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HIGHER CATEGORIFIED ALGEBRAS VERSUS BOUNDED HOMOTOPY ALGEBRAS

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ABSTRACT. We define Lie 3-algebras and prove that these are in 1-to-1 correspondence with the 3-term Lie infinity algebras whose bilinear and trilinear maps vanish in degree $(1, 1)$ and in total degree 1, respectively. Further, we give an answer to a question of [Roy07] pertaining to the use of the nerve and normalization functors in the study of the relationship between categorified algebras and truncated sh algebras.

1. Introduction

Higher structures – infinity algebras and other objects up to homotopy, higher categories, "oidified" concepts, higher Lie theory, higher gauge theory... — are currently intensively investigated. In particular, higher generalizations of Lie algebras have been conceived under various names, e.g. Lie infinity algebras, Lie n-algebras, quasi-free differential graded commutative associative algebras (qfDGCAs for short), n-ary Lie algebras, see e.g. [Dzh05], crossed modules [MP09]... See also [AP10], [GKP11].

More precisely, there are essentially two ways to increase the flexibility of an algebraic structure: homotopification and categorification.

Homotopy, sh or infinity algebras [Sta63] are homotopy invariant extensions of differential graded algebras. This property explains their origin in BRST of closed string field theory. One of the prominent applications of Lie infinity algebras [LS93] is their appearance in Deformation Quantization of Poisson manifolds. The deformation map can be extended from differential graded Lie algebras (DGLAs) to L_∞-algebras and more precisely to a functor from the category $\mathtt{L}_\infty$ to the category $\mathtt{Set}$. This functor transforms a weak equivalence into a bijection. When applied to the DGLAs of polyvector fields and polydifferential operators, the latter result, combined with the formality theorem, provides the 1-to-1 correspondence between Poisson tensors and star products.

On the other hand, categorification [CF94], [Cra95] is characterized by the replacement of sets (resp. maps, equations) by categories (resp. functors, natural isomorphisms).

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Rather than considering two maps as equal, one details a way of identifying them. Categorification is a sharpened viewpoint that leads to astonishing results in TFT, bosonic string theory... Categorified Lie algebras, i.e. Lie 2-algebras (alternatively, semistrict Lie 2-algebras) in the category theoretical sense, have been introduced by J. Baez and A. Crans [BC04]. Their generalization, weak Lie 2-algebras (alternatively, Lie 2-algebras), has been studied by D. Roytenberg [Roy07].

It has been shown in [BC04] that categorification and homotopification are tightly connected. To be exact, Lie 2-algebras and 2-term Lie infinity algebras form equivalent 2-categories. Due to this result, Lie n-algebras are often defined as sh Lie algebras concentrated in the first n degrees [Hen08]. However, this 'definition' is merely a terminological convention, see e.g. Definition 4 in [SS07b]. On the other hand, Lie infinity algebra structures on an $\mathbf{N}$-graded vector space V are in 1-to-1 correspondence with square 0 degree -1 (with respect to the grading induced by V) coderivations of the free reduced graded commutative associative coalgebra $S^c(sV)$, where s denotes the suspension operator, see e.g. [SS07b] or [GK94]. In finite dimension, the latter result admits a variant based on qfDGCAs instead of coalgebras. Higher morphisms of free DGCAs have been investigated under the name of derivation homotopies in [SS07b]. Quite a number of examples can be found in [SS07a].

Besides the proof of the mentioned correspondence between Lie 2-algebras and 2-term Lie infinity algebras, the seminal work [BC04] provides a classification of all Lie infinity algebras, whose only nontrivial terms are those of degree 0 and $n-1$, by means of a Lie algebra, a representation and an $(n+1)$-cohomology class; for a possible extension of this classification, see [Bae07].

In this paper, we give an explicit categorical definition of Lie 3-algebras and prove that these are in 1-to-1 correspondence with the 3-term Lie infinity algebras, whose bilinear and trilinear maps vanish in degree $(1,1)$ and in total degree 1, respectively. Note that a '3-term' Lie infinity algebra implemented by a 4-cocycle [BC04] is an example of a Lie 3-algebra in the sense of the present work.

The correspondence between categorified and bounded homotopy algebras is expected to involve classical functors and chain maps, like e.g. the normalization and Dold-Kan functors, the (lax and oplax monoidal) Eilenberg-Zilber and Alexander-Whitney chain maps, the nerve functor... We show that the challenge ultimately resides in an incompatibility of the cartesian product of linear n-categories with the monoidal structure of this category, thus answering a question of [Roy07].

The paper is organized as follows. Section 2 contains all relevant higher categorical definitions. In Section 3, we define Lie 3-algebras. The fourth section contains the proof of the mentioned 1-to-1 correspondence between categorified algebras and truncated sh algebras — the main result of this paper. A specific aspect of the monoidal structure of the category of linear n-categories is highlighted in Section 5. In the last section, we show that this feature is an obstruction to the use of the Eilenberg-Zilber map in the proof of the correspondence "bracket functor – chain map".

2. Higher linear categories and bounded chain complexes of vector spaces

Let us emphasize that notation and terminology used in the present work originate in [BC04], [Roy07], as well as in [Lei04]. For instance, a linear n-category will be an (a strict) n-category [Lei04] in $\mathtt{Vect}$. Categories in $\mathtt{Vect}$ have been considered in [BC04] and also called internal categories or 2-vector spaces. In [BC04], see Sections 2 and 3, the corresponding morphisms (resp. 2-morphisms) are termed as linear functors (resp. linear natural transformations), and the resulting 2-category is denoted by $\mathtt{VectCat}$ and also by $\mathtt{2Vect}$. Therefore, the $(n+1)$-category made up by linear n-categories (n-categories in $\mathtt{Vect}$ or $(n+1)$-vector spaces), linear n-functors... will be denoted by $\mathtt{Vect}\ n\text{-}\mathtt{Cat}$ or $(n+1)\mathtt{Vect}$.

The following result is known. We briefly explain it here as its proof and the involved concepts are important for an easy reading of this paper.

2.1. Proposition. *The categories* $\mathtt{Vect}\ n\text{-}\mathtt{Cat}$ *of linear* n*-categories and linear* n*-functors and* $\mathtt{C}^{n+1}(\mathtt{Vect})$ *of* $(n+1)$*-term chain complexes of vector spaces and linear chain maps are equivalent.*

We first recall some definitions.

2.2. Definition. *An* **n-globular vector space** L*,* $n \in \mathbf{N}$*, is a sequence*

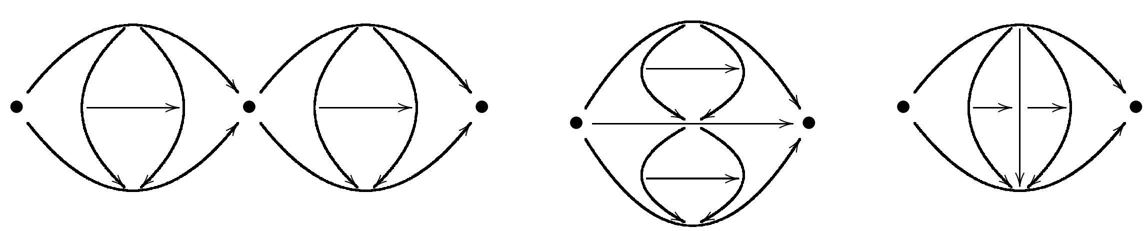
$$L_n \overset{s,t}{\rightrightarrows} L_{n-1} \overset{s,t}{\rightrightarrows} \dots \overset{s,t}{\rightrightarrows} L_0 \rightrightarrows 0, \tag{1}$$

of vector spaces L_m *and linear maps* s, t *such that*

$$s(s(a)) = s(t(a)) \text{ and } t(s(a)) = t(t(a)), \tag{2}$$

for any $a \in L_m$*,* $m \in \{1, \dots, n\}$*. The maps* s, t *are called* **source map** *and* **target map***, respectively, and any element of* L_m *is an* **m-cell***.*

By higher category we mean in this text a **strict** higher category. Roughly, a linear n-category, $n \in \mathbf{N}$, is an n-globular vector space endowed with compositions of m-cells, $0 < m \leq n$, along a p-cell, $0 \leq p < m$, and an identity associated to any m-cell, $0 \leq m < n$. Two m-cells $(a, b) \in L_m \times L_m$ are composable along a p-cell, if $t^{m-p}(a) = s^{m-p}(b)$. The composite m-cell will be denoted by $a \circ_p b$ (the cell that 'acts' first is written on the left) and the vector subspace of $L_m \times L_m$ made up by the pairs of m-cells that can be composed along a p-cell will be denoted by $L_m \times_{L_p} L_m$. The following figure schematizes the composition of two 3-cells along a 0-, a 1-, and a 2-cell.



2.3. Definition. *A* **linear n-category**, $n \in \mathbf{N}$, *is an n-globular vector space L (with source and target maps s, t) together with, for any $m \in \{1, \ldots, n\}$ and any $p \in \{0, \ldots, m-1\}$, a linear composition map $\circ_p : L_m \times_{L_p} L_m \longrightarrow L_m$ and, for any $m \in \{0, \ldots, n-1\}$, a linear identity map $1 : L_m \longrightarrow L_{m+1}$, such that the properties*

- *for $(a, b) \in L_m \times_{L_p} L_m$,*

$$\text{if } \; p = m-1, \; \text{ then } \; s(a \circ_p b) = s(a) \; \text{ and } \; t(a \circ_p b) = t(b),$$

$$\text{if } \; p \leq m-2, \; \text{ then } \; s(a \circ_p b) = s(a) \circ_p s(b) \; \text{ and } \; t(a \circ_p b) = t(a) \circ_p t(b),$$

- $$s(1_a) = t(1_a) = a,$$

- *for any $(a, b), (b, c) \in L_m \times_{L_p} L_m$,*

$$(a \circ_p b) \circ_p c = a \circ_p (b \circ_p c),$$

- $$1^{m-p}_{s^{m-p} a} \circ_p a = a \circ_p 1^{m-p}_{t^{m-p} a} = a$$

are verified, as well as the compatibility conditions

- *for $q < p$, $(a, b), (c, d) \in L_m \times_{L_p} L_m$ and $(a, c), (b, d) \in L_m \times_{L_q} L_m$,*

$$(a \circ_p b) \circ_q (c \circ_p d) = (a \circ_q c) \circ_p (b \circ_q d),$$

- *for $m < n$ and $(a, b) \in L_m \times_{L_p} L_m$,*

$$1_{a \circ_p b} = 1_a \circ_p 1_b.$$

The morphisms between two linear n-categories are the linear n-functors.

2.4. Definition. *A* **linear n-functor** *$F : L \longrightarrow L'$ between two linear n-categories is made up by linear maps $F : L_m \longrightarrow L'_m$, $m \in \{0, \ldots, n\}$, such that the categorical structure — source and target maps, composition maps, identity maps — is respected.*

Linear n-categories and linear n-functors form a category $\texttt{Vect}\ n\text{-}\texttt{Cat}$, see Proposition 2.1. To disambiguate this proposition, let us specify that the objects of $\texttt{C}^{n+1}(\texttt{Vect})$ are the complexes whose underlying vector space $V = \oplus_{i=0}^{n} V_i$ is made up by $n+1$ terms V_i.

