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ON THE DUALITY BETWEEN TREES AND DISKS

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Abstract. A combinatorial category *Disk* was introduced by André Joyal to play a role in his definition of weak ω-category. He defined the category Θ to be dual to *Disks*. In the ensuing literature, a more concrete description of Θ was provided. In this paper we provide another proof of the dual equivalence and introduce various categories equivalent to *Disk* or Θ, each providing a helpful viewpoint.

1. Introduction

André Joyal in order to define weak n-categories introduced Θ which was naturally filtered with the simplicial category Δ being the first term of the filtration. In [Jo97] he defined Θ as the dual of a category *Disk* of disks. He also suggested a more explicit description of Θ involving the trees of Michael Batanin in [BS00]. Michael Makkai and Marek Zawadowski in [MZ01] and Clemens Berger in [Be02] gave explicit proofs that the two version of Θ are equivalent. In this paper we give a third proof which is a conceptual lifting of the duality between ordinals and intervals. In the process, several categories are introduced, each turning out to be equivalent to Θ or *Disk*, and so each providing us with useful new perspectives on Joyal's definitions.

The category Δ_+ is known as the augmented simplicial category and contains a single object (the empty ordinal) in addition to those of Δ. Primarily, we work with *augmented* categories which contain a unique *trivial* object and are more suitable for inductive arguments. Four of these augmented categories have reduced counterparts which are equivalent to the categories *Disk* or Θ.

In Section 2 we recall the definitions of ordinals and intervals and define functors which witness that they are dual. The section ends with two simple results which are used in the proof of Theorem 6.2. In Section 3 we define augmented categories $i\mathcal{I}_+$ and $i\Delta_+$, inductively built from $\mathcal{I}_+$ and Δ_+ (respectively), and prove that they are dual. Their reduced counterparts are denoted $i\mathcal{I}$ and $i\Delta$ (respectively). In Section 4 we recall the definition of Joyal's category *Disk* and demonstrate an equivalence between *Disk* and $i\mathcal{I}$.

In Section 5.2 we recall Street's definition of globular cardinal and define restriction and suspension operations on them. In Section 5.9 we define so called ordinal graphs which

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are inductively defined counterparts to globular cardinals and demonstrate an equivalence between the category of globular cardinals and the category of ordinal graphs. In Section 5.18 we recall the definition of ω-category and define free functors on globular cardinals and on ordinal graphs. We then demonstrate that the free ω-category on a globular cardinal is isomorphic to the free ω-category on the corresponding ordinal graph. The objects of Θ are, by definition, the free ω-categories on globular cardinals. In Section 6 we demonstrate an equivalence between Θ and the inductively defined category $i\Delta$.

In the remaining two sections we provide a more categorical description of the categories *Disk* and Θ by defining so called labeled trees which satisfy specific requirements relevant to our purposes. Two categories of labeled trees, named $t\mathcal{I}_+$ and $t\Delta_+$, are defined and easily shown to be dual. Their reduced counterparts are shown to be equivalent to the categories *Disk* and Θ (respectively).

Restriction and suspension operations are defined on labeled trees with the goal of working with them inductively and constructing equivalences between these two categories of labeled trees and their inductively defined counterparts. The proofs that $i\mathcal{I}_+$ is equivalent to $t\mathcal{I}_+$ (Proposition 8.2) and that $i\Delta_+$ is equivalent to $t\Delta_+$ (Proposition 8.4) are essentially the same; however, we give all the details for clarity.

A disk of dimension $\leq N$ is defined in [Jo97] as a *sequence of length N of bundles of intervals* with extra conditions. If we had been dealing with families, rather than those bundles, of intervals we could have made use of known properties of the finite coproduct completion functor Fam_Σ. In particular, if we have an equivalence $\mathcal{A}^{\text{op}} \simeq \mathcal{B}$, it lifts to an equivalence $Fam_\Sigma(\mathcal{A})^{\text{op}} \simeq Fam_\Pi(\mathcal{B})$ where Fam_Π is the finite product completion functor. Our replacements $t\mathcal{I}$ and $t\Delta$ for *Disk* and Θ are modifications of $Fam_\Sigma(\mathcal{I}_+)$ and $Fam_\Pi(\Delta_+)$. Instead of finite families we have labeled trees.

2. The ordinal/interval duality

The category Δ_+, called *the algebraist's* Δ, has objects $[n] = \{0, .., n\}$ for n in $\{-1, 0, ...\}$ where $[-1] = \{\}$ and morphisms which are order preserving functions. The category $\mathcal{I}_+$ is the sub-category of Δ_+ whose objects, called *intervals*, are non-empty ordinals, i.e. $[0]$, $[1]$, $[2]$, $\ldots$, and whose morphisms preserve the greatest and least elements, i.e. morphism $f\colon [n] \to [m]$ has $f(0) = 0$ and $f(n) = m$. Collectively, the greatest and least elements of $[n]$ are its *end points*.

2.1. Definition. *We define contravariant functors*

$$\begin{aligned} (\_)^\vee &: \mathcal{I}_+ \to \Delta_+ \\ &: [n] \mapsto [n-1] \\ &: f \mapsto f^\vee \end{aligned} \qquad\qquad \begin{aligned} (\_)^\wedge &: \Delta_+ \to \mathcal{I}_+ \\ &: [n] \mapsto [n+1] \\ &: g \mapsto g^\wedge \end{aligned}$$

on morphisms as follows. Given a morphism $f\colon [n] \to [m]$ of $\mathcal{I}_+$ there is a function $f^\vee\colon [m-1] \to [n-1]$ in Δ_+ defined by

$$f^{-1}\{0, 1, \ldots, j\} = \{0, 1, \ldots, f^\vee(j)\}$$

where

$$i \le f^\vee(j) \quad \textit{if and only if} \quad f(i) \le j. \tag{1}$$

Given a morphism $g\colon [n-1] \to [m-1]$ *of* Δ_+ *there is a function* $g^\wedge\colon [m] \to [n]$ *in* $\mathcal{I}_+$ *defined by*

$$g^{-1}\{i, i+1, \dots, m-1\} = \{g^\wedge(i), g^\wedge(i)+1, \dots, n-1\}$$

where

$$g^\wedge(i) \le j \quad \textit{if and only if} \quad i \le g(j). \tag{2}$$

2.2. Theorem. *The functor*

$$(\_)^\vee\colon \mathcal{I}_+^{\mathrm{op}} \to \Delta_+$$

is an isomorphism of categories with inverse $(\_)^\wedge\colon \Delta_+^{\mathrm{op}} \to \mathcal{I}_+$.

Proof. Given a morphism $f\colon [n] \to [m]$ of $\mathcal{I}_+$ and an integer i in $[m]$, then by applying 1 to $f(i) \le f(i)$ we obtain $i \le f^\vee f(i)$ and by applying 2 to this inequality obtain $(f^\vee)^\wedge(i) \le f(i)$. Hence $(f^\vee)^\wedge = f$. Likewise, we have $(g^\vee)^\wedge = g$ for a morphism g of Δ_+. ∎

2.3. Observation. *Let* $\gamma\colon [m] \to [n]$ *be an ordinal morphism. The fiber of* $\gamma^\wedge$ *over* $0 < j \le n$ *is*

$$\{\gamma(j-1)+1, \gamma(j-1)+2, \dots, \gamma(j)\}.$$

This fact is used in Theorem 6.2.

2.4. Observation. *If* $\gamma\colon [m] \to [n]$ *is an ordinal map with* $m \ge 0$ *then* $\gamma^\wedge(i)$ *is an endpoint when either* $i \le \min im\gamma$ *or* $i > \max im\gamma$. *This fact is also used in Theorem 6.2.*

3. Induction on intervals and ordinals

In this section we introduce inductively defined augmented categories $i\mathcal{I}_+$ and $i\Delta_+$ and demonstrate that they are dual.

3.1. Definition. *We define the category* $i\mathcal{I}_+$ *inductively, with objects filtered by height. The object of* height *0 (zero) is the interval* $[0]$ *and is* trivial. *An object* H *of* height n *is an interval* H_* *and for each* i *in* H_* *an object* $H(i)$ *of height strictly less than* n *which is trivial if and only if* i *is an endpoint of* H_*. *The object* H *with* $H_* = [0]$ *is identified with the trivial object.*

For every object H *there is a unique morphism* $H \to [0]$. *Hence* $[0]$ *is terminal. In addition, there are no non-identity morphisms with codomain* $[0]$. *A morphism* $g\colon H \to K$, *for* H *and* K *with non-zero height, consists of an interval map* $g\colon H_* \to K_*$ *and for all* i *in* H_* *a morphism* $g(i)\colon H(i) \to K(gi)$.

Define composition using induction as follows. Let $f\colon H \to K$ *and* $g\colon K \to L$ *be composable morphisms. The object map of* $g \circ f$ *is the composite of the object maps of* g *and* f. *For* $i \in H_*$ *then the composite* $(g \circ f)(i)\colon H(i) \to L(gfi)$ *is* $g(fi) \circ f(i)$. *We have the category* $i\mathcal{I}_+$. *The category* $i\mathcal{I}$ *is the full subcategory of* $i\mathcal{I}_+$ *containing the non-trivial objects.*

3.2. Definition. *We define the category* $i\Delta_+$ *inductively, with objects filtered by height. The object of* height *0 (zero) is the ordinal* $[-1]$ *and is* trivial. *An object* K *of* height n *is an ordinal* K_* *and for each* $j \in (K_*)^\wedge$ *an object* $K(j)$ *of height strictly less than* n *which is trivial if and only if* j *is an endpoint of* $(K_*)^\wedge$. *The object* H *with* $H_* = [-1]$ *is identified with the trivial object.*

For every object K *there is a unique morphism* $[-1] \to K$. *Hence* $[-1]$ *is initial. In addition, there are no non-identity morphisms with domain* $[-1]$. *A morphism* $g\colon H \to K$, *for* H *and* K *of non-zero height, consists of an ordinal map* $g\colon H_* \to K_*$ *and for all* $j \in (K_*)^\wedge$ *a morphism* $g(j)\colon H(g^\wedge j) \to K(j)$.

Define composition using induction as follows. Let $f\colon H \to K$ *and* $g\colon K \to L$ *be composable morphisms. The object map of* $g \circ f$ *is the composite of the object maps of* g *and* f. *For* $i \in L_*$ *then the composite* $(g \circ f)(i)\colon H((g \circ f)^\wedge i) \to L(i)$ *is* $g(i) \circ f(g^\wedge i)$. *We have the category* $i\Delta_+$. *The category* $i\Delta$ *is the full subcategory of* $i\Delta_+$ *containing the non-trivial objects.*

3.3. Definition. *We define functors*

$$\vee\colon {i\mathcal{I}_+}^{\mathrm{op}} \to i\Delta_+$$

and

$$\wedge\colon {i\Delta_+}^{\mathrm{op}} \to i\mathcal{I}_+$$

using the functors $(\_)^\vee$ *and* $(\_)^\wedge$ *of the equivalence between ordinals and intervals.*

Define $\vee$ *on objects using induction on their height. Send the trivial object* $[0]$ *of* $i\mathcal{I}_+$ *to the trivial object* $[-1]$ *of* $i\Delta_+$. *Assume* $\vee$ *is defined for objects of height* n *and let* H *be an object of height* $n+1$. *Define* $\vee H$ *as* $((H_*)^\vee, \vee H(i))$.

Define $\vee$ *on morphisms using induction on the height of their codomain. Send each unique morphism of* $i\mathcal{I}_+$ *into the trivial object* $[0]$ *to the corresponding unique morphism of* $i\Delta_+$ *out of the trivial object* $[-1]$. *Assume* $\vee$ *is defined on morphisms with codomain of height* n *and let* g *be a morphism of* $i\mathcal{I}_+$ *with codomain of height* $n+1$. *Define* $\vee g$ *as* $(g^\vee, \vee g(i))$ *as each morphism* $g(i)$ *has codomain of height* n.

Similarly, define $\wedge$ *on objects (respectively morphisms) using induction on their height (respectively on the height of their domains). Send the trivial object of* $i\Delta_+$ *to the trivial object of* $i\mathcal{I}_+$. *Send morphisms of* $i\Delta_+$ *out of the trivial object to morphisms of* $i\mathcal{I}_+$ *into the trivial object. Let* g *be a morphism of* $i\Delta_+$. *Define* $\wedge g$ *as* $(g^\wedge, \wedge g(i))$.

3.4. Theorem. *The functors* $\vee$ *and* $\wedge$ *are mutually inverse isomorphisms.*

The proof uses induction and follows directly from the mutually inverse functors of the duality between ordinals and intervals.

3.5. Corollary. *The categories* $i\mathcal{I}$ *and* $i\Delta$ *are dual.*

4. Equivalence between $i\mathcal{I}$ and *Disk*

In Section 3 we showed that the categories $i\mathcal{I}$ and $i\Delta$ are dual. Here we demonstrate that the category $i\mathcal{I}$ and Joyal's category *Disk* are equivalent. To do so we construct the category $Disk_+$, the augmented counterpart to *Disk*, and show that it is equivalent to the category $i\mathcal{I}_+$. We begin by defining *forests* and *trees*, and then define operations of restriction and suspension on trees in order to work with them inductively.

