A,A}$ is a monomorphism in $\mathbf{B}(A, A)$

$[iv) \Longrightarrow v)]$ It is a generality that if an object S in a category is subterminal then for any $G \rightarrowtail S$, necessarily unique, $G \leftarrowtail G \rightarrowtail S$ is a product diagram.

$[v) \Longrightarrow iii)]$ is trivial. ∎

3.5. COROLLARY. [Of $iv)$] *For a cartesian bicategory, the axiom* Maps are Comonadic *implies the axiom* Separable.

PROOF. We have $\top_{A,A} = t_A^* t_A$ for the map $t_A : A \rightarrowtail 1$. It follows that the unique $1_A \rightarrowtail t_A^* t_A$ is η_{t_A}. Since t_A is comonadic, η_{t_A} is the equalizer shown:

$$1_A \xrightarrow{\eta_{t_A}} t_A^* t_A \underset{\eta_{t_A} t_A^* t_A}{\overset{t_A^* t_A \eta_{t_A}}{\rightrightarrows}} t_A^* t_A t_A^* t_A$$

and hence a monomorphism. ∎

3.6. COROLLARY. [Of $iv)$] *For separable A in cartesian $\mathbf{B}$, an arrow $G : A \rightarrowtail A$ admits at most one copoint $G \rightarrowtail 1_A$ depending upon whether the unique arrow $G \rightarrowtail \top_{A,A}$ factors through $1_A \rightarrowtail\!\!\!\!\!\rightarrow \top_{A,A}$.* ∎

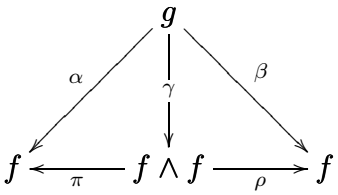
3.7. PROPOSITION. *In a cartesian bicategory, the separable objects are closed under finite products.*

PROOF. If A and B are separable objects then applying the homomorphism $\otimes : \mathbf{B} \times \mathbf{B} \rightarrowtail \mathbf{B}$ we have an adjunction $d_A \times d_B \dashv d_A^* \otimes d_B^*$ with unit $\eta_{d_A} \otimes \eta_{d_B}$ which being an isomorph of the adjunction $d_{A\otimes B} \dashv d_{A\otimes B}^*$ with unit $\eta_{d_{A\otimes B}}$ (via middle-four interchange) shows that the separable objects are closed under binary products. On the other hand, d_I is an equivalence so that I is also separable. ∎

3.8. Corollary. *For a cartesian bicategory, the full subbicategory determined by the separable objects is a cartesian bicategory which satisfies the axiom* Separable. ∎

3.9. Proposition. *If A is a separable object in a cartesian bicategory $\mathbf{B}$, then, for all X in $\mathbf{B}$, the hom-category $\mathbf{M}(X, A)$ is an ordered set, meaning that the category structure forms a reflexive, transitive relation.*

Proof. Suppose that we have arrows $\alpha, \beta : g \rightrightarrows f$ in $\mathbf{M}(X, A)$. In $\mathbf{B}(X, A)$ we have



By Proposition 3.4 we can take $f \wedge f = f$ and $\pi = 1_f = \rho$ so that we have $\alpha = \gamma = \beta$. It follows that $\mathbf{M}(X, A)$ is an ordered set. ∎

3.10. Definition. *An object A in a cartesian bicategory is said to be* discrete *if it is both Frobenius and separable. We write* Dis$\mathbf{B}$ *for the full subbicategory of $\mathbf{B}$ determined by the discrete objects.*

3.11. Remark. Beware that this is quite different to the notion of discreteness in a bicategory. An object A of a bicategory is discrete if each hom-category $\mathbf{B}(X, A)$ is discrete; A is essentially discrete if each $\mathbf{B}(X, A)$ is equivalent to a discrete category. The notion of discreteness for cartesian bicategories defined above turns out to mean that A is essentially discrete in the bicategory Map$\mathbf{B}$.

From Proposition 3.7 above and Proposition 3.4 of [W&W] we immediately have

3.12. Proposition. *For a cartesian bicategory $\mathbf{B}$, the full subbicategory* Dis$\mathbf{B}$ *of discrete objects is a cartesian bicategory in which every object is discrete.* ∎

And from Proposition 3.9 above and Theorem 3.13 of [W&W] we have

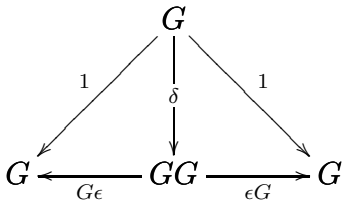
3.13. Proposition. *If A is a discrete object in a cartesian bicategory $\mathbf{B}$ then, for all X in $\mathbf{B}$, the hom category $\mathbf{M}(X, A)$ is an equivalence relation.* ∎

If both the *Frobenius* axiom of [W&W] and the *Separable* axiom of this paper hold for our cartesian bicategory $\mathbf{B}$, then every object of $\mathbf{B}$ is discrete. In this case, because $\mathbf{M}$ is a bicategory, the equivalence relations $\mathbf{M}(X, A)$ are stable under composition from both sides. Thus writing $|\mathbf{M}(X, A)|$ for the set of objects of $\mathbf{M}(X, A)$ we have a mere category, $\mathcal{E}$ whose objects are those of $\mathbf{M}$ (and hence also those of $\mathbf{B}$) and whose hom sets are the quotients $|\mathbf{M}(X, A)|/\mathbf{M}(X, A)$. If the $\mathcal{E}(X, A)$ are regarded as discrete categories, so that $\mathcal{E}$ is a locally discrete bicategory then the functors $\mathbf{M}(X, A) \to |\mathbf{M}(X, A)|/\mathbf{M}(X, A)$ constitute the effect on homs functors for an identity on objects biequivalence $\mathbf{M} \to \mathcal{E}$. To summarize

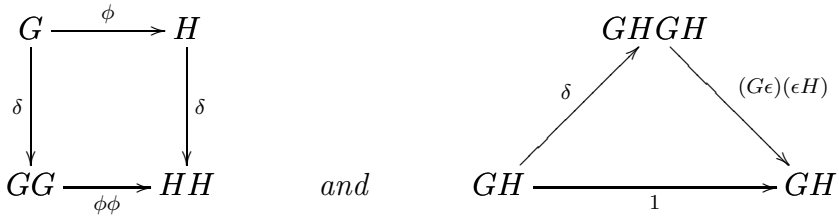
3.14. Theorem. *If a cartesian bicategory $\mathbf{B}$ satisfies both the Frobenius and Separable axioms then the bicategory of maps $\mathbf{M}$ is biequivalent to the locally discrete bicategory $\mathcal{E}$.* ∎

In the following lemma we show that any copointed endomorphism of a discrete object can be made into a comonad; later on, we shall see that this comonad structure is in fact unique.