The proof of Proposition 2.1 is based upon the following result.

2.5. Proposition. *Let L be any n-globular vector space with linear identity maps. If s_m denotes the restriction of the source map to L_m, the vector spaces L_m and $L'_m := \oplus_{i=0}^m V_i$, $V_i := \ker s_i$, $m \in \{0, \ldots, n\}$, are isomorphic. Further, the n-globular vector space with identities can be completed in a unique way by linear composition maps so to form a linear n-category. If we identify L_m with L'_m, this unique linear n-categorical structure reads*

$$s(v_0, \ldots, v_m) = (v_0, \ldots, v_{m-1}), \tag{3}$$

$$t(v_0, \ldots, v_m) = (v_0, \ldots, v_{m-1} + tv_m), \tag{4}$$

$$1_{(v_0, \ldots, v_m)} = (v_0, \ldots, v_m, 0), \tag{5}$$

$$(v_0, \ldots, v_m) \circ_p (v'_0, \ldots, v'_m) = (v_0, \ldots, v_p, v_{p+1} + v'_{p+1}, \ldots, v_m + v'_m), \tag{6}$$

where the two m-cells in Equation (6) are assumed to be composable along a p-cell.

Proof. As for the first part of this proposition, if $m = 2$ e.g., it suffices to observe that the linear maps

$$\alpha_L : L'_2 = V_0 \oplus V_1 \oplus V_2 \ni (v_0, v_1, v_2) \mapsto 1^2_{v_0} + 1_{v_1} + v_2 \in L_2$$

and

$$\beta_L : L_2 \ni a \mapsto (s^2 a, s(a - 1^2_{s^2 a}), a - 1_{s(a - 1^2_{s^2 a})} - 1^2_{s^2 a}) \in V_0 \oplus V_1 \oplus V_2 = L'_2$$

are inverses of each other. For arbitrary $m \in \{0, \ldots, n\}$ and $a \in L_m$, we set

$$\beta_L a = \left(s^m a, \ldots, s^{m-i}(a - \sum_{j=0}^{i-1} 1^{m-j}_{p_j \beta_L a}), \ldots, a - \sum_{j=0}^{m-1} 1^{m-j}_{p_j \beta_L a} \right) \in V_0 \oplus \ldots \oplus V_i \oplus \ldots \oplus V_m = L'_m,$$

where p_j denotes the projection $p_j : L'_m \longrightarrow V_j$ and where the components must be computed from left to right.

For the second claim, note that when reading the source, target and identity maps through the detailed isomorphism, we get $s(v_0, \ldots, v_m) = (v_0, \ldots, v_{m-1})$, $t(v_0, \ldots, v_m) = (v_0, \ldots, v_{m-1} + tv_m)$, and $1_{(v_0, \ldots, v_m)} = (v_0, \ldots, v_m, 0)$. Eventually, set $v = (v_0, \ldots, v_m)$ and let (v, w) and (v', w') be two pairs of m-cells that are composable along a p-cell. The composability condition, say for (v, w), reads

$$(w_0, \ldots, w_p) = (v_0, \ldots, v_{p-1}, v_p + tv_{p+1}).$$

It follows from the linearity of $\circ_p : L_m \times_{L_p} L_m \longrightarrow L_m$ that $(v + v') \circ_p (w + w') = (v \circ_p w) + (v' \circ_p w')$. When taking $w = 1^{m-p}_{t^{m-p} v}$ and $v' = 1^{m-p}_{s^{m-p} w'}$, we find

$$(v_0 + w'_0, \ldots, v_p + w'_p, v_{p+1}, \ldots, v_m) \circ_p (v_0 + w'_0, \ldots, v_p + w'_p + tv_{p+1}, w'_{p+1}, \ldots, w'_m)$$
$$= (v_0 + w'_0, \ldots, v_m + w'_m),$$

so that $\circ_p$ is necessarily the composition given by Equation (6). It is easily seen that, conversely, Equations (3) – (6) define a linear n-categorical structure. ∎

Proof of Proposition 2.1 We define functors $\mathfrak{N} : \texttt{Vect}\ n\text{-}\texttt{Cat} \longrightarrow \texttt{C}^{n+1}(\texttt{Vect})$ and $\mathfrak{G} : \texttt{C}^{n+1}(\texttt{Vect}) \longrightarrow \texttt{Vect}\ n\text{-}\texttt{Cat}$ that are inverses up to natural isomorphisms.

If we start from a linear n-category L, so in particular from an n-globular vector space L, we define an $(n+1)$-term chain complex $\mathfrak{N}(L)$ by setting $V_m = \ker s_m \subset L_m$ and $d_m = t_m|_{V_m} : V_m \longrightarrow V_{m-1}$. In view of the globular space conditions (2), the target space of d_m is actually V_{m-1} and we have $d_{m-1}d_m v_m = 0$.

Moreover, if $F : L \longrightarrow L'$ denotes a linear n-functor, the value $\mathfrak{N}(F) : V \longrightarrow V'$ is defined on $V_m \subset L_m$ by $\mathfrak{N}(F)_m = F_m|_{V_m} : V_m \longrightarrow V'_m$. It is obvious that $\mathfrak{N}(F)$ is a linear chain map.

It is obvious that $\mathfrak{N}$ respects the categorical structures of $\texttt{Vect}\ n\text{-}\texttt{Cat}$ and $\texttt{C}^{n+1}(\texttt{Vect})$.

As for the second functor $\mathfrak{G}$, if (V,d), $V = \oplus_{i=0}^n V_i$, is an $(n+1)$-term chain complex of vector spaces, we define a linear n-category $\mathfrak{G}(V) = L$, $L_m = \oplus_{i=0}^m V_i$, as in Proposition 2.5: the source, target, identity and composition maps are defined by Equations (3) – (6), except that tv_m in the RHS of Equation (4) is replaced by dv_m.

The definition of $\mathfrak{G}$ on a linear chain map $\phi : V \longrightarrow V'$ leads to a linear n-functor $\mathfrak{G}(\phi) : L \longrightarrow L'$, which is defined on $L_m = \oplus_{i=0}^m V_i$ by $\mathfrak{G}(\phi)_m = \oplus_{i=0}^m \phi_i$. Indeed, it is readily checked that $\mathfrak{G}(\phi)$ respects the linear n-categorical structures of L and L'.

Furthermore, $\mathfrak{G}$ respects the categorical structures of $\texttt{C}^{n+1}(\texttt{Vect})$ and $\texttt{Vect}\ n\text{-}\texttt{Cat}$.

Eventually, there exist natural isomorphisms $\alpha : \mathfrak{N}\mathfrak{G} \Rightarrow \mathrm{id}$ and $\gamma : \mathfrak{G}\mathfrak{N} \Rightarrow \mathrm{id}$.

To define a natural transformation $\alpha : \mathfrak{N}\mathfrak{G} \Rightarrow \mathrm{id}$, note that $L' = (\mathfrak{N}\mathfrak{G})(L)$ is the linear n-category made up by the vector spaces $L'_m = \oplus_{i=0}^m V_i$, $V_i = \ker s_i$, as well as by the source, target, identities and compositions defined from $V = \mathfrak{N}(L)$ as in the above definition of $\mathfrak{G}(V)$, i.e. as in Proposition 2.5. It follows that $\alpha_L : L' \longrightarrow L$, defined by $\alpha_L : L'_m \ni (v_0, \ldots, v_m) \mapsto 1^m_{v_0} + \ldots + 1_{v_{m-1}} + v_m \in L_m$, $m \in \{0, \ldots, n\}$, which pulls the linear n-categorical structure back from L to L', see Proposition 2.5, is an invertible linear n-functor. Moreover α is natural in L.

It suffices now to observe that the composite $\mathfrak{G}\mathfrak{N}$ is the identity functor. ■

Next we further investigate the category $\texttt{Vect}\ n\text{-}\texttt{Cat}$.

2.6. Proposition. *The category* $\texttt{Vect}\ n\text{-}\texttt{Cat}$ *admits finite products.*

Let L and L' be two linear n-categories. The product linear n-category $L \times L'$ is defined by $(L \times L')_m = L_m \times L'_m$, $S_m = s_m \times s'_m$, $T_m = t_m \times t'_m$, $I_m = 1_m \times 1'_m$, and $\bigcirc_p = \circ_p \times \circ'_p$. The compositions $\bigcirc_p$ coincide with the unique compositions that complete the n-globular vector space with identities, thus providing a linear n-category. It is straightforwardly checked that the product of linear n-categories verifies the universal property for binary products.

2.7. Proposition. *The category* $\texttt{Vect}\,2\text{-}\texttt{Cat}$ *admits a 3-categorical structure. More precisely, its 2-cells are the linear natural 2-transformations and its 3-cells are the linear 2-modifications.*

This proposition is the linear version (with similar proof) of the well-known result that the category $2\text{-}\texttt{Cat}$ is a 3-category with 2-categories as 0-cells, 2-functors as 1-cells, natural 2-transformations as 2-cells, and 2-modifications as 3-cells. The definitions of n-categories and 2-functors are similar to those given above in the linear context (but they are formulated without the use of set theoretical concepts). As for (linear) natural 2-transformations and (linear) 2-modifications, let us recall their definition in the linear setting:

2.8. Definition. *A* **linear natural 2-transformation** $\theta : F \Rightarrow G$ *between two linear 2-functors* $F, G : \mathcal{C} \longrightarrow \mathcal{D}$*, between the same two linear 2-categories, assigns to any* $a \in \mathcal{C}_0$ *a unique* $\theta_a : F(a) \longrightarrow G(a)$ *in* $\mathcal{D}_1$*, linear with respect to* a *and such that for any* $\alpha : f \Rightarrow g$ *in* $\mathcal{C}_2$*,* $f, g : a \longrightarrow b$ *in* $\mathcal{C}_1$*, we have*

$$F(\alpha) \circ_0 1_{\theta_b} = 1_{\theta_a} \circ_0 G(\alpha)\ . \tag{7}$$

If $\mathcal{C} = L \times L$ is a product linear 2-category, the last condition reads

$$F(\alpha, \beta) \circ_0 1_{\theta_{t^2\alpha, t^2\beta}} = 1_{\theta_{s^2\alpha, s^2\beta}} \circ_0 G(\alpha, \beta),$$

for all $(\alpha, \beta) \in L_2 \times L_2$. As functors respect composition, i.e. as

$$F(\alpha, \beta) = F(\alpha \circ_0 1^2_{t^2\alpha}, 1^2_{s^2\beta} \circ_0 \beta) = F(\alpha, 1^2_{s^2\beta}) \circ_0 F(1^2_{t^2\alpha}, \beta),$$

this naturality condition is verified if and only if it holds true in case all but one of the 2-cells are identities 1^2_{-}, i.e. if and only if the transformation is natural with respect to all its arguments separately.