4.1. Definition. *A* forest *is a functor* $A\colon \omega^{\mathrm{op}} \to Set$

$$\cdots \xrightarrow{p_2} A_2 \xrightarrow{p_1} A_1 \xrightarrow{p_0} A_0.$$

A tree A *is a forest such that* $A_0 \cong \{*\}$*. The vertices of* height n *are the elements of* A_n*. The unique vertex of height* 0 *(zero) of a tree is called the* root*. We often denote the root of a tree* A *as* $*$*. We can regard a tree as a directed graph. There is an edge from vertex* x *to vertex* y *when* $p_n(x) = y$ *for some* $n \in \mathbb{N}$*. We denote the above forest by* (A, p) *or* A*. Define* $p_{n,m}\colon A_{n+m} \to A_n$ *as the composite* $\circ_{i=n}^{n+m-1} p_i$ *which is*

$$A_{n+m} \xrightarrow{p_{n+m-1}} \cdots \xrightarrow{p_n} A_n.$$

A forest has degree n *when* $p_{n,m}$ *is a bijection for all* $m \geq 1$ *and has* finite degree *when it has degree* n *for some* $n \in \mathbb{N}$*.*

4.2. Definition. *A* forest map *is a natural transformation of forests and so is a sequence of set maps*

$$\begin{tikzcd}
\cdots \arrow[r, "p_2"] & A_2 \arrow[r, "p_1"] \arrow[d, "f_2"] & A_1 \arrow[r, "p_0"] \arrow[d, "f_1"] & A_0 \arrow[d, "f_0"] \\
\cdots \arrow[r, "q_2"'] & B_2 \arrow[r, "q_1"'] & B_1 \arrow[r, "q_0"'] & B_0
\end{tikzcd}$$

such that the squares commute. This map is denoted by $f\colon (A, p) \to (B, q)$ *or* $f\colon A \to B$*. A* tree map *is a natural transformation between trees. We have the category* $Forest$ *and its full subcategory* $Tree$*.*

4.3. Definition. *We define a* restriction *operation on trees in order to work with trees as inductive or recursive objects. Let* $u_n\colon \omega \to \omega$ *be the functor defined by* $u_n(i) = i + n$*. Let* A *be a forest and* x *an element of* A_n*. The* restriction of A by x *denoted* $A(x)$ *is the largest subfunctor of* $A \circ u_n\colon \omega^{\mathrm{op}} \to Set$ *such that* $A(x)_0 = \{x\}$*. We sometimes refer to* $A(x)$ *as a* subtree *of* A*.*

The restriction of f by x *denoted* $f(x)$*, where* $f\colon A \to B$ *is a forest map, is the lifting in*

$$\begin{tikzcd} A(x) \arrow[r, dashed, "f(x)"] \arrow[d, "\mathrm{incl}"'] & B(f_n x) \arrow[d, "\mathrm{incl}"] \\ A \circ u_n \arrow[r, "f \cdot u_n"'] & B \circ u_n \end{tikzcd}$$

of $f \cdot u_n$ *along the inclusions* $A(x) \to A \circ u_n$ *and* $B(f_n x) \to B \circ u_n$*.*

4.4. Definition. *Define a* suspension *functor*

$$\mathrm{su}\colon \mathit{Forest} \to \mathit{Tree}$$

as follows. Given a forest A *then its* suspension $\mathrm{su}\, A$ *is*

$$\cdots \xrightarrow{p_1} A_1 \xrightarrow{p_0} A_0 \longrightarrow \{*\}.$$

Given a map $f\colon A \to B$ *of forests then its* suspension $\mathrm{su}\, f$ *is*

$$\begin{tikzcd} \cdots \arrow[r] & A_1 \arrow[r] \arrow[d, "f_1"] & A_0 \arrow[r] \arrow[d, "f_0"] & \{*\} \arrow[d] \\ \cdots \arrow[r] & B_1 \arrow[r] & B_0 \arrow[r] & \{*\}. \end{tikzcd}$$

4.5. Observation. *The coproduct of a collection of trees is a forest and its suspension is a tree. The subtrees of the suspension given by vertices of height 1 (one) are isomorphic to the trees of the original collection. We provide the details below.*

Let $(A(i), p(i))$ *be a tree with* $A(i)_0 = \{x_i\}$ *for each* i *in a set* I*. Let* $A' = \mathrm{su} \sum A(i)$ *and let* $\mathrm{copr}(i)\colon A(i) \to \sum A(i)$ *be coprojections for each* $i \in I$*. The fiber of* $p(i)_{0,n}$ *over* x_i *is* $A(i)_n$*. The fiber of* $[\sum p(i)]_{0,n}$ *over* x_i *lying in* $\sum A(i)_0$ *is* $A'(x_i)_n$*. The coproduct in Set requires that the former is sent by the left coprojection of*

$$\begin{tikzcd} A(i)_n \arrow[r, "p(i)_0^n"] \arrow[d, "\mathrm{copr}"'] & A(i)_0 \arrow[d, "\mathrm{copr}"] \\ \sum A(i)_n \arrow[r, dashed, "\sum p(i)_{0,n}"'] & \sum A(i)_0. \end{tikzcd}$$

onto the latter. As the coprojections are monomorphisms then $A(i)$ *and* $A'(x_i)$ *are isomorphic by* $\mathrm{copr}(i)$*.*

4.6. Definition. *A* disk *as defined by Joyal in [Jo97] is a tree (A, p) of finite degree*

1. *such that the fibers of $p_n \colon A_{n+1} \to A_n$ have interval structure (for $n \in \mathbb{N}$)*
2. *with sections $d_0, d_1 \colon A_n \to A_{n+1}$ of p_n where $p_n^*(x) = [d_0(x), .., d_1(x)]$*
3. *such that the equalizer of $d_0, d_1 \colon A_n \to A_{n+1}$ is $d_0(A_{n-1}) \cup d_1(A_{n-1})$.*

The equalizer of condition 3 is the singular set *of A_n. All fibers are non-empty by condition 2 and the interval $p_0^*(x)$ is strict by condition 3, where x is the single element of A_0.*

A morphism of disks *$f \colon (A, p) \to (B, q)$ as defined by Joyal in [Jo97] as a sequence of set maps $f_n \colon A_n \to B_n$ which commute with the projections p_n and q_n, respect the order of the interval fibers and preserve the endpoints (first and last elements) of the interval fibers. This defines the category $Disk$.*

In addition, we define the augmented category $Disk_+$. An object of $Disk_+$ is a tree of finite degree satisfying 1 and 2 above, but we relax 3 to allow the fiber over the root to be the unique non-strict interval $[0]$. Hence $Disk_+$ contains, in addition to the objects of $Disk$, trees of degree 0 (zero). These additional objects are terminal. The morphisms of $Disk_+$ are defined identically to those of $Disk$.

4.7. Definition. *We define a functor*

$$\Phi \colon Disk_+ \to i\mathcal{I}_+.$$

Define Φ on objects using induction on the degree of disks. Send disks of degree 0 (zero) to $[0]$ the trivial object of $i\mathcal{I}_+$. Assume that Φ is defined on disks of degree n and let (A, p) be a disk of degree $n+1$. We define an object H of $i\mathcal{I}_+$ from the data of (A, p). Let $H_* = p_0^*(x)$, the fiber over the unique element x of A_0, and let $H(i) = \Phi A(i)$ for each $i \in H_*$ where $A(i)$ is the restriction of A by i. Note that H_* is an interval as fibers have interval structure, that $A(i)$ is a disk of degree at most n for each $i \in H_*$ and that $A(i)$ is trivial when i is an endpoint by condition 3. Define ΦA as H.

We define Φ on disk morphisms using induction on the degree of their codomain. The disks of degree 0 (zero) are terminal objects. As Φ preserves the terminal object then disk morphisms with codomain of degree 0 (zero) are sent to the unique morphism into the trivial object of $i\mathcal{I}_+$. Assume Φ is defined on morphisms with codomain of degree n and let $f \colon (A, p) \to (B, q)$ be a disk morphism with codomain of degree $n+1$. We define a morphism g of $i\mathcal{I}_+$ from the data of f. Let $g = f_1$ which is an interval morphism as f preserves order and endpoints. Let $g(i) = \Phi f(i)$ where $f(i)$ is the restriction of f by i for each $i \in A_1$. Define Φf as g.

4.8. Theorem. *The category $Disk_+$ is equivalent to the category $i\mathcal{I}_+$ by*

$$\Phi \colon Disk_+ \to i\mathcal{I}_+$$

which is surjective on objects.

PROOF. **Surjective.** We show that Φ is surjective on objects using induction on the height of objects of $i\mathcal{I}_+$. The trivial disks are sent to the trivial object $[0]$ and so Φ is surjective on objects of height 0 (zero). Assume Φ is surjective on objects of height n and let H be an object of height $n+1$. By induction there exists a disk $A(i)$ with $\Phi A(i) = H(i)$ for each i in H_*. Let $A' = \mathrm{su}\sum A(i)$ where the coproduct is indexed over the elements of H_*. Give the fiber over $*$ an interval structure from that of H_*. For each element x in A'_n with $n > 0$ give the fiber over x an interval structure by pulling back along the coprojections. By Observation 4.5 then $A'(i) \cong A(i)$. Define $\Phi A'$ as H'. As Φ is constant on isomorphism classes then we have $\Phi A'(i) = \Phi A(i)$, equivalently $H'(i) = H(i)$. By construction $H'_* = (p'_0)^*(*)$ is isomorphic to H_* and is in fact identically H_* as Δ is skeletal. Hence Φ is surjective on objects.

Faithful. We show that Φ is faithful using induction on the height of the codomain of morphisms of $i\mathcal{I}_+$. As Φ reflects the terminal object it is faithful on morphisms with codomain of height 0 (zero). Assume that Φ is faithful on morphisms with codomain of height n and let $f, f' \colon (A,p) \to (B,q)$ be parallel disk morphisms such that $\Phi f = \Phi f'$ has codomain of height $n+1$. Then, by induction, $f(i) = f'(i)$ for each $i \in A_1$. As $f(i)$ and $f'(i)$ are defined by liftings on inclusions which are jointly surjective it follows that $f = f'$ and Φ is faithful.

Full. We show Φ is full using induction on the height of the codomain of morphisms of $i\mathcal{I}_+$. As Φ preserves the terminal object it is full on morphisms of height 0 (zero). Assume that Φ is full on morphisms with codomain of height n and let $g \colon \Phi(A,p) \to \Phi(B,q)$ have codomain of height $n+1$. We have an interval morphism g and a morphism $g(i)$ of $i\mathcal{I}_+$ for each $i \in \Phi(A,p)$. By induction there exist disk morphisms $f(i) \colon A(i) \to B(gi)$ such that $\Phi f(i) = g(i)$. In the following diagram

$$\begin{tikzcd}
& A(i)_n \arrow[r, "f(i)_n"] \arrow[dl, "\mathrm{incl}"'] \arrow[d, "\mathrm{copr}"] & B(gi)_n \arrow[d, "\mathrm{copr}"] \arrow[dr, "\mathrm{incl}"] & \\
A_{n+1} \arrow[r, dashed] & \sum A(i)_n \arrow[r, dashed] & \sum B(gi)_n \arrow[r, dashed] & B_{n+1}
\end{tikzcd}$$

the left triangle commutes as the inclusions into A_{n+1} are jointly surjective and the other regions commute by coproduct. Define f' by setting f'_{n+1} to the lower composite for each $n \in \mathbb{N}$. Then $f'(i)$ is (by definition) the upper horizontal morphism and so $f(i) = f'(i)$. The inclusions and commutativity of the diagram imply that f_1 as defined is identically g as required. Hence $\Phi f' = g$. ∎

4.9. COROLLARY. *The category Disk is equivalent to the category $i\mathcal{I}$.*

5. Objects of Θ

From [BS00], an object of Θ is defined as the free ω-category on a non-empty globular cardinal. We define the augmented counterpart to Θ, named Θ_+, as containing objects which are free ω-categories on possibly empty globular cardinals.

In Section 5.2 we recall the definition of globular cardinal and define restriction and suspension operations on them. In Section 5.9 we define ordinal graphs, which are inductively defined counterparts of globular cardinals. An equivalence is demonstrated between the categories of globular cardinals and ordinal graphs. The section closes with the definition of a map that sends objects of $i\Delta_+$ to ordinal graphs. In Section 5.18 we recall the definition of ω-category and adapt Michael Batanin's construction of the free ω-category on a globular set. In addition we define the free ω-category on an ordinal graph and show that it is isomorphic to the free ω-category of the corresponding globular cardinal.

In this and following sections we will use the following shorthand with the intention of providing an uncluttered presentation. In the sequel often indices are given over all elements of a known finite linearly ordered set except the first. In some cases the previous element to the index is also used. Our shorthand is designed to simplify the presentation in these instances.

5.1. Notation. *Let $FinOrd_+$ denote the full subcategory of Ord with objects finite linearly ordered sets and let $FinOrd$ denote the full subcategory containing the non-empty finite linearly ordered sets. Define*

$$\begin{aligned}(\_)\backslash\mathrm{f}\colon\ & \mathrm{Ob}\, FinOrd \to \mathrm{Ob}\, FinOrd_+ \\ &: \{x_0, x_1, \ldots, x_p\} \mapsto \{x_1, x_2, \ldots, x_p\}.\end{aligned}$$

which returns its argument without the first element. The "$\backslash\mathrm{f}$" is intended to indicate that the first element is removed. The operator $\mathrm{p}(\_)$ takes arguments which are elements of finite linearly ordered sets with cardinality at least 2 (two) and returns their predecessor. For example given a finite linearly ordered set $I = \{x_0, x_1, \ldots, x_p\}$ then $\mathrm{p}(x_i) = x_{i-1}$ for all $x_i \in I\backslash\mathrm{f}$.

5.2. Globular cardinals. Globular cardinals were defined by Ross Street in [St00] Section 1. We collect here the definitions and notation related to this concept, as relevant for our purposes. In addition, we define restriction and suspension operations.

5.3. Definition. *We begin by quoting the definition of globular object from [St00]. Let* $\mathbf{G}$ *be the category with objects the natural numbers and non-identity arrows*

$$\sigma_m, \tau_m\colon m \to n \quad \textit{for } m < n$$

such that

$$\begin{tikzcd} k \arrow[rr, "\beta_k"] \arrow[dr, "\beta_k"'] & & m \arrow[dl, "\alpha_k"] \\ & n & \end{tikzcd}$$

commutes for all $k < m < n$ *and all* $\alpha, \beta \in \{\sigma, \tau\}$. *A* globular object *in* $\mathcal{C}$ *is a functor* $X \colon \mathbf{G}^{\mathrm{op}} \to \mathcal{C}$. *A* morphism of globular objects *is a natural transformation between globular objects. Hence a* globular set *is a pair of sequences of set maps*

$$\begin{tikzcd} \cdots \arrow[r, shift left, "s_2"] \arrow[r, shift right, "t_2"'] & X_2 \arrow[r, shift left, "s_1"] \arrow[r, shift right, "t_1"'] & X_1 \arrow[r, shift left, "s_0"] \arrow[r, shift right, "t_0"'] & X_0 \end{tikzcd}$$

such that $s_n s_{n+1} = s_n t_{n+1}$ *and* $t_n s_{n+1} = t_n t_{n+1}$ *for all* $n \in \mathbb{N}$. *The maps of the sequence* s *are* source maps *and the maps of the sequence* t *are* target maps. *The set* X_n *contains the* n-vertices *of* X.

A globular set X *has* dimension n *when* n *is the smallest integer such that* X_m *is empty for all* $m > n$. *The empty globular set has dimension -1 (minus one). A globular set has* finite dimension *when it has dimension* n *for some* $n \geq -1$.

