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TOPOS THEORETIC ASPECTS OF SEMIGROUP ACTIONS

JONATHON FUNK AND PIETER HOFSTRA

ABSTRACT. We define the notion of a torsor for an inverse semigroup, which is based on semigroup actions, and prove that this is precisely the structure classified by the topos associated with an inverse semigroup. Unlike in the group case, not all set-theoretic torsors are isomorphic: we shall give a complete description of the category of torsors. We explain how a semigroup prehomomorphism gives rise to an adjunction between a restrictions-of-scalars functor and a tensor product functor, which we relate to the theory of covering spaces and E-unitary semigroups. We also interpret for semigroups the Lawvere-product of a sheaf and distribution, and finally, we indicate how the theory might be extended to general semigroups, by defining a notion of torsor and a classifying topos for those.

1. Introduction

MOTIVATION. Lawson [9] explains how inverse semigroups may be used to describe *partial symmetries* of mathematical structures in the same way that groups may be used to describe their global symmetries. For instance, the inverse semigroup of partial isomorphisms of a topological space is more informative than just the automorphism group: two spaces may have the same global automorphisms, but different partial automorphisms.

It has recently become clear that there is an interesting and useful connection between inverse semigroups and toposes [4, 5, 8]. Explicitly, for each inverse semigroup S there is a topos $\mathscr{B}(S)$, called the *classifying topos* of S, defined as the category of equivariant sheaves on the associated inductive groupoid of S. This topos is equivalent to the category of presheaves on the (total subcategory of) the idempotent splitting of S. It turns out that many results in semigroup theory have natural interpretations in topos theoretic terms. For instance, it is known that cohomology of an inverse semigroup (Loganathan-Lausch [11]), Morita equivalence of inverse semigroups, the maximum group image, E-unitary inverse semigroups, and even McAlister's P-theorem have natural and canonical topos interpretations.

The notion of a *semigroup action* in a set X [13, 3] goes back at least to the basic representational result in the subject, namely the well-known Wagner-Preston theorem. In fact, there are several related notions; in this paper we shall consider prehomomorphisms $S \longrightarrow I(X)$, where $I(X)$ denotes the symmetric inverse semigroup on X. We shall refer to a homomorphism $S \longrightarrow I(X)$, which is the notion featured in the Wagner-

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Preston theorem, as a *strict action*. A homomorphism $S \longrightarrow I(X)$ that factors through the endomorphism monoid is said to be a *total action*.

The main question we answer is the following: what does the classifying topos of an inverse semigroup actually classify? Put in different terms: what is an S-torsor? We shall see that it is a non-empty object equipped with a transitive and locally free action by S. The theory of semigroup torsors generalizes the group case, but as we shall see it is 'finer' than that of group torsors, and even in the set-theoretic setting provides a useful invariant of semigroups.

On a more general level, the paper aims at unifying three viewpoints of actions by an inverse semigroup: the aforementioned semigroup actions, covariant *Set*-valued functors, and finally *distributions* on the topos $\mathscr{B}(S)$ in Lawvere's sense [2, 10]: thinking of $\mathscr{B}(S)$ as a 'space' associated with S, the distributions may be thought of as measures on this space. Our goal is to give a self-contained exposition of the three perspectives, including how the passage between them, and to illustrate this with some key examples.

Overview. We have tried to make the paper accessible to anyone with a basic familiarity with the language of category theory. At times some topos-theoretic concepts are used without definition; in those cases we provide references. The first section, which describes semigroup actions and torsors in elementary terms, does not require any knowledge of topos theory.

After reviewing some of the basic theory of inverse semigroups, the goal of § 2 is to give an exposition of the elementary notion of a semigroup action. In fact, we identify three related notions depending on the strictness of the action, as well as two different notions of morphism, thus giving rise to various categories of representations. We mention a number of key examples, such as the well-known Wagner-Preston and Munn representations. Then we introduce the notion of a torsor as a special kind of representation. We confine ourselves to proving only a couple of elementary facts here, leaving a more conceptual investigation for § 3. Finally, we study semigroup actions and torsors in categories different from *Set*, in particular in (pre)sheaf toposes.

§ 3 begins by reviewing the classifying topos $\mathscr{B}(S)$ of an inverse semigroup S. Our main goal is to relate the notions of semigroup actions by S to certain classes of functors on $L(S)$. In particular, we obtain an equivalence of categories between the category of S-sets and the category of what we term *torsion-free* functors on $L(S)$ (valued in *Set*). This equivalence specializes to one between *strict* S-sets and pullback-preserving functors, and ultimately between torsors and filtering functors. The latter result gives the desired statement that $\mathscr{B}(S)$ indeed classifies S-torsors in our sense. By general considerations, $\mathscr{B}(S)$ must therefore contain a *generic torsor*; we shall show that this is none other than the well-known presheaf of Schützenberger representations. Finally, we give a complete characterization of all set-theoretic torsors. For the group case, this trivializes, since all torsors are isomorphic, but an inverse semigroup may have non-isomorphic torsors. We give an explicit description of how every S-set, and in particular every torsor, arises as a colimit of *principal torsors*.

In § 4 we explore some aspects of change of base, i.e., how the categories of S-sets and T-sets are related when S and T are connected by a (pre)homomorphism. We first explain how the usual hom-tensor adjunction arises; this essentially follows from the fact that S-sets form a cocomplete category, where it should be noted that, unlike in the group case, coequalizers are not created by the forgetful functor to sets. After that we explicitly calculate the tensor product of torsors; this amounts to unraveling a colimit-extension, but the end result is a bit more complicated than for groups, since the category over which the colimit is taken has more than one object. We apply this to the case of the homomorphism $S \longrightarrow S/\sigma$, the maximum group image of S, and obtain a characterization of E-unitary inverse semigroups: S is E-unitary if and only if its category of torsors is left-cancellative. Finally, we observe that every S-torsor in $Sh(B)$ (which one might call a principal S-bundle) may be completed to a principal S/σ-bundle, in the sense that there is a canonical map from the bundle to its completion, which is injective when S is E-unitary.

§ 5 is concerned with the third perspective on semigroup actions, namely as distributions on the topos $\mathscr{B}(S)$. We recall the definition of distribution, establish a few elementary but useful facts, and establish correspondences between S-sets and torsion-free distributions, and between strict S-sets and what we coin S-distributions. Of course, torsors correspond to left exact distributions, which are the points of $\mathscr{B}(S)$. We explicitly describe some of the leading examples of S-sets in terms of distributions, and also interpret the action of $\mathscr{B}(S)$ in its category of distributions (the Lawvere-product).

Finally, § 6 sketches an approach to a generalization of the subject matter. For a general semigroup T, the topos of presheaves on a category $L(T)$ is not necessarily appropriate as its classifying topos mainly because of fact that general semigroups need not have enough idempotents, or indeed any idempotents at all. Instead, we propose a classifying topos for a general semigroup which plays the same topos-theoretic role as $\mathscr{B}(S)$ does in the inverse case: it classifies torsors. The definition of a semigroup-torsor is straightforward, and for semigroup-torsor pairs it is geometric. The classifying topos for T obtained by pulling back the topos classifier of semigroup-torsor pairs along the point of the semigroup classifier corresponding to T.

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2. Semigroup actions: basic theory

In this section we introduce our basic objects of study, namely semigroup actions and torsors. We review some basic inverse semigroup theory in order to establish some terminology and notation, recall the definition of an action by an inverse semigroup, and give some examples. We also discuss strict and non-strict morphisms between such objects and establish some elementary results. We define torsors for an inverse semigroup, give some examples, and make some basic observations about maps between torsors. Finally, we show how the notion of torsor makes sense in an arbitrary topos.

Background on inverse semigroups. A semigroup S is said to be *inverse* when for every $s \in S$ there exists a unique s^* for which $ss^*s = s$ and $s^*ss^* = s^*$. A canonical example is the inverse semigroup $I(X)$ of partial injective functions from a set X to itself (this is in fact an inverse monoid). More generally, for many mathematical structures it makes sense to consider the inverse semigroup of all partial isomorphisms from that structure to itself.

Elements of the form s^*s and of the form ss^* are evidently idempotent; in fact, all idempotents are of this form. (It is helpful to think of s^*s as the domain of s, and of ss^* as the range.) The subset of S on the idempotents is denoted $E(S)$, or simply E, when S is understood. The set E is in fact endowed with a partial order and binary meets, given by multiplication. In general, it has neither a largest nor a smallest element. In the example $S = I(X)$, the lattice of idempotents is simply the powerset of X with its usual lattice structure.

The well-known partial order in S, which contains E as a subordering, is given by $s \leq t$ if and only if $s = ts^*s$.

We shall consider two notions of morphism between inverse semigroups, homomorphism and prehomomorphism. The weaker notion *prehomomorphism* is a function $\rho : S \longrightarrow T$ between inverse semigroups which satisfies $\rho(st) \leq \rho(s)\rho(t)$. If for all elements s, t we actually have equality, then ρ is a *homomorphism.* It is well-known that a (pre)homomorphism automatically preserves the involution, i.e. that $\rho(s^*) = \rho(s)^*$. Moreover, any (pre)homomorphism preserves the natural ordering, and sends idempotents to idempotents. For more information and explanation concerning these basic concepts, we refer the reader to Lawson's textbook [9].

Actions of inverse semigroups.

2.1. Definition. An *S-set* is a set X and a prehomomorphism $S \xrightarrow{\mu} I(X)$, sometimes written (X, μ). For any $s \in S$ and $x \in X$, we write $s \cdot x$, or sometimes just sx, to mean $\mu(s)(x)$ when defined. Then $\mu(st) \leq \mu(s)\mu(t)$ reads $(st)x = s(tx)$ for all x, which means that if $(st)x$ is defined then so are tx and $s(tx)$ and the given equality holds. An S-set (X, μ) is said to be *strict* when μ is a homomorphism.

For any $e \in E$, $\mu(e)$ is an idempotent of $I(X)$, which amounts to a subset of X that we denote eX. The expression "$x \in eX$" simply means that ex is defined (and $ex = x$). With this notation we may write the partial map $\mu(s) : X \longrightarrow X$ as

$$\begin{tikzcd} eX \arrow[r, tail, "s"] \arrow[d, hook] & X \\ X & \end{tikzcd}$$

where $e = s^*s$.

Let us consider some examples of S-sets.

2.2. Example. A canonical example of an S-set is the *Munn representation* [9] of an inverse semigroup S. This is a well-supported (Def. 2.7) S-set $S \longrightarrow I(E)$ such that $s \cdot e$ is defined if and only if $e \leq s^*s$, in which case $s \cdot e = ses^*$. For any e, we have $eE = \{d \mid d \leq e\}$. The Munn representation is closely related to the *Wagner-Preston representation.* This is the S-set $S \longrightarrow I(S)$ such that $s \cdot t$ is defined if and only if $t = s^*st$, in which case $s \cdot t = st$. For any idempotent e, we have $eS = \{t \mid t = et\}$. The Munn and Wagner-Preston S-sets are strict S-sets in the sense of Def. 2.1.