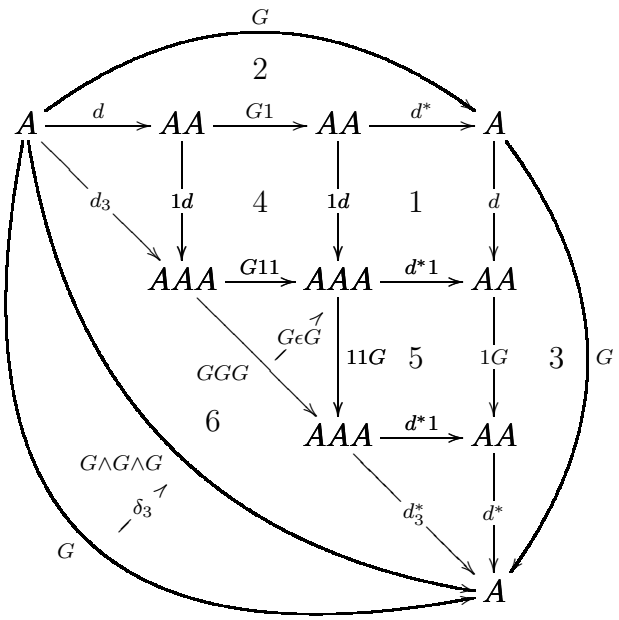
3.15. LEMMA. *If A is a discrete object in a cartesian bicategory $\mathbf{B}$ then, for any copointed endomorphism arrow $\epsilon : G \to 1_A : A \to A$, there is a 2-cell $\delta = \delta_G : G \to GG$ satisfying*



and if both $G, H : A \rightrightarrows A$ are copointed, so that $GH : A \to A$ is also copointed, and $\phi : G \to H$ is any 2-cell, then the δ's satisfy



PROOF. We define $\delta = \delta_G$ to be the pasting composite



wherein $\otimes$ has been abbreviated by juxtaposition and all subregions not explicitly inhabited by a 2-cell are deemed to be inhabited by the obvious invertible 2-cell. A reference number has been assigned to those invertible 2-cells which arise from the hypotheses. As

in [W&W], d_3's denote 3-fold diagonal maps and, similarly, we write δ_3 for a local 3-fold diagonal.

The invertible 2-cell labelled by '1' is that defining A to be Frobenius. The 3-fold composite of arrows in the region labelled by '2' is $G \wedge 1_A$ and, similarly, in that labelled by '3' we have $1_A \wedge G$. Each of these is isomorphic to G because A is separable and G is copointed. The isomorphisms in '4' and '5' express the pseudo-functoriality of $\otimes$ in the cartesian bicategory $\mathbf{B}$. Finally '6' expresses the ternary local product in terms of the ternary $\otimes$ as in [W&W]. Demonstration of the equations is effected easily by pasting composition calculations. ∎

3.16. THEOREM. *If G and H are copointed endomorphisms on a discrete A in a cartesian $\mathbf{B}$ then*

$$G \xleftarrow{G\epsilon} GH \xrightarrow{\epsilon H} H$$

is a product diagram in $\mathbf{B}(A, A)$.

PROOF. If we are given $\alpha : K \nrightarrow G$ and $\beta : K \nrightarrow H$ then K is also copointed and we have

$$K \xrightarrow{\delta} KK \xrightarrow{\alpha\beta} GH$$

as a candidate pairing. That this candidate satisfies the universal property follows from the equations of Lemma 3.15 which are precisely those in the equational description of binary products. We remark that the 'naturality' equations for the projections follow immediately from uniqueness of copoints. ∎

3.17. COROLLARY. *If A is discrete in a cartesian $\mathbf{B}$, then an endo-arrow $G : A \nrightarrow A$ admits a comonad structure if and only if G has the copointed property, and any such comonad structure is unique.*

PROOF. The Theorem shows that the arrow $\delta : G \nrightarrow GG$ constructed in Lemma 3.15 is the product diagonal on G in the category $\mathbf{B}(A, A)$ and, given $\epsilon : G \nrightarrow 1_A$, this is the only comonad comultiplication on G. ∎

3.18. REMARK. It is clear that 1_A is terminal with respect to the copointed objects in $\mathbf{B}(A, A)$.

3.19. PROPOSITION. *If an object B in a bicategory $\mathbf{B}$ has 1_B subterminal in $\mathbf{B}(B, B)$ then, for any map $f : A \nrightarrow B$, f is subterminal in $\mathbf{B}(A, B)$ and f^* is subterminal in $\mathbf{B}(B, A)$. In particular, in a cartesian bicategory in which every object is separable, every adjoint arrow is subterminal.*

PROOF. Precomposition with a map preserves terminal objects and monomorphisms, as does postcomposition with a right adjoint. ∎

4. Bicategories of Comonads

The starting point of this section is the observation, made in the introduction, that a comonad in the bicategory $\mathrm{Span}\,\mathcal{E}$ is precisely a span of the form

$$A \xleftarrow{g} X \xrightarrow{g} A$$

in which both legs are equal.

We will write $\mathbf{C} = \mathrm{Com}\mathbf{B}$ for the bicategory of comonads in $\mathbf{B}$, Com being one of the duals of Street's construction Mnd in [ST]. Thus $\mathbf{C}$ has objects given by the comonads (A, G) of $\mathbf{B}$. The structure 2-cells for comonads will be denoted $\epsilon = \epsilon_G$, for counit and $\delta = \delta_G$, for comultiplication. An arrow in $\mathbf{C}$ from (A, G) to (B, H) is a pair (F, ϕ) as shown in

$$\begin{array}{ccc} A & \xrightarrow{F} & B \\ {\scriptstyle G}\downarrow & \xrightarrow{\phi} & \downarrow{\scriptstyle H} \\ A & \xrightarrow[F]{} & B \end{array}$$

satisfying

$$\begin{array}{ccc} FG & \xrightarrow{\phi} & HF \\ {\scriptstyle F\epsilon}\downarrow & & \downarrow{\scriptstyle \epsilon F} \\ F1_A & \xrightarrow[=]{} & 1_B F \end{array} \qquad \text{and} \qquad \begin{array}{ccccc} FG & & \xrightarrow{\phi} & & HF \\ {\scriptstyle F\delta}\downarrow & & & & \downarrow{\scriptstyle \delta F} \\ FGG & \xrightarrow[\phi G]{} & HFG & \xrightarrow[H\phi]{} & HHF \end{array} \tag{2}$$