A morphism $f \colon X \to Y$ of globular sets *is a sequence* f *of set maps*

$$\begin{tikzcd} \cdots \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & X_2 \arrow[d, "f_2"] \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & X_1 \arrow[d, "f_1"] \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & X_0 \arrow[d, "f_0"] \\ \cdots \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & Y_2 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & Y_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & Y_0 \end{tikzcd}$$

which commute with the source and target maps. We have the category of globular sets.

A globular set X *has a partial order* $\blacktriangleleft$ *generated from the relation*

$$\begin{array}{ccc} x \prec y & \textit{when} & x = s(y) \\ & \textit{or} & y = t(x) \end{array}$$

for $x \in X_n$ *and either* $y \in X_{n+1}$ *or* $y \in X_{n-1}$ *(respectively). When* $x \blacktriangleleft y$ *there exists a (possibly trivial) sequence* $x_0, ..., x_n$ *of vertices in* X *with* $x_0 = x$, $x_n = y$ *and* $x_{i-1} \prec x_i$ *for all* $i \in \{1, .., n\}$. *A* globular cardinal *is a globular set with a finite set of vertices and where the order given above is linear. We have* $GlobCard$ *the category of globular cardinals.*

5.4. Definition. *Vertices* x *and* y *in a globular cardinal* X *are* consecutive in Y *(with* Y *a subset of* X*) when* $\{z \in Y : x \blacktriangleleft z \blacktriangleleft y\} = \{x, y\}$.

5.5. Observation. *Let* $f \colon X \to Y$ *be a morphism of globular cardinals and let* x *and* y *be consecutive in* X_n. *There exists an element* z *either in* X_{n+1} *or* X_{n-1} *such that* $s(z) = x$ *and* $t(z) = y$ *or such that* $s(y) = z$ *and* $t(x) = z$. *In either case* $f_n x$ *and* $f_n y$ *are consecutive in* Y_n *as* f *preserves source and target. Hence* f_n *is injective and the image of* f_n *is an interval.*

5.6. Definition. *A map* $g \colon A \to B$ *of finite linearly ordered sets that is injective and whose image is an interval is called* incremental.

5.7. Definition. *We define a* restriction *operation on globular cardinals. Given two consecutive n-vertices of a globular cardinal X the operation returns the globular cardinal which we might call the hom-set determined by the two n-vertices. Let $u_n\colon \mathbf{G} \to \mathbf{G}$ be the functor defined by $u_n(i) = i + n$. Let X be a globular cardinal with y and z consecutive vertices of X_n. Define the* restriction of X by y, z *denoted $X(y, z)$ as the largest subfunctor of $X \circ u_{n+1}$ such that $y \blacktriangleleft x \blacktriangleleft z$ for all $x \in X(y, z)$. Notice that $X(\sigma)x = y$ and $X(\tau)x = z$ for all $x \in X(y, z)_0 \subseteq X_{n+1}$.*

We have an inclusion $\iota_{y,z}\colon X(y, z) \to X \circ u_{n+1}$ of functors for every pair of consecutive vertices $y, z \in X_n$. The restriction of $f\colon X \to Y$ by y, z *denoted $f(y, z)$ is the lifting in*

$$\begin{tikzcd}
X(y,z) \arrow[r, dashed, "f(y,z)"] \arrow[d] & Y(f_n\, y, f_n\, z) \arrow[d] \\
X \circ u_{n+1} \arrow[r, "f \cdot u_{n+1}"'] & Y \circ u_{n+1}
\end{tikzcd}$$

of $f \cdot u_{n+1}$ along the inclusions $\iota_{y,z}$ and $\iota_{f_n y, f_n z}$.

5.8. Definition. *We define a* suspension *operation on collections of globular cardinals. Let $A = \{x_0, x_1, \ldots, x_p\}$ be a finite linearly ordered set and let $X(i)$ be a globular cardinal for each $i \in A\backslash \mathrm{f}$. We refer to A and the $X(i)$ collectively as a* matched set *below. Define the* suspension of $X(i)$ over A *denoted $\mathrm{su}(X(i), A)$ as follows. Define set maps $s(i), t(i)\colon X(i)_0 \to A$ by $s(i)(y) = \mathrm{p}i$ and $t(i)(y) = i$ for $y \in X(i)_0$ and for each $i \in A\backslash \mathrm{f}$. Then the suspension $\mathrm{su}(X(i), A)$ is*

$$\begin{tikzcd}
\cdots \arrow[r, shift left, "\sum s_1(i)"] \arrow[r, shift right, "\sum t_1(i)"'] & \sum X(i)_1 \arrow[r, shift left, "\sum s_0(i)"] \arrow[r, shift right, "\sum t_0(i)"'] & \sum X(i)_0 \arrow[r, shift left, "\sum s(i)"] \arrow[r, shift right, "\sum t(i)"'] & A
\end{tikzcd}$$

where the coproducts are indexed over all $i \in A\backslash \mathrm{f}$. Notice that $s(i)s_0(i) = s(i)t_0(i)$ as $s(i)$ is constant. In addition as the required identities hold for each $X(i)$ then the universal property of coproduct in Set implies that

$$\sum s(i)_n \sum s(i)_{n+1} = \sum s(i)_n \sum t(i)_{n+1}$$

for $n > 0$. Likewise for the target maps. Hence the source and target identities required of globular cardinals are satisfied by the suspension. The linear order given by the source and target maps is

$$\{x_0, X(1), x_1, X(2), x_2, \ldots, x_{n-1}, X(n), x_n\}.$$

Hence $\mathrm{su}(X(i), A)$ is a globular cardinal.

Similarly, we define a suspension operation on collections of morphisms of globular cardinals. Let $X(i), A$ and $Y(j), B$ be matched sets and let X and Y be their suspensions as defined above. Let $f\colon A \to B$ be an incremental morphism of ordered sets and let

$f(i)\colon X(i) \to Y(fi)$ be a morphism of globular cardinals for each $i \in A\backslash \mathrm{f}$. Define the suspension of $f(i)$ over f *denoted* $\mathrm{su}(f(i), f)$ *as*

$$
\begin{array}{ccccccc}
\cdots \rightrightarrows & X_2 & \rightrightarrows & X_1 & \rightrightarrows & A \\
\cdots & \downarrow{\scriptstyle \sum f(i)_1} & & \downarrow{\scriptstyle \sum f(i)_0} & & \downarrow{\scriptstyle f} \\
\cdots \rightrightarrows & Y_2 & \rightrightarrows & Y_1 & \rightrightarrows & B
\end{array}
$$

where the coproducts are indexed over $A\backslash \mathrm{f}$. The squares commute by universal property of coproduct in Set and the suspension $\mathrm{su}(f(i), f)$ *is a morphism of globular cardinals.*

5.9. Ordinal graphs. The purpose of this section is to define the concept of an enriched graph called a $\mathcal{V}$-graph to allow an inductive definition equivalent to that of globular cardinals. An equivalence is demonstrated between the categories of globular cardinals and ordinal graphs. The section closes with the definition of a map that sends objects of $i\Delta_+$ to ordinal graphs.

5.10. Definition. *A* $\mathcal{V}$-graph *$\mathcal{G}$, for a category $\mathcal{V}$, consists of a set of vertices $\mathcal{G}_*$ and an* edge-object *$\mathcal{G}(x,y)$ in* $\mathrm{Ob}\,\mathcal{V}$ *for every pair of vertices x, y. A* morphism of $\mathcal{V}$-graphs *$f\colon \mathcal{G} \to \mathcal{H}$ is a set map $f\colon \mathcal{G}_* \to \mathcal{H}_*$ and for every edge-object $\mathcal{G}(x,y)$ of $\mathcal{G}$ a morphism*

$$f(x,y)\colon \mathcal{G}(x,y) \to \mathcal{H}(fx, fy)$$

of $\mathcal{V}$. We have the category $\mathcal{V}$-Gph of $\mathcal{V}$-graphs, graphs enriched over $\mathcal{V}$.

5.11. Definition. *Suppose $\mathcal{V}$ has initial object 0 and $\mathcal{U}$ is a subset of* $\mathrm{Ob}\,\mathcal{V}$ *containing 0. A $\mathcal{U}$* ordinal $\mathcal{V}$-graph *$\mathcal{G}$ consists of a finite linearly ordered set $\mathcal{G}_*$, objects $\mathcal{G}(x,y)$ in $\mathcal{U}$ of $\mathcal{V}$ for all pairs x, y in $\mathcal{G}_*$, and such that* $\mathcal{G}(x,y) \neq 0$ *if and only if y is the successor of x. We have the category $(\mathcal{U}, \mathcal{V})$-$Gph_{\mathrm{ord}}$ of $\mathcal{U}$ ordinal $\mathcal{V}$-graphs. Notice that the object maps are incremental.*

5.12. Definition. *We define two categories $Graph_{\mathbb{N}}$ and $OGraph$ of enriched graphs. Let $Graph_0$ denote the category $\{\emptyset\}$-Gph and let $Graph_{n+1}$ denote the category $Graph_n$-Gph for each $n \in \mathbb{N}$. Define $Graph_{\mathbb{N}}$ as the colimit of the diagram*

$$Graph_0 \to Graph_1 \to \cdots \to Graph_n \to \cdots$$

of inclusions. The empty graph $\emptyset$ has dimension *-1 (minus one). A graph of $Graph_n$ has* dimension *n.*

We define $OGraph$ the category of ordinal graphs*, a subcategory of $Graph_{\mathbb{N}}$, which we demonstrate in Theorem 5.15 is equivalent to the category of globular cardinals. Let $OGraph_0$ denote the category $(\emptyset, \emptyset)$-Gph_{ord} and let $OGraph_{n+1}$ denote the category $(OGraph_n, Graph_n)$-Gph for $n \in \mathbb{N}$. Define $OGraph$ as the colimit of the diagram*

$$OGraph_0 \to OGraph_1 \to \cdots \to OGraph_n \to \cdots$$

of inclusions.

5.13. Definition. *We define a functor*

$$\Gamma\colon GlobCard \to OGraph$$

using induction on the dimension of globular cardinals. Let X *be a globular cardinal of dimension -1 (minus one). Then* X *is empty and so an initial object. Define* ΓX *as the empty ordinal graph. Assume* Γ *is defined on globular cardinals of dimension* n *and let* X *have dimension* $n+1$. *Define* ΓX *as* $\mathcal{G}$ *where* $\mathcal{G}_* = X_0$ *and* $\mathcal{G}(\mathrm{p}x, x) = \Gamma X(\mathrm{p}x, x)$ *for each* $x \in X_0 \setminus \mathrm{f}$.

Define Γ *on morphisms using induction on the dimension of their domain. Let* $f\colon X \to Y$ *be a morphism with domain of dimension -1 (minus one). Define* Γf *as* $\emptyset \to \Gamma Y$ *the unique morphism out of the empty graph. Assume* Γ *is defined on morphisms with domain of dimension* n *and let* $f\colon X \to Y$ *be a morphism with domain of dimension* $n+1$. *Define* Γf *as* g *with object map* $f_0\colon X_0 \to Y_0$ *and* $g(\mathrm{p}x, x) = \Gamma f(\mathrm{p}x, x)$ *for each* $x \in X_0 \setminus \mathrm{f}$.

5.14. Definition. *We define a functor*

$$\Gamma'\colon OGraph \to GlobCard$$

using induction on the dimension of ordinal graphs. Let $\mathcal{G}$ *be an ordinal graph of dimension -1 (minus one). Then* $\mathcal{G}$ *is empty and so is initial. Define* $\Gamma'\mathcal{G}$ *as the initial globular cardinal. Assume* Γ' *is defined on ordinal graphs of dimension* n *and let* $\mathcal{G}$ *have dimension* $n+1$. *Define* $\Gamma'\mathcal{G}$ *as* $\mathrm{su}(\Gamma'\mathcal{G}(\mathrm{p}x, x), \mathcal{G}_*)$ *where* $\mathcal{G}(\mathrm{p}x, x)$ *is the collection of non-empty ordinal graphs of* $\mathcal{G}$ *indexed by* $x \in \mathcal{G}_* \setminus \mathrm{f}$.

Define Γ' *on morphisms using induction on the dimension of their domain. Let* $g\colon \mathcal{G} \to \mathcal{H}$ *be a morphism with domain of dimension -1 (minus one). Define* $\Gamma' g$ *as* $\emptyset \to \Gamma'\mathcal{H}$ *the unique morphism out of the empty ordinal graph. Assume* Γ' *is defined on morphisms with domain of dimension* n *and let* g *be a morphism with domain of dimension* $n+1$. *Define* $\Gamma' g$ *as the suspension* $\mathrm{su}(\Gamma' g(\mathrm{p}x, x), g)$ *where* $g(\mathrm{p}x, x)$ *is the collection of morphisms of ordinal graphs of* g *indexed by* $x \in \mathcal{G}_* \setminus \mathrm{f}$.

5.15. Theorem. *The category* $GlobCard$ *is equivalent to the category* $OGraph$ *by*

$$\Gamma\colon GlobCard \to OGraph$$

and its equivalence inverse Γ'.

Proof. We construct natural isomorphisms $\eta\colon \mathrm{Id} \Rightarrow \Gamma'\Gamma$ and $\epsilon\colon \Gamma\Gamma' \Rightarrow \mathrm{Id}$ using induction on the dimension of globular cardinals (respectively ordinal graphs). Let X be a globular cardinal of dimension -1 (minus one). Then X and $\Gamma'\Gamma X$ are both empty globular cardinals. Assume η is a natural isomorphism for globular cardinals of dimension n and let X have dimension $n+1$. Let $X' = \Gamma'\Gamma X$ which is $\mathrm{su}(\Gamma'\Gamma X(\mathrm{p}x, x), X_0)$. We construct an isomorphism $f\colon X \to \Gamma'\Gamma X$. Define f_0 as Id_{X_0} and f_n as the unique map out of the coproduct in

$$\begin{tikzcd}
\Gamma'\Gamma X(\mathrm{p}x,x)_n \arrow[r, "\cong"] \arrow[d, "\mathrm{copr}"'] & X(\mathrm{p}x,x)_n \arrow[d, "\mathrm{incl}"] \\
\sum \Gamma'\Gamma X(\mathrm{p}x,x)_n \arrow[r, dashed] & X_{n+1}
\end{tikzcd}$$

for each $n > 0$. Then f is a bijection as the coprojections are monomorphisms and the composites with the inclusions are monomorphisms and are jointly epi. Naturality arises from the universal property of coproduct.

