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FINITELY PRESENTABLE MORPHISMS IN EXACT SEQUENCES

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ABSTRACT.

Let $\mathcal{K}$ be a locally finitely presentable category. If $\mathcal{K}$ is abelian and the sequence

$$0 \longrightarrow K \overset{k}{\rightarrowtail} X \overset{c}{\twoheadrightarrow} C \longrightarrow 0$$

is short exact, we show that 1) K is finitely generated $\Leftrightarrow$ c is finitely presentable; 2) k is finitely presentable $\Leftrightarrow$ C is finitely presentable. The "$\Leftarrow$" directions fail for semi-abelian varieties. We show that all but (possibly) 2)($\Leftarrow$) follow from analogous properties which hold in all locally finitely presentable categories. As for 2)($\Leftarrow$), it holds as soon as $\mathcal{K}$ is also co-homological, and all its strong epimorphisms are regular. Finally, locally finitely coherent (resp. noetherian) abelian categories are characterized as those for which all finitely presentable morphisms have finitely generated (resp. presentable) kernel objects.

1. Finitely presentable morphisms

Recall (from [GU, 71] or [AR, 94]) that an object X in a category $\mathcal{K}$ is *finitely presentable* (*finitely generated*) if the hom-functor $\mathcal{K}(X, -)\colon X \longrightarrow \mathbf{Set}$ preserves filtered colimits (resp. colimits of filtered diagrams made of monomorphisms). Then, $\mathcal{K}$ is *finitely accessible* if it has a (small) set of finitely presentable objects whose closure under filtered colimits is all of $\mathcal{K}$. Finally, $\mathcal{K}$ is *locally finitely presentable* if it is finitely accessible and cocomplete.

1.1. DEFINITION. Let $f\colon X \longrightarrow Y$ be a morphism in $\mathcal{K}$.

(a) f is *finitely presentable* (resp. *finitely generated*) if it is a finitely presentable (resp. *finitely generated*) object of the slice category $(X \downarrow \mathcal{K})$.

(b) f is *finitary* if X and Y are finitely presentable.

The finitary morphisms of $\mathcal{K}$ are actually the finitely presentable objects of the category of morphisms $\mathcal{K}^{\rightarrow}$. They are precisely those finitely presentable morphisms with a finitely presentable domain (see below).

Finitely presentable morphisms have been first considered in algebraic geometry, where they play an important role (for example in the Chevalley Theorem; see [GD, 64] and [D, 92]). In fact, in the category **CRng** of commutative rings, the definition above

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has a very concrete interpretation, being essentially the structure morphisms $R \longrightarrow R[x_1, ..., x_n]/(p_1, ..., p_m)$ for the finitely presented algebras.

More generally, in finitary varieties, a morphism $f\colon X \longrightarrow Y$ is finitely presentable when f provides a way to "present Y" by adding a finite number of generators and relations to some presentation of X. This follows from 1.3(6) below.

Finitely presentable morphisms are useful outside algebraic geometry, as shown by the following examples.

1.2. Applications. Below, we refer to the infinitary version of the above definitions: for λ a regular infinite cardinal, λ-*presentable* and λ-*ary* are what "finitely presentable" and "finitary" become when "filtered" is replaced by "λ-filtered" throughout in the definitions.

(a) The classes of objects in a locally λ-presentable category $\mathcal{K}$ which are closed under λ-pure subobjects are precisely the cone-injectivity classes with respect to classes of cones made of λ-presentable morphisms (see [H, 98] for a proof and the precise definitions). This was used in [H1, 04] to characterize the classes closed under λ-pure subobjects, products and λ-filtered colimits as the injectivity classes with respect to sets of λ-ary morphisms (a fact which was first proved directly in [RAB, 02]). Both characterizations have obvious syntactic translations when $\mathcal{K}$ is the category of all structures of some signature, generalizing and refining the classical "preservation theorem" of [Ke, 65], which characterizes the (finitary) elementary classes closed under filtered colimits.