(where, as often, we have suppressed the associativity constraints of our normal, cartesian, bicategory $\mathbf{B}$). A 2-cell $\tau{:}(F, \phi) \to (F', \phi'){:}(A, G) \to (B, H)$ in $\mathbf{C}$ is a 2-cell $\tau{:}F \to F'$ in $\mathbf{B}$ satisfying

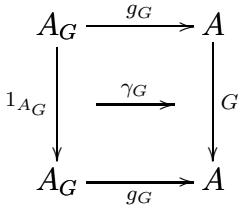
$$\begin{array}{ccc} FG & \xrightarrow{\phi} & HF \\ {\scriptstyle \tau G}\downarrow & & \downarrow{\scriptstyle H\tau} \\ F'G & \xrightarrow[\phi']{} & HF' \end{array} \tag{3}$$

There is a pseudofunctor $I{:}\mathbf{B} \to \mathbf{C}$ given by

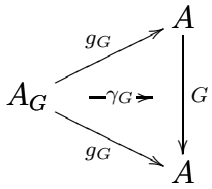
$$I(\tau{:}F \to F'{:}A \to B) = \tau{:}(F, 1_F) \to (F', 1_{F'}){:}(A, 1_A) \to (B, 1_B)$$

From [ST] it is well known that a bicategory $\mathbf{B}$ has Eilenberg-Moore objects for comonads if and only if $I{:}\mathbf{B} \to \mathbf{C}$ has a right biadjoint, which we will denote by $E{:}\mathbf{C} \to \mathbf{B}$. We write

$E(A, G) = A_G$ and the counit for $I \dashv E$ is denoted by



or, using normality of $\mathbf{B}$, better by



with (g_G, γ_G) abbreviated to (g, γ) when there is no danger of confusion. It is standard that each $g = g_G$ is necessarily a map (whence our lower case notation) and the mate $gg^* \to G$ of γ is an isomorphism which identifies ϵ_g and ϵ_G.

We will write $\mathbf{D}$ for the locally full subbicategory of $\mathbf{C}$ determined by all the objects and those arrows of the form (f, ϕ), where f is a map, and write $j\colon \mathbf{D} \to \mathbf{C}$ for the inclusion. It is clear that the pseudofunctor $I\colon \mathbf{B} \to \mathbf{C}$ restricts to give a pseudofunctor $J\colon \mathbf{M} \to \mathbf{D}$. We say that the bicategory $\mathbf{B}$ *has Eilenberg-Moore objects for comonads, as seen by* $\mathbf{M}$, if $J\colon \mathbf{M} \to \mathbf{D}$ has a right biadjoint. (In general, this property does not follow from that of Eilenberg-Moore objects for comonads.)

4.1. Remark. In the case $\mathbf{B} = \operatorname{Span} \mathcal{E}$, a comonad in $\mathbf{B}$ can, as we have seen, be identified with a morphism in E. This can be made into the object part of a biequivalence between the bicategory $\mathbf{D}$ and the category $\mathcal{E}^{\mathbf{2}}$ of arrows in $\mathcal{E}$. If we further identify $\mathbf{M}$ with $\mathcal{E}$, then the inclusion $j : \mathbf{D} \longrightarrow \mathbf{C}$ becomes the diagonal $\mathcal{E} \longrightarrow \mathcal{E}^{\mathbf{2}}$; of course this does have a right adjoint, given by the domain functor.

4.2. Theorem. *If* $\mathbf{B}$ *is a cartesian bicategory in which every object is discrete, the bicategory* $\mathbf{D} = \mathbf{D}(\mathbf{B})$ *admits the following simpler description:*

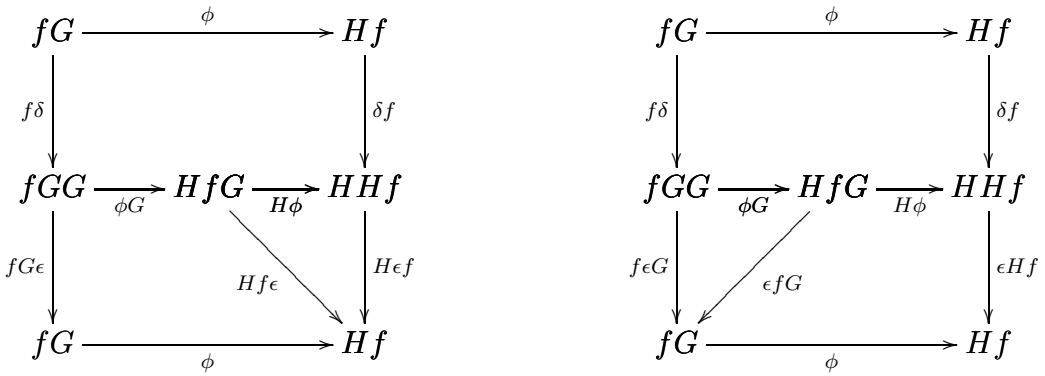
i) An object is a pair (A, G) *where* A *is an object of* $\mathbf{B}$ *and* $G\colon A \to A$ *admits a copoint;*

ii) An arrow $(f, \phi)\colon (A, G) \to (B, H)$ *is a map* $f\colon A \to B$ *and a 2-cell* $\phi\colon fG \to Hf$*;*

iii) A 2-cell $\tau\colon (f, \phi) \to (f', \phi')\colon (A, G) \to (B, H)$ *is a 2-cell satisfying* $\tau\colon f \to f'$ *satisfying equation (3).*

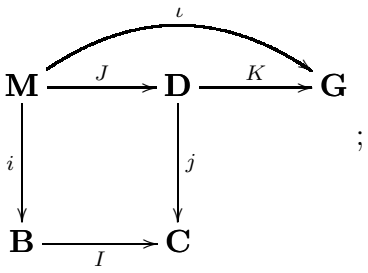
Proof. We have i) by Corollary 3.17 while iii) is precisely the description of a 2-cell in $\mathbf{D}$, modulo the description of the domain and codomain arrows. So, we have only to show ii), which is to show that the equations (2) hold automatically under the hypotheses. For the first equation of (2) we have uniqueness of any 2-cell $fG \to f$ because f is subterminal by Proposition 3.19. For the second, observe that the terminating vertex, HHf, is the product $Hf \wedge Hf$ in $\mathbf{M}(A, B)$ because HH is the product $H \wedge H$ in $\mathbf{M}(B, B)$ by Theorem 3.16 and precomposition with a map preserves all limits. For HHf seen as a product, the projections are, again by Theorem 3.16, $H\epsilon f$ and ϵHf. Thus, it suffices to show that the diagram for the second equation commutes when composed with both $H\epsilon f$

and ϵHf. We have



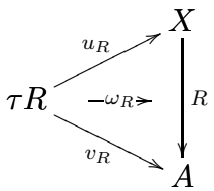
in which each of the lower triangles commutes by the first equation of (2) already established. Using comonad equations for G and H, it is obvious that each composite is ϕ. ∎