2.9. Definition. *Let* $\mathcal{C}, \mathcal{D}$ *be two linear 2-categories. A* **linear 2-modification** $\mu : \eta \Rrightarrow \varepsilon$ *between two linear natural 2-transformations* $\eta, \varepsilon : F \Rightarrow G$*, between the same two linear 2-functors* $F, G : \mathcal{C} \longrightarrow \mathcal{D}$*, assigns to any object* $a \in \mathcal{C}_0$ *a unique* $\mu_a : \eta_a \Rightarrow \varepsilon_a$ *in* $\mathcal{D}_2$*, which is linear with respect to* a *and such that, for any* $\alpha : f \Rightarrow g$ *in* $\mathcal{C}_2$*,* $f, g : a \longrightarrow b$ *in* $\mathcal{C}_1$*, we have*

$$F(\alpha) \circ_0 \mu_b = \mu_a \circ_0 G(\alpha). \tag{8}$$

If $\mathcal{C} = L \times L$ is a product linear 2-category, it suffices again that the preceding modification property be satisfied for tuples (α, β), in which all but one 2-cells are identities 1^2_{-}. The explanation is the same as for natural transformations.

Beyond linear 2-functors, linear natural 2-transformations, and linear 2-modifications, we use below multilinear cells. Bilinear cells e.g., are cells on a product linear 2-category, with linearity replaced by bilinearity. For instance, 2.10. Definition. *Let L, L', and L'' be linear 2-categories. A* **bilinear 2-functor** $F : L \times L' \longrightarrow L''$ *is a 2-functor such that* $F : L_m \times L'_m \longrightarrow L''_m$ *is bilinear for all* $m \in \{0, 1, 2\}$.

Similarly,

2.11. Definition. *Let L, L', and L'' be linear 2-categories. A* **bilinear natural 2-transformation** $\theta : F \Rightarrow G$ *between two bilinear 2-functors* $F, G : L \times L' \longrightarrow L''$, *assigns to any* $(a, b) \in L_0 \times L'_0$ *a unique* $\theta_{(a,b)} : F(a, b) \longrightarrow G(a, b)$ *in* L''_1, *which is bilinear with respect to* (a, b) *and such that for any* $(\alpha, \beta) : (f, h) \Rightarrow (g, k)$ *in* $L_2 \times L'_2$, $(f, h), (g, k) : (a, b) \longrightarrow (c, d)$ *in* $L_1 \times L'_1$, *we have*

$$F(\alpha, \beta) \circ_0 1_{\theta_{(c,d)}} = 1_{\theta_{(a,b)}} \circ_0 G(\alpha, \beta) \,. \tag{9}$$

3. Homotopy Lie algebras and categorified Lie algebras

We now recall the definition of a Lie infinity (strongly homotopy Lie, sh Lie, $L_\infty-$) algebra and specify it in the case of a 3-term Lie infinity algebra.

3.1. Definition. *A* **Lie infinity algebra** *is an* $\mathbf{N}$*-graded vector space* $V = \oplus_{i \in \mathbf{N}} V_i$ *together with a family* $(\ell_i)_{i \in \mathbf{N}^*}$ *of graded antisymmetric i-linear weight $i-2$ maps on V, which verify the sequence of conditions*

$$\sum_{i+j=n+1} \sum_{(i,n-i)\,-\,shuffles\ \sigma} \chi(\sigma)(-1)^{i(j-1)} \ell_j(\ell_i(a_{\sigma_1}, \ldots, a_{\sigma_i}), a_{\sigma_{i+1}}, \ldots, a_{\sigma_n}) = 0, \tag{10}$$

where $n \in \{1, 2, \ldots\}$, *where* $\chi(\sigma)$ *is the product of the signature of* σ *and the Koszul sign defined by* σ *and the homogeneous arguments* $a_1, \ldots, a_n \in V$.

For $n = 1$, the L_∞-condition (10) reads $\ell_1^2 = 0$ and, for $n = 2$, it means that ℓ_1 is a graded derivation of ℓ_2, or, equivalently, that ℓ_2 is a chain map from $(V \otimes V, \ell_1 \otimes \mathrm{id} + \mathrm{id} \otimes \ell_1)$ to (V, ℓ_1).

In particular,

3.2. Definition. *A* **3-term Lie infinity algebra** *is a 3-term graded vector space* $V = V_0 \oplus V_1 \oplus V_2$ *endowed with graded antisymmetric p-linear maps ℓ_p of weight $p-2$,*

$$\begin{array}{ll} \ell_1 : V_i \longrightarrow V_{i-1} & (1 \le i \le 2), \\ \ell_2 : V_i \times V_j \longrightarrow V_{i+j} & (0 \le i+j \le 2), \\ \ell_3 : V_i \times V_j \times V_k \longrightarrow V_{i+j+k+1} & (0 \le i+j+k \le 1), \\ \ell_4 : V_0 \times V_0 \times V_0 \times V_0 \longrightarrow V_2 & \end{array} \tag{11}$$

(all structure maps ℓ_p, $p > 4$, necessarily vanish), which satisfy L_∞-condition (10) (that is trivial for all $n > 5$).

In this 3-term situation, each L_∞-condition splits into a finite number of equations determined by the various combinations of argument degrees, see below.

On the other hand, we have the

3.3. Definition. *A* **Lie 3-algebra** *is a* **linear 2-category** L *endowed with a* **bracket**, *i.e. an antisymmetric bilinear 2-functor* $[-,-] : L \times L \longrightarrow L$, *which verifies the Jacobi identity up to a* **Jacobiator**, *i.e. a skew-symmetric trilinear natural 2-transformation*

$$J_{xyz} : [[x,y],z] \to [[x,z],y] + [x,[y,z]], \tag{12}$$

$x,y,z \in L_0$, *which in turn satisfies the Baez-Crans Jacobiator identity up to an* **Identiator**, *i.e. a skew-symmetric quadrilinear 2-modification*

$$\begin{aligned} \mu_{xyzu} : [J_{x,y,z}, 1_u] \circ_0 (J_{[x,z],y,u} + J_{x,[y,z],u}) \circ_0 ([J_{xzu}, 1_y] + 1) \circ_0 ([1_x, J_{yzu}] + 1) \\ \Rightarrow J_{[x,y],z,u} \circ_0 ([J_{xyu}, 1_z] + 1) \circ_0 (J_{x,[y,u],z} + J_{[x,u],y,z} + J_{x,y,[z,u]}), \end{aligned} \tag{13}$$

$x,y,z,u \in L_0$, *required to verify the* **coherence law**

$$\alpha_1 + \alpha_4^{-1} = \alpha_3 + \alpha_2^{-1}, \tag{14}$$

where α_1 – α_4 *are explicitly given in Definitions 4.21 – 4.24 and where superscript* -1 *denotes the inverse for composition along a 1-cell.*

Just as the *Jacobiator* is a natural transformation between the two sides of the *Jacobi* identity, the *Identiator* is a modification between the two sides of the Baez-Crans Jacobiator *identity.*

In this definition "skew-symmetric 2-transformation" (resp. "skew-symmetric 2-modification") means that, if we identify L_m with $\oplus_{i=0}^m V_i$, $V_i = \ker s_i$, as in Proposition 2.5, the V_1-component of $J_{xyz} \in L_1$ (resp. the V_2-component of $\mu_{xyzu} \in L_2$) is antisymmetric. Moreover, the definition makes sense, as the source and target in Equation (13) are quadrilinear natural 2-transformations between quadrilinear 2-functors from $L^{\times 4}$ to L. These 2-functors are simplest obtained from the RHS of Equation (13). Further, the mentioned source and target actually are natural 2-transformations, since a 2-functor composed (on the left or on the right) with a natural 2-transformation is again a 2-transformation.

4. Lie 3-algebras in comparison with 3-term Lie infinity algebras

4.1. Remark. *In the following, we systematically identify the vector spaces* L_m, $m \in \{0,\ldots,n\}$, *of a linear n-category with the spaces* $L'_m = \oplus_{i=0}^m V_i$, $V_i = \ker s_i$, *so that the categorical structure is given by Equations (3) – (6). In addition, we often substitute common, index-free notations* (*e.g.* $\alpha = (x, \mathbf{f}, \mathbf{a})$) *for our notations* (*e.g.* $v = (v_0, v_1, v_2) \in L_2$).

The next theorem is the main result of this paper.

4.2. Theorem. *There exists a 1-to-1 correspondence between Lie 3-algebras and 3-term Lie infinity algebras* (V, ℓ_p), *whose structure maps* ℓ_2 *and* ℓ_3 *vanish on* $V_1 \times V_1$ *and on triplets of total degree 1, respectively.*

4.3. Example. *There exists a 1-to-1 correspondence between $(n+1)$-term Lie infinity algebras $V = V_0 \oplus V_n$ (whose intermediate terms vanish), $n \geq 2$, and $(n+2)$-cocycles of Lie algebras endowed with a linear representation, see [BC04], Theorem 6.7. A 3-term Lie infinity algebra implemented by a 4-cocycle can therefore be viewed as a special case of a Lie 3-algebra.*

The proof of Theorem 4.2 consists of five lemmas.

4.4. Linear 2-category – three term chain complex of vector spaces First, we recall the correspondence between the underlying structures of a Lie 3-algebra and a 3-term Lie infinity algebra.

4.5. Lemma. *There is a bijective correspondence between linear 2-categories L and 3-term chain complexes of vector spaces (V, ℓ_1).*

Proof. In the proof of Proposition 2.1, we associated to any linear 2-category L a unique 3-term chain complex of vector spaces $\mathfrak{N}(L) = V$, whose spaces are given by $V_m = \ker s_m$, $m \in \{0,1,2\}$, and whose differential ℓ_1 coincides on V_m with the restriction $t_m|_{V_m}$. Conversely, we assigned to any such chain complex V a unique linear 2-category $\mathfrak{G}(V) = L$, with spaces $L_m = \oplus_{i=0}^m V_i$, $m \in \{0,1,2\}$ and target $t_0(x) = 0$, $t_1(x, \mathbf{f}) = x + \ell_1 \mathbf{f}$, $t_2(x, \mathbf{f}, \mathbf{a}) = (x, \mathbf{f} + \ell_1 \mathbf{a})$. In view of Remark 4.1, the maps $\mathfrak{N}$ and $\mathfrak{G}$ are inverses of each other. ■

4.6. Remark. *The globular space condition is the categorical counterpart of L_∞-condition $n = 1$.*

4.7. Bracket – chain map We assume that we already built (V, ℓ_1) from L or L from (V, ℓ_1).

4.8. Lemma. *There is a bijective correspondence between antisymmetric bilinear 2-functors $[-,-]$ on L and graded antisymmetric chain maps $\ell_2 : (V \otimes V, \ell_1 \otimes \mathrm{id} + \mathrm{id} \otimes \ell_1) \longrightarrow (V, \ell_1)$ that vanish on $V_1 \times V_1$.*