2.3. Example. A prehomomorphism of inverse semigroups $S \xrightarrow{\rho} T$ may be construed as an S-set, not strict in general, by restricting the Wagner-Preston T-set to S. Let T_ρ denote this S-set: $s \cdot t$ is defined if and only if $t = \rho(s^*s)t$, in which case $s \cdot t = \rho(s)t$.

2.4. Example. Of course, every inverse semigroup S acts in itself by multiplication; this action is total and as such is an example of a strict S-set. However, it turns out that when we view this as a right action, it is naturally related to the so-called Schützenberger object (1). We shall return to this in § 3.

In general, there may of course be several different actions of S in a given set X; we may say, for two such actions μ, ν, that ν extends μ whenever $\mu(s) \leq \nu(s)$ for all $s \in S$. When $X = \{x\}$ is a singleton, then it is easily seen that an action μ of S in X is determined by specifying a filter $I \subseteq E$, namely $I = \{e \in E \mid ex = x\}$. The terminal S-set is then the singleton set in which S acts totally.

2.5. Definition. A *morphism of S-sets* $(X, \mu) \longrightarrow (Z, \sigma)$ is a map $\psi : X \longrightarrow Y$ such that for all s and x, if sx is defined, then so is $s\psi(x)$ and $\psi(sx) = s\psi(x)$ holds. Of course, it may happen that $s\psi(x)$ is defined when sx is not. If indeed this does not occur, then we say that ψ is *a strict morphism.*

2.6. Example. The Wagner-Preston and Munn representations (Eg. 2.2) are related by the range map $S \longrightarrow E$, $s \mapsto ss^*$, which is a strict morphism of S-sets.

An S-set (X, μ) may also be regarded as a partial map

$$\mu : S \times X \rightharpoonup X \ ,$$

in which case a morphism of S-sets (i.e., an equivariant map) is a map $\psi : X \longrightarrow Y$ such that the square

$$\begin{tikzcd} S \times X \arrow[r, harpoon, "\mu"] \arrow[d, "S\times\psi"'] & X \arrow[d, "\psi"] \\ S \times Z \arrow[r, harpoon, "\sigma"] & Z \end{tikzcd}$$

of partial maps commutes on the nose when ψ is strict, and commutes up to inequality $\psi\mu \leq \sigma(S \times \psi)$ otherwise.

When X is an S-set, then we say that an element $x \in X$ is *supported* by an idempotent $e \in S$, or that e is in the *support of* x, when $x \in eX$. Since mostly we are not interested in unsupported elements of an S-set, we introduce the following terminology.

2.7. Definition. An S-set (X, μ) is *well-supported* if $X = \bigcup_E eX$. (The term "effective" is used in the semigroup literature.)

In any case, if an S-set X is not well-supported, then we can replace it with the well-supported S-set $\bigcup_{e \in E} eX$.

Let $S{-}Set$ denote the category of well-supported S-sets and their morphisms. Of course this includes, but is not limited to, the strict morphisms. The full subcategory of $S{-}Set$ on the strict, well-supported S-sets is denoted $\underline{S{-}Set}$.

Torsors. Having defined strict and general S-sets, we now turn to torsors. Recall that for a group G, a G-torsor X (in *Set*) is a non-empty G-set for which the action is free and transitive. We shall generalize this in the appropriate way to the case of an inverse semigroup, and then explore some examples.

2.8. Definition. A well-supported S-set (X, μ) is *an S-torsor* if:

1. X is non-empty;
2. μ is transitive - for any $x, y \in X$, there are $s, t \in S$ and $z \in X$ such that $sz = x$ and $tz = y$;
3. μ is locally free - for any x and s, t such that $sx = tx$, there is $r \in S$ and $y \in X$ such that $ry = x$ and $sr = tr$.

We denote the full subcategory of $S{-}Set$ on the torsors by $\mathrm{TOR}(S)$.

1. In the inverse case, transitive is equivalent to the following: for any $x, y \in X$, there is $s \in S$ such that $sx = y$.
2. In the inverse case, locally free is equivalent to the following: for any x and s, t such that $sx = tx$, there is an idempotent d such that $x \in dX$ and $sd = td$.

   Moreover, without loss of generality it can be assumed in the locally free requirement that $d \leq s^*s, t^*t$ because otherwise replace d by $e = ds^*st^*t$, noting $eX = dX \cap s^*sX \cap t^*tX$.

In the case of a group G, every set-theoretic torsor is isomorphic to the group G itself. In the inverse semigroup case, this is no longer true, as illustrated by the following example.

2.9. Example. If X is a non-empty set, then X is an $I(X)$-torsor (where of course $I(X)$ acts in X via $f \cdot x = f(x)$). It is clear that this action is transitive; but it is also locally free since if $f \cdot x = g \cdot x$ then f and g agree on the domain $\{x\}$.

At the other end of the spectrum we have the case of meet-semilattices:

2.10. Example. We determine the torsors on an $\wedge$-semilattice D when regarded as an inverse semigroup such that $ab = a \wedge b$ and $a^* = a$. A strict S-set $D \longrightarrow I(X)$ amounts to an action of D in X by partial identities, and such that $aX \cap bX = abX$. If such an action is a torsor, then by transitivity, and since $X \neq \emptyset$, X must be a one-element set. On the other hand, an S-set $D \longrightarrow \{0 \leq 1\} = I(1)$ is necessarily a torsor. In turn, these correspond to filters of D (up-closed and closed under binary $\wedge$), and whence to points of the presheaf topos $PSh(D)$. Thus, D-torsors in the sense of Def. 2.8 coincide with the usual meaning of torsor on an $\wedge$-semilattice (filtering functor on D) [6].

As a final example, here is a torsor for which the action is total:

2.11. Example. Let S/σ be the maximum group of S. Then S acts in S/σ via left multiplication. This action is transitive and free, so S/σ is a torsor. In fact, one may show that if X is an S-torsor for a total action, then $X \cong S/\sigma$.

The last example makes precise the sense in which torsors are a more general invariant of S than the maximum group. We give a structure theorem for general torsors in § 3.10. For now we note a result that is a straightforward generalization of the group case, namely that morphisms of torsors are necessarily isomorphisms:

2.12. Proposition.

1. *An isomorphism of S-sets is strict.*

2. *A strict morphism of S-torsors is an isomorphism.*

3. *Any map of torsors is an epimorphism in $S-Set$.*

Proof. 1. Suppose that $X \xrightarrow{\alpha} Y$ is an isomorphism of S-sets with inverse β. If $s\alpha(x)$ is defined, then so is $\beta(s\alpha(x))$. Hence $s\beta\alpha(x) = sx$ is defined, so α is strict.

2. Suppose that X and Y are torsors, and that α is strict. To see that α is surjective, let $y \in Y$. Choose any $x_0 \in X$, which is possible since torsors are non-empty. By transitivity of Y, there is $s \in S$ such that $s\alpha(x_0) = y$. By strictness, sx_0 is defined, and $\alpha(sx_0) = y$. To see that α is injective, suppose that $\alpha(x) = \alpha(z)$, $x, z \in X$. By transitivity of X, there is s such that $sx = z$. Hence, $s\alpha(x) = \alpha(sx) = \alpha(z) = \alpha(x) = s^*s\alpha(x)$. By the freeness of Y, there is an idempotent $e \leq s^*s$ such that $se = s^*se = e$, and $\alpha(x) \in eX$. By the strictness of α, $x \in eX$, and hence $z = sx = sex = ex = x$.

3. It is easily verified that if two maps $\alpha, \beta : X \longrightarrow Y$ of S-sets, where X is transitive, agree on an element of X, then $\alpha = \beta$. The result follows immediately. ∎

2.13. Corollary. TOR(S) *is an essentially small right-cancellative category.*

Proof. TOR(S) is essentially small because every torsor X admits a surjection $S \twoheadrightarrow X$, $s \mapsto sx$, where x is a fixed element of X. TOR(S) is right-cancellative by 2.12, 3. ∎

Internal semigroup actions. So far we have been working in the category of sets. But as we know from the group case, the theory of torsors becomes much more potent and applicable when we consider it in other categories, such as categories of sheaves. In this section we briefly indicate how to define torsors diagrammatically, and give some examples of torsors in categories other than *Set*.

First of all, it is clear that the notion of a semigroup action makes sense in any category with finite limits. Then the definition of an internal torsor is easily obtained:

2.14. Definition. If S is a semigroup in a topos $\mathscr{E}$, then a well-supported (meaning p below is an epimorphism) S-set (X, μ) in $\mathscr{E}$

$$\begin{tikzcd} U \arrow[r, "\mu"] \arrow[d, hook] & X \\ S \times X & \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r, "q"] \arrow[d, "p"', two heads] & S \\ X & \end{tikzcd}$$

is an S-torsor if:

1. $X \longrightarrow 1$ is an epimorphism;
2. μ is transitive - consider the kernel pair of p (pullback).

$$\begin{tikzcd} H \arrow[r, "\pi_1", two heads] \arrow[d, "\pi_2"', two heads] & U \arrow[d, "p", two heads] \\ U \arrow[r, "p", two heads] & X \end{tikzcd}$$

Then μ is transitive if the map $\langle \mu\pi_1, \mu\pi_2 \rangle : H \longrightarrow X \times X$ is an epimorphism;

3. μ is locally free - consider the following two pullbacks and equalizer.

$$\begin{tikzcd} K \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & U \arrow[d, "\langle \mu, p \rangle"] \\ U \arrow[r, "\langle \mu, p \rangle"] & X \times X \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "f"] \arrow[d, "g"'] & U \arrow[d, "\mu"] \\ K \arrow[r, "p\pi_1 = p\pi_2"] & X \end{tikzcd} \qquad \begin{tikzcd} N \arrow[r, tail] \arrow[dr] & M \arrow[r, bend left, "q\pi_1 k \cdot qu"] \arrow[r, bend right, "q\pi_2 k \cdot qu"'] \arrow[d, "g"] & S \\ & K & \end{tikzcd}$$

In the equalizer, (k, u) is an element of M. Then by definition μ is locally free if the restriction of $g(k, u) = k$ to the equalizer N is an epimorphism.

If S is a semigroup in the topos $\mathscr{E} = Set$, then Defs. 2.8 and 2.14 are equivalent. For instance, when interpreted as a sentence in first order logic, the locally free requirement in Def. 2.14 states

$$\forall s, t \in S, x \in X \ \ (sx = tx) \Rightarrow (\exists r \in S, y \in X(ry = x \wedge sr = tr)) \,.$$

This is precisely the locally free axiom as stated in Def. 2.8. Moreover, this axiom is *geometric* in the sense of geometric logic ([12], page 537).

Because the notion of torsor is geometric, it is evident that inverse image functors of geometric morphisms preserve torsors.

We are mostly concerned with the case that $\mathscr{E}$ is a Grothendieck topos; in that case we can consider any set-theoretic inverse semigroup S as an internal inverse semigroup ΔS, where $\Delta : Set \longrightarrow \mathscr{E}$ is the constant objects functor.

