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ON MODIFIED REEDY AND MODIFIED PROJECTIVE MODEL STRUCTURES

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ABSTRACT. Variations on the notions of Reedy model structures and projective model structures on categories of diagrams in a model category are introduced. These allow one to choose only a subset of the entries when defining weak equivalences, or to use different model categories at different entries of the diagrams. As a result, a bisimplicial model category that can be used to recover the algebraic K-theory for any Waldhausen subcategory of a model category is produced.

1. Introduction

Reedy model structures form the primary means of building model structures on categories of diagrams by imposing additional conditions on the indexing category rather than on the target model category. The traditional goal is to build a model structure where weak equivalences of diagrams are defined by the collection of evaluation functors, such as the projective or standard Reedy structure. The current article provides a method for constructing modified Reedy structures (Theorem 3.7), and modified projective structures (Proposition 6.4), where weak equivalences are defined by only a subset of the evaluation functors. For additional flexibility, one can also consider different model structures at various points in the diagram in both of these results.

After constructing and studying modified Reedy structures, modified projective structures are introduced in order to extend the well-known Quillen equivalence between the standard Reedy and projective structures (Theorem 6.6). As one might expect, the homotopy theory of these modified structures is determined by that of the diagrams indexed on the full subcategory associated to the chosen subset of objects (Proposition 6.8). As the technical conditions for the existence of these model structures on diagrams are different, one suggestion would be viewing them as different means of producing a model for the intended homotopy theory. By choosing appropriate subsets of objects, a variety of (co)localizations of the standard Reedy structure are produced in these left (or right) modified Reedy structures, which again is somewhat surprising because there are no technical restrictions on the target model category. Among other things, two (sometimes three) model structures are given on the simplicial objects $s\mathfrak{M}$ which are each Quillen

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equivalent to $\mathfrak{M}$ itself (Corollary 6.10). Thus the localization of the standard Reedy structure on simplicial objects in $\mathfrak{M}$ considered in [RSS] is Quillen equivalent to, but in general different from, the localization of that same standard Reedy structure constructed here (Remark 4.4 and 6.11), which is a bit surprising. If one considers the Strøm structure [Str] on topological spaces, the model structures on the category of simplicial spaces constructed in this way seem to be new, with nice connections to classical homotopy theory.

Various technical properties of all of these structures are also considered. Among the more obvious of these are the inheritance of various standard conditions, such as being proper, simplicial, or in a very special case cofibrantly generated. It is also shown that strong Quillen pairs, or Quillen equivalences, on the target model category prolong to the same in these structures. In addition, some quite technical refinements allow a cleaner presentation of some of the standard arguments. A purely categorical observation (Lemma 7.5), that for categories with zero object the simplicial structure maps in a categorical nerve all come in adjoint pairs (s_i, d_{i+1}) and (d_i, s_i) for $0 \leq i \leq n-1$ along with new pairs $(\overline{s}_{-1}, d_0)$ and $(d_n, \overline{s}_n)$, is another such technical improvement that seems not to be well-known.

One motivation for this work is the ability to enrich Thomason's approach to Waldhausen's algebraic K-theory construction for a model category. The result (Theorem 7.6) is a bisimplicial object in model categories (so every structure map is a strong left and right Quillen functor) such that applying an 'evaluation functor' for a small full subcategory produces the bisimplicial set for the algebraic K-theory of any Waldhausen subcategory (or even small subcategory of cofibrant objects). This fits in with the approach of [DM], where the additional structure from an enrichment provided a formal approach to the trace map, among other things. One long term hope here would be to understand more of the machinery of algebraic K-theory within the broader context of model categories, a question to be pursued in future joint work with Wojciech Dorabiała. Given the large number of people using model categories right now, it also seems likely that simplifying and generalizing two of the primary approaches to constructing model category structures would lead to improved technical situations in related areas, such as higher categories, as well.

From another viewpoint, there is a reasonable amount of newly found freedom from the ability to consider different model structures at different entries in a diagram, although certain relationships between such structures are necessary at different points in this article. For example, given a model structure on $\mathfrak{M}$, the current techniques produce four distinct model structures on $\mathfrak{M}(\to)$, two of which are Quillen equivalent to the original $\mathfrak{M}$. If $\mathfrak{M}$ also has a (Bousfield) localization $\mathfrak{M}_f$, this leads to an additional five distinct model structures on $\mathfrak{M}(\to)$, four as above starting from $\mathfrak{M}_f$ and the final one, here called a mixed structure, seems to be completely new. This mixed structure relates well to considering just $\mathfrak{M}$ at the source, or $\mathfrak{M}_f$ at the target, so could be used to study localizations in a very structural way. For example, the localization map $X \to L_f X$ is a fibrant replacement in this mixed structure on $\mathfrak{M}(\to)$ for the identity map of any fibrant $X \in \mathfrak{M}$. This and another related example of commutative squares are discussed in more detail in the brief final section.

Several other authors have recently considered extensions of Reedy's original theory, notably [BM], [Ang], and [Bar], in addition to two complete accounts of Reedy structures in [Hir] and [GJ]. It might be interesting to consider how to construct modified versions of the newer results, presumably guided by Proposition 6.8.

1.1. Organization. The point of Section 2 is introducing the standard definitions for Reedy techniques and one new one related to our choice of a subset of the objects. Section 3 then provides the construction of left modified Reedy structures, and the various inheritance properties of these are established in Section 4. As expected from the standard case, it is the 'entrywise' or 'internal' simplicial structure (even when $\mathcal{C} = \Delta^{op}$) which inherits compatibility with the model structures, while the 'external' structure constructed by Quillen for simplicial objects usually does not. For anyone wishing to work with right modified structures, precise definitions and statements are given (without proof) in Section 5. Section 6 is devoted to the existence and properties of modified projective structures and various Quillen equivalence results. Section 7 details how the current theory relates to Waldhausen's algebraic K-theory machine and Thomason's variation thereof. Finally, Section 8 provides a bit more detail about arrow categories and a related discussion of square diagrams, and provides an indication of how to generalize the usual localization square of classical homotopy theory to general model categories.

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2. The Reedy Structure

This section is intended to introduce mostly standard definitions, together with one new condition related to the choice of a full subcategory. First is a bit of motivation for the ideas behind Reedy indexing categories.

Suppose $i : \mathcal{D} \to \mathcal{C}$ is a functor between small categories, and $\mathfrak{M}$ is any category with all small (co)limits. Then i^* induces a (precomposition) functor $\mathfrak{M}^{\mathcal{C}} \to \mathfrak{M}^{\mathcal{D}}$, which has both a left adjoint $\mathbb{L}_i$ and a right adjoint $\mathbb{R}_i$ by the Kan extension formulae. In particular, there are units of adjunction $\mathbb{L}_i i^* X \to X$ and $X \to \mathbb{R}_i i^* X$ for each $X : \mathcal{C} \to \mathfrak{M}$.

The following may seem overly specialized, but instances of both types will be created for each object in a Reedy category. When i is the inclusion of a full subcategory missing only the final object δ in the directed $\mathcal{C}$, the unit of adjunction $\mathbb{L}_i i^* X \to X$ is the identity other than at the final object, where it is the key entry $\operatorname{colim}_{\mathcal{D}} i^* X \to X_\delta$. Similarly, when i is the inclusion of a full subcategory of the directed $\mathcal{C}$ missing only the initial object, $X \to \mathbb{R}_i i^* X$ is the identity other than at the initial object δ, where it is the key entry $X_\delta \to \lim_{\mathcal{D}} i^* X$. Thus, in these two particular instances the units of adjunction instead degenerate to single maps in $\mathfrak{M}$.

Next is the definition of a Reedy indexing category, which is more general than a directed category but still allows a certain form of induction as described below.

2.1. Definition. *A Reedy category is a small category $\mathcal{C}$ together with a whole number valued degree function on objects, and two subcategories, each containing all objects, $\mathcal{C}^+$ and $\mathcal{C}^-$ such that each non-identity morphism of $\mathcal{C}^+$ (resp. $\mathcal{C}^-$) raises (resp. lowers) degree and each morphism in $\mathcal{C}$ has a unique factorization $f = gp$ where $p \in \mathcal{C}^-$ and $g \in \mathcal{C}^+$.*

An important bit of notation is that $F^n\mathcal{C}$ indicates the full subcategory of $\mathcal{C}$ whose objects have degree less than or equal to n. Notice this will always inherit a Reedy category structure from $\mathcal{C}$ itself, since the indicated factorizations pass through an object of degree lower than that of the source or target.

2.2. Latching and Matching Constructions. One can now define certain subcategories of a Reedy category, which should be thought of as allowing attention to focus at a certain object, often acting as if it were the final object of a directed category.

2.3. Definition. *Given $\alpha \in \mathcal{C}$, define the latching category at α, or Latch(α), as the full subcategory of the restricted overcategory $\mathcal{C}^+/\alpha$ (so objects are maps $\beta \to \alpha$ in $\mathcal{C}^+$ with commutative triangles as morphisms) which does not contain 1_α.*

Notice that the restricted overcategory $\mathcal{C}^+/\alpha$ is directed and has 1_α as final object, and there is an obvious inclusion $i_\alpha : \mathrm{Latch}(\alpha) \to \mathcal{C}^+/\alpha$. There is an obvious functor $\mathcal{C}^+/\alpha \to \mathcal{C}$ given by sending $\beta \to \alpha$ in $\mathcal{C}^+$ to β, so any functor $\mathcal{C} \to \mathfrak{M}$ can be 'restricted to $\mathcal{C}^+/\alpha$' by precomposing with this forgetful functor.

In particular, given $X : \mathcal{C} \to \mathfrak{M}$, we restrict X to $\mathcal{C}^+/\alpha$ and then look at the key entry (in $\mathfrak{M}$) of the unit of adjunction associated to the functor i_α (as above). We will denote this $L_\alpha(X) \to X_\alpha$ and call it the absolute latching map, so the absolute latching space $L_\alpha(X) \approx \operatorname{colim}_{\mathrm{Latch}(\alpha)} i_\alpha^* X$.

Given a morphism $f : X \to Y$ of such diagrams in $\mathfrak{M}$, naturality of units of adjunction yields a commutative square

$$\begin{tikzcd} L_\alpha(X) \arrow[r] \arrow[d, "L_\alpha(f)"'] & X_\alpha \arrow[d, "f_\alpha"] \\ L_\alpha(Y) \arrow[r] & Y_\alpha. \end{tikzcd} \tag{1}$$

Next one extends this to a larger commutative diagram

$$\begin{tikzcd} L_\alpha(X) \arrow[r] \arrow[d, "L_\alpha(f)"'] & X_\alpha \arrow[d, "\delta_\alpha(f)"'] \arrow[ddr, "f_\alpha"] & \\ L_\alpha(Y) \arrow[r] \arrow[drr, bend right] & \mathcal{L}_\alpha(f) \arrow[dr, dotted, "\eta_\alpha(f)"'] & \\ & & Y_\alpha \end{tikzcd} \tag{2}$$

where $\mathcal{L}_\alpha(f)$ denotes the pushout of the upper left portion, so the universal property yields the dotted arrow and $\delta_\alpha(f)$ is part of a factorization of f_α as indicated. Here $\mathcal{L}_\alpha(f)$ is the (relative) latching object, and $\eta_\alpha(f)$ is the (relative) latching map (as distinct from the absolute latching object and absolute latching map introduced for a single object above).

The primary reason for this structure is to provide an inductive framework for constructing lifts, which are now defined.

2.4. DEFINITION.

- *Given a (solid) commutative square,*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dotted, "k"] & Y \end{tikzcd} \tag{3}$$

a (dotted) diagonal k making both triangles commute is called a lift.