Proof. Consider first an antisymmetric bilinear "2-map" $[-,-] : L \times L \longrightarrow L$ that verifies all functorial requirements except as concerns composition. This bracket then respects the compositions, i.e., for each pairs $(v, w), (v', w') \in L_m \times L_m$, $m \in \{1, 2\}$, that are composable along a p-cell, $0 \leq p < m$, we have

$$[v \circ_p v', w \circ_p w'] = [v, w] \circ_p [v', w'], \tag{15}$$

if and only if the following conditions hold true, for any $\mathbf{f}, \mathbf{g} \in V_1$ and any $\mathbf{a}, \mathbf{b} \in V_2$:

$$[\mathbf{f}, \mathbf{g}] = [1_{t\mathbf{f}}, \mathbf{g}] = [\mathbf{f}, 1_{t\mathbf{g}}], \tag{16}$$

$$[\mathbf{a}, \mathbf{b}] = [1_{t\mathbf{a}}, \mathbf{b}] = [\mathbf{a}, 1_{t\mathbf{b}}] = 0, \tag{17}$$

$$[1_{\mathbf{f}}, \mathbf{b}] = [1^2_{t\mathbf{f}}, \mathbf{b}] = 0. \tag{18}$$

To prove the first two conditions, it suffices to compute $[\mathbf{f}\circ_0 1_{t\mathbf{f}}, 1_0\circ_0\mathbf{g}]$, for the next three conditions, we consider $[\mathbf{a}\circ_1 1_{t\mathbf{a}}, 1_0\circ_1\mathbf{b}]$ and $[\mathbf{a}\circ_0 1_0^2, 1_0^2\circ_0\mathbf{b}]$, and for the last two, we focus on $[1_{\mathbf{f}}\circ_0 1_{t\mathbf{f}}^2, 1_0^2\circ_0\mathbf{b}]$ and $[1_{\mathbf{f}}\circ_0(1_{t\mathbf{f}}^2+1_{\mathbf{f}'}), \mathbf{b}\circ_0\mathbf{b}']$. Conversely, it can be straightforwardly checked that Equations (16) – (18) entail the general requirement (15).

On the other hand, a graded antisymmetric bilinear weight 0 map $\ell_2 : V\times V\longrightarrow V$ commutes with the differentials ℓ_1 and $\ell_1\otimes\mathrm{id}+\mathrm{id}\otimes\ell_1$, i.e., for all $v,w\in V$, we have

$$\ell_1(\ell_2(v,w)) = \ell_2(\ell_1 v,w)+(-1)^v\ell_2(v,\ell_1 w) \tag{19}$$

(we assumed that v is homogeneous and denoted its degree by v as well), if and only if, for any $y\in V_0$, $\mathbf{f},\mathbf{g}\in V_1$, and $\mathbf{a}\in V_2$,

$$\ell_1(\ell_2(\mathbf{f},y)) = \ell_2(\ell_1\mathbf{f},y), \tag{20}$$

$$\ell_1(\ell_2(\mathbf{f},\mathbf{g})) = \ell_2(\ell_1\mathbf{f},\mathbf{g})-\ell_2(\mathbf{f},\ell_1\mathbf{g}), \tag{21}$$

$$\ell_1(\ell_2(\mathbf{a},y)) = \ell_2(\ell_1\mathbf{a},y), \tag{22}$$

$$0 = \ell_2(\ell_1\mathbf{f},\mathbf{b})-\ell_2(\mathbf{f},\ell_1\mathbf{b}). \tag{23}$$

4.9. Remark. *Note that, in the correspondence $\ell_1\leftrightarrow t$ and $\ell_2\leftrightarrow[-,-]$, Equations (20) and (22) read as compatibility requirements of the bracket with the target and that Equations (21) and (23) correspond to the second conditions of Equations (16) and (18), respectively.*

Proof of Lemma 4.8 (continuation) To prove the announced 1-to-1 correspondence, we first define a graded antisymmetric chain map $\mathfrak{N}([-,-])=\ell_2$, $\ell_2 : V\otimes V\longrightarrow V$ from any antisymmetric bilinear 2-functor $[-,-] : L\times L\longrightarrow L$.

Let $x,y\in V_0$, $\mathbf{f},\mathbf{g}\in V_1$, and $\mathbf{a},\mathbf{b}\in V_2$. Set $\ell_2(x,y)=[x,y]\in V_0$ and $\ell_2(x,\mathbf{g}) = [1_x,\mathbf{g}]\in V_1$. However, we must define $\ell_2(\mathbf{f},\mathbf{g})\in V_2$, whereas $[\mathbf{f},\mathbf{g}]\in V_1$. Moreover, in this case, the antisymmetry properties do not match. The observation

$$[\mathbf{f},\mathbf{g}]=[1_{t\mathbf{f}},\mathbf{g}]=[\mathbf{f},1_{t\mathbf{g}}]=\ell_2(\ell_1\mathbf{f},\mathbf{g})=\ell_2(\mathbf{f},\ell_1\mathbf{g})$$

and Condition (21) force us to *define ℓ_2 on $V_1\times V_1$ as a symmetric bilinear map valued in* $V_2\cap\ker\ell_1$. We further set $\ell_2(x,\mathbf{b})=[1_x^2,\mathbf{b}]\in V_2$, and, as ℓ_2 is required to have weight 0, we must set $\ell_2(\mathbf{f},\mathbf{b})=0$ and $\ell_2(\mathbf{a},\mathbf{b})=0$. It then follows from the functorial properties of $[-,-]$ that the conditions (20) – (22) are verified. In view of Equation (18), Property (23) reads

$$0=[1_{t\mathbf{f}}^2,\mathbf{b}]-\ell_2(\mathbf{f},\ell_1\mathbf{b})=-\ell_2(\mathbf{f},\ell_1\mathbf{b}).$$

In other words, in addition to the preceding requirement, we must *choose ℓ_2 in a way that it vanishes on $V_1\times V_1$ if evaluated on a 1-coboundary.* These conditions are satisfied if we choose $\ell_2=0$ on $V_1\times V_1$.

Conversely, from any graded antisymmetric chain map ℓ_2 that vanishes on $V_1 \times V_1$, we can construct an antisymmetric bilinear 2-functor $\mathfrak{G}(\ell_2) = [-,-]$. Indeed, using obvious notations, we set

$$[x, y] = \ell_2(x, y) \in L_0,\ [1_x, 1_y] = 1_{[x,y]} \in L_1,\ [1_x, \mathbf{g}] = \ell_2(x, \mathbf{g}) \in V_1 \subset L_1.$$

Again $[\mathbf{f}, \mathbf{g}] \in L_1$ cannot be defined as $\ell_2(\mathbf{f}, \mathbf{g}) \in V_2$. Instead, if we wish to build a 2-functor, we must set

$$[\mathbf{f}, \mathbf{g}] = [1_{t\mathbf{f}}, \mathbf{g}] = [\mathbf{f}, 1_{t\mathbf{g}}] = \ell_2(\ell_1\mathbf{f}, \mathbf{g}) = \ell_2(\mathbf{f}, \ell_1\mathbf{g}) \in V_1 \subset L_1,$$

which is possible in view of Equation (21), *if ℓ_2 is on $V_1 \times V_1$ valued in 2-cocycles* (and in particular if it vanishes on this subspace). Further, we define

$$[1^2_x, 1^2_y] = 1^2_{[x,y]} \in L_2,\ [1^2_x, 1_\mathbf{g}] = 1_{[1_x,\mathbf{g}]} \in L_2,$$
$$[1^2_x, \mathbf{b}] = \ell_2(x, \mathbf{b}) \in V_2 \subset L_2,\ [1_\mathbf{f}, 1_\mathbf{g}] = 1_{[\mathbf{f},\mathbf{g}]} \in L_2.$$

Finally, we must set

$$[1_\mathbf{f}, \mathbf{b}] = [1^2_{t\mathbf{f}}, \mathbf{b}] = \ell_2(\ell_1\mathbf{f}, \mathbf{b}) = 0,$$

which is possible in view of Equation (23), *if ℓ_2 vanishes on $V_1 \times V_1$ when evaluated on a 1-coboundary* (and especially if it vanishes on the whole subspace $V_1 \times V_1$), and

$$[\mathbf{a}, \mathbf{b}] = [1_{t\mathbf{a}}, \mathbf{b}] = [\mathbf{a}, 1_{t\mathbf{b}}] = 0,$$

which is possible.

It follows from these definitions that the bracket of $\alpha = (x, \mathbf{f}, \mathbf{a}) = 1^2_x + 1_\mathbf{f} + \mathbf{a} \in L_2$ and $\beta = (y, \mathbf{g}, \mathbf{b}) = 1^2_y + 1_\mathbf{g} + \mathbf{b} \in L_2$ is given by

$$[\alpha, \beta] = (\ell_2(x, y), \ell_2(x, \mathbf{g}) + \ell_2(\mathbf{f}, tg), \ell_2(x, \mathbf{b}) + \ell_2(\mathbf{a}, y)) \in L_2, \tag{24}$$

where $g = (y, \mathbf{g})$. The brackets of two elements of L_1 or L_0 are obtained as special cases of the latter result.

We thus defined an antisymmetric bilinear map $[-,-]$ that assigns an i-cell to any pair of i-cells, $i \in \{0, 1, 2\}$, and that respects identities and sources. Moreover, since Equations (16) – (18) are satisfied, the map $[-,-]$ respects compositions provided it respects targets. For the last of the first three defined brackets, the target condition is verified due to Equation (20). For the fourth bracket, the target must coincide with $[t\mathbf{f}, t\mathbf{g}] = \ell_2(\ell_1\mathbf{f}, \ell_1\mathbf{g})$ and it actually coincides with $t[\mathbf{f}, \mathbf{g}] = \ell_1\ell_2(\ell_1\mathbf{f}, \mathbf{g}) = \ell_2(\ell_1\mathbf{f}, \ell_1\mathbf{g})$, again in view of (20). As regards the seventh bracket, the target $t[1^2_x, \mathbf{b}] = \ell_1\ell_2(x, \mathbf{b}) = \ell_2(x, \ell_1\mathbf{b})$, due to (22), must coincide with $[1_x, t\mathbf{b}] = \ell_2(x, \ell_1\mathbf{b})$. The targets of the two last brackets vanish and $[\mathbf{f}, t\mathbf{b}] = \ell_2(\mathbf{f}, \ell_1\ell_1\mathbf{b}) = 0$ and $[t\mathbf{a}, t\mathbf{b}] = \ell_2(\ell_1\mathbf{a}, \ell_1\ell_1\mathbf{b}) = 0$.

It is straightforwardly checked that the maps $\mathfrak{N}$ and $\mathfrak{G}$ are inverses. ∎

Note that $\mathfrak{N}$ actually assigns to any antisymmetric bilinear 2-functor a class of graded antisymmetric chain maps that coincide outside $V_1 \times V_1$ and whose restrictions to $V_1 \times V_1$ are valued in 2-cocycles and vanish when evaluated on a 1-coboundary. The map $\mathfrak{N}$, with values in chain maps, is well-defined thanks to a canonical choice of a representative of this class. Conversely, the values on $V_1 \times V_1$ of the considered chain map cannot be encrypted into the associated 2-functor, only the mentioned cohomological conditions are of importance. Without the canonical choice, the map $\mathfrak{G}$ would not be injective.