(b) Since every slice $(X \downarrow \mathcal{K})$ in a locally finitely presentable $\mathcal{K}$ is locally finitely presentable, every morphism $f\colon X \longrightarrow Y$ in $\mathcal{K}$ is a colimit in $(X \downarrow \mathcal{K})$ of a filtered diagram of finitely presentable morphisms. What is more surprising is that f is actually a transfinite composition of finitely presentable morphisms (see [H, 06], Example 9). This statement does not hold when restricted either to monomorphisms, or to pure monomorphisms: in fact, one can show that if there exists λ such that every pure monomorphisms is the transfinite composition of λ-presentable pure monos, then $\mathcal{K}$ has enough pure-injectives ([H, 06], Example 13). The same is true for monomorphisms with respect to injectivity if $\mathcal{K}$ has the transferability property (i.e., monos are stable under pushouts). In [BR, 07], what the authors essentially do is to show that this occurs whenever $\mathcal{K}$ has *effective unions* (see [B, 94]) of subobjects (respectively of pure subobjects), a condition which is met in all familiar examples of categories with enough (pure-) injectives, as the authors point out.

For reference, we collect below various properties of finitely presentable and finitely generated morphisms. All are fairly straightforward computations from the definitions, except the last one, which is proved in [H2, 04].

1.3. Properties. In a finitely accessible category $\mathcal{K}$:

(1) Isomorphisms are finitely presentable.

(2) Finitely presentable (resp. generated) morphisms are closed under composition.
(3) Given a commutative diagram

$$\begin{tikzcd} \bullet \arrow[rr, "f"] \arrow[dr, "f_1"'] & & \bullet \\ & \bullet \arrow[ur, "f_2"'] & \end{tikzcd}$$

- f is finitely generated $\implies$ f_2 is finitely generated;
- f_1 is epi and f is finitely presentable $\implies$ f_2 is finitely presentable;
- f_1 is strong epi $\implies$ [f is finitely generated $\Leftrightarrow$ f_2 is finitely generated].

(4) Given a morphism $f\colon X \longrightarrow Y$,
- X is finitely generated and Y is finitely presentable $\implies$ f is finitely presentable;
- X is finitely presentable $\implies$ [Y is finitely presentable $\Leftrightarrow$ f is finitely presentable];
- Y is finitely generated $\implies$ f is finitely generated;
- f is strong epi $\implies$ f is finitely generated.

(5) Pushouts of finitely presentable (resp. generated) morphisms (along any morphism) are finitely presentable (resp. generated).
(6) If $\mathcal{K}$ has connected colimits, then every finitely presentable morphism is the pushout of a finitary morphism (along some morphism).

Note that the coequalizer of a pair of finitely presentable morphisms is not necessarily finitely presentable.

In the category **Mod**-R of all right R-modules, R a ring, the embedding $A \hookrightarrow B$ of a submodule is finitely presentable if and only if the quotient B/A is a finitely presentable module (see [H, 06]). However a finitely presentable morphism may not have a finitely presentable kernel. This raises questions on the behavior of finitely presentable morphisms in exact sequences, which are addressed in the next section.

2. Main results

Theorem 2.1 below is first stated in the narrow context of locally finitely presentable abelian category. It could be more quickly proved directly, but our goal is to get as much as we can of it in the most general context. Hence we will derive much of it from results in locally finitely presentable categories (Proposition 2.5), and then in certain co-homological categories (Proposition 2.7).

Note that a finitely accessible abelian category is necessarily locally finitely presentable, and also that a locally finitely presentable pointed category is abelian if and only if all monos and all epis are *normal* (i.e., are kernels and cokernels respectively; see for example [B, 94], Section 1.4).

2.1. Theorem. *Let $\mathcal{K}$ be a locally finitely presentable abelian category. Then in every short exact sequence*

$$0 \longrightarrow K \overset{k}{\rightarrowtail} X \overset{c}{\twoheadrightarrow} C \longrightarrow 0$$

1) K is finitely generated $\Leftrightarrow$ c is finitely presentable;
2) k is finitely presentable $\Leftrightarrow$ C is finitely presentable.

2.2. Notes. In both 1) and 2), the "$\Leftarrow$" directions fail for semi-abelian varieties (in the sense of [BB, 04]): in the exact sequence

$$0 \longrightarrow [\mathcal{F}_2, \mathcal{F}_2] \rightarrowtail \mathcal{F}_2 \twoheadrightarrow (\mathcal{F}_2)_{ab} \longrightarrow 0$$

in **Grp**, where $\mathcal{F}_2$ is the free group on two generators, and $(\mathcal{F}_2)_{ab} = \mathbb{Z} \times \mathbb{Z}$ is its abelianization, it is well-known that the commutator $[\mathcal{F}_2, \mathcal{F}_2]$ is not finitely generated, and easily shown that its embedding in $\mathcal{F}_2$ is not finitely presentable. However $\mathcal{F}_2$ and $(\mathcal{F}_2)_{ab}$ (and hence $\mathcal{F}_2 \twoheadrightarrow (\mathcal{F}_2)_{ab}$) are obviously finitely presentable.