Finally, let us note that $\mathbf{D}$ is a subbicategory, neither full nor locally full, of the Grothendieck bicategory $\mathbf{G}$ and write $K{:}\mathbf{D} \nrightarrow \mathbf{G}$ for the inclusion. We also write $\iota{:}\mathbf{M} \nrightarrow \mathbf{G}$ for the composite pseudofunctor KJ. Summarizing, we have introduced the following commutative diagram of bicategories and pseudofunctors



;

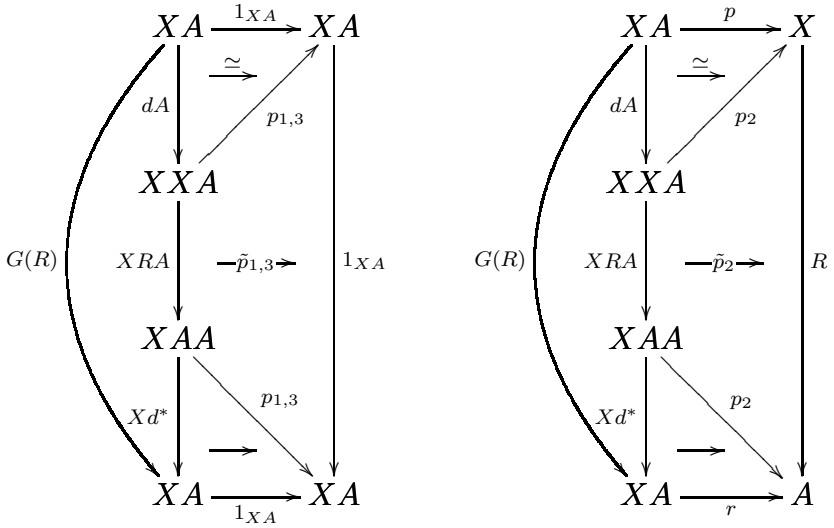
note also that in our main case of interest $\mathbf{B} = \operatorname{Span}\mathcal{E}$, each of $\mathbf{M}$, $\mathbf{D}$, and $\mathbf{G}$ is biequivalent to a mere category. Ultimately, we are interested in having a right biadjoint, say τ, of ι. For such a biadjunction $\iota \dashv \tau$ the counit at an object $R{:}X \nrightarrow A$ in $\mathbf{G}$ will take the form



(4)

(where, as for a biadjunction $I \dashv E{:}\mathbf{C} \nrightarrow \mathbf{B}$, a triangle rather than a square can be taken as the boundary of the 2-cell by the normality of $\mathbf{B}$). In fact, we are interested in the case where we have $\iota \dashv \tau$ and moreover the counit components $\omega_R{:}v_R \nrightarrow Ru_R$ enjoy the property that their mates $v_R u_R^* \nrightarrow R$ with respect to the adjunction $u_R \dashv u_R^*$ are invertible. In this way we represent a general arrow of $\mathbf{B}$ in terms of a span of maps. Since biadjunctions compose we will consider adjunctions $J \dashv F$ and $K \dashv G$ and we begin with the second of these.

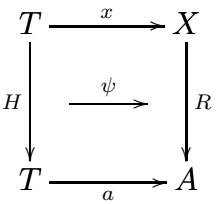
4.3. THEOREM. *For a cartesian bicategory* $\mathbf{B}$ *in which every object is discrete, there is an adjunction* $K \dashv G : \mathbf{G} \to \mathbf{D}$ *where, for* $R : X \to A$ *in* $\mathbf{G}$*, the comonad* $G(R)$ *and its witnessing copoint* $\epsilon : G(R) \to 1_{XA}$ *are given by the left diagram below and the counit* $\mu : KG(R) \to R$ *is given by the right diagram below, all in notation suppressing* $\otimes$*:*



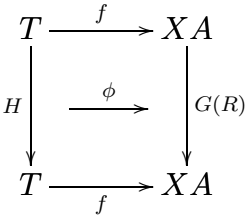
Moreover, the mate $rG(R)p^* \to R$ *of the counit* μ *is invertible.*

In the left diagram, the $p_{1,3}$ *collectively denote projection from the three-fold product in* $\mathbf{G}$ *to the product of the first and third factors. In the right diagram, the* p_2 *collectively denote projection from the three-fold product in* $\mathbf{G}$ *to the second factor. The upper triangles of the two diagrams are the canonical isomorphisms. The lower left triangle is the mate of the canonical isomorphism* $1 \xrightarrow{\simeq} p_{1,3}(Xd)$*. The lower right triangle is the mate of the canonical isomorphism* $r \xrightarrow{\simeq} p_2(Xd)$*.*

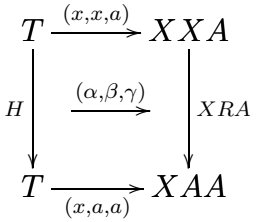
PROOF. Given a comonad $H : T \to T$ and an arrow



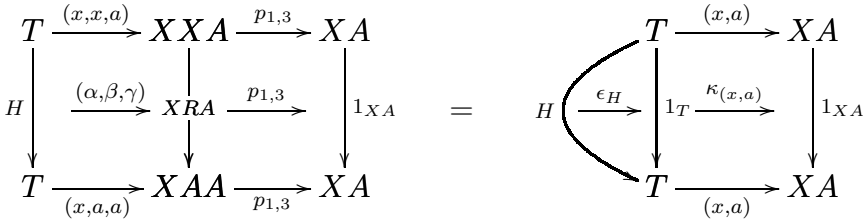
in $\mathbf{G}$, we verify the adjunction claim by showing that there is a unique arrow



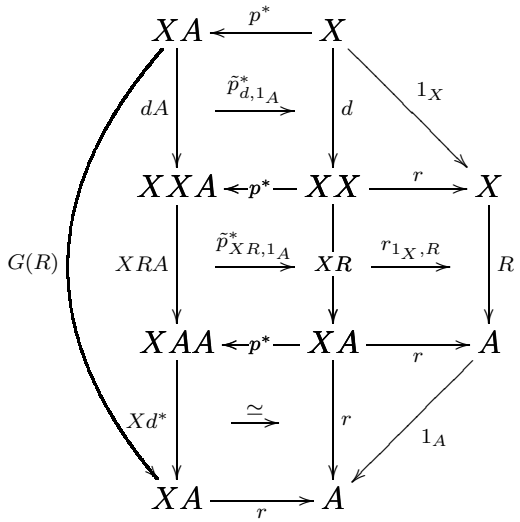
in $\mathbf{D}$, whose composite with the putative counit μ is (x, ψ, a). It is immediately clear that the unique solution for f is (x, a) and to give $\phi{:}(x,a)H \nrightarrow Xd^*(XRA)dA(x,a)$ is to give the mate $Xd(x,a)H \nrightarrow (XRA)dA(x,a)$ which is $(x,a,a)H \nrightarrow (XRA)(x,x,a)$ and can be seen as a $\mathbf{G}$ arrow:



where we exploit the description of products in $\mathbf{G}$. From this description it is clear, since $\tilde{p}_2(\alpha, \beta, \gamma) = \beta$ as a composite in $\mathbf{G}$, that the unique solution for β is ψ. We have seen in Theorem 3.17 that the conditions (2) hold automatically in $\mathbf{D}$ under the assumptions of the Theorem. From the first of these we have:

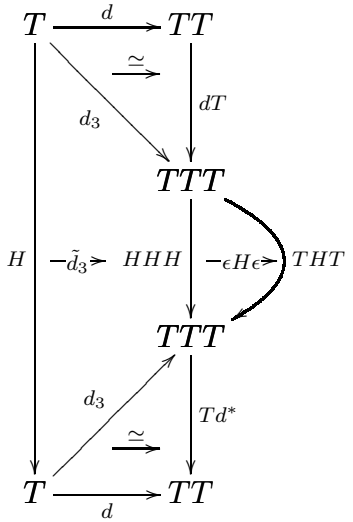


So, with a mild abuse of notation, we have $(\alpha, \gamma) = (1_x\epsilon_H, 1_a\epsilon_H)$, uniquely, and thus the unique solutions for α and γ are $1_x\epsilon_H$ and $1_a\epsilon_H$ respectively. This shows that ϕ is necessarily the mate under the adjunctions considered of $(1_x\epsilon_H, \psi, 1_a\epsilon_H,)$. Since $\mathbf{D}$ and $\mathbf{G}$ are essentially locally discrete this suffices to complete the claim that $K \dashv G$. It only remains to show that the mate $rG(R)p^* \nrightarrow R$ of the counit μ is invertible. In the three middle squares of the diagram



the top two are invertible 2-cells by Proposition 4.18 of [CKWW] while the lower one is the obvious invertible 2-cell constructed from $Xd^*p^* \cong 1_{X,A}$. The right square is an invertible 2-cell by Proposition 4.17 of [CKWW]. This shows that the mate $rG(R)p^* \to R$ of μ is invertible. ∎

4.4. Remark. It now follows that the unit of the adjunction $K \dashv G$ is given (in notation suppressing $\otimes$) by:



where the d_3 collectively denote 3-fold diagonalization $(1,1,1)$ in $\mathbf{G}$. The top triangle is a canonical isomorphism while the lower triangle is the mate of the canonical isomorphism $(T \otimes d)d \xrightarrow{\simeq} d_3$ and is itself invertible, by separability of T.

Before turning to the question of an adjunction $J \dashv F$, we note:

4.5. Lemma. *In a cartesian bicategory in which Maps are Comonadic, if $gF \cong h$ with g and h maps, then F is also a map.*

Proof. By Theorem 3.11 of [W&W] it suffices to show that F is a comonoid homomorphism, which is to show that the canonical 2-cells $\tilde{t}_F : tF \to t$ and $\tilde{d}_F : dF \to (F \otimes F)d$ are invertible. For the first we have:

$$tF \cong tgF \cong th \cong t$$

Simple diagrams show that we do get the right isomorphism in this case and also for the next:

$$(g \otimes g)(dF) \cong dgF \cong dh \cong (h \otimes h)d \cong (g \otimes g)(F \otimes F)d$$

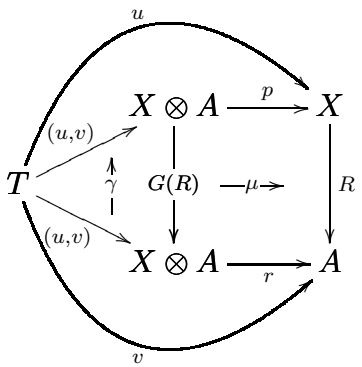
which gives $dF \cong (F \otimes F)d$ since the map $g \otimes g$ reflects isomorphisms. ∎

4.6. THEOREM. *If* $\mathbf{B}$ *is a cartesian bicategory which has Eilenberg-Moore objects for Comonads and for which Maps are Comonadic then* $\mathbf{B}$ *has Eilenberg-Moore objects for Comonads as Seen by* $\mathbf{M}$*, which is to say that* $J:\mathbf{M}\to\mathbf{D}$ *has a right adjoint. Moreover, the counit for the adjunction, say* $JF\to 1_{\mathbf{D}}$*, necessarily having components of the form* $\gamma:g\to Gg$ *with* g *a map, has* $gg^*\to G$ *invertible.*

PROOF. It suffices to show that the adjunction $I\dashv E:\mathbf{C}\to\mathbf{B}$ restricts to $J\dashv F:\mathbf{D}\to\mathbf{M}$. For this it suffices to show that, given $(h,\theta):JT\to(A,G)$, the $F:T\to A_G$ with $gF\cong h$ which can be found using $I\dashv E$ has F a map. This follows from Lemma 4.5. ∎

4.7. THEOREM. *A cartesian bicategory which has Eilenberg-Moore objects for Comonads and for which Maps are Comonadic has tabulation in the sense that the inclusion* $\iota:\mathbf{M}\to\mathbf{G}$ *has a right adjoint* τ *and the counit components* $\omega_R:v_R\to Ru_R$ *as in (4) have the property that the mates* $v_Ru_R^*\to R$*, with respect to the adjunctions* $u_R\dashv u_R^*$*, are invertible.*

PROOF. Using Theorems 4.3 and 4.6 we can construct the adjunction $\iota\dashv\tau$ by composing $J\dashv F$ with $K\dashv G$. Moreover, the counit for $\iota\dashv\tau$ is the pasting composite:



where the square is the counit for $K\dashv G$; and the triangle, the counit for $J\dashv F$, is an Eilenberg-Moore coalgebra for the comonad $G(R)$. The arrow component of the Eilenberg-Moore coalgebra is necessarily of the form (u,v), where u and v are maps, and it also follows that we have $(u,v)(u,v)^*\cong G(R)$. Thus we have

$$vu^*\cong r(u,v)(p(u,v))^*\cong r(u,v)(u,v)^*p^*\cong rG(R)p^*\cong R$$

where the first two isomorphisms are trivial, the third arises from the invertibility of the mate of γ as an Eilenberg-Moore structure, and the fourth is invertibility of μ, as in Theorem 4.3. ∎