For example, when $\mathscr{E} = PSh(\mathbb{C})$, the category of presheaves on a small category $\mathbb{C}$, ΔS-torsors are characterized as follows:

2.15. PROPOSITION. *Let S be a semigroup in Set, and $\mathbb{C}$ a small category. Suppose that a presheaf X is a ΔS-set in $PSh(\mathbb{C})$. Then X is a ΔS-torsor iff for every object c of $\mathbb{C}$, $X(c)$ is an S-torsor.*

PROOF. Suppose that X is a ΔS-torsor. The inverse image functor c^* of the point $Set \xrightarrow{c} PSh(\mathbb{C})$ associated with an object c satisfies $c^*(X) = X(c)$. Now use the fact that c^* preserves torsors.

On the other hand, if every $X(c)$ is an S-torsor, then the torsor conditions are satisfied for X in $PSh(\mathbb{C})$ because finite limits and epimorphisms are determined pointwise in $PSh(\mathbb{C})$. ∎

When $\mathscr{E} = Sh(B)$, where B is a space, then a ΔS-set in $Sh(B)$ is an étale space $X \xrightarrow{p} B$ and a continuous associative semigroup action

$$\mu : S \times X \longrightarrow X$$

over B: $p(sx) = p(x)$, where $\mu(s, x) = sx$. The domain of definition of μ is an open subset of $S \times X$, which simply means that for any $s \in S$, $\{x \mid sx \text{ is defined}\}$ is an open subset of X.

The torsor requirements interpreted in $Sh(B)$ are as follows:

1. $X \xrightarrow{p} B$ is onto;

2. the action is fiberwise transitive - for any $x, y \in X$ such that $p(x) = p(y)$ there are $s, t \in S$ and $u \in X$ such that $su = x$ and $tu = y$. Note that $p(u) = p(su) = p(x) = p(y)$, i.e., u and x (and y) must lie in the same fiber of p.

3. the action is locally free - for any $s, t \in S$ and $x \in X$ such that $sx = tx$, there are $r \in S$ and $u \in X$ such that $ru = x$ and $sr = tr$. Again, u and x must lie in the same fiber of p since $p(u) = p(ru) = p(x)$.

We may call such an étale space a principal S-bundle. We briefly return to these structures and their connections with principal G-bundles in § 4.

3. Torsors and the classifying topos

We have given an elementary definition of S-torsor, but have not motivated this definition, aside from the observation that it indeed generalizes both the group case and the meet-semilattice case. One of the purposes of this section is to show that the classifying topos of S does indeed classify S-torsors, thus justifying the notion at least from the topos point of view. We shall also interpret the notions of strict and well-supported S-sets in more categorical terms, namely as certain functors. Finally, we shall prove a structural result which characterizes all set-theoretic torsors.

The classifying topos of an inverse semigroup. As mentioned in the Introduction the classifying topos of an inverse semigroup S, denoted $\mathscr{B}(S)$, is defined as the category of equivariant sheaves on the inductive groupoid of S. This formulation simplifies to the following: the objects of $\mathscr{B}(S)$ are sets X equipped with a total action by S, which we write on the right, together with a map $X \xrightarrow{p} E$ to the idempotent subset E of S satisfying $xp(x) = x$ and $p(xs) = s^*xs$. Morphisms are S-equivariant maps between such sets over E.

One may think loosely of $\mathscr{B}(S)$ as the 'space' associated with S; technically, the topos $\mathscr{B}(S)$ is an étendue [7].

Let $L(S)$ denote the category whose object set is E, the collection of idempotents of S, and whose morphisms $d \xrightarrow{s} e$ are pairs $(s, e) \in S \times E$ such that $d = s^*s$ and $s = es$. We may think of $L(S)$ as the total map category of the idempotent splitting of S. From another point of view, $L(S)$ is the result of amalgamating the horizontal and vertical compositions of the inductive groupoid of S, regarding it as a double category. It is easily proved that $L(S)$ is left-cancellative in the sense that its morphisms are monomorphisms. Moreover, $L(S)$ has pullbacks: in fact any pullback is built from the following three basic kinds: an isomorphism square, a restriction square, and an inequality square.

$$\begin{tikzcd} e \arrow[r] \arrow[d] & s^*s \arrow[d, "s"] \\ t^*t \arrow[r, "t"] & ss^* = tt^* \end{tikzcd} \qquad \begin{tikzcd} t^*t \arrow[r, phantom, "\leq"] \arrow[d, "t"'] & s^*s \arrow[d, "s"] \\ tt^* \arrow[r, phantom, "\leq"] & ss^* \end{tikzcd} \qquad \begin{tikzcd} b \wedge c \arrow[r, "\leq"] \arrow[d, "\leq"'] & b \arrow[d, "\leq"] \\ c \arrow[r, "\leq"] & d \end{tikzcd}$$

The first two are pullbacks if and only if they commute, and they are preserved by any functor.

The following result is due to Lawson and Steinberg [8].

3.1. Proposition. *$\mathscr{B}(S)$ is equivalent to the category of presheaves on $L(S)$ by an equivalence that associates with a representable presheaf of an idempotent e the étale map $eS \longrightarrow E$, $t \mapsto t^*t$.*

The assignments $S \mapsto L(S) \mapsto \mathscr{B}(S)$ are functorial: a prehomomorphism $\rho : S \longrightarrow T$ defines a functor

$$\rho : L(S) \longrightarrow L(T).$$

whence an (essential) geometric morphism

$$\rho_! \dashv \rho^* \dashv \rho_* : \mathscr{B}(S) \longrightarrow \mathscr{B}(T)$$

of classifying toposes.

The topos $\mathscr{B}(S)$ has a canonical "torsion-free generator," called the *Schützenberger object*, denoted $\mathbf{S}$. In terms of étale maps over E, $\mathbf{S}$ is the domain map $S \xrightarrow{s^*s} E$, where S acts totally on the right by multiplication. As a presheaf on $L(S)$ we have:

$$\mathbf{S}(e) = \{t \mid t^*t = e\} \; ; \qquad \mathbf{S}(s)(t) = ts \; , \tag{1}$$

for any morphism $d \xrightarrow{s} e$ of $L(S)$.

However, $\mathbf{S}$ carries more structure: the operation of postcomposition gives a semigroup action of S, defined pointwise by

$$S \times \mathbf{S}(e) \longrightarrow \mathbf{S}(e) \; ; \qquad (r,t) \mapsto \begin{cases} rt & \text{if } t = r^*rt \\ \text{undefined} & \text{otherwise.} \end{cases}$$

This agrees with the transition maps of $\mathbf{S}$, so that $\mathbf{S}$ is an internal ΔS-action in $\mathscr{B}(S)$. Even better, it is a torsor:

3.2. Proposition. ***S*** *together with its canonical* ΔS*-action is a torsor.*

Proof. We may, by Proposition 2.15, test this pointwise. Each $\mathbf{S}(e)$ is non-empty, as $e \in \mathbf{S}(e)$. Moreover, given $s, t \in \mathbf{S}(e)$, we have $(ts^*)s = t(s^*s) = te = t$, so that the action is transitive. Finally, if $st = st'$ for $t, t' \in \mathbf{S}(e)$, then $t = t'$ follows because $L(S)$ is left-cancellative. ∎

We shall later see that the internal semigroup action $\mathbf{S}$ is the *generic torsor*. Unlike in the group case, the generic torsor $\mathbf{S}$ may not be a representable presheaf. In fact, it is representable if and only if S is an inverse monoid.

Semigroup actions as functors. We relate S-sets, strict S-sets, and S-torsors to three classes of functors on $L(S)$. It should be emphasized that these functors are covariant, whereas the objects of $\mathscr{B}(S)$ are contravariant functors on $L(S)$.

The passage from S-sets to functors is given as follows. Given an S-set (X, μ), define a functor

$$\Phi_\mu : L(S) \longrightarrow Set$$

such that

$$\Phi_\mu(e) = eX = \{x \in X | ex = x\} \; ; \; \Phi_\mu(s)(x) = sx \text{ for } e \xrightarrow{s} d \text{ in } L(S) \; . \tag{2}$$

The action of Φ_μ on morphisms is well-defined: the map $s : e \longrightarrow d$ satisfies $s^*s = e$ and $s = ds$. Thus for $x \in X$ with $ex = x$ we have that s^*sx is defined, whence sx is defined, so that $ss^*(sx) = sx$ and also $d(sx) = sx$.

The assignment $(X,\mu) \mapsto \Phi_\mu$ is the object part of a functor

$$\Phi : S\!-\!Set \longrightarrow \mathrm{Func}[L(S), Set] \ .$$

Explicitly, a morphism of S-sets $\rho : (X,\mu) \longrightarrow (Y,\nu)$ gives a natural transformation $\Phi_\rho : \Phi_\mu \longrightarrow \Phi_\nu$, whose component at e is the function

$$\rho_e : eX \longrightarrow eY; \qquad ex = x \mapsto e\rho(x) = \rho(ex) = \rho(x) \ .$$

Consider the following construction in the other direction. Start with a functor $F : L(S) \longrightarrow Set$, and define

$$\Psi(F) = \varinjlim\ E \longrightarrow L(S) \xrightarrow{F} Set = \coprod_E F(e)/\sim \tag{3}$$

where the equivalence relation is generated by $(e,x) \sim (e', F(e \leq e')(x))$. The set $\Psi(F)$ is in general not an S-set, but the following is a necessary and sufficient condition.

3.3. Definition. A functor $F : L(S) \longrightarrow Set$ is *torsion-free* if for every idempotent e, $F(e) \longrightarrow \Psi(F)$ is injective. $\mathrm{TF}(\mathrm{L(S)}, Set)$ denotes the category of all such torsion-free functors.

3.4. Proposition. *A torsion-free functor $L(S) \longrightarrow Set$ has the property that its transition maps are injective* (said to be transition-injective). *The converse holds if S is an inverse monoid.*

Proof. If F is torsion-free, then clearly for any idempotents d, e, $F(d \leq e)$ is injective. It follows that F is transition-injective because every map in $L(S)$ is the composite of an isomorphism and an inequality. For the converse, if S has a global idempotent 1, then $\Psi(F) \cong F(1)$ identifying the map $F(e) \longrightarrow \Psi(F)$ with $F(e \leq 1)$, which is injective if F is transition-injective. ∎

Since every morphism of $L(S)$ is a monomorphism we may interpret torsion-freeness as a monomorphism preserving property. More importantly, if F is torsion-free, then $\Psi(F)$ is an S-set.