Let $\mathcal{G}$ be an ordinal graph of dimension -1 (minus one). Then $\mathcal{G}$ and $\Gamma\Gamma'\mathcal{G}$ are both empty ordinal graphs. Assume ϵ is a natural isomorphism for ordinal graphs of dimension n and let $\mathcal{G}$ have dimension $n+1$. Define X as

$$\Gamma'\mathcal{G} = \mathrm{su}(\Gamma'\mathcal{G}(x_{i-1}, x_i), \mathcal{G}_*).$$

The globular cardinal $X(\mathrm{p}x, x)$ is the largest subfunctor of X satisfying the requirements of Definition 5.7. The coprojection

$$\Gamma'\mathcal{G}(\mathrm{p}x, x) \to \sum \Gamma'\mathcal{G}(\mathrm{p}x, x)$$

is also such a functor and so is identically $X(\mathrm{p}x, x)$. Then $\mathcal{G}' = \Gamma X = \Gamma\Gamma'\mathcal{G}$ has $\mathcal{G}'_* = \mathcal{G}_*$ and $\mathcal{G}'(\mathrm{p}x, x) = \Gamma\Gamma'\mathcal{G}(\mathrm{p}x, x)$ which by induction is naturally isomorphic to $\mathcal{G}(\mathrm{p}x, x)$. Hence ϵ is a natural isomorphism since for any ordinal graph $\mathcal{G}$ then $\epsilon_{\mathcal{G}}$ consists of an identity and components which are natural isomorphisms. ∎

5.16. Definition. *We define a map*

$$\Upsilon\colon \mathrm{Ob}\, i\Delta_+ \to \mathrm{Ob}\, OGraph$$

by induction on the height of objects of $i\Delta_+$*. Send the initial object* $[-1]$ *to the empty ordinal graph. Assume* Υ *is defined on objects of height* n *and let* H *have height* $n+1$*. Let* ΥH *be the ordinal graph* $\mathcal{G}$ *with* $\mathcal{G}_* = H_*$ *and*

$$\mathcal{G}(i-1, i) = \Upsilon H(i)$$

for i *which are not endpoints of* $(H_*)^\wedge$*. Recall* $H(i)$ *is trivial when* i *is an endpoint of* $(H_*)^\wedge$*. Notice that objects of* $i\Delta_+$ *of height* n *are sent to ordinal graphs of dimension* $n-1$*.*

Define $\Upsilon'\colon \mathrm{Ob}\, OGraph \to \mathrm{Ob}\, i\Delta_+$ *by induction on the dimension of ordinal graphs. Send the initial ordinal graph to the trivial object* $[-1]$*. Assume* Υ' *is defined on ordinal graphs of dimension* n *and let* $\mathcal{G}$ *have dimension* $n+1$*. Suppose* $\mathcal{G}_* = \{x_0, x_1, \ldots, x_p\}$*. Define* $\Upsilon'\mathcal{G}$ *as the object* H *with* $H_* = [p]$*, with* $H(0)$ *and* $H(p+1)$ *the trivial object of* $i\Delta_+$ *and with* $H(i) = \Upsilon'\mathcal{G}(x_{i-1}, x_i))$ *for each* $x \in [p]\setminus\mathrm{f}$*. Notice that the ordinal graphs of dimension* n *are sent to objects of* $i\Delta_+$ *of height* $n+1$*.*

5.17. THEOREM. *The object map* Υ *is surjective.*

PROOF. Clearly Υ is surjective on ordinal graphs of dimension -1 (minus one). Assume that Υ is surjective on ordinal graphs of dimension n, let $\mathcal{G}$ be an ordinal graph of dimension $n+1$ and let $\mathcal{G}' = \Upsilon\Upsilon'\mathcal{G}$. Then $\mathcal{G}_* = \mathcal{G}_*{}'$ and, by induction, for each $i \in \mathcal{G}_* \setminus \mathrm{f}$ we have $\mathcal{G}(\mathrm{p}i, i) = \mathcal{G}'(\mathrm{p}i, i)$. Hence $\mathcal{G} = \mathcal{G}'$ and Υ is surjective. ∎

5.18. ω-CATEGORIES. We start by recalling the definition of ω-category. Definition 5.21 is adapted from Michael Batanin's construction in [Ba98] of the free ω-category on a globular set. In addition we define the free ω-category on an ordinal graph and show that it is isomorphic to the free ω-category on the corresponding globular cardinal.

5.19. DEFINITION. *A* 1-category *is an ordinary category and a* 0-category *is a discrete* 1*-category. An* n-category *is a category enriched over* $(n-1)$*-categories for* $n > 0$*. Then* ωCat*, the category of* ω-categories*, is the colimit of the diagram*

$$0\text{-}Cat \to 1\text{-}Cat \to 2\text{-}Cat \to \cdots \to n\text{-}Cat \to \cdots$$

of inclusions. The initial ω-category has dimension *-1 (minus one). The ω-categories of n-Cat have* dimension n.

5.20. DEFINITION. *Let* Y *be a globular cardinal with* x *and* y *consecutive* n*-vertices. Define* $Y[x,y]$ *as the largest subfunctor of* Y *such that* $Y[x,y]_n = \{x,y\}$*. We have the inclusion* $\iota[x,y]\colon Y[x,y] \to Y$*. Let* γ *be a morphism of globular cardinals. Define* $\gamma[x,y]$ *as the composite* $\gamma \circ \iota[x,y]$*.*

5.21. DEFINITION. *We define a functor*

$$\mathfrak{F}\colon GlobCard \to \omega Cat.$$

Let X *be a globular cardinal. The* n*-cells of* $\mathfrak{F}X$ *are isomorphism classes of objects of* $GlobCard/X$ *which are globular morphisms* $\gamma\colon Y \to X$ *where* Y *has dimension* n*.*

Let Y *be a globular cardinal. We define the* m*-source of* Y *denoted* $\mathrm{s}_m Y$ *(respectively the* m*-target of* Y *denoted* $\mathrm{t}_m Y$*) for* $m \in \mathbb{N}$*. Define* $\mathrm{s}_0 Y$ *(respectively* $\mathrm{t}_0 Y$*) as the smallest subfunctor of* Y *containing the least (respectively greatest) element of* Y*. For* $m \geq 1$ *define* $\mathrm{s}_m Y$ *(respectively* $\mathrm{t}_m Y$*) as the smallest subfunctor of* Y *containing* Y_ℓ *for all* $\ell < m$ *and containing the least (respectively greatest) element of* $Y[\mathrm{p}y, y]_m$ *for all* y *in* $Y_{m-1} \setminus \mathrm{f}$*. Given an* n*-cell* γ *(representing an isomorphism class) then* $\mathrm{dom}_m(\gamma)$ *is (the isomorphism class of) the composite*

$$\mathrm{s}_m Y \xrightarrow{\quad \mathrm{incl} \quad} Y \xrightarrow{\quad \gamma \quad} X.$$

Likewise for $\mathrm{cod}_n(\gamma)$*.*

Composition in $\mathfrak{F}X$ *is given by pushout. Let* $\alpha\colon Y \to X$ *and* $\beta\colon Z \to X$ *be* n*-cells with* $\mathrm{dom}_m(\beta) = \mathrm{cod}_m(\alpha)$*. There is a unique isomorphism* $\delta\colon \mathrm{s}_m Z \to \mathrm{t}_m Y$ *such that*

$\mathrm{dom}_m \beta = \mathrm{cod}_m \alpha \circ \delta$ *where composition and equality is of globular morphisms. Define* $\beta \overset{m}{\circ} \alpha$ *as the unique morphism in*

$$\begin{tikzcd}
\mathrm{s}_m Z \arrow[r, "\mathrm{incl}"] \arrow[d, "\delta"'] & Z \arrow[d] \arrow[ddr, bend left, "\beta"] & \\
Y \arrow[r] \arrow[drr, bend right, "\alpha"'] & P \arrow[dr, dashed] & \\
& & X
\end{tikzcd}$$

out of the pushout P *which is defined as* $Z \backslash \mathrm{s}_m Z + Y$. *Specifically, we have* $P_\ell = Y_\ell = Z_\ell$ *for* $\ell < m$, $P_m = Z_m \backslash (\mathrm{s}_m Z)_m + Y_m$ *and* $P_\ell = Y_\ell + Z_\ell$ *for* $\ell > m$ *where the* $+$ *operation is the ordered union of linearly ordered sets. Let* $\circ$ *denote* 0*-composition.*

An n*-cell* γ *is* indecomposable *when the cardinality of* $Y[\mathrm{p}y, y]_m$ *is 2 (two) for all* $m < n$ *and is 1 (one) when* $m = n$. *An* n*-cell* γ *is* 0-indecomposable *when the cardinality of* Y_0 *is less than or equal to 2 (two). An* n*-cell* γ *is* m-indecomposable *(for* $m \geq 1$*) when the cardinality of* $Y[\mathrm{p}y, y]_m$ *is less than or equal to 2 (two) for all* $y \in Y_{m-1} \backslash \mathrm{f}$.

Observations 5.22, 5.23 and 5.24 are referred to in Definition 5.30.

5.22. Observation. *We identify an arbitrary* n*-cell of* $\mathfrak{F}X$ *with a canonical* 0*-composition of* 0*-indecomposable* n*-cells. Let* $\gamma\colon Y \to X$ *be an* n*-cell where* Y_0 *is* $\{y_0, y_1, \ldots, y_p\}$. *We have*

$$\gamma = \gamma[y_{p-1}, y_p] \circ \ldots \circ \gamma[y_0, y_1].$$

We denote this composite as $\circ_y \gamma[\mathrm{p}y, y]$. *and understand that* y *is an index over* $Y_0 \backslash \mathrm{f}$.

5.23. Observation. *We show* $(\mathrm{dom}_m \gamma)(\mathrm{p}y, y) = \mathrm{dom}_{m-1} \gamma(\mathrm{p}y, y)$ *for an* n*-cell* γ *of* $\mathfrak{F}X$ *where* $m < n$ *and* $y \in Y_0 \backslash \mathrm{f}$. *The lifting in*

$$\begin{tikzcd}
(\mathrm{s}_m Y)(\mathrm{p}y, y) \arrow[r, dashed] \arrow[d] & Y(\mathrm{p}y, y) \arrow[r, dashed, "\gamma(\mathrm{p}y,y)"] \arrow[d] & X(\mathrm{p}\gamma y, \gamma y) \arrow[d] \\
\mathrm{s}_m Y \circ u_1 \arrow[r] & Y \circ u_1 \arrow[r, "\gamma \cdot u_1"'] & X \circ u_1
\end{tikzcd}$$

along the vertical arrows is $(\mathrm{dom}_m \gamma)(\mathrm{p}y, y)$ *where the inclusions are described in Definition 5.7. The composite*

$$\mathrm{s}_{m-1} Y(\mathrm{p}y, y) \xrightarrow{\mathrm{incl}} Y(\mathrm{p}y, y) \xrightarrow{\gamma(\mathrm{p}y,y)} X(\mathrm{p}y, y)$$

is $\mathrm{dom}_{m-1} \gamma(\mathrm{p}y, y)$. *It remains to show that* $(\mathrm{s}_m Y)(\mathrm{p}y, y)$ *and* $\mathrm{s}_{m-1}(Y(\mathrm{p}y, y))$ *are identical. Let* x *be an element of* $(\mathrm{s}_m Y)(\mathrm{p}y, y)$. *Then* $\mathrm{p}y \blacktriangleleft x \blacktriangleleft y$ *and either*

$$x \in Y_\ell \text{ for } 1 < \ell < m$$

or

$$x \text{ is the least element of } Y[\mathrm{p}z, z]_m \text{ for some } z \in Y_{m-1} \backslash \mathrm{f}.$$

Given the first condition then they can be rewritten as

$$x \in Y(\mathrm{p}y, y)_\ell \text{ for } \ell < m-1$$

or

$$x \text{ is the least element of } Y(\mathrm{p}y, y)[\mathrm{p}z, z]_\ell \text{ for some } z \in Y(\mathrm{p}y, y)_{m-2} \backslash \mathrm{f}.$$

Jointly these rewritten conditions state that x *is in* $\mathrm{s}_{m-1}\,(Y(\mathrm{p}y, y))$. *The converse is also true. We have that* $(\mathrm{dom}_m\,\gamma)(\mathrm{p}y, y)$ *and* $\mathrm{dom}_{m-1}\,\gamma(\mathrm{p}y, y)$ *are identical.*

5.24. Observation. *We show* $(\beta \overset{m}{\circ} \alpha)(\mathrm{p}y, y) = \beta(\mathrm{p}y, y) \overset{m-1}{\circ} \alpha(\mathrm{p}y, y)$ *for* m*-composable* n*-cells* α *and* β *of* $\mathfrak{F}X$ *where* $m < n$ *and* $y \in Y_0 \backslash \mathrm{f}$. *Let* $\alpha \colon Y \to X$ *and* $\beta \colon Z \to X$ *be* m*-composable* n*-cells. Their composition* $\beta \overset{m}{\circ} \alpha$ *is given the unique morphism out of the pushout in*

$$\begin{tikzcd}
\mathrm{s}_m Z \arrow[r, "\mathrm{incl}"] \arrow[d, "\delta"'] & Z \arrow[d] \arrow[ddr, bend left, "\beta"] & \\
Y \arrow[r] \arrow[drr, bend right, "\alpha"'] & P \arrow[dr, dashed] & \\
& & X
\end{tikzcd}$$

where $P = Z \backslash \mathrm{s}_m Z + Y$. *Let* y *be an element of* P. *By Observation 5.23 then* $(\mathrm{s}_m Z)(\mathrm{p}y, y)$ *and* $\mathrm{s}_{m-1} Z(\mathrm{p}y, y)$ *are identical. Note that* $Z(\mathrm{p}y, y) \backslash (\mathrm{s}_m Z)(\mathrm{p}y, y)$ *is* $Z(\mathrm{p}y, y) \backslash \mathrm{s}_m Z$ *and further that* $Z(\mathrm{p}y, y) \backslash \mathrm{s}_m Z + Y(\mathrm{p}y, y)$ *is* $(Z \backslash \mathrm{s}_m Z + Y)(\mathrm{p}y, y)$. *Then* $P(\mathrm{p}y, y)$ *is the pushout of*

$$\begin{tikzcd}
\mathrm{s}_{m-1} Z(\mathrm{p}y, y) \arrow[r] \arrow[d] & Z(\mathrm{p}y, y) \arrow[d] \arrow[ddr, bend left, "\beta(\mathrm{p}y, y)"] & \\
Y(\mathrm{p}y, y) \arrow[r] \arrow[drr, bend right, "\alpha(\mathrm{p}y, y)"'] & P(\mathrm{p}y, y) \arrow[dr, dashed] & \\
& & X(\mathrm{p}y, y)
\end{tikzcd}$$

which is the lifting of the above diagram determined by the element $y \in P$. *As all horizontal and vertical maps above are monomorphisms we avoid unnecessary notation by labeling all restrictions with* $(\mathrm{p}y, y)$. *Hence the unique map of the second pushout is* $(\beta \overset{m}{\circ} \alpha)(\mathrm{p}y, y)$ *from the lifting and* $\beta(\mathrm{p}y, y) \overset{m}{\circ} \alpha(\mathrm{p}y, y)$ *by definition.*