4.8. THEOREM. *For a cartesian bicategory* $\mathbf{B}$ *with Eilenberg-Moore objects for Comonads and for which Maps are Comonadic,* Map$\mathbf{B}$ *has pullbacks satisfying the Beck condition*

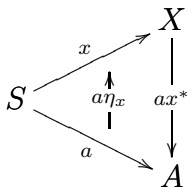
(meaning that for a pullback square

$$\begin{array}{ccc} P & \xrightarrow{r} & M \\ {\scriptstyle p}\downarrow & \cong & \downarrow{\scriptstyle b} \\ N & \xrightarrow[a]{} & A \end{array} \tag{5}$$

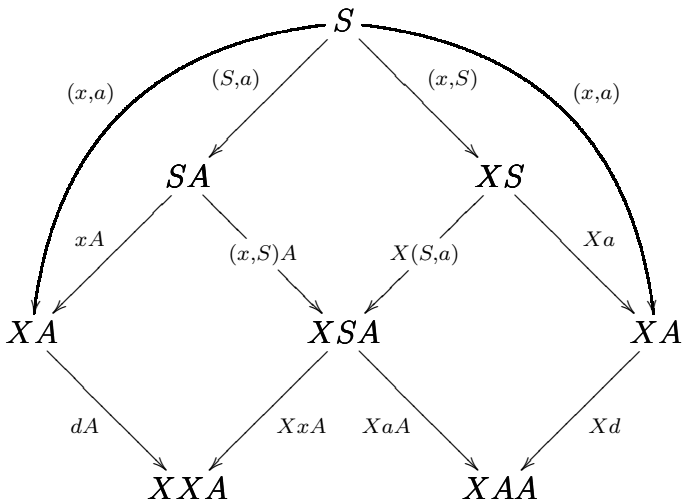
the mate $pr^* \to a^*b$ *of* $ap \cong br$ *in* $\mathbf{B}$*, with respect to the adjunctions* $r \dashv r^*$ *and* $a \dashv a^*$*, is invertible).*

PROOF. Given the cospan $a : N \to A \leftarrow M : b$ in Map$\mathbf{B}$, let P together with (r, σ, p) be a tabulation for $a^*b : M \to N$. Then $pr^* \to a^*b$, the mate of $\sigma : p \to a^*br$ with respect to $r \dashv r^*$, is invertible by Theorem 4.7. We have also $ap \to br$, the mate of $\sigma : p \to a^*br$ with respect to $a \dashv a^*$. Since A is discrete, $ap \to br$ is also invertible and is the only 2-cell between the composite maps in question. If we have also $u : N \leftarrow T \to M : v$, for maps u and v with $au \cong bv$, then the mate $u \to a^*bv$ ensures that the span $u : N \leftarrow T \to M : v$ factors through P by an essentially unique map $w : T \to P$ with $pw \cong u$ and $rw \cong v$. ∎

4.9. PROPOSITION. *In a cartesian bicategory with Eilenberg-Moore objects for Comonads and for which Maps are Comonadic, every span of maps* $x : X \leftarrow S \to A : a$ *gives rise to the following tabulation diagram:*



PROOF. A general tabulation counit $\omega_R : v_R \to Ru_R$ is given in terms of the Eilenberg-Moore coalgebra for the comonad $(u, v)(u, v)^*$ and necessarily $(u, v)(u, v)^* \cong G(R)$. It follows that for $R = ax^*$, it suffices to show that $G(ax^*) \cong (x, a)(x, a)^*$. Consider the diagram (with $\otimes$ suppressed):



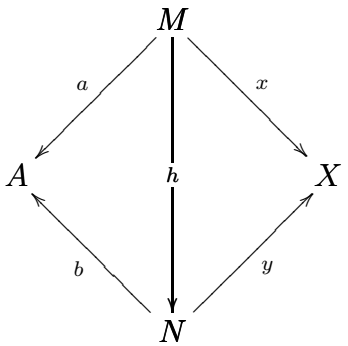
The comonoid $G(ax^*)$ can be read, from left to right, along the 'W' shape of the lower edge as $G(ax^*) \cong Xd^*.XaA.Xx^*A.dA$. But each of the squares in the diagram is a (product-absolute) pullback so that with Proposition 4.8 at hand we can continue:

$$Xd^*.XaA.Xx^*A.dA \cong Xa.X(S,a)^*.(x,S)A.x^*A \cong Xa.(x,S).(S,a)^*.x^*A \cong (x,a)(x,a)^*$$

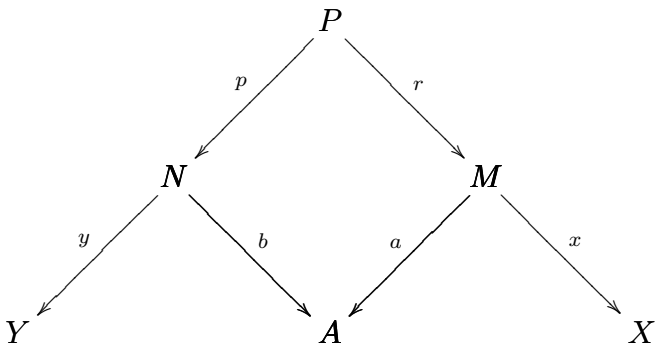
as required. ∎

5. Characterization of Bicategories of Spans

5.1. If $\mathbf{B}$ is a cartesian bicategory with Map$\mathbf{B}$ essentially locally discrete then each slice Map$\mathbf{B}/(X \otimes A)$ is also essentially locally discrete and we can write Span Map$\mathbf{B}(X,A)$ for the categories obtained by taking the quotients of the equivalence relations comprising the hom categories of the Map$\mathbf{B}/(X \otimes A)$. Then we can construct functors $C_{X,A}$:Span Map$\mathbf{B}(X,A) \to \mathbf{B}(X,A)$, where for an arrow in Span Map$\mathbf{B}(X,A)$ as shown,



we define $C(y,N,b) = by^*$ and $C(h) : ax^* = (bh)(yh)^* \cong bhh^*y^* \xrightarrow{b\epsilon_h y^*} by^*$. If Map$\mathbf{B}$ is known to have pullbacks then the Span Map$\mathbf{B}(X,A)$ become the hom-categories for a bicategory Span Map$\mathbf{B}$ and we can consider whether the $C_{X,A}$ provide the effects on homs for an identity-on-objects pseudofunctor C:Span Map$\mathbf{B} \to \mathbf{B}$. Consider