3.5. Proposition. *The assignment $F \mapsto \Psi(F)$ restricts to form the object part of a functor $\Psi : \mathrm{TF}(\mathrm{L(S)}, Set) \longrightarrow S\!-\!Set$.*

Proof. The action by S in $\Psi(F)$ is defined as follows. If $s \in S$ and $\alpha \in \Psi(F)$, then

$$s\alpha = \begin{cases} [ses^*, F(se)(x)] & \text{if } \exists\,(e,x) \in \alpha,\ e \leq s^*s \\ \text{undefined} & \text{otherwise.} \end{cases}$$

This action is well-defined because the maps $F(e) \longrightarrow \Psi(F)$ are injective. The action of Ψ on morphisms is also straightforward and left to the reader. ∎

Moreover, it is easily verified that all functors of the form Φ_μ are torsion-free: given any idempotent e, the map $eX \longrightarrow \Psi(\Phi_\mu)$ is injective because the cocone of subsets $eX \subseteq X$ induces a map $\Psi(\Phi_\mu) \longrightarrow X$. We may now prove:

3.6. PROPOSITION. *The functor* $\Psi : \mathrm{TF}[L(S), \mathit{Set}] \longrightarrow S{-}\mathit{Set}$ *is left adjoint to* $\Phi : S{-}\mathit{Set} \longrightarrow \mathrm{TF}[L(S), \mathit{Set}]$*. Moreover, for any torsion-free* F*,* $\Psi(F)$ *is well-supported, and the unit*

$$F \longrightarrow \Phi\Psi(F)$$

is an isomorphism. For any well-supported S*-set* (X, μ)*, the counit*

$$\Psi\Phi(X, \mu) \longrightarrow (X, \mu)$$

is an isomorphism. Thus, Φ *and* Ψ *establish an equivalence*

$$S{-}\mathit{Set} \;\simeq\; \mathrm{TF}(L(S), \mathit{Set})\;.$$

PROOF. For any idempotent e, the map

$$F(e) \longrightarrow eX,\; x \mapsto [e, x],$$

is an isomorphism, for $X = \Psi(F)$. Indeed, if $(e, x) \sim (e, y)$, then clearly $x = y$, so the map is injective. It is onto because if $e\alpha$ is defined, then by definition there are $d \leq e$ and $y \in F(d)$ such that $\alpha = [d, y]$. But then $\alpha = [e, F_{d\leq e}(y)]$. This isomorphism of sets is natural so that $F \cong \Phi\Psi(F)$. On the other hand, it is not hard to see that a well-supported S-set (X, μ) is recovered from its functor Φ_μ as the colimit $\Psi(\Phi_\mu)$. We omit further details. ■

$\mathscr{B}(S)$ CLASSIFIES TORSORS. We specialize the correspondence between S-sets and torsion-free functors to strict S-sets and torsors, respectively.

Recall that an S-set (X, μ) is called *strict* when μ is in fact a homomorphism (rather than a prehomomorphism). For the proof of the following, recall from § 3 that $L(S)$ has pullbacks, and that a functor $L(S) \longrightarrow \mathit{Set}$ preserves all pullbacks if and only if it preserves inequality pullbacks.

3.7. PROPOSITION. *An* S*-set* (X, μ) *is strict if and only if* Φ_μ *preserves pullbacks.*

PROOF. It is readily checked that the functor Φ_μ preserves the inequality pullbacks if and only if μ is a homomorphism. ■

Let $\mathrm{PB}(\mathbb{C}, \mathit{Set})$ denote the category of functors on a small category $\mathbb{C}$ that preserve any existing pullbacks. The following fact is now evident.

3.8. PROPOSITION. *The equivalence of Prop. 3.6 restricts to one*

$$\underline{S{-}\mathit{Set}} \;\simeq\; \mathrm{PB}(L(S), \mathit{Set})\;.$$

In order to describe torsors as functors we shall say that $F : L(S) \longrightarrow \mathit{Set}$ is *filtering* if its category of elements is a filtered category [12]. The following proposition is then a straightforward generalization of the group case.

3.9. Proposition. *$\mathrm{TOR}(S)$ is equivalent to the category $\mathrm{Filt}(L(S), \mathit{Set})$ of filtering functors on $L(S)$, which is equivalent to the category of finite limit preserving distributions on $\mathscr{B}(S)$* (these are the inverse image functors of the points of $\mathscr{B}(S)$).

Proof. It is relatively straightforward to verify that an S-set (X, μ) satisfies the torsor conditions if and only if the functor Φ_μ is filtering. ∎

Note that a torsor is necessarily a strict S-set because a filtering functor must preserve pullbacks.

In the above result one may replace the category of sets by an arbitrary topos $\mathscr{E}$. In § 3 we already showed that the Schützenberger object $\mathbf{S}$ is a torsor in $\mathscr{B}(S)$. By the above result, it corresponds to a filtering functor $L(S) \longrightarrow \mathscr{B}(S)$, which is easily seen to be the Yoneda embedding. This proves the following because of the well-known correspondence between filtering functors $L(S) \longrightarrow \mathscr{E}$ and geometric morphisms $\mathscr{E} \longrightarrow \mathscr{B}(S)$.

3.10. Theorem. *The functor that associates with a geometric morphism $\gamma : \mathscr{E} \longrightarrow \mathscr{B}(S)$ the torsor $\gamma^* \boldsymbol{S}$ is an equivalence*

$$\boldsymbol{Top}(\mathscr{E}, \mathscr{B}(S)) \simeq \mathrm{TOR}(\mathscr{E}; \Delta S) ,$$

and this equivalence is natural in $\mathscr{E}$. Thus $\mathscr{B}(S)$ classifies S-torsors.

Another way to interpret this is as follows. If X is an S-torsor (in *Set*) with corresponding point p, then we have the following topos pullback.

$$\begin{tikzcd} \mathit{Set}/X \arrow[r, two heads] \arrow[d, "\delta"'] & \mathit{Set} \arrow[d, "p"] \\ \mathscr{B}(S)/\mathbf{S} \arrow[r, two heads] & \mathscr{B}(S) \end{tikzcd} \tag{4}$$

The geometric morphism δ is the support of X, described as a locale morphism in § 3.10.

The covariant representables correspond to torsors (in *Set*) that we call *principal.*

3.11. Example. *Principal torsors.* The covariant representable functor

$$y(e) : L(S) \longrightarrow \mathit{Set}\ ;\ \ y(e)(d) = L(S)(e, d) = \{s \mid s^*s = e,\ s = ds\} = d\mathbf{S}(e)$$

associated with an idempotent e is filtering. The usual colimit extension $e^* : \mathscr{B}(S) \longrightarrow \mathit{Set}$ of $y(e)$ is a finite limit-preserving distribution such that $e^*(P) = P(e)$. As such e^* is the inverse image functor of a point of $\mathscr{B}(S)$. The torsor associated with this point is easily seen to be $e^*(\mathbf{S}) = \mathbf{S}(e)$. The Yoneda lemma asserts in this case that for any S-set X, S-set maps $\mathbf{S}(e) \longrightarrow X$ are in bijective correspondence with the set eX. We thus have a full and faithful functor

$$L(S)^{\mathrm{op}} \longrightarrow \mathrm{TOR}(S)\ ;\ e \mapsto \mathbf{S}(e)\ .$$

A torsor that is isomorphic to $\mathbf{S}(e)$, for some e, is said to be *a principal torsor*. Incidentally, $L(S)^{\mathrm{op}}$ is isomorphic to the category $R(S)$, whose object set is E, and a morphism $d \xrightarrow{s} e$

is an $s \in S$ such that $ss^* = e$ and $s = sd$. Thus, the full subcategory of $\mathrm{TOR}(S)$ on the principal torsors is equivalent to $R(S)$.

We summarize the correspondences explained so far in the following diagram (we treat distributions in § 5):

$$\begin{tikzcd}
\text{rep. points} \arrow[d, "\simeq"] & \subseteq & \text{points} \arrow[d, "\simeq"] & \subseteq & S\text{–distributions} \arrow[d, "\simeq"] & \subseteq & \text{tor. free dist.} \arrow[d, "\simeq"] \\
\text{rep. functors} \arrow[d, "\simeq"] & \subseteq & \mathrm{Filt}(L(S), \mathit{Set}) \arrow[d, "\simeq"] & \subseteq & \mathrm{PB}(L(S), \mathit{Set}) \arrow[d, "\simeq"] & \subseteq & \mathrm{TF}(L(S), \mathit{Set}) \arrow[d, "\simeq"] \\
\text{prin. torsors} & \subseteq & \mathrm{TOR}(S) & \subseteq & \underline{S\text{–}\mathit{Set}} & \subseteq & S\text{–}\mathit{Set}
\end{tikzcd}$$

The structure of torsors. We now return to set-theoretic torsors and their structure. From what we have shown so far, we may conclude the following:

3.12. Proposition. *Any S-set, and in particular any S-torsor, is a colimit of principal S-torsors.*

Proof. A (filtering) functor $L(S) \longrightarrow \mathit{Set}$ is a colimit of representable functors. ■

Although we cannot expect that an arbitrary colimit of (principal) torsors is a torsor, a filtered colimit of torsors is a torsor. We turn now to a closer examination of this aspect, and a more informative version of Prop. 3.12.

3.13. Proposition. *Let $J \subseteq E$ be a filter* (upclosed and closed under binary meets). *Then the colimit of the functor*

$$J^{\mathrm{op}} \subseteq E^{\mathrm{op}} \longrightarrow L(S)^{\mathrm{op}} \longrightarrow \mathit{Set}^{L(S)}\ ;\ d \mapsto y(d)\ ,$$

is a filtering functor. In particular, the colimit preserves pullbacks, and its corresponding S-set, which we denote $\mathbf{S}(J)$, is a torsor.

Proof. A filtered colimit of filtering functors is filtering. J^{op} is filtered (in the category sense), and any representable $y(d)$ is filtering. ■

The torsor associated with a filter $J \subseteq E$ given by Prop. 3.13 is

$$\mathbf{S}(J) = \coprod_{d \in J} \mathbf{S}(d)/\sim\ ,$$

where the equivalence relation is defined as follows: $s \sim t$ if there is $f \in J$ such that $sf = tf$ (without loss of generality we can assume $f \leq s^*s, t^*t$). The action of S in $\mathbf{S}(J)$ is given as follows:

$$s \cdot [t] = \begin{cases} [st] & \text{if there is } r \sim t \text{ such that } r^*r \in J \text{ and } r = s^*sr \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Clearly, this is well-defined. Note: if say $rd = td$, $d \in J$, $d \leq r^*r, t^*t$, then the domain of $std = srd$ is d. We have $std \leq st$, so $d \leq (st)^*st$, whence $(st)^*st \in J$ since J is upclosed.

Thus, $\mathbf{S}(J)$ is a colimit of a diagram of principal torsors $\mathbf{S}(d)$, where d ranges over J and the morphisms are the ones coming from the inequalities $d \leq e$. In general, the maps $\mathbf{S}(d) \longrightarrow \mathbf{S}(J)$ are not strict, and are neither injective nor surjective.

3.14. LEMMA. *Suppose that $t^*t \in J$. Then for any $s \in S$, $s[t]$ is defined and equals $[st]$ iff $(st)^*st \in J$.*

PROOF. We have already seen above that if $s[t]$ is defined, then $(st)^*st \in J$. For the converse, we must produce an $r \sim t$ such that $r^*r \in J$ and $r = s^*sr$. Let $r = s^*st$. Then $r^*r = (st)^*st \in J$ and $r = s^*sr$. Also note

$$r(r^*r) = r = s^*st = s^*stt^*t = tt^*s^*st = t(st)^*st = t(r^*r) ,$$

so that $r \sim t$. ∎

3.15. EXAMPLE. We have the following examples of $\mathbf{S}(J)$.

1. $J = E$: $\mathbf{S}(J) = S/\sigma$, where σ is the minimum group congruence on S.
2. $J =$ the principal filter on $e = \{d \in E \mid e \leq d\}$: $\mathbf{S}(J) = \mathbf{S}(e)$.
3. If S has a zero element 0 ($s0 = 0s = 0$), then 1 is a torsor. In this case, $\mathbf{S}(0) = \mathbf{S}(E) = S/\sigma = 1$.