5.25. DEFINITION. *We define a functor*

$$\mathfrak{F}\colon Graph_{\mathbb{N}} \to \omega Cat$$

inductively on the dimension of ordinal graphs. Let $\mathcal{G}$ *be the enriched graph of dimension -1 (minus one) which is the empty graph. Define* $\mathfrak{F}\mathcal{G}$ *as the empty* ω*-category. Assume* $\mathfrak{F}$ *is defined on enriched graphs of dimension* n *and let* $\mathcal{G}$ *have dimension* $n+1$*. Define the object set of* $\mathfrak{F}\mathcal{G}$ *as the object set of* $\mathcal{G}$*. Its hom-sets are defined by induction for distinct objects* x *and* y *as*

$$(\mathfrak{F}\mathcal{G})\,(x,y) = \sum_{x_0,\ldots,x_n \in \mathcal{G}_*} \mathfrak{F}\,(\mathcal{G}\,(x_{n-1},x_n)) \times ... \times \mathfrak{F}\,(\mathcal{G}\,(x_0,x_1))$$

where $x = x_0$ *and* $y = x_n$*. For all* $x \in \mathcal{G}_*$ *then* $(\mathfrak{F}\mathcal{G})(x,x)$ *is defined as*

$$\mathcal{C}_T + \sum_{x_0,\ldots,x_n \in \mathcal{G}_*} \mathfrak{F}(\mathcal{G}(x_{n-1},x_n)) \times ... \times \mathfrak{F}(\mathcal{G}(x_0,x_1))$$

where $\mathcal{C}_T$ *is the terminal* ω*-category.*

Define $\mathfrak{F}$ *on morphisms as follows. Let* $g\colon \mathcal{G} \to \mathcal{H}$ *be a morphism of enriched graphs with domain of dimension -1 (minus one). Define* $\mathfrak{F}g$ *as the unique* ω*-functor* $\emptyset \to \mathfrak{F}\mathcal{H}$*. Assume* $\mathfrak{F}$ *is defined for morphisms with domain of dimension* n *and let* $g\colon \mathcal{G} \to \mathcal{H}$ *be a morphism of enriched graphs with domain of dimension* $n+1$*. Let* $x_0, x_1, \ldots, x_p$ *denote the objects of* $\mathcal{G}$*. The morphisms of hom-objects are defined for distinct objects* x *and* y *as*

$$(\mathfrak{F}g)_{x,y} = \sum_{x_0,\ldots,x_n \in \mathcal{G}_*} \mathfrak{F}\,(g\,(x_{n-1},x_n)) \times ... \times \mathfrak{F}\,(g\,(x_0,x_1))$$

where $x = x_0$ *and* $y = x_n$*. For all* $x \in \mathcal{G}_*$ *then* $(\mathfrak{F}g)_{x,x}$ *is defined as*

$$g_T + \sum_{x_0,\ldots,x_n \in \mathcal{G}_*} \mathfrak{F}(g(x_{n-1},x_n)) \times ... \times \mathfrak{F}(g(x_0,x_1))$$

where $g_T : \mathcal{C}_T \to \mathcal{C}_T$ *.*

5.26. DEFINITION. *We define a forgetful functor*

$$\mathfrak{U}\colon \omega Cat \to Graph_{\mathbb{N}}$$

using induction on the dimension of ω*-categories. Let* $\mathcal{C}$ *be the* ω*-category of dimension -1 (minus one) which is the empty* ω*-category. Define* $\mathfrak{U}\mathcal{C}$ *as the empty ordinal graph. Assume* $\mathfrak{U}$ *is defined on* ω*-categories of dimension* n *and let* $\mathcal{C}$ *have dimension* $n+1$*. Define* $\mathfrak{U}\mathcal{C}$ *as the ordinal graph with object set* $\mathrm{Ob}\,\mathcal{C}$ *and with edge-object* $(\mathfrak{U}\mathcal{C})(x,y) = \mathfrak{U}(\mathcal{C}(x,y))$ *determined by induction for each pair of objects* x *and* y*.*

Define $\mathfrak{U}$ *on morphisms as follows. Let* $\mathcal{F}\colon \mathcal{C} \to \mathcal{A}$ *be an* ω*-functor with domain of dimension -1 (minus one). Define* $\mathfrak{U}\mathcal{F}$ *as the unique morphism* $\emptyset \to \mathfrak{U}\mathcal{A}$ *of enriched graphs. Assume* $\mathfrak{U}$ *is defined on* ω*-functors with domain of dimension* n *and let* $\mathcal{F}$ *have domain of dimension* $n+1$*. Define* $\mathfrak{U}\mathcal{F}$ *as the ordinal graph with object set morphism identical with that of* $\mathcal{F}$ *and with edge-object morphism* $(\mathfrak{U}\mathcal{F})(x,y) = \mathfrak{U}(\mathcal{F}(x,y))$ *determined by induction for each pair of objects* x *and* y*.*

5.27. Theorem. *The functor* $\mathfrak{F}$ *is left adjoint to* $\mathfrak{U}$.

Proof. We define, given an ordinal graph $\mathcal{G}$ and an ω-category $\mathcal{C}$, a bijection

$$\phi \colon Graph_{\mathbb{N}}(\mathcal{G}, \mathfrak{U}\mathcal{C}) \to \omega Cat(\mathfrak{F}\mathcal{G}, \mathcal{C})$$

using induction on the dimension of ordinal graphs. Let $g \colon \mathcal{G} \to \mathfrak{U}\mathcal{C}$ be a morphism of ordinal graphs with domain of dimension -1 (minus one). Define ϕg as the unique ω-functor $\mathfrak{F}\mathcal{G} \to \mathcal{C}$. Assume ϕ is defined and is a bijection on morphisms of ordinal graphs with domain of dimension n and let g be such a morphism with domain of dimension $n+1$. Define $\phi g = \mathcal{F} \colon \mathfrak{F}\mathcal{G} \to \mathcal{C}$ as follows. The object morphism of $\mathcal{F}$ is that of g. Define, using the induction assumption, the morphism $\mathcal{F}(x, y)$ of hom-sets as $\phi g(x, y)$ for each pair of objects x, y of $\mathcal{G}$.

Suppose that g and g' are morphisms of ordinal graphs such that $\phi g = \phi g'$. Then the object maps of g and g' are identical and by the induction assumption the morphisms of edge-objects are identical. Hence ϕ is injective.

Let $\mathcal{F} \colon \mathfrak{F}\mathcal{G} \to \mathcal{C}$ be an ω-functor. Define a morphism g of ordinal graphs as follows. The object map of g is that of F. The morphisms of edge-objects are defined using the induction assumption. Hence ϕ is surjective and $\mathfrak{F}$ is left adjoint to $\mathfrak{U}$. ■

5.28. Observation. *In the sequel we restrict* $\mathfrak{F}$ *of Definition 5.25 to the category of ordinal graphs. Recall that an ordinal graph* $\mathcal{G}$ *has* $\mathcal{G}(x, y)$ *non-empty if and only if* y *is the successor of* x. *Let* $\mathcal{G}$ *be an ordinal graph. For all objects* x *of* $\mathcal{G}$ *we have the hom-object* $(\mathfrak{F}\mathcal{G})(x, x) = \mathcal{C}_T$. *For every pair* x, y *of distinct objects then*

$$(\mathfrak{F}\mathcal{G})_{x,y} = \mathfrak{F}\left(\mathcal{G}\left(x_{n-1}, x_n\right)\right) \times \ldots \times \mathfrak{F}\left(\mathcal{G}\left(x_0, x_1\right)\right).$$

Given a morphism $g \colon \mathcal{G} \to \mathcal{H}$ *we have*

$$(\mathfrak{F}g)_{x,y} = \mathfrak{F}\left(g\left(x_{n-1}, x_n\right)\right) \times \ldots \times \mathfrak{F}\left(g\left(x_0, x_1\right)\right)$$

where $x = x_0$, $y = x_n$ *and* x_i *is the successor of* x_{i-1} *for* $i = 1, .., n$.

5.29. Observation. *We describe here the* n*-cells, domain, codomain and composition operations of* $\mathfrak{F}\mathcal{G}$ *for an ordinal graph* $\mathcal{G}$ *with objects* $x_0, x_1, \ldots, x_p$. *A* 0*-cell of* $\mathfrak{F}\mathcal{G}$ *is an object of* $\mathcal{G}$. *An* n*-cell* y *of* $\mathfrak{F}\mathcal{G}$ *is a sequence* $(y_i)_{i=h+1}^{k}$ *of* $(n-1)$*-cells, one from each factor, of the product* $\prod_{j=h+1}^{k} \mathfrak{F}\left(\mathcal{G}(x_{i-1}, x_i)\right)$ *for* $h \leq k$ *in* $\{0, 1, \ldots, p\}$. *The* 0*-domain of* y *is* x_h *and the* 0*-codomain of* y *is* x_k. *Composition (*0*-composition) is denoted by* $\circ$ *and is given by concatenation of sequences. Hence every* n*-cell is identified with a unique* 0*-composition.*

The m*-domain and* m*-codomain of an* n*-cell* y *denoted* $\mathrm{dom}_m\, y$ *and* $\mathrm{cod}_m\, y$ *are the* 0*-compositions* $\circ_{i=h+1}^{k} \mathrm{dom}_{m-1}\, y_i$ *and* $\circ_{i=h+1}^{k} \mathrm{cod}_{m-1}\, y_i$ *(respectively) for* $m < n$. *The* m*-composition of* n*-cells* y *and* z *is defined*

$$(y_i)_{i=h+1}^{k} \overset{m}{\circ} (z_i)_{i=h+1}^{k} \quad = \quad \circ_{i=h+1}^{k} \left(y_i \overset{m-1}{\circ} z_i \right)$$

for $m < n$ *where the* $(m-1)$*-composition is in* $\mathfrak{F}\mathcal{G}(x_{i-1}, x_i)$.

5.30. Definition. *We have two* free *functors both denoted* $\mathfrak{F}$, *one for globular cardinals and one for ordinal graphs and the functor* $\Gamma\colon$ *GlobCard* $\to$ *OGraph of Definition 5.13. Let* X *be a globular cardinal. We define an* ω*-functor*

$$L : \mathfrak{F}X \to \mathfrak{F}\Gamma X$$

using induction on the dimension of globular cardinals.

Let X *be a globular cardinal of dimension -1 (minus one). Then* $\mathfrak{F}X$ *and* $\mathfrak{F}\Gamma X$ *are both the empty* ω*-category. Assume that* L *is defined for globular cardinals of dimension* n *and let* X *have dimension* $n+1$. *Define* L *by induction on* m*-cells. Let* $\gamma\colon Y \to X$ *be a* 0*-cell of* $\mathfrak{F}X$. *Define* L *on* γ *as* $\gamma_0(y)$ *where* y *is the unique element of* Y *(and so of* Y_0*). Assume* L *is defined on* m*-cells and let* γ *be an* $(m+1)$*-cell of* $\mathfrak{F}X$. *By Observation 5.22 we have* $\circ_y\gamma[\mathrm{p}y, y]$ *the unique* 0*-decomposition of* γ. *Using the induction assumption define* $L\,\gamma$ *as*

$$(L\,\gamma(\mathrm{p}y, y))_{y\in Y_0\setminus \mathrm{f}} \quad \textit{in} \quad \prod_{y\in\gamma(Y_0\setminus \mathrm{f})} \mathfrak{F}\left(\Gamma X(\gamma\mathrm{p}y, \gamma y)\right).$$

Note that γ *is an* m*-cell if and only if* $L\,\gamma$ *is.*

We show using induction that L *preserves the* ℓ*-domain and* ℓ*-codomain operations for* $\ell < m$. *The* 0*-domain of* $L\,\gamma$ *is* γy_0 *and the* 0*-codomain is* γy_ℓ *where* y_0 *is the first element, respectively* y_ℓ *is the last element, of* Y_0. *Hence* L *preserves* dom_0 *and* cod_0.

Assume that L *preserves the* ℓ*-domain operation. The following derivation begins by replacing* γ *with its* 0*-decomposition, uses a basic property of* ω*-categories to proceed from line 1 to line 2, uses Observation 5.23 to proceed from line 3 to line 4 and uses the induction assumption to proceed from line 4 to line 5. We have*

$$\begin{aligned}
L\,\mathrm{dom}_{\ell+1}\gamma &= L\,\mathrm{dom}_{\ell+1}\ \circ_y \gamma[\mathrm{p}y, y]\\
&= L\ \circ_y \mathrm{dom}_{\ell+1}\gamma[\mathrm{p}y, y]\\
&= \left(L\ (\mathrm{dom}_{\ell+1}\gamma[\mathrm{p}y, y])\,(py, y)\right)_y\\
&= \left(L\,\mathrm{dom}_\ell\ (\gamma[\mathrm{p}y, y](py, y))\right)_y\\
&= \left(\mathrm{dom}_\ell\, L\,\gamma[\mathrm{p}y, y](py, y)\right)_y\\
&= \mathrm{dom}_{\ell+1}\ \left(L\,\gamma(py, y)\right)_y\\
&= \mathrm{dom}_{\ell+1}\, L\,\gamma
\end{aligned}$$

and L *preserves the domain operations. Likewise for the codomain operations.*

We show by induction that L *preserves* ℓ*-composition for* $\ell < m$. *By construction* L *preserves* 0*-composition. Assume that* L *preserves* ℓ*-composition. Let* α *and* β *be* $(\ell+1)$*-composable* n*-cells. Then each has* 0*-decomposition* $\circ_y\alpha[y]$ *and* $\circ_y\beta[y]$ *(respectively). The following derivation begins by replacing* α *and* β *with their* 0*-decompositions, uses a basic property of* ω*-categories to proceed from line 1 to line 2,* *uses Observation 5.24 to proceed from line 3 to line 4 and uses the induction assumption to proceed from line 4 to line 5. We have*

$$\begin{aligned}
L\,(\beta \overset{\ell+1}{\circ} \alpha) &= L\,(\circ_y\beta[\mathrm{p}y,y] \overset{\ell+1}{\circ} \circ_y\alpha[\mathrm{p}y,y]) \\
&= L\,(\circ_y(\beta[\mathrm{p}y,y] \overset{\ell+1}{\circ} \alpha[\mathrm{p}y,y])) \\
&= (L\,(\beta[\mathrm{p}y,y] \overset{\ell+1}{\circ} \alpha[\mathrm{p}y,y])(\mathrm{p}y,y))_y \\
&= (L\,(\beta[\mathrm{p}y,y](\mathrm{p}y,y) \overset{\ell}{\circ} \alpha[\mathrm{p}y,y](\mathrm{p}y,y)))_y \\
&= (L\,\beta(\mathrm{p}y,y) \overset{\ell}{\circ} L\,\alpha(\mathrm{p}y,y))_y \\
&= (L\,\beta(\mathrm{p}y,y))_y \overset{\ell+1}{\circ} (L\,\alpha(\mathrm{p}y,y))_y \\
&= L\,\beta \overset{\ell+1}{\circ} L\,\alpha
\end{aligned}$$

and L *preserves composition.*

5.31. LEMMA. *The* ω*-functor* $L\colon \mathfrak{F}X \to \mathfrak{F}\Gamma X$ *is an isomorphism.*

PROOF. The object set of $\mathfrak{F}X$ is isomorphic to X_0 and the object set of $\mathfrak{F}\Gamma X$ is X_0.