- *One says (f, p) have the lifting property if there exists a lift in every (solid) square diagram of this form. Note this is definitely a property of the ordered pair.*

Assume for the moment that $\mathfrak{M}$ is a model category. One uses the latching constructions to define cofibrations of diagrams by the (relative) latching maps $\eta_\alpha(f)$ being cofibrations. A key step will be the ability to induct along degree to show $\delta_\alpha(f)$ is then a cofibration, so f_α will also be a cofibration in $\mathfrak{M}$. One can also verify lifting properties between f and p in terms of comparing $\eta_\alpha(f)$ and a dual construction outlined below, $\mu_\alpha(p)$. (See Proposition 2.7.)

2.5. EXAMPLES. If $\mathcal{C} = \{0 \to 1\}$ and $\mathfrak{M}$ is a model category, then $\mathfrak{M}^{\mathcal{C}}$ is the arrow category of $\mathfrak{M}$ and a map of arrows $f : X \to Y$ may be viewed as a (distorted and decorated) commutative square.

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d, "f_0"'] & X_1 \arrow[d, "\delta_1(f)"'] \arrow[ddr, "f_1"] & \\ Y_0 \arrow[r] \arrow[drr, bend right] & \mathcal{L}_1(f) \arrow[dr, dotted, "\eta_1(f)"'] & \\ & & Y_1 \end{tikzcd} \tag{4}$$

Consider an entrywise acyclic fibration of arrows, $p : W \to Z$ with p_0 and p_1 both acyclic fibrations in $\mathfrak{M}$, which fits into a (solid) lifting square

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & W \arrow[d, "p"] \\ Y \arrow[r] \arrow[ur, dotted, "k"] & Z. \end{tikzcd} \tag{5}$$

It should be clear that a lift k_0 of f_0 against p_0 and a lift k_1 of f_1 against p_1 may not be sufficiently compatible to define a morphism in the category of arrows, so a lift of f against p. However a (dotted) lift k_0 in the diagram

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d, "f_0"'] & W_0 \arrow[d, "p_0"] \\ Y_0 \arrow[r] \arrow[ur, dotted, "k_0"] & Z_0 \end{tikzcd} \tag{6}$$

does induce a (solid) commutative square in $\mathfrak{M}$

$$\begin{tikzcd} \mathcal{L}_1(f) \arrow[r] \arrow[d, "\eta_1(f)"'] & W_1 \arrow[d, "p_1"] \\ Y_1 \arrow[r] \arrow[ur, dotted, "k_1"] & Z_1 \end{tikzcd} \tag{7}$$

and a (dotted) lift in this second diagram includes precisely the required compatibility with k_0 in order to define a morphism of arrows $k : Y \to W$ which would be a lift of f against p. Thus, it makes more sense to require $\eta_1(f)$ to have some lifting property than to consider only such a property for f_1. In this example the dual notions of matching objects do not occur since $\mathcal{C}$ is already directed (upward).

Now the dual notions of matching constructions are more briefly introduced.

2.6. Definition. *Given $\alpha \in \mathcal{C}$, define the matching category at α, or Match(α), as the full subcategory of the restricted undercategory $\alpha\backslash\mathcal{C}^-$ (so objects are maps $\alpha \to \beta$ in $\mathcal{C}^-$ while commutative triangles are morphisms) which does not contain (the initial object) 1_α.*

Given $X : \mathcal{C} \to \mathfrak{M}$, and $\alpha \in \mathcal{C}$ associated to the inclusion functor $\mathrm{Match}(\alpha) \to \alpha\backslash\mathcal{C}^-$ one has the key entry of the unit of adjunction $X_\alpha \to M_\alpha(X)$ which will be called the absolute matching map, with target the absolute matching object. Given a map $f : X \to Y$ of diagrams, one has the following commutative diagram in $\mathfrak{M}$

$$\begin{tikzcd} X_\alpha \arrow[dr, dotted, "\mu_\alpha(f)"] \arrow[drr, bend left] \arrow[ddr, bend right, "f_\alpha"'] & & \\ & \mathcal{M}_\alpha(f) \arrow[r] \arrow[d, "\sigma_\alpha(f)"] & M_\alpha(X) \arrow[d, "M_\alpha(f)"] \\ & Y_\alpha \arrow[r] & M_\alpha(Y) \end{tikzcd} \tag{8}$$

where $\mathcal{M}_\alpha(f)$ is the pullback of the lower right portion, and the universal property induces the dotted arrow.

Be sure to notice $f_\alpha = \sigma_\alpha(f)\,\mu_\alpha(f)$, just as $f_\alpha = \eta_\alpha(f)\,\delta_\alpha(f)$ earlier.

Next is a convenient presentation of the inductive process, which is slightly more flexible than the standard statements. The added flexibility is what is needed for the current generalizations.

2.7. Proposition. *Suppose f and p are morphisms in $\mathfrak{M}^{\mathcal{C}}$. If $(\eta_\alpha(f), \mu_\alpha(p))$ have the lifting property in $\mathfrak{M}$ for each $\alpha \in \mathcal{C}$, then (f, p) have the lifting property.*

Another important technical result verified by this sort of induction is the following, which will help when characterizing the class of acyclic (co)fibrations in the structures defined below. A complete proof is provided in [Hir, Lemma 15.3.9], although the statement there initially looks a bit different.

2.8. Lemma. *Suppose f is a morphism in $\mathfrak{M}^{\mathcal{C}}$ and g is a morphism in $\mathfrak{M}$. If $(\eta_\beta(f), g)$ has the lifting property whenever $\beta \in Latch(\alpha)$, then $(L_\alpha(f), g)$ has the lifting property. Dually, if $(g, \mu_\beta(f))$ has the lifting property whenever $\beta \in Match(\alpha)$, then $(g, M_\alpha(f))$ has the lifting property.*

2.9. Acceptable Subcategories. Next is the new condition, related to the requirement of choosing a set of objects, or equivalently a full subcategory, in these constructions. Choosing all objects, or equivalently the whole indexing category, recovers the traditional Reedy structure in this context.

2.10. Definition. *Suppose $\mathcal{C}$ is a Reedy category and $\mathfrak{M}$ has all small (co)limits.*

- *The full subcategory $\mathcal{C}_0 \subset \mathcal{C}$ will be called **left acceptable** provided it inherits a Reedy category structure such that the matching objects relative to $\mathcal{C}_0$ and those relative to $\mathcal{C}$ are naturally isomorphic at any object $\alpha \in \mathcal{C}_0$.*

- *The full subcategory $\mathcal{C}_0 \subset \mathcal{C}$ will be called **right acceptable** provided it inherits a Reedy category structure such that the latching objects relative to $\mathcal{C}_0$ and those relative to $\mathcal{C}$ are naturally isomorphic at any object $\alpha \in \mathcal{C}_0$.*

- *The term **acceptable** will apply when $\mathcal{C}_0 \subset \mathcal{C}$ is both left and right acceptable.*

2.11. Examples.

1. It is clear from the definitions that $\mathcal{C}_0 = F^n\mathcal{C}$ is acceptable for each n.

2. Given any object β of degree zero, $\mathcal{C}_0 = \{\beta\}$ is acceptable. In fact, these are generally the only singletons which can be (left or right) acceptable as the latching object for this $\mathcal{C}_0$ would always be an initial object, so unlikely to agree with the latching object with respect to all of $\mathcal{C}$ unless it has degree zero. Similarly, the matching objects with respect to $\mathcal{C}_0$ would always be a final object, so unlikely to agree with the matching object with respect to all of $\mathcal{C}$ unless it has degree zero.

3. To illustrate the general principle of Lemma 2.12 below, let $\mathcal{C}_{m,n}$ denote a 'grid-like' directed category (or $[n] \times [m]$)

$$
\begin{tikzcd}
00 \arrow[r] \arrow[d] & 01 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0n \arrow[d] \\
10 \arrow[r] \arrow[d] & 11 \arrow[r] \arrow[d] & \cdots \arrow[r] & 1n \arrow[d] \\
\vdots & \vdots & \vdots & \vdots \\
m0 \arrow[r] & m1 \arrow[r] & \cdots \arrow[r] & mn
\end{tikzcd}
\tag{9}
$$

As one example, the degree function could be chosen to be the sum of the indices, with $\mathcal{C}^+ = \mathcal{C}$ and $\mathcal{C}^-$ discrete. Taking $\mathcal{C}_0$ to be the first row and first column is then left acceptable by Lemma 2.12. In fact, this example will yield model categories closely related to Waldhausen's algebraic K-theory functor in Section 7.

One can say $\mathcal{C}$ is monotone increasing if there is a degree function where no morphisms decrease degree, or equivalently the decreasing category is discrete. The notion of monotone decreasing is dual.

2.12. LEMMA. *Any full subcategory of a monotone increasing Reedy category is left acceptable. Dually any full subcategory of a monotone decreasing Reedy category is right acceptable.*

PROOF. Whenever the decreasing category is discrete, or equivalently $\mathcal{C}$ is monotone increasing, the matching objects are all limits of empty diagrams, hence the final object. As a consequence, one has $M_\alpha(f)$ an isomorphism (identity of the final object) hence its base change $\sigma_\alpha(f)$ is an isomorphism between $\mu_\alpha(f)$ and f_α. Since the same is true for matching constructions relative to the subcategory, the matching condition for being left acceptable is then satisfied. Choosing any degree function for the whole category, it descends to make the monotone increasing subcategory $\mathcal{C}_0$ a sub-Reedy category as well, and the dual case is similar. ∎

3. Constructing Left Modified Reedy Structures

Here is the construction of the left modified Reedy model category structure for an appropriate choice of diagram category, with detailed definitions and statements for the dual right modified structures included in Section 5. The input throughout this section is a Reedy category $\mathcal{C}$, together with a left acceptable full subcategory $\mathcal{C}_0$ and an $\mathrm{ob}(\mathcal{C})$-indexed collection of model category structures $\mathfrak{M}_?$ on a fixed category $\mathfrak{M}$ which satisfy a compatibility condition as follows. Of course, $Cof(\mathfrak{M})$ indicates the class of cofibrations in $\mathfrak{M}$ and similarly $Fib(\mathfrak{M})$ indicates the class of fibrations.

3.1. Definition. [Left Compatibility Condition] *Suppose $\alpha \in \mathcal{C}$ with $\beta \in Latch(\alpha)$ and $\gamma \in Match(\alpha)$.*

1. $Cof(\mathfrak{M}_\beta) \subset Cof(\mathfrak{M}_\alpha)$
2. $Fib(\mathfrak{M}_\alpha) \subset Fib(\mathfrak{M}_\beta)$
3. $Fib(\mathfrak{M}_\gamma) \subset Fib(\mathfrak{M}_\alpha)$
4. *If both $\alpha, \gamma \in \mathcal{C}_0$ one has $Cof(\mathfrak{M}_\alpha) \subset Cof(\mathfrak{M}_\gamma)$.*

Keep in mind that objects in latching or matching categories have smaller degrees than the indexing object, so for example the first portion says the class of cofibrations is increasing in the degree as one moves along any chain of morphisms. At first glance, it appears the combination of these conditions should be that the model structure remains constant. However, this is far from the case, and there is a variety of interesting examples.

3.2. Examples.

1. Take as $\mathfrak{M}_?$ a fixed model structure on $\mathfrak{M}$ (regardless of the value of ? in ob($\mathcal{C}$)). If, in addition, one chooses $\mathcal{C}_0 = \mathcal{C}$, this section will yield the original Reedy structure on $\mathfrak{M}^{\mathcal{C}}$ (with respect to this structure on $\mathfrak{M}$). All other choices for $\mathcal{C}_0$ will yield colocalizations of the original Reedy structure when the family of model structures on $\mathfrak{M}$ is constant.