In the case of a group G, it is well-known that every G-set decomposes uniquely as a disjoint sum of transitive G-sets, each of which is in turn a quotient of the representable G-set. We now explain how this statement generalizes to the case of an inverse semigroup.

Let X be a non-empty strict S-set, and let $x \in X$ be an arbitrary element. We may consider the set

$$\mathrm{Supp}(x) = \{e \in E \mid ex = x\} ,$$

called the *support* of x. $\mathrm{Supp}(x)$ is easily seen to be a filter of the meet-semilattice of idempotents E. (The strictness of X is needed for closure under binary infima.) If X is a torsor, then there is a locale morphism $\delta : X \longrightarrow E$, occurring in the topos pullback (4), such that if x is regarded as a point $1 \xrightarrow{x} X$ of the discrete locale X, then the point $\delta \cdot x : 1 \longrightarrow X \longrightarrow E$ of the locale E corresponds to the support filter $\mathrm{Supp}(x)$. Indeed, the frame morphism $\mathscr{O}(E) \xrightarrow{\delta^*} P(X)$ associated with δ is given by $\delta^*(e) = eX$. Thus, δ is the support of the torsor X.

Returning to the case of an arbitrary strict S-set X, and $x \in X$, define the S-torsor

$$T_x = \mathbf{S}(\mathrm{Supp}(x)) = \varinjlim_{\mathrm{Supp}(x)} \mathbf{S}(e) .$$

A typical element of T_x is an equivalence class of elements t such that $t^*t \in \mathrm{Supp}(x)$, where two such s and t are equivalent when there exists an $f \in \mathrm{Supp}(x)$ for which $sf = tf$. As before, $[t]$ denotes an equivalence class of such t. The action of S in T_x is defined by

$$s \cdot [t] = \begin{cases} [st] & \text{if } (st)^*st \in \mathrm{Supp}(x) \text{ (Lemma 3.14)} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

There is a canonical map $\nu_x : T_x \longrightarrow X$: $\nu_x[t] = tx$. Note that tx is defined since $(t^*t)x = x$ is, and that this is independent of the choice of representative. ν_x is the map from the colimit induced by the cocone of maps $\mathbf{S}(e) \xrightarrow{x} X$, where $e \in \mathrm{Supp}(x)$ (corresponding by Yoneda to $x \in eX$).

3.16. LEMMA. *For any strict S-set X, ν_x is a strict map of S-sets.*

PROOF. Let $s \in S$, and $[t] \in T_x$. If $s[t] = [st]$ is defined, then

$$\nu_x(s[t]) = \nu_x[st] = (st)x$$

is defined. Therefore, tx and $s(tx) = s\nu_x[t]$ are defined, and the latter equals the above. On the other hand, if $s\nu_x[t] = s(tx)$ is defined, then since X is strict $(st)x = \nu_x[st]$ is also defined. But this says that $(st)^*st \in \mathrm{Supp}(x)$, so that by Lemma 3.14 $s[t]$ is defined and equals $[st]$. ■

By the *orbit* of x we mean the S-set

$$O_x = \{y \in X \mid \exists s \in S, sx = y\}\ .$$

3.17. LEMMA. *The image of $\nu_x : T_x \longrightarrow X$ is precisely the orbit O_x. The map ν_x is surjective precisely when the action of S in X is transitive.*

PROOF. If $y = sx \in O_x$, then $s^*s \in \mathrm{Supp}(x)$, and $\nu_x[s] = sx = y$. Conversely, elements of the image are clearly in the orbit. For the other statement note that the action is transitive if and only if there is precisely one orbit, which equals the whole of X. ■

Any strict S-set X can be written as the coproduct (disjoint sum) of its orbits. (If X is not strict, then the orbits may not be disjoint.) Let I be an indexing set for this decomposition, so that we have

$$X \cong \coprod_{i \in I} O_{x_i}\ .$$

Then the maps $\nu_{x_i} : T_{x_i} \longrightarrow O_{x_i}$ assemble to form a covering of X:

$$\coprod_{i \in I} T_{x_i} \twoheadrightarrow \coprod_{i \in I} O_{x_i} \cong X\ .$$

In particular, this shows how X is canonically a colimit of S-torsors, and ultimately a colimit of principal S-torsors.

3.18. COROLLARY. *If x is an element of an S-torsor X, then the map $T_x \xrightarrow{\nu_x} X$ is an isomorphism of S-sets. I.e., $T_x \cong O_x = X$ in this case.*

PROOF. ν_x is strict (Lemma 3.16), whence an isomorphism (Prop. 2.12). ■

The above results are stated for strict S-sets, but it is equally true that an arbitrary S-set admits a decomposition as a colimit of principal torsors (Prop. 3.12). We do not give an explicit description, which would be more involved since in general $\mathrm{Supp}(x)$ may not be a filter.

4. Change of base

So far we have been working with a fixed inverse semigroup S; in this section we examine what happens when we vary S. We begin by showing that a prehomomorphism $S \longrightarrow T$ induces an adjunction between the categories $S{-}Set$ and $T{-}Set$; the right adjoint is restriction of scalars, and the left adjoint is a tensor product with T. In general, the right adjoint does not restrict to the subcategories of torsors (just as in the group case), but when X is an S-torsor, then $T \otimes_S X$ is a T-torsor, and thus $\mathrm{TOR}(-)$ is a covariant functor. We give an explicit description of the tensor product $T \otimes_S X$, which is more involved than for groups. We apply this to the homomorphism $S \longrightarrow S/\sigma$, obtaining a torsor characterization of E-unitary semigroups. Finally, we relate principal S-bundles on a space to principal bundles for the maximum group of S.

THE HOM-TENSOR ADJUNCTION. We fix a prehomomorphism $\rho : S \longrightarrow T$. If $T \longrightarrow I(Z)$ is a T-set, then the composite prehomomorphism

$$S \xrightarrow{\rho} T \longrightarrow I(Z)$$

is an S-set (with the same underlying set Z) that we denote Z_ρ. This gives a functor

$$T{-}Set \longrightarrow S{-}Set \qquad Z \mapsto Z_\rho\ ,$$

which we call restriction of scalars.

Clearly, if we wish to restrict this functor to a functor from strict S-sets to strict T-sets, we need to require that ρ is in fact a homomorphism.

Restriction of scalars has a left adjoint because categories of the form $S{-}Set$ are cocomplete. We first prove this fact:

4.1. LEMMA. *The category $S{-}Set$ is cocomplete.*

PROOF. Coproducts of S-sets are set-theoretic. Concretely, for a family (X_i, μ_i) of S-sets, form the set $\coprod_i X_i$, and define a semigroup action via

$$s \cdot (x, i) = (sx, i)$$

where this is defined if and only if sx is defined. It is readily checked that this has the correct property. This same construction is valid for strict S-sets.

Coequalizers are not set-theoretic. Consider two maps $\alpha, \beta : X \longrightarrow Y$ of S-sets (where α, β need not be strict). Define an equivalence relation $\sim$ on Y generated by the following two clauses:

$$\begin{array}{rl} y \sim y' & \text{if} \quad \exists x \in X.\, \alpha(x) = y \text{ and } \beta(x) = y' \\ sy \sim sy' & \text{if} \quad y \sim y' \text{ (and } sy,\ sy' \text{ are defined).} \end{array}$$

Now define an action in $Y/\!\sim$ by putting

$$t[y] = [ty'] \text{ for some } y' \sim y \text{ with } ty' \text{ defined.}$$

This is well-defined on representatives, and gives an action of S in $Y/\sim$. Clearly the quotient function $Y \longrightarrow Y/\!\sim$ is equivariant. The verification of the coequalizer property is left to the reader. ■

4.2. PROPOSITION. *The restriction-of-scalars functor $T{-}Set \longrightarrow S{-}Set$ along a prehomomorphism $\rho : S \longrightarrow T$ has a left adjoint.*

PROOF. We have proved in § 3.10 that every S-set is canonically a colimit of principal torsors (where the diagram generally contains non-strict maps). Since S-torsors are the same as points of the topos $\mathscr{B}(S)$, it is easy to see how to pass forward an S-torsor X along a prehomomorphism $S \xrightarrow{\rho} T$: if p is the point of $\mathscr{B}(S)$ corresponding to X, then the composite geometric morphism

$$Set \xrightarrow{p} \mathscr{B}(S) \xrightarrow{\rho} \mathscr{B}(T)$$

is a point of $\mathscr{B}(T)$ whose corresponding T-torsor we denote $T \otimes_S X$. In the special case of a principal torsor $X = \mathbf{S}(e)$, one easily verifies that $T \otimes_S \mathbf{S}(e) \cong \mathbf{T}(\rho(e))$, where $\mathbf{T}$ is the Schützenberger object of T. Putting this together, the left adjoint to $(-)_\rho$ may be taken to be the following: if $X \cong \varinjlim_e \mathbf{S}(e)$, then

$$T \otimes_S X \cong T \otimes_S \big(\varinjlim_e \mathbf{S}(e)\big) \cong \varinjlim_e \big(T \otimes_S \mathbf{S}(e)\big) \cong \varinjlim_e \mathbf{T}(\rho(e))\ .$$

The last colimit is taken in $T{-}Set$. ■

TENSOR PRODUCT OF TORSORS. We have described the functor $X \mapsto T \otimes_S X$ in an abstract way. In this section we calculate an explicit description for the case where X is a torsor.

Consider the subset of

$$L(T) \times E(S) \times X$$

consisting of those 4-tuples (t, d, e, x) such that $t = dt$, $t^*t = \rho(e)$, and $x \in eX$. In other words, $\rho(e) \xrightarrow{t} d$ is a morphism of $L(T)$, and x is an element of eX. Let $T \otimes_S X$ denote the quotient of this subset given by the equivalence relation generated by equating

$$(t, c, e, x) \sim (t, d, e, x)$$

whenever $c \leq d$, and

$$(t\rho(s), d, f, x) \sim (t, d, e, sx)$$

whenever $f \xrightarrow{s} e$ is a morphism of $L(S)$. Let $[t, d, e, x]$ denote the equivalence class of an element (t, d, e, x). A semigroup action by T is defined in $T \otimes_S X$ as follows: if $r \in T$ and $\alpha \in T \otimes_S X$, then

$$r\alpha = \begin{cases} [rt, rr^*, e, x] & \exists (t, d, e, x) \in \alpha \text{ such that } d \leq r^* r \\ \text{undefined} & \text{otherwise.} \end{cases}$$

From yet another point of view, if X is an S-torsor, with corresponding point p, we may regard $T \otimes_S X$ and the connecting map $X \longrightarrow T \otimes_S X$ as arising from the following diagram of topos pullbacks.

$$\begin{tikzcd} \mathit{Set}/X \arrow[r] \arrow[d, "\delta"'] & \mathit{Set}/T \otimes_S X \arrow[r] \arrow[d] & \mathit{Set} \arrow[d, "p"] \\ \mathscr{B}(S)/\mathbf{S} \arrow[r] & \mathscr{B}(S)/\rho^*\mathbf{T} \arrow[r] & \mathscr{B}(S) \end{tikzcd}$$

The Schützenberger object $\mathbf{T}$ is the generic ΔT-torsor in $\mathscr{B}(T)$. The connecting map commutes with support.