(6)

where the square is a pullback. In somewhat abbreviated notation, what is needed further are coherent, invertible 2-cells $\widetilde{C}$:$CN.CM \to C(NM) = CP$, for each composable pair of

spans M, N, and coherent, invertible 2-cells $C^\circ : 1_A \nrightarrow C(1_A)$, for each object A. Since the identity span on A is $(1_A, A, 1_A)$, and $C(1_A) = 1_A.1_A^* \cong 1_A.1_A \cong 1_A$ we take the inverse of this composite for C°. To give the $\widetilde{C}$ though is to give 2-cells $yb^*ax^* \nrightarrow ypr^*x^*$ and since spans of the form $(1_N, N, b)$ and $(a, M, 1_M)$ arise as special cases, it is easy to verify that to give the $\widetilde{C}$ it is necessary and sufficient to give coherent, invertible 2-cells $b^*a \nrightarrow pr^*$ for each pullback square in Map$\mathbf{B}$. The inverse of such a 2-cell $pr^* \nrightarrow b^*a$ is the mate of a 2-cell $bp \nrightarrow aq$. But by discreteness a 2-cell $bp \nrightarrow aq$ must be essentially an identity. Thus, definability of $\widetilde{C}$ is equivalent to the invertibility in $\mathbf{B}$ of the mate $pr^* \nrightarrow b^*a$ of the identity $bp \nrightarrow ar$, for each pullback square as displayed in (6). In short, if Map$\mathbf{B}$ has pullbacks and these satisfy the Beck condition as in Proposition 4.8 then we have a canonical pseudofunctor $C : \mathrm{Span}\,\mathrm{Map}\mathbf{B} \nrightarrow \mathbf{B}$.

5.2. Theorem. *For a bicategory $\mathbf{B}$ the following are equivalent:*

i) There is a biequivalence $\mathbf{B} \simeq \mathrm{Span}\,\mathcal{E}$, for $\mathcal{E}$ a category with finite limits;

ii) The bicategory $\mathbf{B}$ is cartesian, each comonad has an Eilenberg-Moore object, and every map is comonadic.

iii) The bicategory Map$\mathbf{B}$ *is an essentially locally discrete bicategory with finite limits, satisfying in $\mathbf{B}$ the Beck condition for pullbacks of maps, and the canonical*

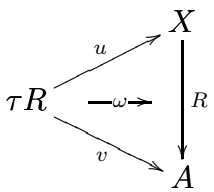
$$C : \mathrm{Span}\,\mathrm{Map}\mathbf{B} \nrightarrow \mathbf{B}$$

is a biequivalence of bicategories.

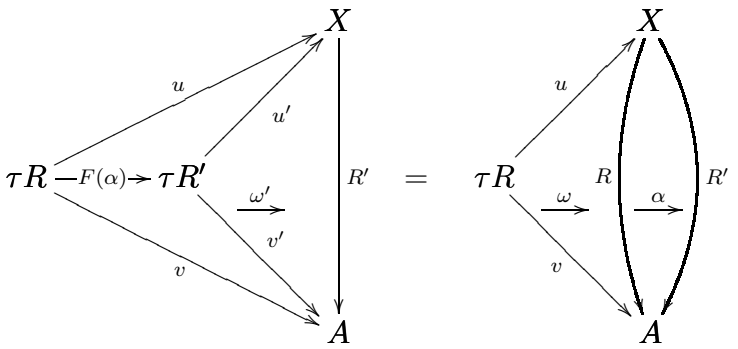
Proof. That *i)* implies *ii)* follows from our discussion in the Introduction. That *iii)* implies *i)* is trivial so we show that *ii)* implies *iii)*.

We have already observed in Theorem 3.14 that, for $\mathbf{B}$ cartesian with every object discrete, Map$\mathbf{B}$ is essentially locally discrete and we have seen by Propositions 2.3 and 3.5 that, in a cartesian bicategory in which Maps are Comonadic, every object is discrete. In Theorem 4.8 we have seen that, for $\mathbf{B}$ satisfying the conditions of *ii)*, Map$\mathbf{B}$ has pullbacks, and hence all finite limits and, in $\mathbf{B}$ the Beck condition holds for pullbacks. Therefore we have the canonical pseudofunctor $C : \mathrm{Span}\,\mathrm{Map}\mathbf{B} \nrightarrow \mathbf{B}$ developed in 5.1. To complete the proof it suffices to show that the $C_{X,A} : \mathrm{Span}\,\mathrm{Map}\mathbf{B}(X, A) \nrightarrow \mathbf{B}(X, A)$ are equivalences of categories.

Define functors $F_{X,A} : \mathbf{B}(X, A) \nrightarrow \mathrm{Span}\,\mathrm{Map}\mathbf{B}(X, A)$ by $F(R) = F_{X,A}(R) = (u, \tau R, v)$ where



is the R-component of the counit for $\iota \dashv \tau : \mathbf{G} \to \mathrm{Map}\mathbf{B}$. For a 2-cell $\alpha : R \to R'$ we define $F(\alpha)$ to be the essentially unique map satisfying



(We remark that essential uniqueness here means that $F(\alpha)$ is determined to within unique invertible 2-cell.) Since $\omega : v \to Ru$ has mate $vu^* \to R$ invertible, because $(v, \tau R, u)$ is a tabulation of R, it follows that we have a natural isomorphism $CFR \to R$. On the other hand, starting with a span (x, S, a) from X to A we have as a consequence of Theorem 4.9 that (x, S, a) is part of a tabulation of $ax^* : X \to A$. It follows that we have a natural isomorphism $(x, S, a) \to FC(x, S, a)$, which completes the demonstration that $C_{X,A}$ and $F_{X,A}$ are inverse equivalences. ∎

6. Direct sums in bicategories of spans

In the previous section we gave a characterization of those (cartesian) bicategories of the form $\mathrm{Span}\,\mathcal{E}$ for a category $\mathcal{E}$ with finite limits. In this final section we give a refinement, showing that $\mathrm{Span}\,\mathcal{E}$ has direct sums if and only if the original category $\mathcal{E}$ is lextensive [CLW].

Direct sums are of course understood in the bicategorical sense. A *zero object* in a bicategory is an object which is both initial and terminal. In a bicategory with finite products and finite coproducts in which the initial object is also terminal there is a canonical induced arrow $X + Y \longrightarrow X \times Y$, and we say that the bicategory has *direct sums* when this map is an equivalence.

6.1. Remark. Just as in the case of ordinary categories, the existence of direct sums gives rise to a calculus of matrices. A morphism $X_1 + \ldots + X_m \longrightarrow Y_1 + \ldots + Y_n$ can be represented by an $m \times n$ matrix of morphisms between the summands, and composition can be represented by matrix multiplication.