2. Suppose $\mathcal{C}$ is monotone increasing and $\mathfrak{M}$ is a left proper, cellular model category. Choose as $\mathfrak{M}_?$ various (left Bousfield) localizations of this model structure in such a way as to localize more and more as the degree of the objects increases along any chain of maps. Then the class of cofibrations remains that of the original $\mathfrak{M}$, the class of fibrations decreases as we localize so as to make more cofibrations acyclic, and the matching categories are all empty so the two conditions related to them are vacuously satisfied. Hence, this family will satisfy the left compatibility condition.

3. As a special case of (2), consider $\mathcal{C}$ a commutative square
$$\begin{tikzcd} 00 \arrow[r] \arrow[d] & 01 \arrow[d] \\ 10 \arrow[r] & 11 \end{tikzcd} \tag{10}$$
with degree function given by the sum of the indices (monotone increasing so $\mathcal{C}^+ = \mathcal{C}$ and $\mathcal{C}^-$ discrete). Then choose an appropriate model category $\mathfrak{M}_{00}$ and two localizations for $\mathfrak{M}_{01}$ and $\mathfrak{M}_{10}$. Now think of forming the combined localization if possible, inverting any cofibration inverted under either of the initial localizations, and allow this to be $\mathfrak{M}_{11}$. Varying the choice of $\mathcal{C}_0$, the model structures constructed here will be readily comparable to the original or any of the indicated localizations (see section 8 below).

Now the definition of left modified Reedy structure can be made in terms of this input.

3.3. Definition. *Given a Reedy category $\mathcal{C}$, a left acceptable subcategory $\mathcal{C}_0 \subset \mathcal{C}$ and an* $\mathrm{ob}(\mathcal{C})$*-indexed family of model structures $\mathfrak{M}_?$ on $\mathfrak{M}$ satisfying the left compatibility condition, the **left modified Reedy structure** on $\mathfrak{M}^{\mathcal{C}}$, or **Left**$(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}})$, consists of the classes of:*

- *weak equivalences, defined as those morphisms f where f_α is a weak equivalence in $\mathfrak{M}_\alpha$ whenever $\alpha \in \mathcal{C}_0$;*
- *fibrations, defined as those morphisms f where $\mu_\alpha(f)$ is a fibration in $\mathfrak{M}_\alpha$ for each α; and*
- *cofibrations, defined as those morphisms f where $\eta_\alpha(f)$ is a cofibration in $\mathfrak{M}_\alpha$ for each α, which must also be acyclic whenever $\alpha \notin \mathcal{C}_0$.*

3.4. Remark. The fibrations in $\mathbf{Left}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}})$ are precisely those of the standard Reedy structure (when $\mathfrak{M}_?$ is constant), while the cofibrations (which appear on the left in lifting diagrams) have been modified, hence the terminology.

Notice there are now two alternative possible formulations of acyclic cofibrations, which must coincide if the result is to be a model category structure. Dually, a flexible characterization of acyclic fibrations is also necessary, which is actually the only point where the left acceptable condition is required. Since the next two results are standard for the ordinary Reedy structure, a proof is included only for the more difficult of them, in order to make clear the dependence upon the various pieces of the compatibility assumption as well as the left acceptable condition.

3.5. Lemma. *Suppose $\mathfrak{M}_?$ satisfies the left compatibility condition (parts 1 and 2). Then the class of cofibrations in **Left**$(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}})$ which are also weak equivalences is characterized by $\eta_\alpha(f)$ an acyclic cofibration in $\mathfrak{M}_\alpha$ for each α. Furthermore, any cofibration f satisfies f_α a cofibration in $\mathfrak{M}_\alpha$ for each α.*

3.6. Lemma. *Suppose $\mathcal{C}_0 \subset \mathcal{C}$ is left acceptable and $\mathfrak{M}_?$ satisfies the left compatibility condition (parts 3 and 4). Then in **Left**$(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}})$ the class of fibrations which are also weak equivalences is characterized by $\mu_\alpha(f)$ a fibration in $\mathfrak{M}_\alpha$ for each α, which must be acyclic if $\alpha \in \mathcal{C}_0$. Furthermore, any fibration p satisfies p_α a fibration in $\mathfrak{M}_\alpha$ for each α.*

Proof. First, suppose p a morphism in $\mathfrak{M}^{\mathcal{C}}$ with each $\mu_\alpha(p)$ a fibration in $\mathfrak{M}_\alpha$. In particular, $\mu_\gamma(f)$ is then a fibration in $\mathfrak{M}_\alpha$ whenever $\gamma \in \mathrm{Match}(\alpha)$ by (3) of the left compatibility condition. Then to see p_α is a fibration in $\mathfrak{M}_\alpha$, it suffices to see $\sigma_\alpha(p)$ is a fibration in $\mathfrak{M}_\alpha$, or by closure under cobase change, that $M_\alpha(p)$ is a cofibration in $\mathfrak{M}_\alpha$. However, this follows from Lemma 2.8 by considering an arbitrary acyclic cofibration f in $\mathfrak{M}_\alpha$, and observing that $(f, \mu_\gamma(p))$ has the lifting property.

If p has each $\mu_\alpha(p)$ a fibration in $\mathfrak{M}_\alpha$ that must also be acyclic when $\alpha \in \mathcal{C}_0$, then suppose $\gamma \in \mathrm{Match}(\alpha)$ with $\alpha, \gamma \in \mathcal{C}_0$. By (4) of the left compatibility condition, $f \in Cof(\mathfrak{M}_\alpha)$ implies $(f, \mu_\gamma(p))$ has the lifting property. Due to the left acceptable condition, it would be equivalent to consider lifting against the matching map formed in the subcategory $\mu_\gamma(p)^{\mathcal{C}_0}$, so by applying Lemma 2.8 with respect to the smaller indexing category $\mathcal{C}_0$ it follows that $(f, M_\alpha^{\mathcal{C}_0}(p))$ has the lifting property. Again using the left acceptable condition, one concludes $(f, M_\alpha(p))$ also has the lifting property. Since f was arbitrary, it follows that $M_\alpha(p)$, and as a consequence its base change $\sigma_\alpha(p)$, is an acyclic fibration in $\mathfrak{M}_\alpha$.

Finally, suppose p is a fibration as well as a weak equivalence. Then by definition each $\mu_\alpha(p)$ is a fibration, which must be acyclic whenever $\alpha \notin \mathcal{C}_0$ and each p_α with $\alpha \in \mathcal{C}_0$ is a weak equivalence in $\mathfrak{M}_\alpha$. Proceed by induction on the degree of α to verify that $\mu_\alpha(p)$ is an acyclic fibration in $\mathfrak{M}_\alpha$ even when $\alpha \in \mathcal{C}_0$. If $\alpha \in \mathcal{C}_0$ with $|\alpha| = 0$, then $\mu_\alpha(p) \approx p_\alpha$ so the claim follows from the assumption that p is a weak equivalence. Now assume $\mu_\gamma(p)$ is an acyclic fibration in $\mathfrak{M}_\gamma$ whenever $\gamma \in \mathrm{Match}(\alpha)$, so as in the previous paragraph $M_\alpha(p)$, hence also its base change $\sigma_\alpha(p)$, is an acyclic fibration in $\mathfrak{M}_\alpha$ (using Lemma 2.8 and (4) of the left compatibility condition). Then the decomposition $p_\alpha = \sigma_\alpha(p) \circ \mu_\alpha(p)$ and the 2 of 3 property for weak equivalences in $\mathfrak{M}_\alpha$ implies $\mu_\alpha(p)$ is a weak equivalence in $\mathfrak{M}_\alpha$ as well. ∎

Next is the existence theorem for the left modified Reedy structure. With all of the technical details handled already, the proof is now relatively short.

3.7. THEOREM. *Suppose $\mathcal{C}_0$ is a left acceptable full subcategory of $\mathcal{C}$, and $\mathfrak{M}_?$ is a family of model structures satisfying the left compatibility condition. Then $\boldsymbol{Left}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}})$ is a Quillen model category.*

PROOF. The existence of (co)limits is well-known in this case, as they are built 'entrywise' (e.g. [MacL, Cor. to V.3.1]). The 2 of 3 property and closure of each class under retracts is a consequence of the definitions, the fact that latching and matching constructions preserve retracts, and the same property in each $\mathfrak{M}_\alpha$.

Suppose f is an acyclic cofibration and p is a fibration. By Lemma 3.5 and the definition of fibration, each $(\eta_\alpha(f), \mu_\alpha(p))$ has the lifting property, so by Proposition 2.7 (f, p) also has the lifting property. If instead f is a cofibration and p is an acyclic fibration, $(\eta_\alpha(f), \mu_\alpha(p))$ still has the lifting property for each α by Lemma 3.6 and the definition of cofibration, since $\eta_\alpha(f)$ is acyclic if $\alpha \notin \mathcal{C}_0$, whereas $\mu_\alpha(p)$ is acyclic if $\alpha \in \mathcal{C}_0$. As a consequence, Proposition 2.7 still implies (f, p) has the lifting property.

Suppose $g : X \to Y$ is an arbitrary morphism of $\mathfrak{M}^{\mathcal{C}}$, and inductively produce a factorization $g = pf$ with p an acyclic fibration and f a cofibration.

Since $F^0\mathcal{C}$ is discrete, if $\alpha \in \mathcal{C}_0$ one simply chooses an appropriate factorization of each g_α, as a cofibration $f_\alpha : X_\alpha \to Z_\alpha$ followed by an acyclic fibration $p_\alpha : Z_\alpha \to Y_\alpha$ in $\mathfrak{M}_\alpha$, or if instead $\alpha \notin \mathcal{C}_0$ with f_α an acyclic cofibration and p_α a fibration in $\mathfrak{M}_\alpha$.

Now suppose a factorization as $F^{n-1}\mathcal{C}$-indexed diagrams has been chosen. Given α of degree n, if $\alpha \in \mathcal{C}_0$, factor the induced map $\mathcal{L}_\alpha(f) \to \mathcal{M}_\alpha(p)$ as a cofibration $\eta_\alpha(f)$ followed by an acyclic fibration $\mu_\alpha(p)$ in $\mathfrak{M}_\alpha$, or if instead $\alpha \notin \mathcal{C}_0$ with $\eta_\alpha(f)$ an acyclic cofibration followed by a fibration $\mu_\alpha(p)$. As for the standard Reedy induction argument, these choices suffice to define a factorization $X \to Z \to Y$ as $F^n\mathcal{C}$-indexed diagrams, completing the induction step for producing a factorization. Notice f so constructed is a cofibration by definition, while Lemma 3.6 implies p is an acyclic fibration.

For the other factorization, one factors in each instance as an acyclic cofibration followed by a fibration, and refers to Lemma 3.5 rather than 3.6. ∎

4. Inheritance Properties of Left Modified Reedy Structures

The point of this section is to indicate that many of the commonly used conditions in model categories are inherited under this construction. The property of being cofibrantly generated is surprisingly technical, so might have been avoided other than for its potential usefulness in algebraic K-theory applications. In order to avoid an Eilenberg swindle forcing algebraic K-theory to vanish, one needs to impose some sort of finiteness condition, which can sometimes be phrased nicely using cofibrant generation (see [Sag]). Thus, a special case sufficient for these applications is included near the end of this section, but cofibrant generation is not discussed for the right modified Reedy structures at all.

The remaining conditions are relatively straightforward, so are handled first.