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\delta"'] & T \otimes_S X \arrow[d, "\delta"] \\ E(S) \arrow[r, "\rho"] & E(T) \end{tikzcd}$$

E-UNITARY SEMIGROUPS. We now apply some of the topos machinery to characterize some well-known concepts from semigroup theory.

First recall that a prehomomorphism of inverse semigroups is said to be *idempotent-pure* if it reflects idempotents [9].

4.3. PROPOSITION. *If ρ is idempotent-pure, then for any S-torsor X, the connecting map $X \longrightarrow T \otimes_S X$ is a monomorphism.*

PROOF. A semigroup prehomomorphism ρ is idempotent-pure iff the connecting map of generic torsors $\mathbf{S} \longrightarrow \rho^*\mathbf{T}$ is a monomorphism. ∎

As previously mentioned, if ρ is a homomorphism, then the restriction-of-scalars functor $T{-}\mathit{Set} \longrightarrow S{-}\mathit{Set}$ preserves strictness. For example, the map

$$S \longrightarrow S/\sigma = G\ ; \qquad s \mapsto \overline{s}\ ,$$

to the maximum group is a homomorphism.

4.4. COROLLARY. *If S is E-unitary with maximum group image G* (so that the homomorphism $S \longrightarrow G$ is idempotent-pure)*, then every S-torsor is isomorphic to an S-subset of the S-torsor G.*

PROOF. If X is an S-torsor, then $G \otimes_S X$ is a G-torsor, whence isomorphic to G. Therefore, by Prop. 4.3, X is isomorphic to an S-subset of G regarded as an S-set. ∎

4.5. EXAMPLE. In the E-unitary case a principal torsor $\mathbf{S}(e)$ is isomorphic to the S-subset $\{\overline{s} \mid s^*s = e\}$ of G, where the action by S is given by:

$$t\overline{s} = \begin{cases} \overline{ts} & \text{if } s = t^*ts \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Generally, if $J \subseteq E$ is a filter, then the S-subset $\{\overline{s} \mid s^*s \in J\}$ of G is a torsor, where the action by S is given in just the same way. This describes, in the E-unitary case, every torsor up to isomorphism. It simultaneously generalizes the $\wedge$-semilattice (Eg. 2.10) and group cases.

For the proof of the following corollary, we shall need the (essentially unique) connected universal locally constant object $\overline{\mathbf{S}}$ of $\mathscr{B}(S)$ - universal in the sense that it splits all locally constant objects [5]. (The notation is meant to suggest that this object is a kind of closure of $\mathbf{S}$, which we do not need to discuss here.) Explicitly, the presheaf $\overline{\mathbf{S}}$ may be given as $\overline{\mathbf{S}}(e) = G$, and transition along a morphism $d \xrightarrow{s} e$ of $L(S)$ is given by $g \mapsto g\overline{s}$. There is a canonical natural transformation $\eta : \mathbf{S} \longrightarrow \overline{\mathbf{S}}$, such that $\eta_e(s) = \overline{s}$. The geometric morphism $\mathscr{B}(S)/\overline{\mathbf{S}} \longrightarrow \mathscr{B}(S)$ is the universal locally constant covering of $\mathscr{B}(S)$.

4.6. COROLLARY. *The following are equivalent for an inverse semigroup S:*

1. *S is E-unitary;*
2. *for every Grothendieck topos $\mathscr{E}$, $\mathrm{TOR}(\mathscr{E}; \Delta S)$ is left-cancellative and the forgetful functor*
$$\mathrm{TOR}(\mathscr{E}; \Delta S) \longrightarrow \mathscr{E}$$
preserves monomorphisms;
3. *$\mathrm{TOR}(S)$ is left-cancellative and the forgetful functor*
$$\mathrm{TOR}(S) \longrightarrow \mathit{Set}$$
preserves monomorphisms.

PROOF. $1 \Rightarrow 2$. A map $X \xrightarrow{m} Y$ of torsors in $\mathscr{E}$ corresponds to a natural transformation $\tau : p^* \longrightarrow q^*$ of their classifying points $p, q : \mathscr{E} \longrightarrow \mathscr{B}(S)$. If $\eta : \mathbf{S} \longrightarrow \overline{\mathbf{S}}$ is the canonical morphism in $\mathscr{B}(S)$, then the following square in $\mathscr{E}$ commutes.

$$\begin{tikzcd} p^*\mathbf{S} \cong X \arrow[r, "p^*\eta"] \arrow[d, "\tau_{\mathbf{S}} = m"'] & p^*\overline{\mathbf{S}} \arrow[d, "\tau_{\overline{\mathbf{S}}}"] \\ q^*\mathbf{S} \cong Y \arrow[r, "q^*\eta"] & q^*\overline{\mathbf{S}} \end{tikzcd}$$

If S is E-unitary, then η is a monomorphism, whence so are $p^*\eta$ and $q^*\eta$. $\tau_{\overline{\mathbf{S}}}$ is an isomorphism because it is a map of ΔG-torsors, $G = S/\sigma$. Therefore, m is a monomorphism in $\mathscr{E}$.

$2 \Rightarrow 3$. This is trivial.

$3 \Rightarrow 1$. If $\mathrm{TOR}(S)$ is left-cancellative, then in particular for any idempotent e, the map of torsors $\mathbf{S}(e) \longrightarrow G$ is a monomorphism in $\mathrm{TOR}(S)$. If $\mathrm{TOR}(S) \longrightarrow \mathit{Set}$ preserves monomorphisms, then this map is injective, which says that S is E-unitary. ∎

Principal Bundles. We return to the topos $\mathit{Sh}(B)$ of sheaves on a space B. We regard sheaves as étale spaces over B; as explained in § 2.12, an étale space $X \longrightarrow B$ equipped with a semigroup action of ΔS is a torsor when (i) every fiber is nonempty, (ii) the action is fiberwise transitive and (iii) the action is fiberwise locally free.

Let $S \longrightarrow G = S/\sigma$ denote again the maximum group image of S, and consider a ΔS-torsor $X \xrightarrow{\psi} B$ over B. We may now use the tensor product to form a ΔG-torsor $G \otimes_S X = Y \xrightarrow{\varphi} B$, which is a locally constant covering of B. It may be interesting to examine the connecting map $X \longrightarrow G \otimes_S X$ over B. To this end, let $\mathit{Sh}(B) \xrightarrow{p} \mathscr{B}(S)$ denote the geometric morphism associated with a ΔS-torsor $X \xrightarrow{\psi} B$, so that $p^*(\mathbf{S}) = \psi$. Consider the following diagram of topos pullbacks.

$$\begin{tikzcd}
\mathit{Sh}(X) \arrow[r] \arrow[d] \arrow[rr, bend left, "\psi"] & \mathit{Sh}(Y) \arrow[r, "\varphi"] \arrow[d] & \mathit{Sh}(B) \arrow[d, "p"] \\
\mathscr{B}(S)/\mathbf{S} \arrow[r, "\eta"] & \mathscr{B}(S)/\overline{\mathbf{S}} \arrow[r] \arrow[d] & \mathscr{B}(S) \arrow[d] \\
 & \mathit{Set} \arrow[r, "\gamma"] & \mathscr{B}(G)
\end{tikzcd}$$

Then $Y \xrightarrow{\varphi} B$ is a ΔG-torsor. It is also a locally constant covering because it is a pullback of the universal covering γ (which is the unique point of $\mathscr{B}(G)$). Thus, $G \otimes_S$ maps the category of ΔS-torsors over B to the category of ΔG-torsors over B, which in turn maps to the category of locally constant coverings of B. Moreover, the two torsors are related by a map over B.

$$\begin{tikzcd}
X \arrow[r] \arrow[dr, "\psi"'] & Y \arrow[d, "\varphi = \text{loc. constant}"] \\
 & B
\end{tikzcd}$$

For example, if S is E-unitary, which is characterized by the condition that η is a monomorphism, then the connecting map $X \longrightarrow Y$ is an (open) inclusion.

5. Distributions

We now complete the picture of S-sets and S-torsors by introducing the third viewpoint, namely as distributions on the topos $\mathscr{B}(S)$.

Distributions on a topos. We rehearse some standard definitions concerning topos distributions [2], and characterize the category of pullback-preserving functors on $\mathbb{C}$ in these terms. This in turn yields a description of the category of strict S-sets in terms of distributions (Prop. 5.7).

5.1. Definition. (Lawvere) A *distribution* on a Grothendieck topos $\mathscr{E}$ (with values in *Set*) is a colimit-preserving functor $\mathscr{E} \longrightarrow \mathit{Set}$. If $\mathbb{C}$ is a small category and $\mathit{PSh}(\mathbb{C})$ denotes the category of presheaves on $\mathbb{C}$, then we shall refer to a distribution on $\mathit{PSh}(\mathbb{C})$ simply as a *distribution on* $\mathbb{C}$.

The inverse image functor of a point of $\mathscr{E}$, i.e. of a geometric morphism $\mathit{Set} \longrightarrow \mathscr{E}$, is a distribution. A distribution has a right adjoint, but in general it need not be the inverse image functor of a point of the topos since it need not preserve finite limits. Thus, we may think of distributions as generalized points.

It is well-known that the category of distributions on $\mathbb{C}$ (with natural transformations) is equivalent to the category of (covariant) functors $\mathbb{C} \longrightarrow \mathit{Set}$. The equivalence is given on the one hand by composing with the Yoneda functor $\mathbb{C} \longrightarrow \mathit{PSh}(\mathbb{C})$, and on the other by a colimit extension formula along the same Yoneda functor: if F is a functor on $\mathbb{C}$, then

$$\lambda(P) = \varinjlim \mathbb{P} \longrightarrow \mathbb{C} \xrightarrow{F} \mathit{Set}$$

is a distribution, where $\mathbb{P} \longrightarrow \mathbb{C}$ is the discrete fibration corresponding to a presheaf P. For any object c of $\mathbb{C}$, we have $\lambda(c) = F(c)$, where typically we use the same symbol c to denote an object of $\mathbb{C}$ and the corresponding representable presheaf.

The following probably well-known fact helps in the study of S-sets. $\mathrm{PB}(\mathbb{C}, \mathit{Set})$ denotes the category of pullback-preserving functors on $\mathbb{C}$.

5.2. Proposition. *If* $\mathbb{C}$ *has pullbacks, then* $\mathrm{PB}(\mathbb{C}, \mathit{Set})$ *is equivalent to the full subcategory of distributions on* $\mathbb{C}$ *that preserve pullbacks of the form*

$$\begin{tikzcd} P \arrow[r] \arrow[d] & Q \arrow[d] \\ c \arrow[r, "m"] & d \end{tikzcd}$$

in $\mathit{PSh}(\mathbb{C})$*, where* m *is a morphism of* $\mathbb{C}$.