Before proving Theorem 2.1, we prove the following consequence. Recall (from [F, 75]) that a locally finitely presentable category is *locally finitely coherent* if every mono with finitely generated domain and finitely presentable codomain has its domain finitely presentable, and is *locally finitely noetherian* if every finitely generated object is finitely presentable.

2.3. Theorem. *Let $\mathcal{K}$ be a locally finitely presentable abelian category. Then*
A) in every exact sequence

$$0 \longrightarrow K \overset{k}{\rightarrowtail} X \xrightarrow{f} Y \overset{c}{\twoheadrightarrow} C \longrightarrow 0 \qquad (*)$$

1) [K is finitely generated and C is finitely presentable] $\Rightarrow$ f is finitely presentable;

2) f is finitely presentable $\Rightarrow$ C is finitely presentable.

B) Furthermore,

3) [f is finitely presentable $\Rightarrow$ K is finitely generated] holds for every exact sequence (*) iff $\mathcal{K}$ is locally finitely coherent;

4) [f is finitely presentable $\Rightarrow$ K is finitely presentable] holds for every exact sequence (*) iff $\mathcal{K}$ is locally finitely noetherian.

Proof. Let $f = f_2 f_1$ be the image (= (Epi, Mono)) factorization of f.

1) Since $f_2 = \mathit{ker}\, c$ and $f_1 = \mathit{coker}\, k$, both f_1 and f_2 are finitely presentable under the hypotheses (by Theorem 2.1), and hence f too (by 1.3).

2) Given f finitely presentable, f_2 is finitely presentable by 1.3 (3). Hence C is finitely presentable by Theorem 2.1.

3) ($\Rightarrow$) Given a monomorphism $g\colon A \rightarrowtail B$ with A finitely generated and B finitely presentable, there exists an epimorphism $h\colon D \twoheadrightarrow A$ with D finitely presentable ([AR, 94]). Now gh is finitary, hence finitely presentable. By the assumption, its kernel object is then finitely generated, so that h is finitely presentable, by Theorem 2.1. But then A must be finitely presentable, by 1.3 (4).

($\Leftarrow$) Given $f\colon X \longrightarrow Y$ finitely presentable in (*), there exists, by 1.3 (6), a pushout diagram

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "u"'] & Y' \arrow[d, "v"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

with X' and Y' finitely presentable. Take the image factorization $g = g_2 g_1$ of g and consider the pushout u' of u along g_1, and then the pushout u'' of u' along g_2. Then we can assume $u'' = v$ and $f = f_2' f_1'$:

$$\begin{tikzcd} X' \arrow[r, two heads, "g_1"] \arrow[d, "u"'] & Z' \arrow[r, tail, "g_2"] \arrow[d, "u'"] & Y' \arrow[d, "v"] \\ X \arrow[r, "f_1'"] \arrow[rr, bend right, "f"'] & Z \arrow[r, "f_2'"] & Y \end{tikzcd}$$

Then f_1' is epi and f_2' is mono, since $\mathcal{K}$ is abelian. Now Z' is finitely generated because g_1 is (strongly) epi and X' is finitely presentable, and so it is finitely presentable, from the assumption. Therefore g_1 is finitely presentable, and so is its pushout f_1'. But $\mathit{ker}\, f_1' = \mathit{ker}\, f$ (since f_2' is mono), hence K is finitely generated by Theorem 2.1, as required.

4) ($\Rightarrow$) Given X finitely generated, consider the short exact sequence

$$0 \longrightarrow X \overset{i_1}{\rightarrowtail} X + X \cong X \times X \overset{\pi_2}{\twoheadrightarrow} X \longrightarrow 0$$

where i_1 and π_2 are the canonical injection and projection respectively. Then π_2 is finitely presentable, by Theorem 2.1, so that X is finitely presentable, by the assumption.

($\Leftarrow$) Clear from Theorem 2.1. ■

2.4. Problems. Theorem 2.3 raises the question of the characterization of the finitely presentable morphisms by completing (in the exact sequence (*)) the equation

$$[\ f \ \mathit{finitely\ presentable}\ ] \iff [\ ??\ +\ C \ \mathit{finitely\ presentable}\ ].$$

Note that "K finitely generated" would do precisely for the locally finitely coherent categories.