4.10. Remark. *The categorical counterpart of L_∞-condition $n = 2$ is the functor condition on compositions.*

4.11. Remark. *A 2-term Lie infinity algebra (resp. a Lie 2-algebra) can be viewed as a 3-term Lie infinity algebra (resp. a Lie 3-algebra). The preceding correspondence then of course reduces to the correspondence of [BC04].*

4.12. Jacobiator – third structure map We suppose that we already constructed (V, ℓ_1, ℓ_2) from $(L, [-,-])$ or $(L, [-,-])$ from (V, ℓ_1, ℓ_2).

4.13. Lemma. *There exists a bijective correspondence between skew-symmetric trilinear natural 2-transformations $J : [[-,-],\bullet] \Rightarrow [[-,\bullet],-] + [-,[-,\bullet]]$ and graded antisymmetric trilinear weight 1 maps $\ell_3 : V^{\times 3} \longrightarrow V$ that verify L_∞-condition $n = 3$ and vanish in total degree 1.*

Proof. A skew-symmetric trilinear natural 2-transformation $J : [[-,-],\bullet] \Rightarrow [[-,\bullet],-]+$ $[-,[-,\bullet]]$ is a map that assigns to any $(x,y,z) \in L_0^{\times 3}$ a unique $J_{xyz} : [[x,y],z] \longrightarrow [[x,z],y]+$ $[x,[y,z]]$ in L_1, such that for any $\alpha = (z, \mathbf{f}, \mathbf{a}) \in L_2$, we have

$$[[1_x^2, 1_y^2], \alpha] \circ_0 1_{J_{x,y,t^2\alpha}} = 1_{J_{x,y,s^2\alpha}} \circ_0 \left([[1_x^2, \alpha], 1_y^2] + [1_x^2, [1_y^2, \alpha]]\right)$$

(as well as similar equations pertaining to naturality with respect to the other two variables). A short computation shows that the last condition decomposes into the following two requirements on the V_1- and the V_2-component:

$$\mathbf{J}_{x,y,t\mathbf{f}} + [1_{[x,y]}, \mathbf{f}] = [[1_x, \mathbf{f}], 1_y] + [1_x, [1_y, \mathbf{f}]], \tag{25}$$

$$[1^2_{[x,y]}, \mathbf{a}] = [[1_x^2, \mathbf{a}], 1_y^2] + [1_x^2, [1_y^2, \mathbf{a}]]. \tag{26}$$

A graded antisymmetric trilinear weight 1 map $\ell_3 : V^{\times 3} \longrightarrow V$ verifies L_∞-condition $n = 3$ if

$$\ell_1(\ell_3(u,v,w)) + \ell_2(\ell_2(u,v),w) - (-1)^{vw}\ell_2(\ell_2(u,w),v) + (-1)^{u(v+w)}\ell_2(\ell_2(v,w),u)$$
$$+\ell_3(\ell_1(u),v,w) - (-1)^{uv}\ell_3(\ell_1(v),u,w) + (-1)^{w(u+v)}\ell_3(\ell_1(w),u,v) = 0, \tag{27}$$

for any homogeneous $u, v, w \in V$. This condition is trivial for any arguments of total degree $d = u + v + w > 2$. For $d = 0$, we write $(u,v,w) = (x,y,z) \in V_0^{\times 3}$, for $d = 1$, we consider $(u,v)=(x,y)\in V_0^{\times 2}$ and $w=\mathbf{f}\in V_1$, for $d=2$, either $(u,v)=(x,y)\in V_0^{\times 2}$ and $w=\mathbf{a}\in V_2$, or $u=x\in V_0$ and $(v,w)=(\mathbf{f},\mathbf{g})\in V_1^{\times 2}$, so that Equation (27) reads

$$\ell_1(\ell_3(x,y,z))+\ell_2(\ell_2(x,y),z)-\ell_2(\ell_2(x,z),y)+\ell_2(\ell_2(y,z),x)=0, \tag{28}$$

$$\ell_1(\ell_3(x,y,\mathbf{f}))+\ell_2(\ell_2(x,y),\mathbf{f})-\ell_2(\ell_2(x,\mathbf{f}),y)+\ell_2(\ell_2(y,\mathbf{f}),x)+\ell_3(\ell_1(\mathbf{f}),x,y)=0, \tag{29}$$

$$\ell_2(\ell_2(x,y),\mathbf{a})-\ell_2(\ell_2(x,\mathbf{a}),y)+\ell_2(\ell_2(y,\mathbf{a}),x)+\ell_3(\ell_1(\mathbf{a}),x,y)=0, \tag{30}$$

$$\ell_2(\ell_2(x,\mathbf{f}),\mathbf{g})+\ell_2(\ell_2(x,\mathbf{g}),\mathbf{f})+\ell_2(\ell_2(\mathbf{f},\mathbf{g}),x)-\ell_3(\ell_1(\mathbf{f}),x,\mathbf{g})-\ell_3(\ell_1(\mathbf{g}),x,\mathbf{f})=0. \tag{31}$$

It is easy to associate to any such map ℓ_3 a unique Jacobiator $\mathfrak{G}(\ell_3)=J$: it suffices to set $J_{xyz}:=([[x,y],z],\ell_3(x,y,z))\in L_1$, for any $x,y,z\in L_0$. Equation (28) means that J_{xyz} has the correct target. Equations (25) and (26) exactly correspond to Equations (29) and (30), respectively, *if we assume that in total degree $d=1$, ℓ_3 is valued in 2-cocycles and vanishes when evaluated on a 1-coboundary.* These conditions are verified if we start from a structure map ℓ_3 that vanishes on any arguments of total degree 1.

4.14. Remark. *Remark that the values $\ell_3(x,y,\mathbf{f})\in V_2$ cannot be encoded in a natural 2-transformation $J: L_0^{\times 3}\ni(x,y,z)\longrightarrow J_{xyz}\in L_1$ (and that the same holds true for Equation (31), whose first three terms are zero, since we started from a map ℓ_2 that vanishes on $V_1\times V_1$).*

Proof of Lemma 4.13 (continuation). Conversely, to any Jacobiator J corresponds a unique map $\mathfrak{N}(J)=\ell_3$. Just set $\ell_3(x,y,z):=\mathbf{J}_{xyz}\in V_1$ and $\ell_3(x,y,\mathbf{f})=0$, for all $x,y,z\in V_0$ and $\mathbf{f}\in V_1$ (as ℓ_3 is required to have weight 1, it must vanish if evaluated on elements of degree $d\geq 2$).

Obviously the composites $\mathfrak{N}\mathfrak{G}$ and $\mathfrak{G}\mathfrak{N}$ are identity maps. ∎

4.15. Remark. *The naturality condition is, roughly speaking, the categorical analogue of the L_∞-condition $n=3$.*

4.16. Identiator – fourth structure map For $x,y,z,u\in L_0$, we set

$$\eta_{xyzu}:=[J_{x,y,z},1_u]\circ_0(J_{[x,z],y,u}+J_{x,[y,z],u})\circ_0([J_{xzu},1_y]+1)\circ_0([1_x,J_{yzu}]+1)\in L_1 \tag{32}$$

and

$$\varepsilon_{xyzu}:=J_{[x,y],z,u}\circ_0([J_{xyu},1_z]+1)\circ_0(J_{x,[y,u],z}+J_{[x,u],y,z}+J_{x,y,[z,u]})\in L_1, \tag{33}$$

see Definition 3.3. The identities 1 are uniquely determined by the sources of the involved factors. The quadrilinear natural 2-transformations η and ε are actually the left and right hand composites of the Baez-Crans octagon that pictures the coherence law of a Lie 2-algebra, see [BC04], Definition 4.1.3. They connect the quadrilinear 2-functors $F,G: L\times L\times L\times L\longrightarrow L$, whose values at (x,y,z,u) are given by the source and the target of the 1-cells η_{xyzu} and ε_{xyzu}, as well as by the top and bottom sums of triple brackets of the mentioned octagon.

4.17. LEMMA. *The skew-symmetric quadrilinear 2-modifications $\mu : \eta \Rrightarrow \varepsilon$ are in 1-to-1 correspondence with the graded antisymmetric quadrilinear weight 2 maps $\ell_4 : V^{\times 4} \longrightarrow V$ that verify the L_∞-condition $n = 4$.*

PROOF. A skew-symmetric quadrilinear 2-modification $\mu : \eta \Rrightarrow \varepsilon$ maps every tuple $(x, y, z, u) \in L_0^{\times 4}$ to a unique $\mu_{xyzu} : \eta_{xyzu} \Rightarrow \varepsilon_{xyzu}$ in L_2, such that, for any $\alpha = (u, \mathbf{f}, \mathbf{a}) \in L_2$, we have

$$F(1_x^2, 1_y^2, 1_z^2, \alpha) \circ_0 \mu_{x,y,z,u+t\mathbf{f}} = \mu_{xyzu} \circ_0 G(1_x^2, 1_y^2, 1_z^2, \alpha) \tag{34}$$

(as well as similar results concerning naturality with respect to the three other variables). If we decompose $\mu_{xyzu} \in L_2 = V_0 \oplus V_1 \oplus V_2$,

$$\mu_{xyzu} = (F(x, y, z, u), \mathbf{h}_{xyzu}, \mathbf{m}_{xyzu}) = 1_{\eta_{xyzu}} + \mathbf{m}_{xyzu},$$

Condition (34) reads

$$F(1_x, 1_y, 1_z, \mathbf{f}) + \mathbf{h}_{x,y,z,u+t\mathbf{f}} = \mathbf{h}_{xyzu} + G(1_x, 1_y, 1_z, \mathbf{f}), \tag{35}$$

$$F(1_x^2, 1_y^2, 1_z^2, \mathbf{a}) + \mathbf{m}_{x,y,z,u+t\mathbf{f}} = \mathbf{m}_{xyzu} + G(1_x^2, 1_y^2, 1_z^2, \mathbf{a}). \tag{36}$$

On the other hand, a graded antisymmetric quadrilinear weight 2 map $\ell_4 : V^{\times 4} \longrightarrow V$, and more precisely $\ell_4 : V_0^{\times 4} \longrightarrow V_2$, verifies L_∞-condition $n = 4$, if

$$\begin{aligned}
&\ell_1(\ell_4(a, b, c, d)) - \ell_2(\ell_3(a, b, c), d) + (-1)^{cd}\ell_2(\ell_3(a, b, d), c) \\
&\quad - (-1)^{b(c+d)}\ell_2(\ell_3(a, c, d), b) + (-1)^{a(b+c+d)}\ell_2(\ell_3(b, c, d), a) \\
&\quad + \ell_3(\ell_2(a, b), c, d) - (-1)^{bc}\ell_3(\ell_2(a, c), b, d) + (-1)^{d(b+c)}\ell_3(\ell_2(a, d), b, c) \\
&\quad + (-1)^{a(b+c)}\ell_3(\ell_2(b, c), a, d) - (-1)^{ab+ad+cd}\ell_3(\ell_2(b, d), a, c) \\
&\quad + (-1)^{(a+b)(c+d)}\ell_3(\ell_2(c, d), a, b) - \ell_4(\ell_1(a), b, c, d) + (-1)^{ab}\ell_4(\ell_1(b), a, c, d) \\
&\quad - (-1)^{c(a+b)}\ell_4(\ell_1(c), a, b, d) + (-1)^{d(a+b+c)}\ell_4(\ell_1(d), a, b, c) = 0,
\end{aligned} \tag{37}$$

for all homogeneous $a, b, c, d \in V$. The condition is trivial for $d \geq 2$. For $d = 0$, we write $(a, b, c, d) = (x, y, z, u) \in V_0^{\times 4}$, and, for $d = 1$, we take $(a, b, c, d) = (x, y, z, \mathbf{f}) \in V_0^{\times 3} \times V_1$, so that — since ℓ_2 and ℓ_3 vanish on $V_1 \times V_1$ and for $d = 1$, respectively — Condition (37) reads

$$\ell_1(\ell_4(x, y, z, u)) - \mathbf{h}_{xyzu} + \mathbf{e}_{xyzu} = 0, \tag{38}$$

$$\ell_4(\ell_1(\mathbf{f}), x, y, z) = 0, \tag{39}$$

where $\mathbf{h}_{xyzu}$ and $\mathbf{e}_{xyzu}$ are the V_1-components of η_{xyzu} and ε_{xyzu}, see Equations (32) and (33).