Proof. Let λ denote the colimit extension of a functor $F : \mathbb{C} \longrightarrow \mathit{Set}$. We have two functors:

$$\mathit{PSh}(\mathbb{C}/d) \;\simeq\; \mathit{PSh}(\mathbb{C})/d \longrightarrow \mathit{Set}/F(c)\ .$$

One functor carries $Q \longrightarrow d$ first to its pullback $P \longrightarrow c$ along m and then to $\lambda(P) \longrightarrow F(c)$, and the other carries $Q \longrightarrow d$ first to $\lambda(Q) \longrightarrow F(d)$ and then to the pullback along $F(m)$. Since F preserves pullbacks (by assumption), we see that the two functors are isomorphic when composed with Yoneda $\mathbb{C}/d \longrightarrow \mathit{PSh}(\mathbb{C}/d)$. But both functors preserve colimits, so they must be isomorphic, which says that λ preserves pullbacks of the specified form. ∎

Torsion-free distributions. We wish to interpret general S-sets as distributions on $\mathscr{B}(S)$. According to Def. 5.1 and Prop. 3.1, this is equivalent to what we call a distribution on $L(S)$, but in some cases (such as the Wagner-Preston and Munn) it is beneficial to regard $\mathscr{B}(S)$ as the category of étale maps $X \longrightarrow E$.

5.3. Definition. A distribution $\lambda : \mathscr{B}(S) \longrightarrow \mathit{Set}$ is *torsion-free* if for every $s \in \mathbf{S}(e)$, $\lambda(s) : \lambda(e) \longrightarrow \lambda(\mathbf{S})$ is injective.

This definition is in agreement with Def. 3.3 in the following sense.

5.4. Proposition. *A functor $L(S) \longrightarrow \mathit{Set}$ is torsion-free iff its colimit extension $\mathscr{B}(S) \longrightarrow \mathit{Set}$ is torsion-free.*

Proof. A distribution λ on $\mathscr{B}(S)$ is torsion-free iff for every idempotent e, the map $\lambda(e)$: $\lambda(e) \longrightarrow \lambda(\mathbf{S})$ is injective because an arbitrary element $e \xrightarrow{s} \mathbf{S}$ factors as $e \xrightarrow{s} ss^* \longrightarrow \mathbf{S}$, where $e \xrightarrow{s} ss^*$ is an isomorphism. If $\lambda = F$ on $L(S)$, then $\lambda(e)$ is the canonical map $F(e) \longrightarrow \Psi(F)$. Note that $\lambda(\mathbf{S})$ is isomorphic to

$$\varinjlim \ \mathbb{S} \longrightarrow L(S) \xrightarrow{F} \mathit{Set} \ ,$$

where $\mathbb{S} \longrightarrow L(S)$ is the discrete fibration corresponding to $\mathbf{S}$. This discrete fibration is equivalent to $E \longrightarrow L(S)$ [4], so the above colimit is isomorphic to $\Psi(F)$. ■

Prop. 5.5, which is a distribution version of Prop. 3.6, requires what we call *the generic singleton of a set.* Let X be a set and consider $I(X)$, the symmetric inverse semigroup on X. Write $L(X)$ for $L(I(X))$; explicitly, the objects of this category are the subsets of X and the morphisms are the injective maps between them. Also, let $\mathscr{B}(X)$ denote $\mathscr{B}(I(X))$. Consider the functor

$$x^* : L(X) \longrightarrow \mathit{Set} \ ; \ x^*(A) = A \ .$$

If X is non-empty, then x^* is filtering, so that it corresponds to a geometric morphism

$$x : \mathit{Set} \longrightarrow \mathscr{B}(X) \ ,$$

which we call the generic singleton of X. If an element $a \in X$ is regarded as a singleton subset $\{a\} \subseteq X$, whence an object of $L(X)$, then its corresponding (representable) point of $\mathscr{B}(X)$ is isomorphic to the generic singleton x, by a unique isomorphism. If $X = \emptyset$, then $\mathscr{B}(X) = \mathit{Set}$ and $x^* : \mathit{Set} \longrightarrow \mathit{Set}$ is the 0-distribution: $x^*(A) = \emptyset$. In this case, x^* is not (the inverse image functor of) a point.

For any S-set (X, μ), the composite functor

$$L(S) \longrightarrow \mathscr{B}(S) \xrightarrow{\mu_!} \mathscr{B}(X) \xrightarrow{x^*} \mathit{Set}$$

is precisely Φ_μ, where x is the generic singleton of X.

5.5. Proposition. *For any S-set (X,μ), the distribution $x^*\cdot\mu_!$ is torsion-free* (by Props. 3.6 and 5.4, or by Prop. 5.6)*. The category $S-Set$ is equivalent to the full subcategory of torsion-free distributions on $\mathscr{B}(S)$. The equivalence associates with an S-set (X,μ) the torsion-free distribution $x^*\cdot\mu_!$, and with a torsion-free distribution λ the S-set $\lambda(\boldsymbol{S})$.*

The restriction-of-scalars functor has a distribution interpretation because if a T-set Z corresponds to torsion-free distribution λ, then Z_ρ corresponds to $\lambda\cdot\rho_!$. Thus, $\lambda\cdot\rho_!$ is torsion-free. A direct proof of this fact is probably noteworthy.

5.6. Proposition. *If $\lambda:\mathscr{B}(T)\longrightarrow Set$ is a torsion-free distribution, then so is*

$$\mathscr{B}(S)\xrightarrow{\rho_!}\mathscr{B}(T)\xrightarrow{\lambda} Set\,.$$

Proof. $\rho_!(e\longrightarrow\mathbf{S})$ equals $\rho(e)\longrightarrow\rho_!(\mathbf{S})$. But the composite of this with the transpose $\rho_!(\mathbf{S})\longrightarrow\mathbf{T}$ of $\mathbf{S}\longrightarrow\rho^*(\mathbf{T})$ is the monomorphism $\rho(e)\longrightarrow\mathbf{T}$, which is taken by λ to an injective map, where $\mathbf{T}$ denotes the Schützenberger object of $\mathscr{B}(T)$. Therefore, λ carries $\rho(e)\longrightarrow\rho_!(\mathbf{S})$ to an injective map. Hence, $\lambda\cdot\rho_!$ is torsion-free. ■

We should point out that the left adjoint to restriction-of-scalars (as described in § 4) is in general not obtained by taking a colimit in the category of distributions on $\mathscr{B}(T)$: if X is an S-set with associated (torsion-free) distribution λ, then the distribution associated with $T\otimes_S X$ is different from $\lambda\cdot\rho^*$.

S-distributions. We may apply Prop. 5.2 to the category $L(S)$. $\underline{S-Set}$ denotes the category of strict, well-supported S-sets and equivariant maps.

5.7. Proposition. *$\underline{S-Set}$ is equivalent to the full subcategory of distributions $\mathscr{B}(S)\longrightarrow Set$* (which we shall call S-distributions) *that preserve pullbacks of the form*

$$\begin{tikzcd} P \arrow[r, hook] \arrow[d] & Q \arrow[d, "m"] \\ d \arrow[r, phantom, "\leq"] & e \end{tikzcd}$$

where $d\leq e$ in E, and $P(c)=\{x\in Q(c)\mid dm_c(x)=m_c(x)\}$. The equivalence associates with an S-set (X,μ) the distribution $x^*\cdot\mu_!$, which is an S-distribution, and with an S-distribution λ the S-set $\lambda(\mathbf{S})$.

Proof. Prop. 5.2 says that the statement holds for pullbacks of the form

$$\begin{tikzcd} P \arrow[r, tail] \arrow[d] & Q \arrow[d] \\ d \arrow[r, tail, "s"] & e \end{tikzcd}$$

where $d\xrightarrow{s}e$ is a morphism of $L(S)$. But the isomorphism factor $d\xrightarrow{s}ss^*$ of this morphism is irrelevant. ■

EXAMPLES. We give explicit descriptions of the distributions associated with some of the key examples of S-sets.

5.8. EXAMPLE. Consider the terminal S-set 1, where every $s \in S$ acts as the unique total function $1 \longrightarrow 1$. The pullback-preserving functor $L(S) \longrightarrow Set$ associated with this S-set is identically 1. Its S-distribution is the connected components functor $\pi_0 : \mathscr{B}(S) \longrightarrow Set$. Incidentally, π_0 is terminal amongst all distributions on $\mathscr{B}(S)$, not just the S-distributions.

5.9. EXAMPLE. *The Wagner-Preston distribution.* Consider again the Wagner-Preston representation of S (Eg. 2.2): the corresponding pullback-preserving functor on $L(S)$ is

$$W(e) = eS = \{t \mid t = et\} .$$

Transition in W along $e \xrightarrow{s} f$ is given by $t \mapsto st$. The S-distribution associated with the Wagner-Preston S-set is

$$W(Y \longrightarrow E) = Y ,$$

where $Y \longrightarrow E$ is an étale right S-set. For instance, if $\mathbb{G} = (G_0, G_1) = (S, E)$ is the inductive groupoid associated with S, then $W(E \xrightarrow{1} E) = E = G_0$, the set of objects of $\mathbb{G}$.

5.10. EXAMPLE. *The Munn distribution.* The pullback-preserving functor on $L(S)$ associated with the Munn S-set is

$$M(e) = eE = \{d \mid d \leq e\} .$$

Transition in M along a morphism $e \xrightarrow{s} f$ of $L(S)$ sends $d \leq e$ to $sds^* \leq ss^* \leq f$. Its S-distribution sends an étale right S-set $Y \xrightarrow{p} E$ to its set of 'Munn-orbits:'

$$M(p) = O_M(Y) = Y/\sim ,$$

where $\sim$ is the equivalence relation generated by relating $y \sim ys$ whenever $p(y) \leq ss^*$ (without loss of generality, by replacing s with $p(y)s$ we can insist that $p(y) = ss^*$, for then $ys = yp(y)s$ and $p(y) = p(y)s(p(y)s)^*$). Note that $p(ys) = s^*p(y)s \leq s^*ss^*s = s^*s$ (with equality if $p(y) = ss^*$). One easily sees that $M(E \xrightarrow{1} E) = \pi_0(\mathbb{G})$, the set of connected components of $\mathbb{G}$.

$\mathscr{B}(S)$ ACTS IN $\underline{S\!-\!Set}$. Lawvere observes that a topos $\mathscr{E}$ acts in its category of distributions by the formula:

$$P \cdot \lambda(E) = \lambda(P \times E) \text{ , or } \int E \, d(P \cdot \lambda) = \int P \cdot E \, d\lambda ,$$

where P and E are objects of $\mathscr{E}$, and λ is a distribution on $\mathscr{E}$. The presheaf case $\mathscr{E} = PSh(\mathbb{C})$ of this formula yields

$$P \cdot F(c) = \varinjlim \mathbb{X}(c) \longrightarrow \mathbb{C} \xrightarrow{F} Set \ ,$$

where

$$\begin{tikzcd} \mathbb{X}(c) \arrow[r] \arrow[d] & \mathbb{C}/c \arrow[d] \\ \mathbb{P} \arrow[r] & \mathbb{C} \end{tikzcd}$$

is a pullback of discrete fibrations.