Faithful. We show that L is faithful using induction on the n-cells of $\mathfrak{F}\Gamma X$. Consider morphisms $\gamma\colon Y \to X$ and $\gamma'\colon Y' \to X$ of $\mathfrak{F}X$. Suppose that $L\,\gamma = L\,\gamma'$ is a 0-cell of $\mathfrak{F}\Gamma X$. Then Y and Y' are singletons, γ and γ' have identical image and so are isomorphic in $GlobCard/X$. Hence they are identical in $\mathfrak{F}X$.

Assume that L is faithful on n-cells and let $L\,\gamma = L\,\gamma'$ be an $(n{+}1)$-cell. We construct an isomorphism $\alpha\colon Y \to Y'$. By Observation 5.22 then γ and γ' have canonical 0-decompositions $\circ_y\gamma(y)$ and $\circ_{y'}\gamma'(y')$ for $y \in Y_0\setminus$f and $y' \in Y_0'\setminus$f respectively. By the construction of these compositions and by definition of L then Y_0 Y_0' are isomorphic. Such an isomorphism is unique and we have $\alpha_0\colon Y_0 \to Y_0'$ as Y_0 and Y_0' are linear orders.

Uniqueness of the 0-decompositions in $\mathfrak{F}\Gamma X$ implies that $L\,\gamma(y) = L\,\gamma'(\delta_0 y)$ for each $y \in Y_0\setminus$f. By definition of L then $(L\,\gamma(y)(\mathrm{p}y,y))$ and $(L\,\gamma'(\delta_0 y)(\mathrm{p}\delta_0 y', \delta_0 y'))$ are identical and by induction the restrictions $\gamma(y)(\mathrm{p}y,y)$ and $\gamma'(\delta_0 y)(\mathrm{p}\delta_0 y, \delta_0 y)$ are identical. Then there is a (unique) isomorphism $Y(\mathrm{p}y,y) \cong Y'(\mathrm{p}\delta_0 y, \delta_0 y)$ for each $y \in Y_0\setminus$f. As the corresponding inclusions into $Y \circ u_1$ and $Y' \circ u_1$ (respectively) are jointly epi then we have isomorphisms $\delta_n\colon Y_n \cong Y_n'$ for $n \geq 1$. The source and target operations are preserved and we have $\delta\colon Y \cong Y'$. Hence $\gamma = \gamma'$ and L is faithful.

Full. We show that L is full using induction on the n-cells of $\mathfrak{F}\Gamma X$. Let x be a 0-cell. Then x is an element of X_0. Let Y be a globular cardinal with a single element y and define a globular morphism $\gamma\colon Y \to X$ by $\gamma_0\, y = x$. Then $L\gamma = x$.

Assume that L is full on n-cells and let $x = (x_i)_{i=h+1}^k$ be an $(n+1)$-cell. By induction we have n-cells $\gamma(i)$ such that $x_i = L\,\gamma(i)$. Then as L preserves composition we have $L\,(\circ_{i=h+1}^k\gamma(i)) = \circ_{i=h+1}^k x_i$ which is $(x_i)_{i=h+1}^k$.

Hence $L : \mathfrak{F}X \to \mathfrak{F}\Gamma X$ is an isomorphism. ∎

6. Equivalence between $i\Delta$ and Θ

Michael Makkai and Marek Zawadowski demonstrate, in [MZ01], a duality between the category *Disk* as defined by André Joyal in [Jo97] and the category, denoted $\mathcal{S}$ in [MZ01], of *simple* ω-categories. We refer the reader to [MZ01] for the details, but quote their definition below. Note that $[G]$ is the free ω-category on G an ω-graph which we call a globular set.

> Let G be an ω-graph. Let us call an element (cell) a of $[G]$ *maximal* if it is *proper*, that is, not an identity cell, and if the only monomorphisms $m\colon H \to G$ for which a belongs to the image of $[m]$ are isomorphisms. Intuitively, an element is maximal if it is proper, and the whole graph G is needed to generate it. We call G *composable* if $[G]$ has a *unique* maximal element; in that case, the maximal element may be called the *composite* of the graph.
>
> ...
>
> An ω-category is *simple* if it is of the form $[G]$ for a composable ω-graph. The category $\mathcal{S}$ is defined as the full subcategory of ωCat on the simple ω-category as objects.

In Proposition 4.8 of [MZ01] they demonstrate that an ω-graph is composable if and only if it is a globular cardinal. Hence the objects of $\mathcal{S}$ are ω-categories which are isomorphic to the free ω-category $\mathfrak{F}X$ for some globular cardinal X.

6.1. Definition. *We define a functor*

$$\Psi\colon i\Delta_+ \to \omega Cat.$$

Define Ψ on objects as the composite $\mathfrak{F}\circ\Upsilon$. See Definitions 5.16 and 5.25 and Observation 5.28, but we make it more explicit here. Let H be an object of $i\Delta_+$. Let $\mathcal{G} = \Upsilon H$. Then the objects of $\mathcal{G}$ are those of H and the edge-object $\mathcal{G}(\mathrm{pi}, i)$ is $\Upsilon H(i)$ for each $i \in \mathcal{G}_* \setminus \mathrm{f}$. The remaining edge-objects are empty ordinal graphs. Let $\mathcal{A} = \mathfrak{F}\mathcal{G}$. Then $\mathcal{A}$ is an ω-category with object set $\mathcal{G}_*$ and hom-objects

$$\mathcal{A}(i,j) = \prod_{k=i+1}^{j} \mathfrak{F}\,\Upsilon H(k)$$

when $i < j$ in $\mathcal{G}_*$. For all $i \in \mathcal{G}_*$ then $\mathcal{A}(x,x)$ is the terminal ω-category. For $i > j$ then $\mathcal{A}(i,j)$ is the empty ω-category. Notice that the trivial object $[-1]$ of $i\Delta_+$ is sent to the empty ω-category which is also initial.

We define Ψ on morphisms by induction on the height of their domain. Send morphisms with domain of dimension -1 (minus one) to the unique morphism out of the empty ω-category. Assume that Ψ is defined on morphisms with domain of height n and let

$g\colon H \to K$ have domain of height $n+1$. We construct an ω-functor $\mathcal{F}\colon \Psi H \to \Psi K$ from the data of g. Let $\mathcal{C} = \Psi H$ and $\mathcal{A} = \Psi K$. Then $\mathcal{C}$ has object set H_* and $\mathcal{A}$ has object set K_*. Define the object map of $\mathcal{F}$ as the object map of g. We construct an ω-functor $\mathcal{F}_{\mathrm{p}i,i}\colon \mathcal{C}\,(\mathrm{p}i, i) \to \mathcal{A}\,(\mathcal{F}(\mathrm{p}i), \mathcal{F}(i))$ for each $i \in \mathrm{Ob}\,\mathcal{C} \setminus \mathrm{f}$. From the definition of Ψ on objects we have $\mathcal{C}\,(\mathrm{p}i, i) = \mathfrak{F}\Upsilon H(i)$ and

$$\mathcal{A}\,(\mathcal{F}\,(\mathrm{p}i)\,, \mathcal{F}\,(i)) = \prod_{j=\mathcal{F}(\mathrm{p}i)+1}^{\mathcal{F}(i)} \mathfrak{F}\Upsilon K(j)$$

where $J_i = \{\mathcal{F}\,(\mathrm{p}i) + 1, ..., \mathcal{F}\,(i)\}$ is the index set of the product. From Observation 2.3 we have $J_i = (g^\wedge)^*\,(i)$ and so $g^\wedge(j) = i$ for all $j \in J_i$. We have a morphism $g(j)\colon H(i) \to K(j)$ of $i\Delta_+$ with domain of height n for each $j \in J_i$ by definition of g. By induction, there are ω-functors $\Psi g(j)\colon \mathfrak{F}\Upsilon H(i) \to \mathfrak{F}\Upsilon K(j)$ which by the universal property of the product give the required morphism

$$\mathcal{F}_{\mathrm{p}i,i}\colon \mathfrak{F}\Upsilon H(i) \to \prod_{i=\mathcal{F}(\mathrm{p}i)+1}^{\mathcal{F}(i)} \mathfrak{F}\Upsilon K(j). \tag{3}$$

We have $\mathcal{F}_{i,j} = \prod_{k=i+1}^{j} \mathcal{F}_{\mathrm{p}k,k}$ by Definition 5.25 and Observation 5.28 for $i < j$. For i an object of $\mathcal{C}$ then $\mathcal{F}_{i,i}$ is the unique morphism into the terminal ω-category. For $i > j$ then $\mathcal{F}_{i,j}$ has domain the initial (empty) ω-category.

Preserves composition. *Let $g'\colon H \to K$ and $g''\colon K \to L$ be composable morphisms of $i\Delta_+$. Put $\mathcal{F}' = \Psi(g')$, $\mathcal{F}'' = \Psi(g'')$ and $\mathcal{F} = \Psi(g)$ where $g = g'' \circ g'$. Let i be an element of H_*, let $J_i = \left(g'^\wedge\right)^*(i)$ and let $L_j = \left(g''^\wedge\right)^*(j)$. Then $L_i = \left(g'' \circ g'^\wedge\right)^*(i)$ is identically $\bigcup_{j \in J_i} L_j$.*

We show $\mathcal{F} = \mathcal{F}'' \circ \mathcal{F}'$ by showing that the upper horizontal composite of

$$\begin{tikzcd}
\mathfrak{F}\Upsilon H(i) \arrow[r, "{\mathcal{F}'}_{\mathrm{p}i,i}"] \arrow[dr, "\Psi g'(j)"'] & \prod_{j=J_i} \mathfrak{F}\Upsilon K(j) \arrow[r, "\prod_j {\mathcal{F}''}_{\mathrm{p}j,j}"] \arrow[d, "\mathrm{pr}"] & \prod_{\ell=L_i} \mathfrak{F}\Upsilon L(\ell) \arrow[d, "\mathrm{pr}"] \\
& \mathfrak{F}\Upsilon K(j) \arrow[r, "{\mathcal{F}''}_{\mathrm{p}j,j}"] \arrow[dr, "\Psi g''(\ell)"'] & \prod_{\ell=L_j} \mathfrak{F}\Upsilon L(\ell) \arrow[d, "\mathrm{pr}"] \\
& & \mathfrak{F}\Upsilon L(\ell)
\end{tikzcd}$$

is identically $\mathcal{F}_{\mathrm{p}i,i}$. For each l in L_* then $g(\ell)\colon H(g^\wedge \ell) \to L(\ell)$ is $g''(\ell) \circ g'(j)$ by definition of composition in $i\Delta_+$ where $j = g''^\wedge \ell$. By induction then $\Psi g(\ell)$ is the diagonal composite. The construction ending at line 3 gives $\mathcal{F}_{\mathrm{p}i,i}$ as the unique map $\mathfrak{F}\Upsilon H(i) \to \prod_{\ell=L_i} \mathfrak{F}\Upsilon L(\ell)$ which is $\prod_j {\mathcal{F}''}_{\mathrm{p}j,j} \circ {\mathcal{F}'}_{\mathrm{p}i,i}$ as required.

6.2. THEOREM. *The category* $i\Delta_+$ *is equivalent to the category* Θ_+ *by*

$$\Psi\colon i\Delta_+ \to \Theta_+.$$

PROOF. The functor Ψ is essentially surjective by Lemma 5.31.

Faithful. We show Ψ is faithful using induction on the dimension of the domain of ω-functors. As the initial object of $i\Delta_+$ is sent to the initial ω-category then Ψ is faithful on morphisms with domains of dimension -1 (minus one) as their domain is the initial object. Assume Ψ is faithful on morphisms with domain of dimension n and that $g, g'\colon H \to K$ are parallel morphisms of $i\Delta_+$ with domains of dimension $n+1$ such that $\mathcal{F} = \Psi g$ and $\mathcal{F}' = \Psi g'$ are identical. Then $g = g'$ as the object maps of $\mathcal{F}$ and $\mathcal{F}'$ are identical.

Let $\mathcal{C} = \Psi H$. For each $i \in \mathrm{Ob}\,\mathcal{C} \setminus \mathrm{f}$ the construction ending at line 3 gives

$$\mathcal{F}_{\mathrm{p}i,i} = \mathcal{F}'_{\mathrm{p}i,i}\colon \mathfrak{F}\Upsilon H(i) \to \prod_{j=\mathcal{F}(\mathrm{p}i)+1}^{\mathcal{F}(i)} \mathfrak{F}\Upsilon K(j)$$

where the index set of the product is $J_i = \{\mathcal{F}(\mathrm{p}j)+1, \mathcal{F}(\mathrm{p}j)+2, \ldots, \mathcal{F}(j)\}$. By construction, composition with the projections out of the product gives

$$\Psi g(j) = \Psi g'(j)\colon \mathfrak{F}\Upsilon H(i) \to \mathfrak{F}\Upsilon K(j)$$

and by the induction assumption $g(j) = g'(j)$ for each $j \in J_i$. Let $J = \bigcup_{i \in \mathrm{Ob}\,\mathcal{C}\setminus \mathrm{f}} J_i$. It remains to show for $j \notin J$ that the morphisms $g(j)$ and $g'(j)$ determined by j are identical.