We can associate to any such map ℓ_4 a unique 2-modification $\mathfrak{G}(\ell_4) = \mu$, $\mu : \eta \Rrightarrow \varepsilon$. It suffices to set, for $x, y, z, u \in L_0$,

$$\mu_{xyzu} = (F(x, y, z, u), \mathbf{h}_{xyzu}, -\ell_4(x, y, z, u)) \in L_2.$$

In view of Equation (38), the target of this 2-cell is

$$t\mu_{xyzu} = (F(x, y, z, u), \mathbf{h}_{xyzu} - \ell_1(\ell_4(x, y, z, u))) = \varepsilon_{xyzu} \in L_1.$$

Note now that the 2-naturality equations (25) and (26) show that 2-naturality of $\eta : F \Rightarrow G$ means that

$$F(1_x, 1_y, 1_z, \mathbf{f}) + \mathbf{h}_{x,y,z,u+t\mathbf{f}} = \mathbf{h}_{xyzu} + G(1_x, 1_y, 1_z, \mathbf{f}),$$
$$F(1_x^2, 1_y^2, 1_z^2, \mathbf{a}) = G(1_x^2, 1_y^2, 1_z^2, \mathbf{a}).$$

When comparing with Equations (35) and (36), we conclude that μ is a 2-modification if and only if $\ell_4(\ell_1(\mathbf{f}), x, y, z) = 0$, which is exactly Equation (39).

Conversely, if we are given a skew-symmetric quadrilinear 2-modification $\mu : \eta \Rrightarrow \varepsilon$, we define a map $\mathfrak{N}(\mu) = \ell_4$ by setting $\ell_4(x, y, z, u) = -\mathbf{m}_{xyzu}$, with self-explaining notations. L_∞-condition $n = 4$ is equivalent with Equations (38) and (39). The first means that μ_{xyzu} must have the target ε_{xyzu} and the second requires that $\mathbf{m}_{t\mathbf{f},x,y,z}$ vanish — a consequence of the 2-naturality of η and of Equation (36).

The maps $\mathfrak{N}$ and $\mathfrak{G}$ are again inverses. ■

4.18. Coherence law – L_∞-condition $n = 5$

4.19. Lemma. *Coherence law (14) is equivalent to L_∞-condition $n = 5$.*

Proof. The sh Lie condition $n = 5$ reads,

$$\begin{aligned}
&\ell_2(\ell_4(x, y, z, u), v) - \ell_2(\ell_4(x, y, z, v), u) + \ell_2(\ell_4(x, y, u, v), z) - \ell_2(\ell_4(x, z, u, v), y)\\
&+\ell_2(\ell_4(y, z, u, v), x) + \ell_4(\ell_2(x, y), z, u, v) - \ell_4(\ell_2(x, z), y, u, v) + \ell_4(\ell_2(x, u), y, z, v)\\
&-\ell_4(\ell_2(x, v)y, z, u) + \ell_4(\ell_2(y, z), x, u, v) - \ell_4(\ell_2(y, u), x, z, v) + \ell_4(\ell_2(y, v), x, z, u)\\
&+\ell_4(\ell_2(z, u), x, y, v) - \ell_4(\ell_2(z, v), x, y, u) + \ell_4(\ell_2(u, v), x, y, z)\\
&= 0, \qquad (40)
\end{aligned}$$

for any $x, y, z, u, v \in V_0$. It is trivial in degree $d \geq 1$. Let us mention that it follows from Equation (28) that (V_0, ℓ_2) is a Lie algebra up to homotopy, and from Equation (30) that ℓ_2 is a representation of V_0 on V_2. Condition (40) then requires that ℓ_4 be a Lie algebra 4-cocycle of V_0 represented upon V_2.

The coherence law for the 2-modification μ corresponds to four different ways to re-bracket the expression $F([x, y], z, u, v) = [[[[x, y], z], u], v]$ by means of μ, J, and $[-,-]$. More precisely, we define, for any tuple $(x, y, z, u, v) \in L_0^{\times 5}$, four 2-cells

$$\alpha_i : \sigma_i \Rightarrow \tau_i,$$

$i \in \{1, 2, 3, 4\}$, in L_2, where $\sigma_i, \tau_i : A_i \longrightarrow B_i$. Dependence on the considered tuple is understood. We omit temporarily also index i. Of course, σ and τ read $\sigma = (A, \mathbf{s}) \in L_1$ and $\tau = (A, \mathbf{t}) \in L_1$.

$$\text{If } \alpha = (A, \mathbf{s}, \mathbf{a}) \in L_2, \text{ we set } \alpha^{-1} = (A, \mathbf{t}, -\mathbf{a}) \in L_2,$$

which is, as easily seen, the inverse of α for composition along 1-cells.

4.20. DEFINITION. *The* **coherence law** *for the 2-modification μ of a Lie 3-algebra $(L, [-,-], J, \mu)$ reads*

$$\alpha_1 + \alpha_4^{-1} = \alpha_3 + \alpha_2^{-1}, \tag{41}$$

where α_1 – α_4 are detailed in the next definitions.

4.21. DEFINITION. *The* **first 2-cell** α_1 *is given by*

$$\alpha_1 = 1_{11} \circ_0 \left(\mu_{x,y,z,[u,v]} + [\mu_{xyzv}, 1_u^2]\right) \circ_0 1_{12} \circ_0 \left(\mu_{[x,v],y,z,u} + \mu_{x,[y,v],z,u} + \mu_{x,y,[z,v],u} + 1^2\right), \tag{42}$$

where

$$1_{11} = 1_{J_{[[x,y],z],u,v}}, 1_{12} = 1_{[J_{x,[z,v],y},1_u]+[J_{[x,v],z,y},1_u]+[J_{x,z,[y,v]},1_u]} + 1^2, \tag{43}$$

and where the 1^2 are the identity 2-cells associated with the elements of L_0 provided by the composability condition.

For instance, the squared target of the second factor of α_1 is

$$G(x, y, z, [u, v]) + [G(x, y, z, v), u],$$

whereas the squared source of the third factor is

$$[[[x, [z, v]], y], u] + [[[[x, v], z], y], u] + [[[x, z], [y, v]], u] + \ldots.$$

As the three first terms of this sum are three of the six terms of $[G(x, y, z, v), u]$, the object "...", at which 1^2 in 1_{12} is evaluated, is the sum of the remaining terms and $G(x, y, z, [u, v])$.

4.22. DEFINITION. *The* **fourth 2-cell** α_4 *is equal to*

$$\alpha_4 = [\mu_{xyzu}, 1_v^2] \circ_0 1_{41} \circ_0 \left(\mu_{[x,u],y,z,v} + \mu_{x,[y,u],z,v} + \mu_{x,y,[z,u],v}\right) \circ_0 1_{42}, \tag{44}$$

where

$$1_{41} = 1_{[J_{[x,u],z,y},1_v]+[J_{x,z,[y,u]},1_v]+[J_{x,[z,u],y},1_v]} + 1^2,$$

$$1_{42} = 1_{[[J_{xuv},1_z],1_y]+[J_{xuv},1_{[y,z]}]+[1_x,[J_{yuv},1_z]]+[[1_x,J_{zuv}],1_y]+[1_x,[1_y,J_{zuv}]]+[1_{[x,z]},J_{yuv}]} + 1^2. \tag{45}$$

4.23. DEFINITION. *The* **third 2-cell** α_3 *reads*

$$\alpha_3 = \mu_{[x,y],z,u,v} \circ_0 1_{31} \circ_0 \left([\mu_{xyuv}, 1_z^2] + 1^2\right) \circ_0 1_{32} \circ_0 1_{33}, \tag{46}$$

where

$$1_{31} = 1_{[J_{[x,y],v,u},1_z]} + 1^2,$$

$$1_{32} = 1_{[J_{xyv},1_{[z,u]}]+J_{x,y,[[z,v],u]}+J_{x,y,[z,[u,v]]}+J_{[[x,v],u],y,z}+J_{[x,v],[y,u],z}+J_{[x,u],[y,v],z}+J_{x,[[y,v],u],z}+J_{[x,[u,v]],y,z} +J_{x,[y,[u,v]],z}+[J_{xyu},1_{[z,v]}]}$$

$$1_{33} = 1_{J_{x,[y,v],[z,u]}+J_{[x,v],y,[z,u]}+J_{x,[y,u],[z,v]}+J_{[x,u],y,[z,v]}} + 1^2. \tag{47}$$

4.24. Definition. *The* **second 2-cell** α_2 *is defined as*

$$\alpha_2 = 1_{21} \circ_0 \left(\mu_{[x,z],y,u,v} + \mu_{x,[y,z],u,v}\right) \circ_0 1_{22} \circ_0 \left([1^2_x, \mu_{yzuv}] + [\mu_{xzuv}, 1^2_y] + 1^2\right) \circ_0 1_{23}, \quad (48)$$

where

$$1_{21} = 1_{[[J_{xyz}, 1_u], 1_v]}, 1_{22} = 1_{[1_x, J_{[y,z],v,u}] + [J_{[x,z],v,u}, 1_y]} + 1^2, \quad (49)$$
$$1_{23} = 1_{[J_{xzv}, 1_{[y,u]}] + [J_{xzu}, 1_{[y,v]}] + [1_{[x,v]}, J_{yzu}] + [1_{[x,u]}, J_{yzv}]} + 1^2$$

To get the component expression

$$(A_1 + A_4, \mathbf{s}_1 + \mathbf{t}_4, \mathbf{a}_1 - \mathbf{a}_4) = (A_3 + A_2, \mathbf{s}_3 + \mathbf{t}_2, \mathbf{a}_3 - \mathbf{a}_2) \quad (50)$$

of the coherence law (41), we now comment on the computation of the components $(A_i, \mathbf{s}_i, \mathbf{a}_i)$ (resp. $(A_i, \mathbf{t}_i, -\mathbf{a}_i)$) of α_i (resp. α_i^{-1}).