5.11. Proposition. *Suppose that $\mathbb{C}$ has pullbacks. If a functor F on $\mathbb{C}$ preserves pullbacks, then for any presheaf P on $\mathbb{C}$, $P \cdot F$ also preserves pullbacks.*

Proof. We must show that the distribution corresponding to F preserves pullbacks of the following form.

$$\begin{tikzcd} P \times a \arrow[r] \arrow[d] & P \times b \arrow[d] \\ P \times c \arrow[r] & P \times d \end{tikzcd}$$

This is a consequence of Prop. 5.2. ∎

5.12. Corollary. *If λ is an S-distribution, then so is $P \cdot \lambda$, for any object P of $\mathscr{B}(S)$.*

Consequently, $\mathscr{B}(S)$ acts in $\underline{S{-}Set}$. If a strict S-set X corresponds to S-distribution λ, then let us write $P \otimes X$ for the strict S-set corresponding to the S-distribution $P \cdot \lambda$: we have

$$P \otimes X \cong P \cdot \lambda(\mathbf{S}) = \lambda(P \times \mathbf{S}) \ .$$

We call $P \otimes X$ the Lawvere tensor product.

If P is interpreted as an étale map $Y \xrightarrow{p} E$, then we denote $P \otimes X$ by $Y \otimes X$. We have

$$Y \otimes X \cong \lambda(S \times_E Y \longrightarrow E) \ , \tag{5}$$

where

$$\begin{tikzcd} S \times_E Y \arrow[r] \arrow[d] & Y \arrow[d, "p"] \\ S \arrow[r, "s \mapsto s^* s"] & E \end{tikzcd}$$

is a pullback of étale maps.

A description of the S-set $Y \otimes X$ may be given that is similar to the one given for the tensor product $T \otimes_S X$. For any étale map $Y \xrightarrow{p} E$, we may construct $Y \otimes X$ as the quotient of

$$\{(y, t, x) \mid p(y) = t^* t, \ tx \text{ is defined}\}$$

by the equivalence relation that identifies $(ys, ts, x) \sim (y, t, sx)$. If $[y, t, x]$ denotes an element of $Y \otimes X$, then the partial (left) action by S is given by

$$s[y, t, x] = [y, st, x] \ ,$$

provided $t = s^*st$, from which it follows that $(st)x = s(tx)$ is defined.

5.13. Example. Let 1 denote the terminal S-set, which corresponds to the S-distribution π_0. Then by (5) we have

$$\mathbf{S} \otimes 1 \cong \pi_0(\mathbf{S} \times \mathbf{S}) = \pi_0(S \times_E S \longrightarrow E) \cong S/\sigma \ .$$

In other words, the S-sets $\mathbf{S} \otimes 1$ and S/σ are isomorphic.

5.14. Example. Consider again the S-sets S and E, the Wagner-Preston and Munn representations (Eg. 2.2). By (5)

$$\mathbf{S} \otimes S \cong W(\mathbf{S} \times \mathbf{S}) = W(S \times_E S \longrightarrow E) = S \times_E S \ ,$$

and

$$\mathbf{S} \otimes E \cong M(\mathbf{S} \times \mathbf{S}) = M(S \times_E S \longrightarrow E) = O_M(S \times_E S) \ .$$

A Munn-orbit of $S \times_E S$ is given by relating a pair (r, t), such that $r^*r = t^*t$, with $(r, t)s = (rs, ts)$ whenever $ss^* = r^*r$ $(= t^*t)$. Let $[r, t]$ denote the Munn-orbit of (r, t). We have the following commutative triangle of S-set morphisms.

$$\begin{tikzcd}
S \times_E S \arrow[r, "(r,t)\mapsto tr^*"] \arrow[d, "(r,t)\mapsto[r,t]"'] & S \\
O_M(S \times_E S) \arrow[ur, "[r,t]\mapsto tr^*"'] &
\end{tikzcd}$$

The morphism $(r, t) \mapsto tr^*$ factors through the S-set of Munn-orbits by a (well-defined) isomorphism of S-sets. This shows that the Lawvere tensor of the Schützenberger object and the Munn S-set equals the Wagner-Preston S-set: $\mathbf{S} \otimes E \cong S$. The same equation in terms of S-distributions,

$$\mathbf{S} \cdot M \cong W \ ,$$

or even its 'integral' form

$$\int \mathbf{S} \times P \, dM = \int P \, dW \ ,$$

may appeal to the reader.

6. The classifying topos of an arbitrary semigroup

In this section, we define a topos $\mathscr{B}(T)$ associated with an arbitrary semigroup T. We make no assumptions on T, although if T is inverse, then the topos $\mathscr{B}(T)$ obtained is equivalent to the usual one. Instead of defining first a category $L(T)$ and then taking for $\mathscr{B}(T)$ the category of presheaves on that category, what seems like a reasonable and viable alternative is to define $\mathscr{B}(T)$ as the topos classifier of T-torsors. In any case, while it is unclear to us that the "presheaves on $L(T)$" approach does not degenerate (because a general semigroup may not have 'enough' idempotents), or that $\mathscr{B}(T)$ should even be a presheaf topos in general, we also cannot be sure that reasonable and viable generalizations of $L(T)$, or of the inductive groupoid, do not exist.

Semigroup torsors. Let $M(X)$ denote the set of partial maps $X \rightharpoonup X$. $M(X)$ is an ordered semigroup (not inverse). More generally, if X is an object of a topos, then let $M(X) = \widetilde{X}^X$, where $\widetilde{X}$ denotes the classifier of partial maps into X.

In the case of a general semigroup, we must upgrade Def. 2.1 by replacing the inverse semigroup $I(X)$ with the ordered semigroup $M(X)$: a T-set (X, μ) of T is thus a semigroup prehomomorphism

$$\mu : T \longrightarrow M(X) \ ; \ \mu(s)(x) = sx \ .$$

If T is inverse, then a T-set $T \longrightarrow M(X)$ necessarily factors through $I(X) \subseteq M(X)$.

We may now observe that Defs. 2.8 and 2.14 make sense for an arbitrary semigroup, not just inverse ones. This gives us a category $\mathrm{TOR}(T)$ of torsors and equivariant maps for T.

Construction of $\mathscr{B}(T)$. The construction fo $\mathscr{B}(T)$ begins with a simpler classifying topos, namely the topos classifier of semigroups, which we shall denote $\mathscr{S}$. It can be constructed as the topos of functors on the category of finitely presented semigroups. The generic semigroup in $\mathscr{S}$ is the underlying set functor, which we denote $\mathbf{R}$. A sketch approach for semigroups is also known [1].

Consider next the topos classifier $\mathscr{T}$ of pairs (S, X), where S is a semigroup and X is an S-torsor. Its existence can be explained using the syntactic site associated with a geometric theory. This theory has two sorts X and S, a binary associative operation symbol on S, and also a relation symbol

$$R \subseteq S \times X \times X$$

that is functional (but not total) in the first two arguments:

$$\forall s, x, y, z. R(s, x, y) \wedge R(s, x, z) \ \Rightarrow \ y = z \ ,$$

and well-supported

$$\forall x \exists s, y. R(s, x, y) \ .$$

We require that R is (strictly) associative:

$$\forall s, t, x, y. R(st, x, y) \Rightarrow \exists z. R(t, x, z) \wedge R(s, z, y) \ ,$$

$$\forall s,t,x,y \exists z. R(t,x,z) \wedge R(s,z,y) \Rightarrow R(st,x,y) \; .$$

We require that X is non-empty:

$$\exists x . x = x \; ,$$

that the action is transitive:

$$\forall x, y \exists s, t, z. R(s,z,x) \wedge R(t,z,y) \; ,$$

and locally free:

$$\forall s,t,x,y. R(s,x,y) \wedge R(t,x,y) \Rightarrow \exists r, z. (R(r,z,x) \wedge sr = tr) \; .$$

If $(\mathbf{S}, \mathbf{X})$ denotes the generic semigroup-torsor pair of $\mathscr{T}$, then since $\mathscr{S}$ (together with $\mathbf{R}$) classifies semigroups there is a geometric morphism $\gamma : \mathscr{T} \longrightarrow \mathscr{S}$ corresponding to $\mathbf{S}$, where $\gamma^*(\mathbf{R}) = \mathbf{S}$. If T is a semigroup in *Set*, with corresponding point $Set \xrightarrow{p} \mathscr{S}$, so that $p^*(\mathbf{R}) = T$, then the topos pullback of γ and p classifies T-torsors.

$$\begin{tikzcd}
\mathscr{B}(T) \arrow[r, "\rho"] \arrow[d, "\Delta \dashv \Gamma"'] & \mathscr{T} \arrow[d, "\gamma"] \\
Set \arrow[r, "p"] & \mathscr{S}
\end{tikzcd}$$

Moreover, we have

$$\rho^*(\mathbf{S}) = \rho^* \gamma^* (\mathbf{R}) \cong \Delta p^*(\mathbf{R}) = \Delta T \; ,$$

and the generic ΔT-torsor in $\mathscr{B}(T)$ is $\rho^*(\mathbf{X})$. We call $\mathscr{B}(T)$ the classifying topos of T, just as in the inverse case. We have thus proved the following.

6.1. THEOREM. *An arbitrary semigroup T has a topos, denoted $\mathscr{B}(T)$, which classifies T-torsors. If T is inverse, then $\mathscr{B}(T)$ is equivalent to the usual classifying topos of T.*

We must admit that outside the inverse case we know very little about $\mathscr{B}(T)$. A closer examination of γ may be revealing.

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Ezra Getzler, Northwestern University: `getzler(at)northwestern(dot)edu`
Martin Hyland, University of Cambridge: `M.Hyland@dpmms.cam.ac.uk`
P. T. Johnstone, University of Cambridge: `ptj@dpmms.cam.ac.uk`
Anders Kock, University of Aarhus: `kock@imf.au.dk`
Stephen Lack, University of Western Sydney: `s.lack@uws.edu.au`
F. William Lawvere, State University of New York at Buffalo: `wlawvere@acsu.buffalo.edu`
Tom Leinster, University of Glasgow, `T.Leinster@maths.gla.ac.uk`
Jean-Louis Loday, Université de Strasbourg: `loday@math.u-strasbg.fr`
Ieke Moerdijk, University of Utrecht: `moerdijk@math.uu.nl`
Susan Niefield, Union College: `niefiels@union.edu`
Robert Paré, Dalhousie University: `pare@mathstat.dal.ca`
Jiri Rosicky, Masaryk University: `rosicky@math.muni.cz`
Brooke Shipley, University of Illinois at Chicago: `bshipley@math.uic.edu`
James Stasheff, University of North Carolina: `jds@math.unc.edu`
Ross Street, Macquarie University: `street@math.mq.edu.au`
Walter Tholen, York University: `tholen@mathstat.yorku.ca`
Myles Tierney, Rutgers University: `tierney@math.rutgers.edu`
Robert F. C. Walters, University of Insubria: `robert.walters@uninsubria.it`
R. J. Wood, Dalhousie University: `rjwood@mathstat.dal.ca`