Theory and Applications of Categories, Vol. 24, No. 5, 2010, pp. 114–116.

ON A CONJECTURE BY J.H.SMITH

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ABSTRACT. We show that the class of weak equivalences of a combinatorial model category can be detected by an accessible functor into simplicial sets.

The purpose of this short note is to prove the following result that was conjectured by J.H. Smith in [[2],p. 460].

0.1. THEOREM. *For every combinatorial model category $\mathcal{M}$, there is an accessible functor $F : \mathcal{M} \to \mathcal{S}Set$ that detects the weak equivalences, i.e., a morphism f in $\mathcal{M}$ is a weak equivalence if and only if $F(f)$ is a weak homotopy equivalence.*

PROOF. By [[3], Theorem 1.1], there is a small category C, a set of morphisms S in $\mathcal{S}Set^C$ and a Quillen equivalence $F : \mathcal{L}_S\mathcal{S}Set^C \rightleftarrows \mathcal{M} : G$, where $\mathcal{L}_S\mathcal{S}Set^C$ denotes the left Bousfield localisation of $\mathcal{S}Set^C$ with the projective model structure at the set of morphisms S (see [4]). By [[3], Proposition 7.1], there is a fibrant replacement functor $R : \mathcal{M} \to \mathcal{M}$ that is accessible. Let $u : ObC \to C$ denote the inclusion of the objects of C (as a discrete category) into C. Then let $F : \mathcal{M} \to \mathcal{S}Set$ be the following composition of functors

$$\mathcal{M} \xrightarrow{R} \mathcal{M} \xrightarrow{G} \mathcal{S}Set^C \xrightarrow{u^*} \mathcal{S}Set^{ObC} \xrightarrow{\prod} \mathcal{S}Set.$$

F is accessible because it is a composition of accessible functors. The functors G, u^* and $\prod$ are accessible because they are right adjoints between locally presentable categories (see [[1], 1.66]). Since G is a right Quillen equivalence, a morphism f in $\mathcal{M}$ is a weak equivalence if and only if $GR(f)$ is a weak equivalence in $\mathcal{L}_S\mathcal{S}Set^C$. The functor GR maps into the category of fibrant (or S-local) objects, therefore $GR(f)$ is a weak equivalence in $\mathcal{L}_S\mathcal{S}Set^C$ (i.e. S-local equivalence) if and only if it is a weak equivalence in $\mathcal{S}Set^C$, i.e., a pointwise weak equivalence (see [[4], Theorem 3.2.13]). The morphism $GR(f)$ is a pointwise weak equivalence if and only if $u^*GR(f)$ is a pointwise weak equivalence in $\mathcal{S}Set^{ObC}$. By the combinatorial definition of homotopy groups, the product functor $\prod$ detects the pointwise weak equivalences between pointwise fibrant objects. Since u^*GR takes values in pointwise fibrant objects, it follows that F detects the weak equivalences. ∎

As an immediate corollary, we have that the class of weak equivalences of a combinatorial model category is accessible and accessibly embedded. This was proved by different methods in [[5], Corollary A.2.6.6] and [[6], Theorem 4.1].

Received by the editors 2010-02-08 and, in revised form, 2010-04-13.
Transmitted by J. Rosicky. Published on 2010-04-05.
2000 Mathematics Subject Classification: 55U35, 18C35.
Key words and phrases: combinatorial model category, accessible category, simplicial sets.

0.2. COROLLARY. *The class of weak equivalences $\mathcal{W}$ of a combinatorial model category $\mathcal{M}$ is accessible and accessibly embedded in $\mathcal{M}^{\to}$.*

PROOF. By the previous theorem, there is an accessible functor $F : \mathcal{M} \to \mathcal{SS}et$ that detects the class of weak equivalences $\mathcal{W}$. The full inverse image of an accessible, accessibly embedded subcategory of an accessible category by an accessible functor is again accessible and accessibly embedded [[1], Remark 2.50]. Hence the corollary follows, because the class of weak homotopy equivalences in $\mathcal{SS}et$ is known to be accessible and accessibly embedded in $\mathcal{SS}et^{\to}$ by [[2], Example 3.1]. As pointed out by the referee, there is an alternative way to see this as follows. A map $f : X \to Y$ between Kan-fibrant simplicial sets is a weak homotopy equivalence if and only if (i) $\pi_0(f)$ is a bijection of sets, and (ii) for every $n > 0$, there is a pullback square

$$\begin{tikzcd} \pi_n(X) \arrow[r] \arrow[d] & \pi_n(Y) \arrow[d] \\ X_0 \arrow[r] & Y_0 \end{tikzcd}$$

where $\pi_n : \mathcal{SS}et \to \mathcal{S}et^{\to}$ is the functor that takes a simplicial set X to the bundle of n-th (combinatorial) homotopy groups, viewed as a set that is indexed by the set of vertices:

$$\pi_n(X) = \coprod_{x \in X_0} \pi_n(X, x) \to X_0.$$

The functor π_0 and the functors π_n, for $n > 0$, are accessible. The category of pullback squares of sets and natural transformations is accessible and it is accessibly embedded in the category of squares of sets because directed colimits commute with finite limits in $\mathcal{S}et$. Also, Kan's fibrant replacement functor $Ex^{\infty} : \mathcal{SS}et \to \mathcal{SS}et$ is accessible. Thus the full subcategory of weak homotopy equivalences in $\mathcal{SS}et^{\to}$ is the intersection of the full inverse images of accessible, accessibly embedded subcategories of accessible categories by a set of accessible functors, and the result follows by [[1], Corollary 2.37] and [[1], Remark 2.50] as before. ∎

Acknowledgements. The author would like to thank J. Rosický and the anonymous referee for their comments.

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