4.1. Inheriting Properness, Being Simplicial, and Quillen Pairs. The three conditions which are inherited without undue difficulty are properness, compatibility with the 'internal' simplicial structure, and the existence of strong Quillen pairs or even further strong Quillen equivalences. Properness can be split into two pieces, and either piece will be inherited, but only the combined statement is given here.

4.2. Lemma. *If each model category $\mathfrak{M}_\alpha$ is proper, then $\boldsymbol{Left}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}})$ is proper.*

Proof. Lemma 3.5 implies a cofibration f has each f_α a cofibration in $\mathfrak{M}_\alpha$, and pushouts are defined entrywise in $\mathfrak{M}^{\mathcal{C}}$, so left properness follows. Right properness is dual. ∎

Another property one would like to inherit would be compatibility with a simplicial structure. Here we use what Goerss-Jardine [GJ, just above VII.2.13] call the "internal" structure, sometimes known as the "entrywise" structure, which differs from the "external" structure [GJ, II.2.5 and above] used by Quillen in the special case where $\mathcal{C} = \Delta^{op}$. For the sake of clarity, the following is to remind the reader how the relevant operations are defined, assuming a fixed simplicial structure $(\otimes_{\mathfrak{M}}, \hom_{\mathfrak{M}})$ on $\mathfrak{M}$ has already been chosen. Given X, $Y \in \mathfrak{M}^{\mathcal{C}}$ and $K \in \mathcal{S}$, define $X \otimes K \in \mathfrak{M}^{\mathcal{C}}$ by $(X \otimes K)_\alpha = X_\alpha \otimes_{\mathfrak{M}} K$ and similarly $\hom(K, Y) \in \mathfrak{M}^{\mathcal{C}}$ by $\hom(K, Y)_\alpha = \hom_{\mathfrak{M}}(K, Y_\alpha)$, while $\mathrm{map}(X, Y) \in \mathcal{S}$ is defined by $\mathrm{map}(X, Y)_n = \mathfrak{M}^{\mathcal{C}}(X \otimes \Delta[n], Y)$. Then the triple adjunction relationship is expressed by the following natural isomorphisms of simplicial sets

$$\mathcal{S}(K, \mathrm{map}(X, Y)) \approx \mathrm{map}(X \otimes K, Y) \approx \mathrm{map}(X, \hom(K, Y)).$$

4.3. Proposition. *If $\mathfrak{M}$ has a simplicial structure in which each $\mathfrak{M}_\alpha$ is a simplicial model category, then the "internal" simplicial structure described above makes any left modified Reedy structure (which exists) into a simplicial model category.*

Proof. Suppose $f : X \to Y$ is a left modified Reedy cofibration in $\mathfrak{M}^{\mathcal{C}}$ and $j : K \to L$ is a cofibration in $\mathcal{S}$. One must show the induced map in $\mathfrak{M}^{\mathcal{C}}$

$$X \otimes L \coprod_{X \otimes K} Y \otimes K \to Y \otimes L$$

is a left modified Reedy cofibration, which is acyclic provided either f or j is acyclic. Evaluating at an object α of degree zero, notice every construction is entrywise, so one simply has the pushout-product in $\mathfrak{M}_\alpha$ of f_α and j. Thus, $\mathfrak{M}_\alpha$ simplicial implies the result is a cofibration which is acyclic provided either f_α or j is acyclic.

Now, consider an α of non-zero degree. Then the (relative) latching map under consideration

$$\left[(X \otimes L \coprod_{X \otimes K} Y \otimes K)_\alpha \coprod_{L_\alpha(X \otimes L \coprod_{X \otimes K} Y \otimes K)} L_\alpha(Y \otimes L) \right] \to (Y \otimes L)_\alpha$$

is isomorphic to the pushout-product in $\mathfrak{M}$ of $\eta_\alpha(f)$ and the map j

$$\left[(Y_\alpha \otimes K) \coprod_{(L_\alpha Y \coprod_{L_\alpha X} X_\alpha) \otimes K} ((L_\alpha Y \coprod_{L_\alpha X} X_\alpha) \otimes L) \right] \to Y_\alpha \otimes L$$

by compatibility of colimits (see [Hir, end of 15.3.16]). Since each $\mathfrak{M}_\alpha$ is a simplicial model category, the result is thus a cofibration in $\mathfrak{M}_\alpha$ which is acyclic whenever either j or $\eta_\alpha(f)$ is acyclic. Now the claim follows from the definition of cofibration and Lemma 3.5. ■

4.4. Remark. In modified Reedy structures, the external simplicial structure will rarely be fully compatible. The standard reason given is that tensoring with the acyclic cofibration of simplicial sets given by 'the lowest' $d^0 : \Delta[0] \to \Delta[1]$ would require that for every cofibrant $Z \in \mathfrak{M}^{\mathcal{C}}$ and $n \in \mathcal{C}_0$

$$Z_n \approx (Z \otimes \Delta[0])_n \to (Z \otimes \Delta[1])_n \approx \coprod_{\Delta[1]_n} Z_n$$

is a weak equivalence in $\mathfrak{M}_n$, which should rarely hold. This property is important, as it distinguishes $\mathbf{Right}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}})$ with $\mathcal{C} = \Delta^{op}$ and $\mathcal{C}_0 = [0]$ from another localization of the standard Reedy structure considered by [RSS], even though they are (indirectly) Quillen equivalent (see Remark 6.11).

The next inheritance question considered involves prolonging strong Quillen pairs and Quillen equivalences from the target model category. The interesting point here is that one only needs the Quillen equivalence condition at entries of the subcategory $\mathcal{C}_0$ in order to deduce a prolonged Quillen equivalence for left modified structures.

4.5. Proposition. *Suppose (F, G) forms an adjoint pair between $\mathfrak{M}$ and $\mathcal{N}$, such that they become a strong Quillen pair between the model categories $\mathfrak{M}_\alpha$ and $\mathcal{N}_\alpha$ for each $\alpha \in \mathcal{C}$. Then their prolongations (F_*, G_*) induce a strong Quillen pair between $\textbf{\textit{Left}}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}})$ and $\textbf{\textit{Left}}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}}_{\mathcal{N}})$ (if both exist). If, in addition, (F, G) is a strong Quillen equivalence for each $\alpha \in \mathcal{C}_0$, then (F_*, G_*) is a strong Quillen equivalence as well.*

Proof. The left adjoint F must preserve colimits, cofibrations, and acyclic cofibrations as a strong left Quillen functor. In this case, the prolongation F_* will commute with latching constructions and so preserve cofibrations by definition and acyclic cofibrations by Lemma 3.5.

For the Quillen equivalence condition, suppose X is cofibrant in $\mathbf{Left}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}})$ and Y is fibrant in $\mathbf{Left}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}}_{\mathcal{N}})$. Then for each $\alpha \in \mathcal{C}_0$, X_α is cofibrant in $\mathfrak{M}_\alpha$ by Lemma 3.5 and Y_α is fibrant in $\mathcal{N}_\alpha$ by Lemma 3.6. Now the claim follows from the definition of weak equivalence and the assumption of (F, G) a Quillen equivalence for each $\alpha \in \mathcal{C}_0$. ∎

4.6. A Special Case of Inheriting Cofibrant Generation. The final property whose inheritance is considered is being cofibrantly generated, which is clearly *not* a self-dual condition by its nature. In fact, it becomes quite technical to pursue this condition in general, which will be avoided here, so only the minimum necessary for potential applications to algebraic K-theory will be handled in this subsection. As a consequence, the focus will be on the case of a single cofibrantly generated model category structure on $\mathfrak{M}$, with $\mathcal{C}$ monotone increasing, throughout this subsection. As is customary, I will denote a set of generating cofibrations, and J a set of generating acyclic cofibrations in $\mathfrak{M}$, but it will also be convenient to make the (often satisfied but) non-standard additional assumption that $J \subset I$. This is really just a way of hiding the technical assumption that the domains of J are small with respect to the subcategory of I-cofibrations, as noted near the end of the proof of Proposition 4.7.

Notice each evaluation functor $ev_\alpha : \mathfrak{M}^{\mathcal{C}} \to \mathfrak{M}$ has a left adjoint, defined by

$$(\mathbf{F}^\alpha X)_\beta = \begin{cases} \sqcup_{\alpha \to \beta} X, & \text{if } \mathcal{C}(\alpha, \beta) \neq \emptyset; \\ \emptyset, & \text{otherwise.} \end{cases}$$

with the relevant initial map or summand identity for structure maps. Similarly, there are also right adjoints to evaluations defined as either products of the given object "before" the chosen entry, or the final object otherwise. Both are instances of the usual Kan extension formula [MacL, X.3].

4.7. Proposition. *Suppose $\mathcal{C}$ is monotone increasing and $\mathfrak{M}$ is a cofibrantly generated model category, with $\mathcal{C}_0$ any full subcategory of $\mathcal{C}$ and $J \subset I$. Then $\textbf{\textit{Left}}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}})$ is a cofibrantly generated model category with*

$$I_L = \cup_{\alpha \in \mathcal{C}_0} \mathbf{F}^\alpha(I) \bigcup \cup_{\alpha \notin \mathcal{C}_0} \mathbf{F}^\alpha(J) \text{ and } J_L = \cup_{\alpha \in \mathcal{C}} \mathbf{F}^\alpha(J)$$

as set of generating (acyclic) cofibrations.

Proof. First, notice this model structure exists by Theorem 3.7, since Lemma 2.12 says $\mathcal{C}_0$ is left acceptable. The monotone increasing assumption also implies $ev_\alpha \approx \mu_\alpha\,(?)$ by $M_\alpha\,(?)$ constant on the final object. Thus, if these sets admit the small object argument, their role as generating (acyclic) cofibrations is essentially a restatement of Lemma 3.6 and the definition of fibrations via the $(\mathbf{F}^\alpha, ev_\alpha)$ adjunction.

To see these sets permit the small object argument, notice that $\mathbf{F}^\beta C$ is small with respect to a set of maps S in $\mathfrak{M}^{\mathcal{C}}$ provided C is small in $\mathfrak{M}$ with respect to $ev_\beta S$. As a consequence, J_L will permit the small object argument provided $ev_\beta(J_L)$ consists of acyclic cofibrations in $\mathfrak{M}$, since the domains of the maps in J are small with respect to the whole class of acyclic cofibrations in $\mathfrak{M}$ by [Hir, 10.5.27] (rather than just with respect to the relative cell complexes built using J). However, $ev_\beta \mathbf{F}^\alpha j$ is either the identity of the initial object, or else a coproduct of copies of j. Hence, $ev_\beta \mathbf{F}^\alpha j$ is an acyclic cofibration in $\mathfrak{M}$ whenever $j \in J$, which suffices.

The argument for I_L allowing the small object argument is similar, but complicated by the fact that domains of J need not be small with respect to I-cofibrations in general. However, this follows from the stronger assumption that $J \subset I$ (and I allows the small object argument). ■

4.8. Remark. In the special case of $\mathcal{C} = \Delta^{op}$, one of the two initial cases of interest in Bousfield-Kan [BK] and then Reedy [Ree], a different left adjoint to matching objects also allows one to give an explicit set of generating (acyclic) cofibrations.

5. Statements for Right Modified Reedy Structures

Rather than trying to state each definition and result to this point (with the exception of Proposition 4.7) in two parts, this section serves to include clear statements for anyone working with right modified structures.