There is a formulation for morphisms: if $f_2 f_1$ is the image factorization of f, the question is how to complete the equivalence

$$[\ f \ \mathit{is}\ \ \mathit{finitely\ presentable}\ ] \iff [\ ??\ +\ f_2 \ \mathit{is\ finitely\ presentable}\ ].$$

Here, "f_1 is finitely generated" is too weak (it is true of every epimorphism!), and "f_1 is finitely presentable" is too strong, since the equivalence

$$[\ f \textit{ is finitely presentable}\ ] \iff [\ f_1 \textit{ and } f_2 \textit{ are finitely presentable}\ ]$$

also characterizes the locally finitely coherent categories, as the proof of part 3)($\Leftarrow$) of Theorem 2.3 actually shows.

Recall the celebrated theorem of G. Higman ([Hi, 61]), which states that a finitely generated group can be embedded in a finitely presentable one iff it is recursively presented, i.e., it has a presentation made of recursively enumerable sets of generators and relations. Because f_1 is always finitely generated, and f_2 is automatically finitely presentable when f is, our question is really one of embeddability, in the slice category $(X \downarrow \mathcal{K})$, of a finitely generated object into a finitely presentable one. Whether anything useful can be extracted from this connection might be worth investigating.

Much of Theorem 2.1 is an immediate consequence of things already happening in every locally finitely presentable category, as we will now see. Recall that a *relation* on an object X is just a pair $u, v\colon R \rightrightarrows X$ such that $\langle u, v\rangle\colon R \longrightarrow X \times X$ is mono.

2.5. PROPOSITION. *Let $\mathcal{K}$ be locally finitely presentable. Then*
(a) the finitely presentable regular epis are precisely the coequalizers of relations with finitely generated domains.
(b) If $\mathcal{K}$ is also pointed, then
(i) the finitely presentable normal epis are precisely the cokernels of monomorphisms with finitely generated domains, and
(ii) the cokernel of every finitely presentable morphism has a finitely presentable codomain.

PROOF. (a) This follows readily from Proposition 2.11 in [AH, 09], which states that the finitely presentable regular epimorphisms are precisely the coequalizers of pairs of morphisms from finitely presentable domains. Indeed, given a coequalizer diagram

$$K \underset{v}{\overset{u}{\rightrightarrows}} X \xrightarrow[coeq(u,v)]{c\,=} C$$

with c finitely presentable, we can then assume that K is also finitely presentable. We consider the diagram

$$K \xrightarrow{e} R \overset{m}{\rightarrowtail} X \times X \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} X \xrightarrow[c]{} C$$

where me is the (Strong Epi, Mono)-factorization of $\langle u, v\rangle$ (which always exists in a locally finitely presentable category). Then R is finitely generated, and e being epi, c is also the coequalizer of the relation $\langle \pi_1 m, \pi_2 m\rangle$.

Conversely, given a relation $u, v\colon R \rightrightarrows X$ with R finitely generated, there must exist a strong epi $e\colon K \twoheadrightarrow R$ with K finitely presentable, hence the coequalizer of (ue, ve) is finitely presentable. But again, $coeq(u, v) = coeq(ue, ve)$.

(b) (i) A normal epi $c\colon X \twoheadrightarrow C$ is the coequalizer of some pair of the form $0, v\colon K \rightrightarrows X$. If c is finitely presentable, the proof of Proposition 2.11 of [AH, 09] shows the existence of a morphism $f\colon K_i \longrightarrow K$ such that K_i is finitely presentable and c is the coequalizer of $0f = 0$ and vf. Hence c is the cokernel of vf. As in (a), the (Strong Epi, Mono)-factorization me of vf will give us c as the cokernel of m, which has a finitely generated domain, as required.

For the converse, we argue exactly as in (a).