As concerns α_1, it is straightforwardly seen that all compositions make sense, that its V_0-component is

$$A_1 = F([x, y], z, u, v),$$

and that the V_2-component is

$$\mathbf{a}_1 = -\ell_4(x, y, z, \ell_2(u, v)) - \ell_2(\ell_4(x, y, z, v), u) - \ell_4(\ell_2(x, v), y, z, u) - \ell_4(x, \ell_2(y, v), z, u)$$
$$-\ell_4(x, y, \ell_2(z, v), u).$$

When actually examining the composability conditions, we find that 1^2 in the fourth factor of α_1 is $1^2_{G(x,y,z,[u,v])}$ and thus that the target $t^2\alpha_1$ is made up by the 24 terms

$$G([x, v], y, z, u) + G(x, [y, v], z, u) + G(x, y, [z, v], u) + G(x, y, z, [u, v]).$$

The computation of the V_1-component $\mathbf{s}_1$ is tedious but simple — it leads to a sum of 29 terms of the type "$\ell_3\ell_2\ell_2$, $\ell_2\ell_3\ell_2$, or $\ell_2\ell_2\ell_3$". We will comment on it in the case of α_4^{-1}, which is slightly more interesting.

The V_0-component of α_4^{-1} is

$$A_4 = [F_{xyzu}, v] = F([x, y], z, u, v)$$

and its V_2-component is equal to

$$-\mathbf{a}_4 = \ell_2(\ell_4(x, y, z, u), v) + \ell_4(\ell_2(x, u), y, z, v) + \ell_4(x, \ell_2(y, u), z, v) + \ell_4(x, y, \ell_2(z, u), v).$$

The V_1-component $\mathbf{t}_4$ of α_4^{-1} is the V_1-component of the target of α_4. This target is the composition of the targets of the four factors of α_4 and its V_1-component is given by

$$\mathbf{t}_4 = [\mathbf{e}_{xyzu}, 1_v] + [\mathbf{J}_{[x,u],z,y}, 1_v] + [\mathbf{J}_{x,z,[y,u]}, 1_v] + [\mathbf{J}_{x,[z,u],y}, 1_v] + \mathbf{e}_{[x,u],y,z,v} + \mathbf{e}_{x,[y,u],z,v} + \mathbf{e}_{x,y,[z,u],v}$$

$$+[[\mathbf{J}_{xuv}, 1_z], 1_y] + [\mathbf{J}_{xuv}, 1_{[y,z]}] + [1_x, [\mathbf{J}_{yuv}, 1_z]] + [[1_x, \mathbf{J}_{zuv}], 1_y] + [1_x, [1_y, \mathbf{J}_{zuv}]] + [1_{[x,z]}, \mathbf{J}_{yuv}].$$

The definition (33) of ε immediately provides its V_1-component $\mathbf{e}$ as a sum of 5 terms of the type "$\ell_3\ell_2$ or $\ell_2\ell_3$". The preceding V_1-component $\mathbf{t}_4$ of α_4^{-1} can thus be explicitly written as a sum of 29 terms of the type "$\ell_3\ell_2\ell_2$, $\ell_2\ell_3\ell_2$, or $\ell_2\ell_2\ell_3$". It can moreover be checked that the target $t^2\alpha_4^{-1}$ is again a sum of 24 terms — the same as for $t^2\alpha_1$.

The V_0-component of α_3 is

$$A_3 = F([x,y],z,u,v),$$

the V_1-component $\mathbf{s}_3$ can be computed as before and is a sum of 25 terms of the usual type "$\ell_3\ell_2\ell_2$, $\ell_2\ell_3\ell_2$, or $\ell_2\ell_2\ell_3$", whereas the V_2-component is equal to

$$\mathbf{a}_3 = -\ell_4(\ell_2(x,y),z,u,v) - \ell_2(\ell_4(x,y,u,v),z).$$

Again $t^2\alpha_3$ is made up by the same 24 terms as $t^2\alpha_1$ and $t^2\alpha_4^{-1}$.

Eventually, the V_0-component of α_2^{-1} is

$$A_2 = F([x,y],z,u,v),$$

the V_1-component $\mathbf{t}_2$ is straightforwardly obtained as a sum of 27 terms of the form "$\ell_3\ell_2\ell_2$, $\ell_2\ell_3\ell_2$, or $\ell_2\ell_2\ell_3$", and the V_2-component reads

$$-\mathbf{a}_2 = \ell_4(\ell_2(x,z),y,u,v) + \ell_4(x,\ell_2(y,z),u,v) + \ell_2(x,\ell_4(y,z,u,v)) + \ell_2(\ell_4(x,z,u,v),y).$$

The target $t^2\alpha_2^{-1}$ is the same as in the preceding cases.

Coherence condition (41) and its component expression (50) can now be understood. The condition on the V_0-components is obviously trivial. The condition on the V_2-components is nothing but L_∞-condition $n = 5$, see Equation (40). The verification of triviality of the condition on the V_1-components is lengthy: 6 pairs (resp. 3 pairs) of terms of the LHS $\mathbf{s}_1 + \mathbf{t}_4$ (resp. RHS $\mathbf{s}_3 + \mathbf{t}_2$) are opposite and cancel out, 25 terms of the LHS coincide with terms of the RHS, and, finally, 7 triplets of LHS-terms combine with triplets of RHS-terms and provide 7 sums of 6 terms, e.g.

$$\ell_3(\ell_2(\ell_2(x,y),z),u,v) + \ell_2(\ell_3(x,y,z),\ell_2(u,v)) + \ell_2(\ell_2(\ell_3(x,y,z),v),u)$$
$$-\ell_2(\ell_2(\ell_3(x,y,z),u),v) - \ell_3(\ell_2(\ell_2(x,z),y),u,v) - \ell_3(\ell_2(x,\ell_2(y,z)),u,v).$$

Since, for $\mathbf{f} = \ell_3(x,y,z) \in V_1$, we have

$$\ell_1(\mathbf{f}) = t\mathbf{J}_{xyz} = \ell_2(\ell_2(x,z),y) + \ell_2(x,\ell_2(y,z)) - \ell_2(\ell_2(x,y),z),$$

the preceding sum vanishes in view of Equation (29). Indeed, if we associate a Lie 3-algebra to a 3-term Lie infinity algebra, we started from a homotopy algebra whose term ℓ_3 vanishes in total degree 1, and if we build an sh algebra from a categorified algebra, we already constructed an ℓ_3-map with that property. Finally, the condition on V_1-components is really trivial and the coherence law (41) is actually equivalent to L_∞-condition $n = 5$. ■

5. Monoidal structure of the category $\texttt{Vect}\ n\text{-}\texttt{Cat}$

In this section we exhibit a specific aspect of the natural monoidal structure of the category of linear n-categories.

5.1. Proposition. *If L and L' are linear n-categories, a family $F_m : L_m \longrightarrow L'_m$ of linear maps that respects sources, targets, and identities, commutes automatically with compositions and thus defines a linear n-functor $F : L \longrightarrow L'$.*

Proof. If $v = (v_0, \ldots, v_m), w = (w_0, \ldots, w_m) \in L_m$ are composable along a p-cell, then $F_m v = (F_0 v_0, \ldots, F_m v_m)$ and $F_m w = (F_0 w_0, \ldots, F_m w_m)$ are composable as well, and $F_m(v \circ_p w) = (F_m v) \circ_p (F_m w)$ in view of Equation (6). ■

5.2. Proposition. *The category $\texttt{Vect}\ n\text{-}\texttt{Cat}$ admits a canonical symmetric monoidal structure $\boxtimes$.*

Proof. We first define the product $\boxtimes$ of two linear n-categories L and L'. The n-globular vector space that underlies the linear n-category $L \boxtimes L'$ is defined in the obvious way, $(L \boxtimes L')_m = L_m \otimes L'_m$, $S_m = s_m \otimes s'_m$, $T_m = t_m \otimes t'_m$. Identities are clear as well, $I_m = 1_m \otimes 1'_m$. These data can be completed by the unique possible compositions that then provide a linear n-categorical structure.

If $F : L \longrightarrow M$ and $F' : L' \longrightarrow M'$ are two linear n-functors, we set

$$(F \boxtimes F')_m = F_m \otimes F'_m \in \mathrm{Hom}_{\mathbb{K}}(L_m \otimes L'_m, M_m \otimes M'_m),$$

where $\mathbb{K}$ denotes the ground field. Due to Proposition 5.1, the family $(F \boxtimes F')_m$ defines a linear n-functor $F \boxtimes F' : L \boxtimes L' \longrightarrow M \boxtimes M'$.

It is immediately checked that $\boxtimes$ respects composition and is therefore a functor from the product category $(\texttt{Vect}\ n\text{-}\texttt{Cat})^{\times 2}$ to $\texttt{Vect}\ n\text{-}\texttt{Cat}$. Further, the linear n-category K, defined by $K_m = \mathbb{K}$, $s_m = t_m = \mathrm{id}_{\mathbb{K}}$ $(m > 0)$, and $1_m = \mathrm{id}_{\mathbb{K}}$ $(m < n)$, acts as identity object for $\boxtimes$. Its is now clear that $\boxtimes$ endows $\texttt{Vect}\ n\text{-}\texttt{Cat}$ with a symmetric monoidal structure. ■

5.3. Proposition. *Let L, L', and L'' be linear n-categories. For any bilinear n-functor $F : L \times L' \longrightarrow L''$, there exists a unique linear n-functor $\tilde{F} : L \boxtimes L' \longrightarrow L''$, such that $\boxtimes \tilde{F} = F$. Here $\boxtimes : L \times L' \longrightarrow L \boxtimes L'$ denotes the family of bilinear maps $\boxtimes_m : L_m \times L'_m \ni (v, v') \mapsto v \otimes v' \in L_m \otimes L'_m$, and juxtaposition denotes the obvious composition of the first with the second factor.*

Proof. The result is a straightforward consequence of the universal property of the tensor product of vector spaces. ■

The next remark is essential.

5.4. Remark. *Proposition 5.3 is not a Universal Property for the tensor product $\boxtimes$ of* `Vect` n-`Cat`*, since $\boxtimes : L \times L' \longrightarrow L \boxtimes L'$ is not a bilinear n-functor. It follows that bilinear n-functors on a product category $L \times L'$ cannot be identified with linear n-functors on the corresponding tensor product category $L \boxtimes L'$.*

The point is that the family $\boxtimes_m$ of bilinear maps respects sources, targets, and identities, but not compositions (in contrast with a similar family of linear maps, see Proposition 5.1). Indeed, if $(v, v'), (w, w') \in L_m \times L'_m$ are two p-composable pairs (note that this condition is equivalent with the requirement that $v, w \in L_m$ and $v', w' \in L'_m$ be p-composable), we have

$$\boxtimes_m((v, v') \circ_p (w, w')) = (v \circ_p w) \otimes (v' \circ_p w') \in L_m \otimes L'_m, \tag{51}$$

and

$$\boxtimes_m(v, v') \ \circ_p \ \boxtimes_m(w, w') = (v \otimes v') \circ_p (w \otimes w') \in L_m \otimes L'_m. \tag{52}$$

As the elements (51) and (52) arise from the compositions in $L_m \times L'_m$ and $L_m \otimes L'_m$, respectively, — which are forced by linearity and thus involve the completely different linear structures of these spaces — it can be expected that the two elements do not coincide.