5.1. Definition. [Right Compatibility Condition] *Suppose $\alpha \in \mathcal{C}$ with $\beta \in Latch(\alpha)$ and $\gamma \in Match(\alpha)$.*

1. $Fib(\mathfrak{M}_\gamma) \subset Fib(\mathfrak{M}_\alpha)$
2. $Cof(\mathfrak{M}_\alpha) \subset Cof(\mathfrak{M}_\gamma)$
3. $Cof(\mathfrak{M}_\beta) \subset Cof(\mathfrak{M}_\alpha)$
4. *If both $\alpha, \beta \in \mathcal{C}_0$ one has $Fib(\mathfrak{M}_\alpha) \subset Fib(\mathfrak{M}_\beta)$.*

The analog of Example 3.2(2) in this context is as follows. Suppose one chooses a monotone decreasing $\mathcal{C}$ and successively colocalize, or take right Bousfield localizations, as the degree increases. Then the class of fibrations remains fixed and the class of cofibrations gradually shrinks as more fibrations are made acyclic, while the latching categories are all empty. Thus, the Right Compatibility Condition would be satisfied in this case.

A commutative square remains a special case of potential interest, comparing two colocalizations, but the common target would have degree zero, the intermediate objects have degree one, and the common source have degree two. Thus, $\mathcal{C}^- = \mathcal{C}$ and $\mathcal{C}^+$ is discrete, with any choice of $\mathcal{C}_0$ right acceptable by Lemma 2.12.

5.2. Definition. *Given a Reedy category $\mathcal{C}$, a right acceptable subcategory $\mathcal{C}_0 \subset \mathcal{C}$ and an* $\mathrm{ob}(\mathcal{C})$*-indexed family of model structures $\mathfrak{M}_?$ on $\mathfrak{M}$ satisfying the right compatibility condition, the* ***right modified Reedy structure*** *on $\mathfrak{M}^{\mathcal{C}}$, or* ***Right***$(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}})$*, consists of the classes of:*

- *weak equivalences, defined as those morphisms f where f_α is a weak equivalence in $\mathfrak{M}_\alpha$ whenever $\alpha \in \mathcal{C}_0$;*
- *cofibrations, defined as those morphisms f where $\eta_\alpha(f)$ is a cofibration in $\mathfrak{M}_\alpha$ for each α; and*
- *fibrations, defined as those morphisms f where $\mu_\alpha(f)$ is a fibration in $\mathfrak{M}_\alpha$ for each α, which must also be acyclic whenever $\alpha \notin \mathcal{C}_0$.*

5.3. Lemma. *Suppose $\mathfrak{M}_?$ satisfies the right compatibility condition (parts 1 and 2). Then the class of fibrations in* ***Right***$(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}})$ *which are also weak equivalences is characterized by $\mu_\alpha(p)$ an acyclic fibration in $\mathfrak{M}_\alpha$ for each α. Furthermore, any fibration p satisfies p_α a fibration in $\mathfrak{M}_\alpha$ for each α.*

In the right modified case, identifying the acyclic cofibrations is where the right acceptable condition is necessary.

5.4. Lemma. *Suppose $\mathcal{C}_0 \subset \mathcal{C}$ is right acceptable and $\mathfrak{M}_?$ satisfies the right compatibility condition (parts 3 and 4). Then in* ***Right***$(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}})$ *the class of cofibrations which are also weak equivalences is characterized by $\eta_\alpha(f)$ a cofibration in $\mathfrak{M}_\alpha$ for each α, which must be acyclic if $\alpha \in \mathcal{C}_0$. Furthermore, any cofibration f satisfies f_α a cofibration in $\mathfrak{M}_\alpha$ for each α.*

5.5. Theorem. *Suppose $\mathcal{C}_0$ is a right acceptable full subcategory of $\mathcal{C}$, and $\mathfrak{M}_?$ is a family of model structures satisfying the right compatibility condition. Then* ***Right***$(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}})$ *is a Quillen model category.*

5.6. Lemma. *If each model category $\mathfrak{M}_\alpha$ is proper, then* ***Right***$(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}})$ *is proper.*

5.7. Proposition. *If $\mathfrak{M}$ has a simplicial structure in which each $\mathfrak{M}_\alpha$ is a simplicial model category, then the "internal" simplicial structure described above makes any right modified Reedy structure (which exists) into a simplicial model category.*

Once again, notice the Quillen equivalence assumption for just the entries in the subcategory suffices to produce a Quillen equivalence between right modified structures.

5.8. Proposition. *Suppose* (F, G) *forms an adjoint pair between* $\mathfrak{M}$ *and* $\mathcal{N}$*, such that they become a strong Quillen pair between the model categories* $\mathfrak{M}_\alpha$ *and* $\mathcal{N}_\alpha$ *for each* $\alpha \in \mathcal{C}$*. Then their prolongations* (F_*, G_*) *induce a strong Quillen pair between* $\boldsymbol{Right}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}})$ *and* $\boldsymbol{Right}(\mathcal{C}_0, \mathcal{N}^{\mathcal{C}})$ *(if both exist). If, in addition,* (F, G) *is a strong Quillen equivalence for each* $\alpha \in \mathcal{C}_0$*, then* (F_*, G_*) *is a strong Quillen equivalence as well.*

6. Modified Projective Structures

Another familiar fact is that the standard Reedy structure is Quillen equivalent to the projective (or diagram) model structure on $\mathfrak{M}^{\mathcal{C}}$ when both exist. In order to generalize this fact, one first needs to introduce modified projective structures, after a small technical digression.

6.1. Remark. It is easy to show the intersection of the three distinguished classes in a model category are precisely the isomorphisms, characterized as those f where (f, f) has the lifting property. This is useful to keep in mind when working with various 'trivial' model category structures. For example, together with the lifting properties, it implies weak equivalences in a model category are precisely the isomorphisms if and only if all maps are both cofibrations and fibrations, hence a rigidity result for the most commonly used trivial model structure.

6.2. Definition. *Let* $\mathfrak{M}_\emptyset$ *denote the (co)complete category* $\mathfrak{M}$ *equipped with the following rather trivial model category structure. All maps are both fibrations and weak equivalences, while the cofibrations are simply the isomorphisms. In fact, this is cofibrantly generated with the empty set of generating (acyclic) cofibrations, hence the notation.*

Notice there is also a dual trivial model structure on any (co)complete category, with all maps acyclic cofibrations and with fibrations characterized as the isomorphisms, but it would be naturally fibrantly, rather than cofibrantly, generated.

Next is the existence theorem for modified projective structures. Such structures can be used in various places to provide flexibility in comparing model structures. As one example, in [JY] it is shown that (with a fixed model structure on the target category) a modified projective structure on colored PROPs is Quillen equivalent to the usual projective structure on colored operads, hence the full projective model structure on PROPs is in some sense a refinement of the projective structure on operads. The current definition is more general than is common, in allowing various different model structures rather than a fixed one for $\mathfrak{M}$. The point is to be able to generalize the usual close relationship between Reedy and projective model structures, as well as allowing much more flexibility in studying the homotopy theory of diagrams.

6.3. Definition. *Say a collection of model structures* $\mathfrak{M}_\alpha$ *on a fixed* $\mathfrak{M}$ *has fibrations which decrease along the indexing subcategory* $\mathcal{C}_0$ *if* $\mathcal{C}_0(\beta, \alpha)$ *non-empty implies* $Fib(\mathfrak{M}_\alpha) \subset Fib(\mathfrak{M}_\beta)$*.*

Notice in this case the acyclic cofibrations increase along the subcategory, in other words $AcycCofs(\mathfrak{M}_\beta) \subset AcycCofs(\mathfrak{M}_\alpha)$ if $\mathcal{C}_0(\beta, \alpha)$ non-empty, by the lifting characterization with respect to fibrations. If $\mathcal{C}$ is monotone increasing, this condition is implied by the left compatibility condition, since $\mathcal{C}_0(\beta, \alpha)$ non-empty and $\beta \neq \alpha$ must then imply $\beta \in \mathrm{Latch}(\alpha)$.

The proof below essentially comes from [Hir, 11.6.1], but is included mainly for convenience and to clarify notation. Here I_α will indicate the set of generating cofibrations for $\mathfrak{M}_\alpha$, and similarly with J_α the generating acyclic cofibrations, which conflicts with the notation of [Hir].

6.4. Proposition. *Suppose each $\mathfrak{M}_\alpha$ is a cofibrantly generated model category, and the collection has fibrations which decrease along the indexing subcategory $\mathcal{C}_0$. Then there is a modified projective model structure $\boldsymbol{Proj}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}})$ on $\mathfrak{M}^{\mathcal{C}}$, with fibrations (resp. weak equivalences) defined as those maps sent to fibrations (resp. weak equivalences) by each ev_α with $\alpha \in \mathcal{C}_0$. Furthermore, the sets of generating (acyclic) cofibrations are*

$$I_{\mathcal{C}_0} = \cup_{\alpha \in \mathcal{C}_0} \mathbf{F}^\alpha(I_\alpha) \textit{ and } J_{\mathcal{C}_0} = \cup_{\alpha \in \mathcal{C}_0} \mathbf{F}^\alpha(J_\alpha).$$

Proof. First, notice [Hir, Props. 7.1.7 and 11.1.10]

$$\prod_{\alpha \in \mathrm{Ob}(\mathcal{C}_0)} \mathfrak{M}_\alpha \times \prod_{\alpha \notin \mathrm{Ob}(\mathcal{C}_0)} \mathfrak{M}_\emptyset$$

is itself a cofibrantly generated model category, with generating sets

$$I_1 = \cup_{\alpha \in \mathcal{C}} (I_\alpha \times \prod_{\beta \neq \alpha} 1_\beta) \text{ and } J_1 = \cup_{\alpha \in \mathcal{C}} (J_\alpha \times \prod_{\beta \neq \alpha} 1_\beta)$$

where 1_β is the identity of the initial object of $\mathfrak{M}_\beta$. If $\mathbf{F}$ is the left adjoint to the forgetful functor $\mathbf{U} : \mathfrak{M}^{\mathcal{C}} \to \mathfrak{M}^{\mathcal{C}^{\mathrm{disc}}}$, then the image of the generating cofibrations $\mathbf{F}(I_1) = I_{\mathcal{C}_0}$ since I_α is empty for $\alpha \notin \mathcal{C}_0$ by construction and similarly $J_{\mathcal{C}_0} = \mathbf{F}(J_1)$. Thus, it will suffice to show one can lift this model structure from $\mathfrak{M}^{\mathcal{C}^{\mathrm{disc}}}$ to $\mathfrak{M}^{\mathcal{C}}$ over the adjoint pair $(\mathbf{F}, \mathbf{U})$ to complete the proof.

Now notice these sets allow the small object argument, just as for J_L in the proof of Proposition 4.7. Also, if $\beta \in \mathcal{C}_0$, $ev_\beta(j)$ for $j \in J_{\mathcal{C}_0}$ is an acyclic cofibration in $\mathfrak{M}_\beta$. This follows since $ev_\beta \mathbf{F}^\alpha(j_\alpha) = \coprod_{\mathcal{C}_0(\alpha,\beta)} j_\alpha$ or 1_β. By construction, this is an acyclic cofibration in $\mathfrak{M}_\alpha$, so by the assumption of fibrations decreasing along the subcategory $\mathcal{C}_0$, an acyclic cofibration in $\mathfrak{M}_\beta$. As a consequence, $\mathbf{U}$ takes relative $J_{\mathcal{C}_0}$-cell complexes to weak equivalences, and one can apply [Hir, 11.3.2]. ∎

6.5. Remark. If $\mathcal{C}$ is monotone increasing, the relative matching maps are isomorphic to the entries by triviality of the absolute matching objects (as limits over empty categories). Hence, one has $\mathbf{Left}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}})$ isomorphic (not just equivalent) to $\mathbf{Proj}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}})$, since they have precisely the same fibrations and weak equivalences. This is well-known for the standard Reedy and projective structures, in this language the case $\mathcal{C}_0 = \mathcal{C}$.

Now one has the anticipated comparison result.