For $j \notin J$ then $g^\wedge j$ is an endpoint by Observation 2.4. Hence $H(g^\wedge j)$ is trivial and $g(j), g'(j)\colon H(g^\wedge j) \to K(j)$ are identical. Therefore Ψ is faithful.

Full. We show that Ψ is full using induction on the dimension of the domain of ω-functors. Let $\mathcal{F}\colon \Psi H \to \Psi K$ be an ω-functor where $\mathcal{C} = \Psi H$. We construct a morphism g of $i\Delta_+$ from the data of $\mathcal{F}$ such that $\Psi g = \mathcal{F}$. Define the object map of g as the object map of $\mathcal{F}$.

Let $\mathcal{F}$ be an ω-functor with domain of dimension -1 (minus one). Then $\mathcal{F}$ has domain an initial ω-category and so g is the unique morphism $[-1] \to K$. Assume that Ψ is full for ω-functors with domain of dimension n and suppose $\mathcal{F}$ has domain of dimension $n+1$. We have for each object $j \in \mathrm{Ob}\,\mathcal{C} \setminus \mathrm{f}$ morphisms (ω-functors)

$$\mathcal{F}_{\mathrm{p}i,i}\colon \mathcal{C}(\mathrm{p}i, i) \to \mathcal{A}(\mathcal{F}(\mathrm{p}i), \mathcal{F}(i))$$

with domains of dimension n which are identically

$$\mathcal{F}_{\mathrm{p}i,i}\colon \Psi H(i) \to \prod_{j=\mathcal{F}(\mathrm{p}i)+1}^{\mathcal{F}(i)} \Psi K(j).$$

The product has index set $J_i = \{\mathcal{F}(\mathrm{p}j)+1, \mathcal{F}(\mathrm{p}j)+2, \ldots, \mathcal{F}(i)\}$. Composition with the projections of the product gives for each $j \in J_i$ an ω-functor

$$\mathcal{F}(i,j)\colon \Psi H(i) \to \Psi K(j).$$

By induction there is a morphism $g(i,j)$ such that $\Psi g(i,j) = \mathcal{F}(i,j)$ for each $j \in J_i$ and each $i \in \mathrm{Ob}\,\mathcal{C} \backslash \mathrm{f}$. Let $J = \bigcup J_i$ where the union is indexed over $\mathrm{Ob}\,\mathcal{C} \backslash \mathrm{f}$. It remains to define, for each $j \notin J$, morphisms $g(j)\colon H(g^\wedge j) \to K(j)$. For such j then $g^\wedge j$ is an endpoint by Observation 2.4. Hence $H(g^\wedge j)$ is initial and $g(i)$ is determined. Therefore Ψ is full. ∎

6.3. Corollary. *The category $i\Delta$ is equivalent to the category $\mathcal{S}$.*

We have demonstrated that the categories *Disk* and Θ are dual using the well-known equivalence between intervals and ordinals. The category $i\mathcal{I}$ has been shown equivalent to *Disk* and provides a description of disks as inductively defined intervals. Similarly $i\Delta$ has been shown equivalent to Θ and provides a description of Θ in terms of inductively defined ordinals.

7. Labeled trees

In this section we define dual categories, named $t\mathcal{I}$ and $t\Delta$, which are equivalent to the category *Disk* and the category Θ (respectively). To do so we construct categories named $t\mathcal{I}_+$ and $t\Delta_+$, which are augmented counterparts to $t\mathcal{I}$ and $t\Delta$ and show that they are dual using the equivalence between ordinals and intervals. They are constructed from trees whose vertices are labeled with intervals and ordinals (respectively) and which satisfy certain conditions.

We begin by defining the concept of labeled tree with appropriate restriction and suspension operations. After defining the concept of constrained tree we state and prove a general theorem which gives mild conditions which allow an equivalence of categories to be lifted to an equivalence between certain categories of constrained trees labeled by the categories of the original equivalence.

We then define dual categories of constrained trees labeled by $\mathcal{I}_+$ and Δ_+ which satisfy the conditions of the general theorem. After defining the concept of cropped tree the above equivalence is then restricted to the full subcategories with objects the cropped trees of positive degree.

7.1. Definition. *A* forest *(A, F)* labeled in a category *$\mathcal{C}$ is a forest equipped with functors $F_n\colon A_n \to \mathcal{C}$ for all $n \in \mathbb{N}$ where A_n is considered a discrete category. A* tree *(A, F)* labeled in a category *$\mathcal{C}$ has A a tree.*

7.2. Definition. *A* forest morphism *$(f, \alpha)\colon (A, F) \to (B, G)$* in *$\mathcal{C}$ is given by a tree map $f\colon A \to B$ of trees and a set of natural transformations*

$$\alpha_n\colon F_n \Rightarrow G_n \circ f_n\colon A \to \mathcal{C}$$

for all $n \in \mathbb{N}$*. A* tree morphism $(f, \alpha)\colon A \to B$ in $\mathcal{C}$ *has both* A *and* B *labeled trees.*

Identity morphisms are given by an identity set map and identity natural transformations. Let $(f\colon A \to B, \alpha\colon F \Rightarrow Gf)$ *and* $(g\colon B \to C, \beta\colon G \Rightarrow Hg)$ *be composable morphisms. Then* $(g, \beta) \circ (f, \alpha)$ *is defined as*

$$(g \circ f\colon A \to C, \beta_f \circ \alpha\colon F \Rightarrow Hgf).$$

A forest op-morphism $(f, \alpha)\colon (B, G) \to (A, F)$ in $\mathcal{C}$ *is a tree morphism in* $\mathcal{C}^{\mathrm{op}}$ *and so is a tree map* $f\colon A \to B$ *and a set of natural transformations*

$$\alpha_n\colon G_n \circ f_n \Rightarrow F_n\colon A \to \mathcal{C}$$

for all $n \in \mathbb{N}$*. Let* $(g\colon B \to C, \beta\colon Hg \Rightarrow G)$ *and* $(f\colon A \to B, \alpha\colon Gf \Rightarrow F)$ *be composable op-morphisms. Then* $(f, \alpha) \circ (g, \beta)$ *is defined as*

$$(g \circ f\colon A \to C, \alpha \circ \beta_f\colon Hgf \Rightarrow F).$$

We have the category $Forest(\mathcal{C})$ *of labeled forests and morphisms in* $\mathcal{C}$ *and its full subcategory* $Tree(\mathcal{C})$ *with objects labeled trees in* $\mathcal{C}$*.*

7.3. Definition. *We define a* restriction *operation on labeled trees. The* restriction of (A, F) by x *is denoted* $(A(x), F(x))$ *where* (A, F) *is a labeled forest and* x *is an element of* A_n *has* $A(x)$ *given by Definition 4.3 and* $F(x)_m$ *defined as the composite*

$$A(x)_m \xrightarrow{\text{incl.}} A_{n+m} \xrightarrow{F_{n+m}} \mathcal{C}.$$

The restriction of (f, α) by x *is denoted* $(f(x), \alpha(x))$ *where* $(f, \alpha)\colon A \to B$ *is a forest morphism in* $\mathcal{C}$ *and* x *is an element of* A_n *has* $f(x)$ *given by Definition 4.3 and* $\alpha(x)_m$ *given by the composite pasting diagram*

$$\begin{tikzcd}
A(x)_m \arrow[r, "f(x)_m"] \arrow[d, "\text{incl}"'] & B(f_n x)_m \arrow[d, "\text{incl}"] \\
A_{n+m} \arrow[r, "f_{n+m}"] \arrow[d, "F_{n+m}"'] \arrow[dr, phantom, "\overset{\alpha_{n+m}}{\Rightarrow}"] & B_{n+m} \arrow[d, "G_{n+m}"] \\
\mathcal{C} \arrow[r, "1"'] & \mathcal{C}
\end{tikzcd}$$

where the two vertical composites are $F(i)_m$ *and* $G(f_n x)_m$ *(respectively).*

The restriction of (f, α) *by* x *is denoted* $(f(x), \alpha(x))$ *where* (f, α) *is a forest op-morphism (with* $f\colon A \to B$ *and* $\alpha\colon Gf \Rightarrow F$*) and* x *is an element of* A_n *has* $f(x)$ *given by Definition 4.3 and has* $\alpha(x)_m$ *given by the composite pasting diagram*

$$\begin{tikzcd}
B(f_n x)_m \arrow[d, "\text{incl}"'] & A(x)_m \arrow[l, "f(x)_m"'] \arrow[d, "\text{incl}"] \\
B_{n+m} \arrow[d, "G_{n+m}"'] \arrow[dr, phantom, "\overset{\alpha_{n+m}}{\Rightarrow}"] & A_{n+m} \arrow[l, "f_{n+m}"'] \arrow[d, "F_{n+m}"] \\
\mathcal{C} \arrow[r, "1"'] & \mathcal{C}
\end{tikzcd}$$

where the two vertical composites are $G(f_n x)_m$ *and* $F(x)_m$ *(respectively).*

7.4. DEFINITION. *Define a* suspension *functor*

$$\mathrm{su}\colon \mathit{Forest}(\mathcal{C}) \times \mathcal{C} \to \mathit{Tree}(\mathcal{C})$$

as follows. Given a forest (A, F) *in* $\mathcal{C}$ *and an object* c *in* $\mathcal{C}$ *then the* suspension of A over c *denoted* $\mathrm{su}(A, c)$ *is* (A', F') *where* $A' = \mathrm{su}\, A$ *and* F' *is defined by* $F'_0(*) = c$ *and* $F'_{n+1} = F_n$ *for* $n \in \mathbb{N}$.

Given a forest morphism $(f, \alpha)\colon (A, F) \to (B, G)$ *and a morphism* $g\colon c \to d$ *of* $\mathcal{C}$ *then the* suspension of f, α over g *denoted* $\mathrm{su}(f, g)$ *is* $(f', \alpha')\colon \mathrm{su}(A, c) \to \mathrm{su}(B, d)$ *where* $f' = \mathrm{su}\, f$ *and* α' *is defined by* $\alpha'_0(*) = g$ *and* $\alpha'_{n+1} = \alpha_n$ *for* $n \in \mathbb{N}$.

7.5. OBSERVATION. *This is the "labeled" counterpart to Observation 4.5. The coproduct of a collection of labeled trees is a labeled forest and its suspension is a labeled tree. The labeled subtrees of the suspension are isomorphic to the trees of the original collection. We provide the details below.*

Let c *be an object of* $\mathcal{C}$ *and* $(A(i), F(i))$ *be a labeled tree in* $\mathcal{C}$ *with* $A(i)_0 = \{x_i\}$ *for each* i *in a set* I. *Let* (A', F') *be the suspension of* $(\sum A(i), \sum F(i))$ *over* c. *By Observation 4.5 then* $\mathrm{copr}(i)\colon A(i) \cong A'(x_i)$ *and the left triangle of*

$$\begin{tikzcd}
 & A(i)_n \arrow[dl, "\cong"'] \arrow[d, "\mathrm{copr}"] \arrow[dr, "F(i)_n"] & \\
A'(x_i)_n \arrow[r, "\text{incl.}"'] & \sum A(i)_n \arrow[r, dashed] & \mathcal{C}
\end{tikzcd}$$

commutes. The lower composite is $F'(x_i)_n$ *and we have an isomorphism*

$$(\mathrm{copr(i)}, \mathrm{Id}\colon F(i) \Rightarrow F'(x_i) \circ \mathrm{copr}(i))$$

between $(A(i), F(i))$ *and* $(A'(x_i), F'(x_i))$.

7.6. DEFINITION. *A labeled forest* (A, p, F) *in* $\mathcal{C}$ *is said to be* constrained by the functor $\mathcal{U}\colon \mathcal{C} \to \mathit{Set}$ *when for each* $n \in \mathbb{N}$ *we have an isomorphism*

$$\lambda_n\colon A_{n+1} \cong \mathit{el}(\mathcal{U}F_n)$$

such that $\lambda_n x \in \mathcal{U}F_n(p_n x)$ *where* $\mathit{el}(\mathcal{U}F_n)$ *is the category of elements of* $\mathcal{U}F_n$ *and consists of pairs* (y, ξ) *with* $y \in A_n$ *and* $\xi \in \mathcal{U}F_n(y)$. *Then* $\lambda_n x = (y, \xi)$ *where* $y = p_n(x)$. *The set* A_{n+1} *of* $(n+1)$*-dimensional vertices of* A *is determined by the labels on its* n*-dimensional vertices.*

7.7. DEFINITION. *A morphism* $(f,\alpha)\colon (A,p,F) \to (B,q,G)$ *between labeled trees in* $\mathcal{C}$ *is said to be* constrained by $\mathcal{U}\colon \mathcal{C} \to \mathit{Set}$ *when for each* $n \in \mathbb{N}$

$$\begin{tikzcd} A_{n+1} \arrow[r, "\lambda_n"] \arrow[d, "f_{n+1}"'] & \mathrm{el}(\mathcal{U}F_n) \arrow[d, "\mathrm{el}(\mathcal{U}\cdot\alpha_n)"] \\ B_{m+1} \arrow[r, "\eta_n"'] & \mathrm{el}(\mathcal{U}G_n) \end{tikzcd}$$

commutes where the horizontal arrows are the isomorphisms of Definition 7.6. An op-morphism $(f\colon A \to B, \alpha\colon Gf \Rightarrow F)\colon (B,q,G) \to (A,p,F)$ between labeled trees in $\mathcal{C}$ constrained by $\mathcal{U}\colon \mathcal{C}^{\mathrm{op}} \to \mathit{Set}$ *also has that the above diagram commutes. As the horizontal maps above are isomorphisms, the set map* f_{n+1} *is determined by the data from lower dimensions. We have the category* $\mathit{Con}\,(\mathcal{C},\mathcal{U})$ *of labeled forests in* $\mathcal{C}$ *constrained by* $\mathcal{U}$*.*