Indeed, when confining ourselves, to simplify, to the case $n = 1$ of linear categories, we easily check that

$$(v \circ w) \otimes (v' \circ w') = (v \otimes v') \circ (w \otimes w') + (v - 1_{tv}) \otimes \mathbf{w}' + \mathbf{w} \otimes (v' - 1_{tv'}). \tag{53}$$

Observe also that the source spaces of the linear maps

$$\circ_L \otimes \circ'_{L'} : (L_1 \times_{L_0} L_1) \otimes (L'_1 \times_{L'_0} L'_1) \ni (v, w) \otimes (v', w') \mapsto (v \circ w) \otimes (v' \circ w') \in L_1 \otimes L'_1$$

and

$$\circ_{L \boxtimes L'} : (L_1 \otimes L'_1) \times_{L_0 \otimes L'_0} (L_1 \otimes L'_1) \ni ((v \otimes v'), (w \otimes w')) \mapsto (v \otimes v') \circ (w \otimes w') \in L_1 \otimes L'_1$$

are connected by

$$\ell_2 : (L_1 \times_{L_0} L_1) \otimes (L'_1 \times_{L'_0} L'_1) \ni (v, w) \otimes (v', w') \mapsto (v \otimes v', w \otimes w') \in (L_1 \otimes L'_1) \times_{L_0 \otimes L'_0} (L_1 \otimes L'_1) \tag{54}$$

— a linear map with nontrivial kernel.

6. Discussion

We continue working in the case $n = 1$ and investigate a more conceptual approach to the construction of a chain map $\ell_2 : \mathfrak{N}(L) \otimes \mathfrak{N}(L) \longrightarrow \mathfrak{N}(L)$ from a bilinear functor $[-, -] : L \times L \longrightarrow L$.

When denoting by $[-,-] : L \boxtimes L \longrightarrow L$ the induced linear functor, we get a chain map $\mathfrak{N}([-,-]) : \mathfrak{N}(L \boxtimes L) \longrightarrow \mathfrak{N}(L)$, so that it is natural to look for a second chain map

$$\phi : \mathfrak{N}(L) \otimes \mathfrak{N}(L) \longrightarrow \mathfrak{N}(L \boxtimes L).$$

The informed reader may skip the following subsection.

6.1. Nerve and normalization functors, Eilenberg-Zilber chain map The objects of the simplicial category Δ are the finite ordinals $n = \{0, \ldots, n-1\}$, $n \geq 0$. Its morphisms $f : m \longrightarrow n$ are the order respecting functions between the sets m and n. Let $\delta_i : n \rightarrowtail n+1$ be the injection that omits image i, $i \in \{0, \ldots, n\}$, and let $\sigma_i : n+1 \twoheadrightarrow n$ be the surjection that assigns the same image to i and $i+1$, $i \in \{0, \ldots, n-1\}$. Any order respecting function $f : m \longrightarrow n$ reads uniquely as $f = \sigma_{j_1} \ldots \sigma_{j_h} \delta_{i_1} \ldots \delta_{i_k}$, where the j_r are decreasing and the i_s increasing. The application of this epi-monic decomposition to binary composites $\delta_i\delta_j$, $\sigma_i\sigma_j$, and $\delta_i\sigma_j$ yields three basic commutation relations.

A simplicial object in the category $\texttt{Vect}$ is a functor $S \in [\Delta^{+\,\mathrm{op}}, \texttt{Vect}]$, where Δ^+ denotes the full subcategory of Δ made up by the nonzero finite ordinals. We write this functor $n{+}1 \mapsto S(n{+}1) =: S_n$, $n \geq 0$, (S_n is the vector space of n-simplices), $\delta_i \mapsto S(\delta_i) =: d_i : S_n \longrightarrow S_{n-1}$, $i \in \{0, \ldots, n\}$ (d_i is a face operator), $\sigma_i \mapsto S(\sigma_i) =: s_i : S_n \longrightarrow S_{n+1}$, $i \in \{0, \ldots, n\}$ (s_i is a degeneracy operator). The d_i and s_j verify the duals of the mentioned commutation rules. The simplicial data (S_n, d_i^n, s_i^n) (we added superscript n) of course completely determine the functor S. Simplicial objects in $\texttt{Vect}$ form themselves a category, namely the functor category $s(\texttt{Vect}) := [\Delta^{+\,\mathrm{op}}, \texttt{Vect}]$, for which the morphisms, called simplicial morphisms, are the natural transformations between such functors. In view of the epi-monic factorization, a simplicial map $\alpha : S \longrightarrow T$ is exactly a family of linear maps $\alpha_n : S_n \longrightarrow T_n$ that commute with the face and degeneracy operators.

The nerve functor

$$\mathcal{N} : \texttt{VectCat} \longrightarrow s(\texttt{Vect})$$

is defined on a linear category L as the sequence $L_0, L_1, L_2 := L_1 \times_{L_0} L_1, L_3 := L_1 \times_{L_0} L_1 \times_{L_0} L_1 \ldots$ of vector spaces of $0, 1, 2, 3 \ldots$ simplices, together with the face operators "composition" and the degeneracy operators "insertion of identity", which verify the simplicial commutation rules. Moreover, any linear functor $F : L \longrightarrow L'$ defines linear maps $F_n : L_n \ni (v_1, \ldots, v_n) \longrightarrow (F(v_1), \ldots, F(v_n)) \in L'_n$ that implement a simplicial map.

The normalized or Moore chain complex of a simplicial vector space $S = (S_n, d_i^n, s_i^n)$ is given by $N(S)_n = \cap_{i=1}^n \ker d_i^n \subset S_n$ and $\partial_n = d_0^n$. Normalization actually provides a functor

$$N : s(\texttt{Vect}) \leftrightarrow \texttt{C}^+(\texttt{Vect}) : \Gamma$$

valued in the category of nonnegatively graded chain complexes of vector spaces. Indeed, if $\alpha : S \longrightarrow T$ is a simplicial map, then $\alpha_{n-1} d_i^n = d_i^n \alpha_n$. Thus, $N(\alpha) : N(S) \longrightarrow N(T)$, defined on $c_n \in N(S)_n$ by $N(\alpha)_n(c_n) = \alpha_n(c_n)$, is valued in $N(T)_n$ and is further a chain map. Moreover, the Dold-Kan correspondence claims that the normalization functor N admits a right adjoint Γ and that these functors combine into an equivalence of categories.

It is straightforwardly seen that, for any linear category L, we have

$$N(\mathcal{N}(L)) = \mathfrak{N}(L). \tag{55}$$

The categories $s(\texttt{Vect})$ and $\texttt{C}^+(\texttt{Vect})$ have well-known monoidal structures (we denote the unit objects by I_s and $I_{\texttt{C}}$, respectively). The normalization functor $N : s(\texttt{Vect}) \to \texttt{C}^+(\texttt{Vect})$ is lax monoidal, i.e. it respects the tensor products and unit objects up to coherent chain maps $\varepsilon : I_{\texttt{C}} \longrightarrow N(I_s)$ and

$$EZ_{S,T} : N(S) \otimes N(T) \longrightarrow N(S \otimes T)$$

(functorial in $S, T \in s(\texttt{Vect})$), where $EZ_{S,T}$ is the Eilenberg-Zilber map. Functor N is lax comonoidal or oplax monoidal as well, the chain morphism being here the Alexander-Whitney map $AW_{S,T}$. These chain maps are inverses of each other up to chain homotopy, $EZ\ AW = 1,\ AW\, EZ \sim 1$.

The Eilenberg-Zilber map is defined as follows. Let $a \otimes b \in N(S)_p \otimes N(T)_q \subset S_p \otimes T_q$ be an element of degree $p + q$. The chain map $EZ_{S,T}$ sends $a \otimes b$ to an element of $N(S \otimes T)_{p+q} \subset (S \otimes T)_{p+q} = S_{p+q} \otimes T_{p+q}$. We have

$$EZ_{S,T}(a \otimes b) = \sum_{(p,q)\text{–shuffles } (\mu,\nu)} \operatorname{sign}(\mu,\nu)\ \ s_{\nu_q}(\dots(s_{\nu_1}a)) \otimes s_{\mu_p}(\dots(s_{\mu_1}b)) \in S_{p+q} \otimes T_{p+q},$$

where the shuffles are permutations of $(0, \dots, p+q-1)$ and where the s_i are the degeneracy operators.

6.2. Monoidal structure and obstruction We now come back to the construction of a chain map $\phi : \mathfrak{N}(L) \otimes \mathfrak{N}(L) \longrightarrow \mathfrak{N}(L \boxtimes L)$.

For $L' = L$, the linear map (54) reads

$$\ell_2 : (\mathcal{N}(L) \otimes \mathcal{N}(L))_2 \ni (v, w) \otimes (v', w') \mapsto (v \otimes v', w \otimes w') \in \mathcal{N}(L \boxtimes L)_2.$$

If its obvious extensions ℓ_n to all other spaces $(\mathcal{N}(L) \otimes \mathcal{N}(L))_n$ define a simplicial map $\ell : \mathcal{N}(L) \otimes \mathcal{N}(L) \longrightarrow \mathcal{N}(L \boxtimes L)$, then

$$N(\ell) : N(\mathcal{N}(L) \otimes \mathcal{N}(L)) \longrightarrow N(\mathcal{N}(L \boxtimes L))$$

is a chain map. Its composition with the Eilenberg-Zilber chain map

$$EZ_{\mathcal{N}(L),\mathcal{N}(L)} : N(\mathcal{N}(L)) \otimes N(\mathcal{N}(L)) \longrightarrow N(\mathcal{N}(L) \otimes \mathcal{N}(L))$$

finally provides the searched chain map ϕ, see Equation (55).

However, the ℓ_n do not commute with all degeneracy and face operators. Indeed, we have for instance

$$\ell_2((d_2^3 \otimes d_2^3)((u, v, w) \otimes (u', v', w'))) = (u \otimes u', (v \circ w) \otimes (v' \circ w')),$$

whereas

$$d_2^3(\ell_3((u,v,w)\otimes(u',v',w'))) = (u\otimes u', (v\otimes v')\circ(w\otimes w')).$$

Equation (53), which means that $\boxtimes : L\times L' \longrightarrow L\boxtimes L'$ is not a functor, shows that these results do not coincide.

A natural idea would be to change the involved monoidal structures $\boxtimes$ of `VectCat` or $\otimes$ of $\mathtt{C}^+(\mathtt{Vect})$. However, even if we substitute the Loday-Pirashvili tensor product $\otimes_{\mathrm{LP}}$ of 2-term chain complexes of vector spaces, i.e. of linear maps [LP98], for the usual tensor product $\otimes$, we do not get $\mathfrak{N}(L)\otimes_{\mathrm{LP}}\mathfrak{N}(L) = \mathfrak{N}(L\boxtimes L)$.

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