6.6. Theorem. *Suppose $\mathfrak{M}_\alpha$ is a collection of model category structures and $\mathcal{C}$ is a Reedy category equipped with a choice of full subcategory $\mathcal{C}_0$.*

- *If both structures exist, then the identity $1 : \boldsymbol{Right}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}}) \to \boldsymbol{Left}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}})$ is the right half of a strong Quillen equivalence.*
- *If the structures exist and $\beta \in Match(\alpha)$ implies $Fib(\mathfrak{M}_\beta) \subset Fib(\mathfrak{M}_\alpha)$, then the identity $1 : \boldsymbol{Left}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}}) \to \boldsymbol{Proj}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}})$ is the right half of a strong Quillen equivalence, as is $1 : \boldsymbol{Right}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}}) \to \boldsymbol{Proj}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}})$.*

Proof. In each case, the model structures being compared have the same class of weak equivalences, so it suffices to show the identity preserves fibrations when considered as a functor in the appropriate direction. For the first claim, this follows from the definitions and for the second claim this follows from the entrywise fibration portion of Lemma 3.6, which requires only this one part of the compatibility assumption. ∎

6.7. Remark. Keeping in mind that the three structures considered in Theorem 6.6 exist under different technical assumptions, this result might be viewed as providing alternative existence criteria for a convenient model of the common homotopy category. The large number of different model structures which may be constructed by these methods should make this additional flexibility quite useful.

Notice $\mathbf{Proj}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}})$ obviously inherits the right proper condition, since pullbacks, fibrations, and weak equivalences are defined in terms of (certain) entries. Once cofibrations are shown to be preserved by evaluations in $\mathcal{C}_0$ as before by considering the generating cofibrations, $\mathbf{Proj}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}})$ also inherits the left proper condition since pushouts are also defined entrywise. For the "internal" simplicial structure each entry of the pullback-product construction for diagrams is isomorphic to the pullback-product construction for that entry (see [Hir, 11.7.3]) and so $\mathbf{Proj}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}})$ will also be simplicial when each $\mathfrak{M}_\alpha$ is a simplicial model category. As for modified Reedy structures, the "external" simplicial structure will rarely be fully compatible with modified projective structures.

Next is the rather appealing fact that $\mathfrak{M}^{\mathcal{C}_0}$ really determines the homotopy theory of $\mathbf{Left}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}})$. Similar results hold for $\mathbf{Right}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}})$ and $\mathbf{Proj}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}})$ as well, although forgetful functors are normally not strong left Quillen functors for modified projective structures. In some sense, the proposition says $\mathbf{Left}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}})$ is essentially just lifting the standard Reedy structure from $\mathfrak{M}^{\mathcal{C}_0}$, without any of the technical conditions on the target model category normally associated with lifting techniques.

6.8. Proposition. *If $\mathcal{C}_0$ is left acceptable, then the forgetful functor $U : \boldsymbol{Left}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}})$ to $\boldsymbol{Left}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}_0})$ (the standard Reedy structure on the smaller diagram category) is the right half of a strong Quillen equivalence. If $\mathcal{C}_0$ is acceptable, then U is also the left half of a strong Quillen equivalence*

PROOF. Recall the left Kan extension formula gives a left adjoint L to the forgetful functor $U : \mathfrak{M}^{\mathcal{C}} \to \mathfrak{M}^{\mathcal{C}_0}$. The left acceptable condition says that the forgetful functor preserves fibrations in this context, and it preserves (in fact it reflects) weak equivalences (between fibrant objects) by definition. Thus, it is a strong Quillen pair which, by [Hov, Cor.1.3.16], is a Quillen equivalence provided the derived unit $X \to UQLX$ is a weak equivalence for each cofibrant object $X \in \mathbf{Left}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}_0})$, where Q indicates a fibrant replacement. Since U preserves all weak equivalences, it is enough to instead consider the unit of adjunction $X \to ULX$, which is an isomorphism by $\mathcal{C}_0$ a full subcategory (see [MacL, Cor. X.3.3]).

If, in addition, $\mathcal{C}_0$ is right acceptable, then the forgetful functor also preserves cofibrations by construction, so the dual argument applies. ∎

Next is an observation about a special case, which allows one to recover the homotopy theory of the original category within the context of a diagram category in many instances. This should be particularly useful in combination with choosing appropriate (co)localizations for the different $\mathfrak{M}_\alpha$, or for simplicial objects over a model category which is not cofibrantly generated, such as the Strøm structure [Str] on topological spaces.

6.9. REMARK. Suppose $\mathcal{C}_0$ consists of a singleton β which taken alone is left (resp. right) acceptable and has no non-trivial endomorphisms, e.g. where $|\beta| = 0$ as in Example 2.11(2). Then $(\mathbf{F}^\beta, ev_\beta)$ yields a strong Quillen equivalence between $\mathfrak{M}_\beta$ and $\mathbf{Left}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}})$ (resp. $\mathbf{Right}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}})$ or $\mathbf{Proj}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}})$).

One application is related to the construction of [RSS], the current result being somewhat more general, but much weaker by missing the key property for their application. Keep in mind that $F^0\mathcal{C}$ is always an acceptable subcategory.

6.10. COROLLARY. *For any model category $\mathfrak{M}$, there are two different model structures, $\boldsymbol{Left}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}})$ and $\boldsymbol{Right}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}})$ with $\mathcal{C}_0 = [0]$, on the simplicial objects $\mathfrak{M}^{\Delta^{op}}$ for which $(const, ev_0)$ and (ev_0, R_0) each form a strong Quillen equivalence with $\mathfrak{M}$. If, in addition, $\mathfrak{M}$ is cofibrantly generated, then a third is given by $\boldsymbol{Proj}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}})$.*

It may be helpful to recall the right adjoint R_0 to ev_0 can be written explicitly as a power object related to the cosimplicial set $\Delta([0], ?)$. Here each degeneracy is built from a product of diagonals and identities, while the face map d_i comes from projection to those factors whose indices lie in the image of d^i.

6.11. REMARK. While the structures considered here are compatible with the internal simplicial structure whenever $\mathfrak{M}$ itself is simplicial, the different localization of the Reedy structure considered by [RSS] is always simplicial in Quillen's 'external' structure. Hence, their structure provides a simplicial model category Quillen equivalent to $\mathfrak{M}$ whenever it exists, even if $\mathfrak{M}$ itself were not simplicial. Thus $\mathbf{Right}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}})$ and the localization of the Reedy structure considered by [RSS] may provide the first interesting example of two localizations of the same model structure (the standard Reedy structure on simplicial objects) which are (indirectly) strongly Quillen equivalent but rarely coincide (as $\mathbf{Right}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}})$ is not simplicial in the 'external' structure in most cases as discussed in Remark 4.4).

7. An Enrichment of Algebraic K-theory for a pointed model category

One complication for this section is the problem of deciding how to define 'finiteness' so as to avoid the Eilenberg swindle, or to keep the notion of algebraic K-theory non-trivial. Working with the cofibrant, homotopy finite objects, following [Sag], requires some hypotheses to be sure one is working with a Waldhausen category. The advantage is that the structure is uniquely associated to the model category. Working with a choice of complete Waldhausen subcategory, following [DS1], is quite flexible, but a canonical choice is only given for stable model categories. Here the latter will be pursued, although the reader is warned that the resulting algebraic K-theory space could, at this point, depend on more choices than just the underlying pointed model category.

7.1. Dugger-Shipley Approach to Finiteness. First, a brief review of the relevant details from [DS1]. Given a subcategory $\mathcal{U}$ of $\mathfrak{M}$, let $\overline{\mathcal{U}}$ denote the full subcategory of $\mathfrak{M}$ consisting of cofibrant objects weakly equivalent to objects of $\mathcal{U}$. If all objects of $\mathcal{U}$ are cofibrant, call $\mathcal{U}$ *complete* if $\mathcal{U} = \overline{\mathcal{U}}$. A *Waldhausen subcategory* of $\mathfrak{M}$ will denote a pointed (i.e. including the zero object) full subcategory $\mathcal{U}$ of cofibrant objects which is closed under homotopy pushouts, in the sense that the pushout P formed in $\mathfrak{M}$ of

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d] \\ C \arrow[r] & P \end{tikzcd} \tag{11}$$

lies in $\mathcal{U}$ provided $A, B, C \in \mathcal{U}$ and at least one of f or g is a cofibration.

It is then shown in [DS1] that for a strong Quillen equivalence $L : \mathfrak{M} \leftrightarrow \mathcal{N} : R$ and a complete Waldhausen subcategory $\mathcal{U}$ of $\mathfrak{M}$, $\overline{L\,\mathcal{U}}$ is a complete Waldhausen subcategory of $\mathcal{N}$. In fact, they go on to show the induced map is an isomorphism on algebraic K-theory, and that result could be recovered as an application of the theory which follows.

The following slight extension of [DS1, Prop. 3.6] is implicit in [DS1, Rem. 3.7 & Lemma A.1].

7.2. Proposition. *Given Quillen equivalent model categories $\mathfrak{M}$ and $\mathcal{N}$, together with a complete Waldhausen subcategory $\mathcal{U} \subset \mathfrak{M}$, there is an 'image' complete Waldhausen subcategory $\mathcal{V} \subset \mathcal{N}$ and weakly equivalent algebraic K-theory spaces $K(\mathcal{U})$ and $K(\mathcal{V})$.*

Once again, if the model category $\mathfrak{M}$ is stable, there is a canonical choice of complete Waldhausen subcategory, coming from the compact objects (defined with respect to the triangulated homotopy category of $\mathfrak{M}$). If both $\mathfrak{M}$ and $\mathcal{N}$ are stable, with $\mathcal{U}$ the compact objects of $\mathfrak{M}$ and $L : \mathfrak{M} \to \mathcal{N}$, then $\overline{L\,\mathcal{U}}$ agrees with the compact objects of $\mathcal{N}$ by [DS1, Cor. 3.9], so any syzygy of Quillen equivalences will be compatible with the canonical choice of compact objects. However, without the assumption of stability, the situation remains less clear.

7.3. The $T_\bullet$ bisimplicial model category. Throughout this subsection, the categories under consideration will be $\mathbf{Left}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}_{n,m}})$ with $\mathcal{C}_{n,m}$ as in Example 2.11(3) and $\mathcal{C}_0$ chosen to be the combination of the first row and first column. The idea is to use Thomason's variant of Waldhausen's construction to build a bisimplicial model category from which one recovers the algebraic K-theory of $\mathcal{U}$ by choosing the set (at least after an appropriately large choice of universe) of cofibrant objects in $\mathbf{Left}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}_{n,m}})$ with entries in $\mathcal{U}$. This choice of universe business is reasonably convenient now, since no significant cardinality arguments were involved in developing $\mathbf{Left}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}_{n,m}})$. One may object that the assumption that model categories contain all small (co)limits forbids us from changing the universe here, but within this one section it is instead convenient to expand to Quillen's original assumption, that only finite (co)limits are necessary.

When working with Waldhausen's $S_\bullet$ construction, it is technically important that the simplicial face and degeneracy maps are all exact functors. Recall that a left adjoint is an exact functor precisely when it preserves the distinguished classes of cofibrations and of weak equivalences. Thus, the natural generalization of this condition to a bisimplicial model category would be that each face and degeneracy map is a strong left Quillen functor. In fact, even more is true here, as each face and degeneracy map is also a strong right Quillen functor. As a consequence, the entire bisimplicial structure descends to the level of homotopy categories, all of the bisimplicial structure maps preserve arbitrary weak equivalences as well as all homotopy (co)limits, and restricting to strong left (or right) Quillen functors as maps still yields a bisimplicial model category. Notice this ability is technically vital here, since the stated goal is to recover a bisimplicial set by restricting to objects in $\mathbf{Left}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}_{n,m}})$ with entries in $\mathcal{U}$ that are in addition cofibrant. There would be no reason to expect cofibrancy to be preserved by all of these face and degeneracy maps without something akin to the fact that the structure maps are all strong left Quillen functors.