(ii) Given $cf\colon X \longrightarrow Y \twoheadrightarrow C$ with $c = \mathit{coker}\, f$ and f finitely presentable, let $h\colon C \longrightarrow D$, where $d_i\colon D_i \longrightarrow D$ is the colimit of a filtered diagram $(d_{ij})_I$. Then $hcf = 0$, and it is the colimit (object) of the diagram $(d_{ij}\colon 0_i \longrightarrow 0_j)_{ij}$ in $(X \downarrow \mathcal{K})$, with $0_i = 0\colon X \longrightarrow D_i$ for all i. Hence there exists $i \in I$ and $g\colon Y \longrightarrow D_i$ such that $gf = 0$ and $d_i g = hc$.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "0_i=0"'] \arrow[drr, "0"] & Y \arrow[r, two heads, "c"] \arrow[dl, dashed, "g"] \arrow[dr, "hc"] & C \arrow[d, "h"] \\
D_i \arrow[rr, "d_i"'] & & D
\end{tikzcd}$$

Then, $gf = 0$ implies there exists $h'\colon C \longrightarrow D_i$, unique such that $h'c = g$. That $d_i h' = h$ follows from c being epi. Finally, for $h''\colon C \longrightarrow D_i$ such that $d_i h'' = d_i h'$, we have $d_i h'' c = hc = d_i g$, so that $d_{ij} h'' c = d_{ij} g = d_{ij} h' c$ for some j, hence $d_{ij} h'' = d_{ij} h'$, as required. ∎

2.6. Notes. Concerning part a) of Proposition 2.5, examples of semi-abelian varieties are easily constructed where the domain of the kernel pair of a finitely presentable (even finitary) morphism is not finitely generated. The case of **Grp** is special here: it follows from [Gd, 78] that the domain of the kernel pair of a finitary morphism is finitely generated (interestingly, [Gd, 78] shows that it is finitely presentable if and only if the codomain of the finitary morphism is finite.) For example, the domain of the kernel pair of $\mathcal{F}_2 \twoheadrightarrow (\mathcal{F}_2)_{ab}$ in 2.2 is generated by $(a, a), (b, b), (1, aba^{-1}b^{-1})$, and $(aba^{-1}b^{-1}, 1)$, where a and b are the free generators of $\mathcal{F}_2$. Whether this extends to finitely presentable morphisms seems unlikely.

Proof of Theorem 2.1

1) follows immediately from Proposition 2.5 (b)(i), since all monos are normal in an abelian category. 2)($\Rightarrow$) is 2.5 (b)(ii). 2)($\Leftarrow$) holds in a slightly more general context, as the following Proposition 2.7 will show. ∎

Recall from [BB, 04] that a category is called *homological* if it is pointed, regular, and *protomodular*. The latter is also defined in [BB, 04], but we will only need the observation there (Theorem 4.1.10) that in the presence of the first two conditions, $\mathcal{K}$ is homological iff is satisfies the following Short Five Lemma: given a commutative diagram with short exact rows,

$$\begin{tikzcd}
0 \arrow[r] & K' \arrow[r, tail, "k'"] \arrow[d, "\alpha"'] & X' \arrow[r, two heads, "c'"] \arrow[d, "\beta"'] & C' \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & K \arrow[r, tail, "k"'] & X \arrow[r, two heads, "c"'] & C \arrow[r] & 0
\end{tikzcd}$$

if α and γ are isomorphisms, then so is β. (In this context, that

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, tail, "k"'] & X \arrow[r, two heads, "c"'] & C \arrow[r] & 0
\end{tikzcd}$$

is *short exact* just means that k is the kernel of c and c is the cokernel of k). Then we have:

2.7. PROPOSITION. *Let $\mathcal{K}$ be a locally finitely presentable category. If $\mathcal{K}$ is co-homological and all its strong epimorphisms are normal, then part 2) of Theorem 2.1 holds: in every short exact sequence*

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, tail, "k"] & X \arrow[r, two heads, "c"] & C \arrow[r] & 0
\end{tikzcd}$$

k is finitely presentable if and only if C is finitely presentable.

PROOF. The "$\Rightarrow$" direction follows from Proposition 2.5.

For the converse, first note that in any locally finitely presentable category $\mathcal{K}$, for any strong epi $c\colon X \twoheadrightarrow C$ with C finitely presentable, there exists $\beta\colon X' \longrightarrow X$ with X' finitely presentable such that $c\beta$ is a strong epi. Indeed, one can verify (using in particular [AR, 94] 1.62) that every strong epi is the colimit $(f_i, g_i)\colon c_i \longrightarrow c$ in $\mathcal{K}^{\rightarrow}$ of a filtered diagram $(c_i)_I$ of finitary strong epis. In particular $(g_i)_I$ is a colimit cone in $\mathcal{K}$, so that 1_C must factorize through some g_i. Then take $\beta\colon X' \longrightarrow X$ to be f_i.