6.2. Theorem. *Let $\mathcal{E}$ be a category with finite limits, and $\mathbf{B} = \mathrm{Span}\,\mathcal{E}$. Then the following are equivalent:*

i) **B** *has direct sums;*

ii) **B** *has finite coproducts;*

iii) $\mathbf{B}$ *has finite products;*

iv) $\mathcal{E}$ *is lextensive.*

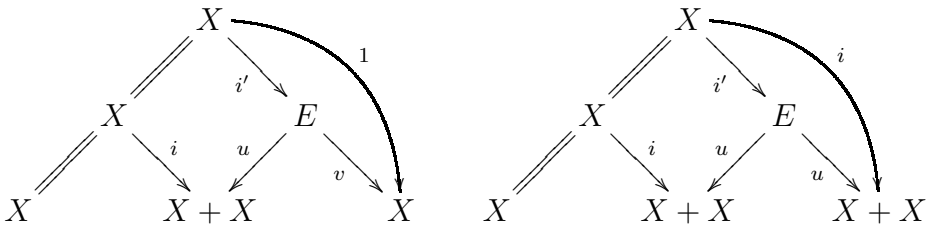
PROOF. [*i*) $\Longrightarrow$ *ii*)] is trivial.

[*ii*) $\Longleftrightarrow$ *iii*)] follows from the fact that $\mathbf{B}^{\mathrm{op}}$ is biequivalent to $\mathbf{B}$.

[*ii*) $\Longrightarrow$ *iv*)] Suppose that $\mathbf{B}$ has finite coproducts, and write 0 for the initial object and $+$ for the coproducts.

For every object X there is a unique span $0 \leftarrow D \rightarrow X$. By uniqueness, any map into D must be invertible, and any two such with the same domain must be equal. Thus when we compose the span with its opposite, as in $0 \leftarrow D \rightarrow X \leftarrow D \rightarrow 0$, the resulting span is just $0 \leftarrow D \rightarrow 0$. Now by the universal property of 0 once again, this must just be $0 \leftarrow 0 \rightarrow 0$, and so $D \cong 0$, and our unique span $0 \longrightarrow X$ is a map.

Clearly coproducts of maps are coproducts, and so the coproduct injections $X + 0 \longrightarrow X + Y$ and $0 + Y \longrightarrow X + Y$ are also maps. Thus the coproducts in $\mathbf{B}$ will restrict to $\mathcal{E}$ provided that the codiagonal $u{:}X + X \leftarrow E \rightarrow X{:}v$ is a map for all objects X. Now the fact that the codiagonal composed with the first injection $i : X \longrightarrow X + X$ is the identity tells us that we have a diagram as on the left below



in which the square is a pullback; but then the diagram on the right shows that the composite of $u{:}X+X \leftarrow E \rightarrow X+X{:}u$ with the injection $i : X \longrightarrow X+X$ is just i. Similarly its composite with the other injection $j : X \longrightarrow X+X$ is j, and so $u{:}X+X \leftarrow E \rightarrow X+X{:}u$ is the identity. This proves that the codiagonal is indeed a map, and so that $\mathcal{E}$ has finite coproducts; we have already assumed that it has finite limits. To see that $\mathcal{E}$ is lextensive observe that we have equivalences

$$\mathcal{E}/(X + Y) \simeq \mathbf{B}(X + Y, 1) \simeq \mathbf{B}(X, 1) \times \mathbf{B}(Y, 1) \simeq \mathcal{E}/X \times \mathcal{E}/Y.$$

[*iv*) $\Longrightarrow$ *i*)] Suppose that $\mathcal{E}$ is lextensive. Then in particular, it is distributive, so that $(X + Y) \times Z \cong X \times Z + X \times Y$, and we have

$$\begin{aligned}\mathbf{B}(X + Y, Z) &\simeq \mathcal{E}/\big((X + Y) \times Z\big) \simeq \mathcal{E}/(X \times Z + Y \times Z) \\ &\simeq \mathcal{E}/(X \times Z) \times \mathcal{E}/(Y \times Z) \simeq \mathbf{B}(X, Z) \times \mathbf{B}(Y, Z)\end{aligned}$$

which shows that $X + Y$ is the coproduct in $\mathbf{B}$; but a similar argument shows that it is also the product. ∎

6.3. Remark. The implication *iv*) $\Rightarrow$ *i*) was proved in [P&S, Section 3].

6.4. Remark. The equivalence *ii*) $\Leftrightarrow$ *iv*) can be seen as a special case of a more general result [H&S] characterizing colimits in $\mathcal{E}$ which are also (bicategorical) colimits in $\operatorname{Span}\mathcal{E}$.

6.5. Remark. There is a corresponding result involving partial maps in lextensive categories, although the situation there is more complicated as one does not have direct sums but only a weakened relationship between products and coproducts, and a similarly weakened calculus of matrices. See [C&L, Section 2].

There is also a nullary version of the theorem. We simply recall that an initial object in a category $\mathcal{E}$ is said to be *strict*, if any morphism into it is invertible, and then leave the proof to the reader. Once again the equivalence *ii*) $\Leftrightarrow$ *iv*) is a special case of [H&S].

6.6. Theorem. *Let $\mathcal{E}$ be a category with finite limits, and $\mathbf{B} = \operatorname{Span}\mathcal{E}$. Then the following are equivalent:*

i) $\mathbf{B}$ *has a zero object;*

ii) $\mathbf{B}$ *has an initial object;*

iii) $\mathbf{B}$ *has a terminal object;*

iv) $\mathcal{E}$ *has a strict initial object.*

References

[CKWW] A. Carboni, G.M. Kelly, R.F.C. Walters, and R.J. Wood. Cartesian bicategories II, *Theory Appl. Categ.* 19 (2008), 93–124.

[CLW] A. Carboni, Stephen Lack, and R.F.C. Walters. Introduction to extensive and distributive categories. *J. Pure Appl. Algebra* 84 (1993), 145–158.

[C&W] A. Carboni and R.F.C. Walters. Cartesian bicategories. I. *J. Pure Appl. Algebra* 49 (1987), 11–32.

[C&L] J.R.B. Cockett and Stephen Lack. Restriction categories III: colimits, partial limits, and extensivity, *Math. Struct. in Comp. Science* 17 (2007), 775–817.

[LSW] I. Franco Lopez, R. Street, and R.J. Wood, Duals Invert, *Applied Categorical Structures*, to appear.

[H&S] T. Heindel and P. Sobociński, Van Kampen colimits as bicolimits in Span, *Lecture Notes in Computer Science,* (CALCO 2009), 5728 (2009), 335–349.

[P&S] Elango Panchadcharam and Ross Street, Mackey functors on compact closed categories, *J. Homotopy and Related Structures* 2 (2007), 261–293.

[ST] R. Street. The formal theory of monads, *J. Pure Appl. Algebra* 2 (1972), 149–168.

[W&W] R.F.C. Walters and R.J. Wood. Frobenius objects in cartesian bicategories, *Theory Appl. Categ.* 20 (2008), 25–47.

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