First, the required construction of 'extra degeneracy functors' for a categorical nerve.

7.4. Definition. *Suppose $\mathfrak{M}$ is a category with both initial ($\emptyset$) and final ($*$) objects. Then in the categorical nerve with $\mathbb{N}_n(\mathfrak{M}) = Fun([n], \mathfrak{M})$ there are two additional functors $\overline{s}_{-1}, \overline{s}_n : N_{n-1}(\mathfrak{M}) \to N_n(\mathfrak{M})$ given by*

$$\overline{s}_{-1}(X_0 \to X_1 \to \cdots \to X_{n-1}) = \emptyset \to X_0 \to X_1 \to \cdots \to X_{n-1} \text{ and}$$
$$\overline{s}_n(X_0 \to X_1 \to \cdots \to X_{n-1}) = X_0 \to X_1 \to \cdots \to X_{n-1} \to *$$

Now, the following purely categorical observation, that does not seem to be well-known, establishes the requisite underlying adjunctions.

7.5. Lemma. *If $\mathfrak{M}$ is a category with both initial and final objects, then the simplicial category $\mathbb{N}_n(\mathfrak{M})$ (or categorical nerve) has the property that for $0 \le i \le n-1$ both (d_i, s_i) and (s_i, d_{i+1}), in addition to $(\overline{s}_{-1}, d_0)$ and $(d_n, \overline{s}_n)$, form adjoint pairs as indicated. Furthermore, the only new entries introduced by any of these functors are the initial and final objects.*

Proof. First recall that d_i is defined by removing the object i, through composition if i is neither 0 nor n. In the same manner, s_i for $0 \leq i \leq n-1$ is defined by inserting the identity on the i-th object. In all cases, it is straightforward to verify the adjoint property directly. For example, to see that (d_i, s_i) and (s_i, d_{i+1}) for $0 \leq i \leq n-1$ are adjoint pairs between $\mathbb{N}_n(\mathfrak{M})$ and $\mathbb{N}_{n-1}(\mathfrak{M})$, observe that the existence of a commutative diagram of the form

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d] & \cdots \arrow[r] & X_{i-1} \arrow[r] \arrow[d] & X_i \arrow[r] \arrow[d] & X_{i+1} \arrow[r] \arrow[d] & X_{i+2} \arrow[r] \arrow[d] & \cdots \arrow[r] & X_n \arrow[d] \\
Y_0 \arrow[r] \arrow[d] & \cdots \arrow[r] & Y_{i-1} \arrow[r] \arrow[d] & Y_i \arrow[r, "="] \arrow[d] & Y_i \arrow[r] \arrow[d] & Y_{i+1} \arrow[r] \arrow[d] & \cdots \arrow[r] & Y_{n-1} \arrow[d] \\
Z_0 \arrow[r] & \cdots \arrow[r] & Z_{i-1} \arrow[r] & Z_i \arrow[r] & Z_{i+1} \arrow[r] & Z_{i+2} \arrow[r] & \cdots \arrow[r] & Z_n
\end{tikzcd} \tag{12}$$

where the second row corresponds to $s_i(Y)$, is equivalent to the existence of a commutative subdiagram

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d] & \cdots \arrow[r] & X_{i-1} \arrow[r] \arrow[d] & X_{i+1} \arrow[r] \arrow[d] & X_{i+2} \arrow[r] \arrow[d] & \cdots \arrow[r] & X_n \arrow[d] \\
Y_0 \arrow[r] \arrow[d] & \cdots \arrow[r] & Y_{i-1} \arrow[r] \arrow[d] & Y_i \arrow[r] \arrow[d] & Y_{i+1} \arrow[r] \arrow[d] & \cdots \arrow[r] & Y_{n-1} \arrow[d] \\
Z_0 \arrow[r] & \cdots \arrow[r] & Z_{i-1} \arrow[r] & Z_i \arrow[r] & Z_{i+2} \arrow[r] & \cdots \arrow[r] & Z_n
\end{tikzcd} \tag{13}$$

where the top row corresponds to $d_i(X)$ and the bottom row corresponds to $d_{i+1}(Z)$.

To see that $(d_n, \overline{s}_n)$ forms an adjoint pair, observe the existence of a commutative diagram of the form

$$\begin{tikzcd}
Y_0 \arrow[r] \arrow[d] & Y_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & Y_{n-2} \arrow[r] \arrow[d] & Y_{n-1} \arrow[r] \arrow[d] & Y_n \arrow[d] \\
Z_0 \arrow[r] & Z_1 \arrow[r] & \cdots \arrow[r] & Z_{n-2} \arrow[r] & Z_{n-1} \arrow[r] & *
\end{tikzcd} \tag{14}$$

where the bottom row corresponds to $\overline{s}_n(Z)$, is equivalent to the existence of a commutative subdiagram

$$\begin{tikzcd}
Y_0 \arrow[r] \arrow[d] & Y_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & Y_{n-2} \arrow[r] \arrow[d] & Y_{n-1} \arrow[d] \\
Z_0 \arrow[r] & Z_1 \arrow[r] & \cdots \arrow[r] & Z_{n-2} \arrow[r] & Z_{n-1}
\end{tikzcd} \tag{15}$$

where the top row corresponds to $d_n(Y)$.

Similarly, to see that $(\overline{s}_{-1}, d_0)$ forms an adjoint pair, observe the existence of a commutative diagram of the form

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & Y_0 \arrow[r] \arrow[d] & Y_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & Y_{n-1} \arrow[d] \\ Z_0 \arrow[r] & Z_1 \arrow[r] & Z_2 \arrow[r] & \cdots \arrow[r] & Z_n \end{tikzcd} \tag{16}$$

where the top row corresponds to $\overline{s}_{-1}(Y)$, is equivalent to the existence of a commutative subdiagram

$$\begin{tikzcd} Y_0 \arrow[r] \arrow[d] & Y_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & Y_{n-1} \arrow[d] \\ Z_1 \arrow[r] & Z_2 \arrow[r] & \cdots \arrow[r] & Z_n \end{tikzcd} \tag{17}$$

where the bottom row corresponds to $d_0(Z)$. ■

Given a model category of diagrams $\mathfrak{M}^{\mathcal{C}}$ and a small subcategory $\mathcal{U} \subset \mathfrak{M}$, let $ev_{\mathcal{U}}$ denote the set of diagrams whose entries all lie in $\mathcal{U}$, while $ev_{\mathcal{U}}^{cof}$ is the subset of such diagrams which are also cofibrant in $\mathfrak{M}^{\mathcal{C}}$.

7.6. THEOREM. *For any pointed model category $\mathfrak{M}$, there is a bisimplicial (pointed) model category* $\textbf{\textit{Left}}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}_{n,m}})$ *where for each n (and choice of horizontal or vertical) the structure maps for $0 \leq i \leq n-1$ both (d_i, s_i) and (s_i, d_{i+1}), in addition to $(\overline{s}_{-1}, d_0)$ and $(d_n, \overline{s}_n)$, form strong Quillen pairs as indicated. Furthermore, for any (small) complete Waldhausen subcategory $\mathcal{U}$ inside $\mathfrak{M}$, applying $ev_{\mathcal{U}}^{cof}$ everywhere yields a bisimplicial set which is a model for the algebraic K-theory of $\mathcal{U}$.*

PROOF. First, observe that from the Cartesian closed property for Cat, and the fact that the indexing category $\mathcal{C}_{n,m} \approx [n] \times [m]$, Lemma 7.5 implies each face and degeneracy map of this bisimplicial category in either direction is part of an adjoint pair as indicated in the statement.

To show these are all strong Quillen pairs, first consider the case $(\overline{s}_{-1}, d_0)$. In this case, the latching maps at previously existing objects remain unchanged, since for any object α formerly in $\mathcal{C}_0$ (the first row or column) one has $L_\alpha(X)$ the initial object. As all latching maps at the newly added $\mathcal{C}_0$ objects are identities, it follows that $\overline{s}_{-1}$ preserves (acyclic) cofibrations. Having handled the exceptional case of $(\overline{s}_{-1}, d_0)$, it will now suffice to show each right adjoint other than d_0, but including $\overline{s}_n$, (vertical or horizontal) preserves (acyclic) fibrations.

Notice fibrations are defined entrywise for these monotone increasing indexing categories. Hence, both omitting and repeating entries, or inserting the identity on final objects, will preserve fibrations. Similarly, repeating entries, inserting the identity on final objects, or omitting entries other than the first row or first column will preserve weak equivalences. Since d_0 (omitting the first row or column) is excluded at this point, the result is that each right adjoint currently under consideration is a strong right Quillen functor, so each left adjoint is also a strong left Quillen functor.

The second statement now follows from Proposition 7.10 below, keeping in mind that Lemma 7.5 together with the first statement implies the simplicial structure maps all commute with $ev_{\mathcal{U}}^{cof}$, which thereby yields a bisimplicial set. ■

Notice that each simplicial structure map above must preserve all weak equivalences as both strong left and strong right Quillen functors, since one can factor an arbitrary weak equivalence as an acyclic cofibration followed by an acyclic fibration. It also follows that the derived functors remain adjoints at the level of the homotopy categories, so this structure is fairly rigid.

7.7. Remark. In fact, the construction of a bisimplicial set above works just as well with $\mathcal{U}$ the full subcategory on any set of cofibrant objects, without assuming it is a Waldhausen subcategory. No claims are made here about the properties of such an extension, but it could provide some flexibility in working with algebraic K-theory slightly outside of Waldhausen's original context.

7.8. Models for algebraic K-theory. Now the topic shifts to recovering the relation between the construction above and Thomason's variant of Waldhausen's construction. The complications of dealing with Waldhausen's construction directly are avoided in this way, although they are outlined in Remark 7.11. For a Waldhausen subcategory $\mathcal{U}$ of $\mathfrak{M}$, let $T_n\mathcal{U}$ denote the full subcategory of $\mathbb{N}_n(\mathcal{U})$ whose objects have each $X_i \to X_{i+1}$ a cofibration (in $\mathfrak{M}$), or equivalently, the (standard) Reedy cofibrant objects as defined in $\mathbb{N}_n(\mathfrak{M})$ whose entries all lie in $\mathcal{U}$. Here a morphism of $T_n\mathcal{U}$ is called a cofibration if it is a (standard) Reedy cofibration (considered in $\mathbb{N}_n(\mathfrak{M})$. Then Thomason's notion of weak equivalence w is the class of maps with $\eta_i(f)$ a weak equivalence for each $i > 0$, so his acyclic cofibrations $\overline{w}$ in this case are precisely the cofibrations from $\mathbf{Left}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}})$ where $\mathcal{C} = [n]$ and $\mathcal{C}_0 = [0]$. Just to be clear, $\overline{w}$ consists of natural transformations between functors $[n] \to \mathfrak{M}$ where the zero entry is a cofibration and all higher latching maps are acyclic cofibrations in $\mathfrak{M}$.

One now needs to observe that Waldhausen's proof (using Quillen's Theorem A) applies in this case as well to see $\overline{w}$ is 'big enough' to lead to the algebraic K-theory space.