$$\begin{tikzcd}
X_i \arrow[r, two heads, "c_i"] \arrow[d, "f_i"'] & C_i \arrow[d, "g_i"'] & C \arrow[l, dashed] \arrow[dl, "1_C"] \\
X \arrow[r, two heads, "c"'] & C &
\end{tikzcd}$$

Now, $g = c\beta$ is a strong epi, hence normal. Being also finitary, Proposition 2.5 (b) (i) implies it is the cokernel of some monomorphism $k'\colon K' \rightarrowtail X'$ with K' finitely generated. This gives us the commutative diagram:

$$\begin{tikzcd}
 & K' \arrow[r, tail, "k'"] \arrow[d, "\alpha"'] & X' \arrow[r, two heads, "g"] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, equal, "1_C"] & 0 \\
0 \arrow[r] & K \arrow[r, tail, "k"'] & X \arrow[r, two heads, "c"'] & C \arrow[r] & 0
\end{tikzcd}$$

where α is induced by $c\beta k' = 0$. Now, according to (the dual of) Theorem IV 4.2 of [PT, 04] (the "normalized Barr-Kock" property), this is enough here to conclude that the left hand square is a pushout. Using the Properties 1.3, we see that k' is finitely presentable, and hence its pushout k as well, as required. ∎

2.8. Notes. Among the locally finitely presentable categories, the co-homological regular ones satisfy the conditions of Proposition 2.7. In fact, the Short Five Lemma being self-dual, it will be satisfied by these categories, which will then be homological. This in turn implies that all strong epis are normal (see [PT, 04] IV.4.4).

Hence we have that the conditions
(i) $\mathcal{K}$ is co-homological and regular;
(ii) $\mathcal{K}$ is homological and co-regular;
(iii) $\mathcal{K}$ is homological and co-homological
are all equivalent. It was recently observed by G. Janelidze (see [RT, 02] for a proof) that the latter condition actually implies that $\mathcal{K}$ is additive, and hence that it is equivalent to the following two:
(iv) $\mathcal{K}$ is additive, regular and co-regular;
(v) $\mathcal{K}$ is *quasi-abelian* (= *almost abelian* = *Raikov-semiabelian*).
The quasi-abelian categories are defined as the additive categories in which all kernels and cokernels exist and are stable under pushouts and pullbacks respectively. The category of torsion-free abelian groups is a (non-abelian) example. They have been studied in several papers, together with the slightly weaker notion of *(Gruson-) semiabelian* categories (see [Gn, 66], [Ra, 69], [Ru, 01], [Ko, 05]). The latter should not be confused with the semi-abelian categories mentioned in 2.2 and 2.6 above, which were introduced in [JMT, 02]. Semiabelian categories can be characterized as those additive categories in which every morphism can be factorized as a normal epimorphism, followed by a bimorphism, followed by a normal monomorphism (in well-powered and finitely complete categories, this just means that strong epimorphisms and strong monomorphisms are normal). Rump ([Ru, 08]) recently showed that they are not all regular nor co-regular (and hence not almost abelian); this paper also gives a detailed account of the confusing history of these various terms.

We don't know whether the conditions of Proposition 2.7 imply that $\mathcal{K}$ is regular, and hence almost abelian.

2.9. Problems. (1) *Is part 2) of Theorem 2.1 also a trivial consequence of something happening in every locally finitely presentable category?*

In particular, in a pointed locally finitely presentable category (or at least in a semi-abelian category), is every normal epi with a finitely presentable codomain the cokernel of some finitely presentable (mono)morphism? Is this even the case in **Grp**, for that matter?

(2) *Does part 1) of Theorem 2.1 hold under the conditions of Proposition 2.7?*

(3) *Does part 2) of Theorem 2.1 hold in all locally finitely presentable additive categories?*

Question (3) is motivated by the following. A pure mono in a locally finitely presentable category $\mathcal{K}$ can be defined as a morphism $f\colon A \longrightarrow B$ such that for all commutative square

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d] & Y \arrow[d] \arrow[dl, dashed, "d"'] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

with g finitely presentable, there exists a diagonal d making the upper triangle commute ([H, 98]). Theorem 5.2 of [P, 09] states that if $\mathcal{K}$ is also additive, a pure mono f satisfies this diagonalization property for every morphism g with the codomain of its cokernel finitely presentable. (The proof there appears to be incorrect, but one can prove the statement using the argument in [AR, 04], Example 3). Note that if $\mathcal{K}$ is (locally finitely presentable additive and) regular and coregular, this follows trivially from our Proposition 2.7.

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