7.8. DEFINITION. *The* 2*-category* $\mathit{Cat}/_{\mathit{Set}}$ *is the* 2*-category of small categories over* Set*. We define a* 2*-functor*

$$\mathit{Con} : \mathit{Cat}/_{\mathit{Set}} \to \mathit{Cat}$$

as follows. Let $\mathcal{U}\colon \mathcal{C} \to \mathit{Set}$ *be an object of* $\mathit{Cat}/_{\mathit{Set}}$*, which we also write as* $(\mathcal{C},\mathcal{U})$*, and define* $\mathit{Con}\,(\mathcal{C},\mathcal{U})$ *as the category with objects and morphisms given by Definitions 7.6 and 7.7 (respectively). Let* $F\colon \mathcal{C} \to \mathcal{A}$ *be a morphism of* $\mathit{Cat}/_{\mathit{Set}}$ *and define*

$$\begin{aligned} \mathit{Con}\,F\colon \mathit{Con}\,(\mathcal{C},\mathcal{U}_{\mathcal{C}}) &\to \mathit{Con}\,(\mathcal{A},\mathcal{U}_{\mathcal{A}}) \\ : (A,H) &\mapsto (A, F\circ H) \\ : (f,\alpha) &\mapsto (f, F\cdot\alpha) \end{aligned}$$

by post-composition with F*. Let* (A,H) *be a constrained tree. Then* A_{n+1} *is isomorphic to* $\mathrm{el}(\mathcal{U}_{\mathcal{C}}H_n)$ *which is identically* $\mathrm{el}(\mathcal{U}_{\mathcal{A}}FH_n)$ *by the commutativity required of morphisms in comma categories, in this case* $\mathcal{U}_{\mathcal{C}} = \mathcal{U}_{\mathcal{A}}F$*. Then* (A,FH) *is a constrained tree. Let* (f,α) *be a constrained morphism. Then* $\mathrm{el}(\mathcal{U}_{\mathcal{C}}\cdot\alpha)$ *is identically* $\mathrm{el}(\mathcal{U}_{\mathcal{A}}F\cdot\alpha)$ *and* $\mathit{Con}\,(F)$ *is well-defined.*

Let $\gamma\colon F \to G$ *be a* 2*-cell (natural transformation) of* $\mathit{Cat}/_{\mathit{Set}}$ *and define*

$$\begin{aligned} \mathit{Con}\,\gamma\colon \mathit{Con}\,F \Rightarrow \mathit{Con}\,G\colon \mathit{Con}\,(\mathcal{C},\mathcal{U}_{\mathcal{C}}) &\to \mathit{Con}\,(\mathcal{A},\mathcal{U}_{\mathcal{A}}) \\ : (A,H) &\mapsto (1_A, \gamma_H) \end{aligned}$$

by post-composition with γ*. Naturality of* $\mathit{Con}\,(\alpha)$ *follows directly from that of* α*.*

It is easy to see that Con *preserves identities and composition of* 1*-cells and of* 2*-cells.*

7.9. THEOREM. *Let* $F\colon \mathcal{C} \to \mathcal{A}$ *and* $G\colon \mathcal{A} \to \mathcal{C}$ *be* 1*-cells of* $\mathit{Cat}/_{\mathit{Set}}$*. If* F *and* G *are an adjoint pair then so are* $\mathit{Con}\,F$ *and* $\mathit{Con}\,G$*. Moreover, if* F *and* G *are mutual inverse equivalences (respectively isomorphisms) then* $\mathit{Con}\,F$ *and* $\mathit{Con}\,G$ *are mutual inverse equivalences (respectively isomorphisms).*

PROOF. Since Con is a 2-functor, it preserves adjunctions, equivalences and isomorphisms. ■

Two cases of labeled trees are of interest here: labeling by Δ_+ and by $\mathcal{I}_+$. In the case of Δ_+ we constrain by the functor $\mathcal{U} = U \circ (\_)^\wedge$ and for $\mathcal{I}_+$ by $\mathcal{U} = U$ where U is the underlying set functor. We define an additional requirement on such trees and call the trees satisfying this additional requirement *cropped*.

7.10. DEFINITION. *Let (A, p, F) be a labeled tree in Δ_+, respectively a labeled tree in $\mathcal{I}_+$, constrained as stated above. An* end element $x \in A_{n+1}$ *is one for which $\lambda_{n,x}$ is either a greatest or least element of $F_n(p_n x)^\wedge$ (respectively $F_n(p_n x)$). We call (A, p, F)* cropped *when, for all $n \in \mathbb{N}$,*

$$x \text{ is an end element of } A_{n+1} \text{ if and only if } \mathcal{U}F_{n+1}(x) \text{ is a singleton.}$$

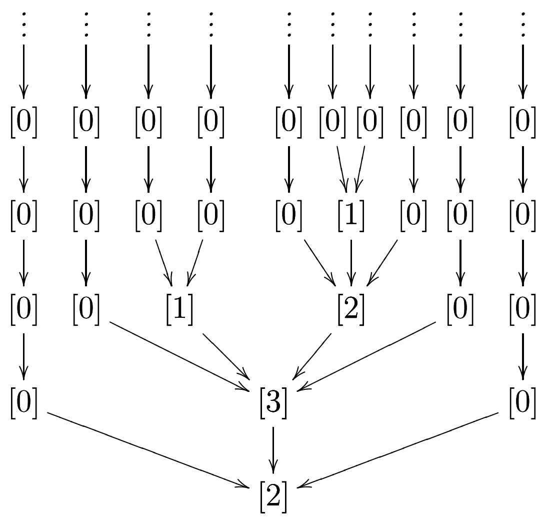
If (A, p) is a tree of degree 0 (zero), so that all the p_n are bijections, then the labeling F of a cropped tree (A, p, F) is unique. We call these the trivial *cropped trees.*

7.11. DEFINITION. *The category $t\mathcal{I}_+$ is the full subcategory of $Con(\mathcal{I}_+, U)$ whose objects are cropped labeled trees in $\mathcal{I}_+$ of finite degree. The category $t\mathcal{I}$ is the full subcategory of $t\mathcal{I}_+$ containing the trees of positive degree.*

7.12. DEFINITION. *The category $t\Delta_+$ is the full subcategory of $Con(\Delta_+, U(\_)^\wedge)$ whose objects are cropped labeled trees in Δ_+^{op} of finite degree. The category $t\Delta$ is the full subcategory of $t\Delta_+$ containing the trees of positive degree.*

7.13. OBSERVATION. *An object (A, F) of $t\mathcal{I}_+$ is trivial if and only if $F_0(*) = [0]$. An object (A, F) of $t\Delta_+$ is trivial if and only if $F_0(*) = [-1]$.*

7.14. OBSERVATION. *We provide an example of an object of $t\mathcal{I}_+$ of degree 3 (three).*



7.15. COROLLARY. *The categories $t\mathcal{I}_+$ (resp. $t\mathcal{I}$) and $t\Delta_+$ (resp. $t\Delta$) are dual.*

PROOF. The functors $(\_)^\wedge$ and $(\_)^\vee$ of the ordinal/interval equivalence and the constraining functors of $t\mathcal{I}_+$ and $t\Delta_+$ satisfy the hypothesis of Theorem 7.9. ■

The categories $t\mathcal{I}_+$ and $t\Delta_+$ of labeled trees were designed for their similarity with the categories $Fam_\Sigma(\mathcal{I}_+)$ and $Fam_\Pi(\Delta_+)$ (respectively). The objects of $t\mathcal{I}_+$ and $t\Delta_+$ are trees of intervals, respectively of ordinals, with extra structure and their morphisms are essentially tree morphisms which respect the additional structure.

8. Equivalence of new definitions

In this last section we demonstrate an equivalence between $t\mathcal{I}_+$ and $i\mathcal{I}_+$ and, in a parallel proof, between $t\Delta_+$ and $i\Delta_+$.

8.1. OBSERVATION. *A constrained tree is a labeled tree with the requirement (see Definition 7.6) that the fiber of a vertex and the underlying set of its label are isomorphic. The restriction operation reflects the fibers of all vertices. Hence the restriction of a constrained (respectively cropped) tree is constrained (respectively cropped) and the restriction of a constrained morphism is constrained.*

We show that the coproduct of constrained trees is constrained. Given a collection of constrained trees $(A(i), F(i))$ *with comma objects* $\lambda(i)$ *then, by the universal property of coproduct,* $\sum \lambda_n \colon \sum A_{n+1} \to \sum \mathrm{el}(\mathcal{U}F(i)_n)$ *is an isomorphism. As the functor* $\mathrm{el}(\_)$ *preserves coproducts then* $\sum \mathrm{el}(\mathcal{U}F(i)_n)$ *is isomorphic to* $\mathrm{el}(\sum \mathcal{U}F(i)_n)$. *Hence* $\sum A(i)_{n+1} \cong \sum \mathrm{el}(\mathcal{U}F(i)_n)$ *and* $\sum A(i)$ *is constrained.*

We show that the coproduct of constrained morphisms is constrained. Given a collection $(f(i), \alpha(i))$ *of constrained morphisms then in*

$$\begin{tikzcd}
\sum A(i)_{n+1} \arrow[r, "\cong"] \arrow[d, "\sum f(i)_{n+1}"'] & \sum \mathrm{el}(\mathcal{U}F(i)_n) \arrow[r, "\cong"] \arrow[d, "\sum \mathrm{el}(\mathcal{U}\cdot\alpha_n)"] & \mathrm{el}(\sum \mathcal{U}F(i)_n) \arrow[d, "\mathrm{el}(\sum \mathcal{U}\cdot\alpha_n)"] \\
\sum B(j)_{n+1} \arrow[r, "\cong"] & \sum \mathrm{el}(\mathcal{U}G(j)_n) \arrow[r, "\cong"'] & \mathrm{el}(\sum \mathcal{U}F(i)_n)
\end{tikzcd}$$

the left square commutes by functoriality of coproduct and the right square commutes by naturality. Hence the coproduct of constrained morphisms (and constrained trees) is constrained. The coproduct of cropped trees is cropped as the coprojections in Set are jointly surjective monomorphisms.

The suspension $\mathrm{su}(A, c)$ *is constrained by* $\mathcal{U}$ *if* $A_0 \cong \mathcal{U}c$ *and* A *is constrained by* $\mathcal{U}$. *The suspension* $\mathrm{su}((f, \alpha), g)$ *is constrained by* $\mathcal{U}$ *if* $f_0 \cong \mathcal{U}g$ *and* (f, α) *is constrained by* $\mathcal{U}$. *Similarly,* $\mathrm{su}(A, c)$ *is cropped if* A *is cropped and* $A(i)$ *is trivial when* i *is an endpoint of* $\mathcal{U}c$.

8.2. PROPOSITION. *The category* $t\mathcal{I}_+$ *is equivalent to the category* $i\mathcal{I}_+$ *by a functor*

$$\Xi_{\mathcal{I}} \colon t\mathcal{I}_+ \to i\mathcal{I}_+$$

which is surjective on objects.

PROOF. Let (A, F) be a cropped tree of height n in $t\mathcal{I}_+$. Then we have an object $C = F_0(*)$ of $\mathcal{I}_+$ and a set map

$$\begin{aligned} H\colon U(C) &\to \operatorname{Ob} Con(\mathcal{I}_+, U) \\ : i &\to A(i) \end{aligned}$$

such that $H(i)$ is trivial if and only if i is an endpoint of C. Hence a tree of height n in $t\mathcal{I}_+$ has an alternative description as a pair (C, H) consisting of an object C of $\mathcal{I}_+$ and function $H\colon U(C) \to \operatorname{Ob} Con(\mathcal{I}_+, U)$ such that $H(i)$ is trivial if only if i is an endpoint of C and has height strictly less than n, for all $i \in U(C)$. Similarly, a morphism of $t\mathcal{I}_+$ has an alternative description as a pair $(\gamma, h)\colon (C, H) \to (C', H')$ consisting of a morphism $\gamma\colon C \to C'$ of $\mathcal{I}_+$ and function $h\colon U(C) \to \operatorname{Ar} Con(\mathcal{I}_+, U)$ with $h(i)\colon H(i) \to H'(\gamma i)$.

This data is, by design, that required by Definition 3.1 for the objects and morphisms of $i\mathcal{I}_+$. We define a functor

$$\Xi_{\mathcal{I}}\colon t\mathcal{I}_+ \to i\mathcal{I}_+$$

by sending the trivial tree of $t\mathcal{I}_+$ to the trivial object of $i\mathcal{I}_+$, and by sending morphisms into the trivial tree to morphisms into the trivial object. The rest follows by induction. ∎

8.3. COROLLARY. *The category $t\mathcal{I}$ is equivalent to the category $i\mathcal{I}$.*

The proofs of propositions 8.2 and 8.4 are nearly identical. We include both and list here the two ways in which the categories $i\mathcal{I}_+$ and $i\Delta_+$ differ which affect the details of the two proofs. First, the trivial object of $i\mathcal{I}_+$ is the terminal object $[0]$ and the trivial object of $i\Delta_+$ is the initial object $[-1]$. Second, the subtrees of an object H of $i\mathcal{I}_+$ are indexed by the elements of H_* where the subtrees of an object K of $i\Delta_+$ are indexed by the elements of $(K_*)^\wedge$.

8.4. PROPOSITION. *The category $t\Delta_+$ is equivalent to the category $i\Delta_+$ by*

$$\Xi_\Delta\colon t\Delta_+ \to i\Delta_+$$

which is surjective on objects.

PROOF. Let (A, F) be a cropped tree of height n in $t\Delta_+$. Then we have an object $C = F_0(*)$ of Δ_+ and a set map

$$\begin{aligned} H\colon U(C^\wedge) &\to \operatorname{Ob} Con(\Delta_+, U(\_)^\wedge) \\ : i &\to A(i) \end{aligned}$$

such that $H(i)$ is trivial if and only if i is an endpoint of $C^\wedge$. Hence a tree of height n in $t\Delta_+$ has an alternative description as a pair (C, H) consisting of an object C of Δ_+ and function $H\colon U(C^\wedge) \to \operatorname{Ob} Con(\Delta_+, U(\_)^\wedge)$ such that $H(i)$ is trivial if only if i is an endpoint of $C^\wedge$ and has height strictly less than n, for all $i \in U(C^\wedge)$. Similarly, a morphism of $t\Delta_+$ has an alternative description as a pair $(\gamma, h)\colon (C', H') \to (C, H)$ consisting of a morphism $\gamma\colon C' \to C$ of Δ_+ and function $h\colon U(C^\wedge) \to \operatorname{Ar} Con(\Delta_+, U)$ with $h(i)\colon H'(\gamma^\wedge i) \to H(i)$.

This data is, by design, that required by Definition 3.2 for the objects and morphisms of $i\Delta_+$. We define a functor

$$\Xi_\Delta \colon t\Delta_+ \to i\Delta_+$$

by sending the trivial tree of $t\Delta_+$ to the trivial object of $i\Delta_+$, and by sending morphisms out of the trivial tree to morphisms out of the trivial object. The rest follows by induction. ∎

8.5. Corollary. *The category $t\Delta$ is equivalent to the category $i\Delta$.*

The category $t\mathcal{I}$ provides a description of disks as trees with vertices labeled by intervals. Similarly $t\Delta$ provides a description of Θ in terms of trees with vertices labeled by ordinals.

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