7.9. Lemma. *There is a homotopy equivalence between the bisimplicial sets $N_*\overline{w}T_\bullet\mathcal{U}$ and $N_*wT_\bullet\mathcal{U}$. Thus, the former also yields a model for the algebraic K-theory space of $\mathcal{U}$.*

Proof. The first statement follows the same proof as for Lemma 1.6.3 of [Wald], using Quillen's Theorem A. The second then follows from the end of section 1.3 of [Wald], where $wT_n\mathcal{U}$ is discussed (although $T_n\mathcal{U}$ is never made explicit). ■

Now one can show the model category approach yields an enrichment of Thomason's approach.

7.10. Proposition. *There is a homotopy equivalence between the two bisimplicial sets $ev_{\mathcal{U}}^{cof}\,\boldsymbol{Left}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}_{*,\bullet}})$ and $N_*wT_\bullet\mathcal{U}$, so the former yields a model for the algebraic K-theory space of $\mathcal{U}$.*

PROOF. In light of Lemma 7.9, it suffices to show $ev_{\mathcal{U}}^{cof}\mathbf{Left}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}_{n,m}})$ is isomorphic, as a bisimplicial set, to $N_*\overline{w}T_\bullet\mathcal{U}$. Since in all cases the bisimplicial structure maps are inherited from $\mathbb{N}_*\mathbb{N}_\bullet\mathfrak{M}$, which contains both, it will suffice to observe that for a fixed (n, m) the subsets coincide.

Consider the diagrams of the following form (all squares indicated, even if distorted, are pushouts used to define the latching objects)

$$\begin{tikzcd}
X_{i0} \arrow[r] \arrow[d] & X_{i1} \arrow[r] \arrow[d] & X_{i2} \arrow[r] \arrow[d] & \cdots \arrow[r] & X_{im} \arrow[d] \\
X_{(i+1)0} \arrow[r] & \mathcal{L}_{(i+1)1} \arrow[d, "\sim"] & \mathcal{L}_{(i+1)2} \arrow[d, "\sim"] & \cdots & \mathcal{L}_{(i+1)m} \arrow[d, "\sim"] \\
 & X_{(i+1)1} \arrow[ur] & X_{(i+1)2} \arrow[ur] & \cdots \arrow[ur] & X_{(i+1)m}
\end{tikzcd} \tag{18}$$

where objects are all in $\mathcal{U}$, all maps are cofibrations, and those labeled with $\sim$ are also weak equivalences. By definition of $\overline{w}$, this set serves for $0 \leq i < n$ as the vertical map $i \to i+1$ of an entry of $N_n\overline{w}T_m\mathcal{U}$. However, keeping in mind that for cofibrant objects in $\mathbf{Left}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}_{n,m}})$ the latching maps are all cofibrations and those whose target is outside the first row and first column are acyclic cofibrations, this same form of diagram serves as any row of a cofibrant object of $\mathbf{Left}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}_{n,m}})$ with entries in $\mathcal{U}$. Hence by induction up to n, the sets in question coincide. ∎

7.11. REMARK. It is also possible to approximate Waldhausen's $S_\bullet$ construction directly, by working with $\mathcal{C}_0$ the top row and shifting $\mathbf{Left}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}_{n,m}})$ to the entry $(n+1, m)$. Of course, this runs into the usual difficulty of how to define d_0^h for $S_\bullet$. In fact, in line with Lemma 7.5, one can view d_0^h as simply a left adjoint to s_0^h (with some additional technical conditions, see [Joh]). As a consequence, Waldhausen's introduction of quotients becomes quite natural, as one can see from the following smaller example in a categorical nerve. The key observation is that a commutative diagram of the form

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d] & X_1 \arrow[r] \arrow[d] & X_2 \arrow[d] \\
* \arrow[r] & Y_0 \arrow[r] & Y_1
\end{tikzcd} \tag{19}$$

where the second row represents Waldhausen's s_0Y, is equivalent to a commutative diagram

$$\begin{tikzcd}
X_1/X_0 \arrow[r] \arrow[d] & X_2/X_0 \arrow[d] \\
Y_0 \arrow[r] & Y_1
\end{tikzcd} \tag{20}$$

so one chooses the top row of this second diagram as $d_0(X)$ to get the left adjoint property. To justify this statement notice that the first commutative square in the top diagram is equivalent to the existence of the first vertical map in the bottom diagram. Then the large commutative rectangle in the top diagram is equivalent to the existence of the second vertical map in the bottom diagram. Finally, the commutativity of the second square in the top diagram is now equivalent to commutativity of the bottom diagram, since both vertical maps in the second square factor through the quotients and the top square in

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d] & X_2 \arrow[d] \\ X_1/X_0 \arrow[r] \arrow[d] & X_2/X_0 \arrow[d] \\ Y_0 \arrow[r] & Y_1 \end{tikzcd} \tag{21}$$

commutes (is even a pushout) by the fact that $X_0 \to X_2$ factors through $X_0 \to X_1$.

Of course, a left adjoint is only unique (some might say only defined) up to natural isomorphism, so this would lead to something not quite a simplicial category, where d_0 is only well-defined up to 'homotopy'. However, Waldhausen's requirement of keeping track of choices of quotients for $S_\bullet$ may be viewed as a special case of an explicit rectification functor, which minimally modifies this construction to produce an actual simplicial object in small categories or even in Waldhausen categories with exact functors as morphisms. This small rectification construction is described concretely using descending sequences of objects together with certain choices of isomorphisms, in an inductive manner. Of course, one could instead appeal to modern technology for rectifying all kinds of pseudo-diagrams, but that would lose both the explicit nature of the construction and the historical context. See [Joh] for complete details.

8. Examples of Applications to Localizations

This brief, informal section is mainly intended to discuss two very simple indexing categories and the many model structures one can produce by the techniques of this article.

8.1. ARROW CATEGORIES. Begin by considering $\mathfrak{M}(\to)$, so $\mathcal{C} = \{0 \to 1\} = [1]$, for an arbitrary model category $\mathfrak{M}$. Notice one has the standard Reedy structure (with $\mathcal{C}_0 = \mathcal{C}$), and $\mathcal{C}$ is monotone increasing so any full subcategory is left acceptable. Thus, one also has $\mathbf{Left}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}})$ with $\mathcal{C}_0$ either 0 or 1. Finally, $F^0\mathcal{C}$ is always acceptable, so one has $\mathbf{Right}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}})$ with $\mathcal{C}_0 = 0$ as well. The last is the only one of these which is unusual in most cases, while both $\mathbf{Left}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}})$ and $\mathbf{Right}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}})$ for $\mathcal{C}_0 = 0$ are Quillen equivalent to the original $\mathfrak{M}$ by Remark 6.9. If $\mathfrak{M}$ happens to be cofibrantly generated, one might be tempted to add to this list the modified projective structures, but they are already here by Remark 6.5.

Changing the choice of $\mathcal{C}_0$ has not been considered until now, but the following observation makes it tractable for certain comparisons.

8.2. Lemma. *If $\mathcal{C}_1 \subset \mathcal{C}_0$, then (when both exist) the identity forms a strong right Quillen functor $\textbf{\textit{Left}}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}}) \to \textbf{\textit{Left}}(\mathcal{C}_1, \mathfrak{M}^{\mathcal{C}})$. Dually, the identity forms a strong left Quillen functor $\textbf{\textit{Right}}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}}) \to \textbf{\textit{Right}}(\mathcal{C}_1, \mathfrak{M}^{\mathcal{C}})$ (when both exist).*

Proof. Since $\mathbf{Left}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}})$ and $\mathbf{Left}(\mathcal{C}_1, \mathfrak{M}^{\mathcal{C}})$ both use the Reedy notion of fibration, it suffices to observe that weak equivalences in the first are always weak equivalences in the second by definition. The other case is dual. ∎

Thus, one has four distinct model structures on $\mathfrak{M}(\to)$, and there are five more if one assumes there is also a localization $\mathfrak{M}_f$ around. As above, one would get four more model structures by only considering $\mathfrak{M}_f$ and the techniques of this article. However, by combining the two structures, there is also one more. Let $\mathfrak{M}_0 = \mathfrak{M}$ and $\mathfrak{M}_1 = \mathfrak{M}_f$. Then $\mathbf{Left}(\mathcal{C}, \mathfrak{M}^{\mathcal{C}})$ (notice $\mathcal{C}_0 = \mathcal{C}$ as for the standard Reedy structure) gives something here referred to as a 'mixed structure' which is a convenient place to study the localization map. For example, if $X \in \mathfrak{M}$ is fibrant, then the localization map $X \to L_f X$ becomes a fibrant replacement in the mixed structure for the identity $1 : X \to X$. The mixed structure also relates nicely with looking at $\mathbf{Left}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}})$ for $\mathfrak{M}$ with $\mathcal{C}_0 = 0$ and for $\mathfrak{M}_f$ with $\mathcal{C}_0 = 1$ by Lemma 8.2 and Proposition 6.8.

In fact, the mixed structure is the only reasonable way to combine the two structures using the techniques of this article, since it coincides with the associated $\mathbf{Proj}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}})$ by $\mathcal{C}$ monotone increasing, any choice of $\mathcal{C}_0 \neq \mathcal{C}$ is covered by the eight cases related to a single structure on $\mathfrak{M}$, and the reversed choice ($\mathfrak{M}_0 = \mathfrak{M}_f$ and $\mathfrak{M}_1 = \mathfrak{M}$) does not seem to satisfy the compatibility condition.

8.3. Commutative Squares. One can also proceed as above for $\mathcal{C}$ a commutative square

$$\begin{tikzcd} 00 \arrow[r] \arrow[d] & 01 \arrow[d] \\ 10 \arrow[r] & 11 \end{tikzcd} \tag{22}$$

as well, which is again clearly monotone increasing. However, a new variant is now available, if one has two different localizations of $\mathfrak{M}$, say $\mathfrak{M}_f$ and $\mathfrak{M}_g$, together with the 'common localization' $\mathfrak{M}_h$. It will not be necessary to be precise about what $\mathfrak{M}_h$ should mean or when it exists. However, it should represent localization with respect to both f and g at the same time, so the idea is that any cofibration which is acyclic in either $\mathfrak{M}_f$ or $\mathfrak{M}_g$ should now be acyclic in $\mathfrak{M}_h$. Now take $\mathcal{C}_0 = \mathcal{C}$ (again like the standard Reedy structure) with $\mathfrak{M}_{00} = \mathfrak{M}$, $\mathfrak{M}_{01} = \mathfrak{M}_f$, $\mathfrak{M}_{10} = \mathfrak{M}_g$ and $\mathfrak{M}_{11} = \mathfrak{M}_h$ to form a 'highly mixed structure'. In this case, one has a 'localization square'

$$\begin{tikzcd} X \arrow[r] \arrow[d] & L_f X \arrow[d] \\ L_g X \arrow[r] & L_h X \end{tikzcd} \tag{23}$$

which arises as a fibrant replacement in the highly mixed structure for the constant square on a fibrant object $X \in \mathfrak{M}$. Perhaps the usual localization square for topological spaces, with the far corner the rationalization and one near corner a p-localization could lead to some valuable insights via this highly mixed structure. By Lemma 8.2, it also relates well with the two mixed structures associated to $\mathbf{Left}(\mathcal{C}_0, \mathfrak{M}^{\mathcal{C}})$ by picking $\mathcal{C}_0 = \{00 \to 01\}$ or $\{00 \to 10\}$ and these same choices for $\mathfrak{M}_{ij}$. Among other things, these are model structures on the category of squares which are closely related to just one of the two original localizations by Proposition 6.8.

It seems likely that further examples along these lines could be used to understand successive localizations in a highly structured form. Perhaps the relationship between the highly mixed structure for two localizations and the mixed structure for just one of them could lead to an inductive framework for studying successive localizations. One interesting source of examples could be the smashing localizations of the stable homotopy category, where localization squares already appear prominently.

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