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SYMMETRY AND CAUCHY COMPLETION OF QUANTALOID-ENRICHED CATEGORIES

HANS HEYMANS AND ISAR STUBBE

ABSTRACT. We formulate an elementary condition on an involutive quantaloid $\mathcal{Q}$ under which there is a distributive law from the Cauchy completion monad over the symmetrisation comonad on the category of $\mathcal{Q}$-enriched categories. For such quantaloids, which we call Cauchy-bilateral quantaloids, it follows that the Cauchy completion of any symmetric $\mathcal{Q}$-enriched category is again symmetric. Examples include Lawvere's quantale of non-negative real numbers and Walters' small quantaloids of closed cribles.

1. Introduction

A quantaloid $\mathcal{Q}$ is a category enriched in the symmetric monoidal closed category Sup of complete lattices and supremum-preserving functions. Viewing $\mathcal{Q}$ as a bicategory, it is natural to study categories, functors and distributors enriched in $\mathcal{Q}$. If $\mathcal{Q}$ comes equipped with an involution, it makes sense to consider symmetric $\mathcal{Q}$-enriched categories. An important application of quantaloid-enriched categories was discovered by R.F.C. Walters [1981, 1982]: he proved that the topos of sheaves on a small site $(\mathcal{C}, J)$ is equivalent to the category of *symmetric and Cauchy complete* categories enriched in a suitable "small quantaloid of closed cribles" $\mathcal{R}(\mathcal{C}, J)$. A decade earlier, F.W. Lawvere [1973] had already pointed out that the category of generalised metric spaces and non-expansive maps is equivalent to the category of categories enriched in the quantale (that is, a one-object quantaloid) $([0,\infty], \bigwedge, +, 0)$ of extended non-negative real numbers. This is a symmetric quantale, hence it is trivially involutive; and here too the *symmetric and Cauchy complete* $[0,\infty]$-enriched categories are important, if only to connect with the classical theory of metric spaces. Crucial in both examples is thus the use of categories enriched in an involutive quantaloid $\mathcal{Q}$ which are both symmetric and Cauchy complete. R. Betti and R.F.C. Walters [1982] therefore raised the question "whether the Cauchy completion of a symmetric [quantaloid-enriched] category is again symmetric". That is to say, they ask whether it is possible to *restrict* the Cauchy completion functor $(-)_{\mathsf{cc}}\colon \mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q})$ along the embedding $\mathsf{SymCat}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q})$ of symmetric $\mathcal{Q}$-categories. They show that the answer to their question is affirmative for both $\mathcal{R}(\mathcal{C}, J)$ and $[0,\infty]$, by giving an *ad hoc* proof in each case; they also give an example of an involutive quantale for which the answer to their question is negative. Thus, it depends on the base quantaloid $\mathcal{Q}$ whether

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or not the Cauchy completion of a symmetric $\mathcal{Q}$-category is again symmetric.

In this paper we address this issue in a slightly different manner to produce a single, simple argument for both Walters' small quantaloids of closed cribles and Lawvere's quantale of non-negative real numbers, thus giving perhaps a more decisive answer to Betti and Walters' question. The embedding $\mathsf{SymCat}(\mathcal{Q})\longrightarrow\mathsf{Cat}(\mathcal{Q})$ has a right adjoint $(-)_{\mathsf{s}}\colon\mathsf{Cat}(\mathcal{Q})\longrightarrow\mathsf{SymCat}(\mathcal{Q})$, which we call 'symmetrisation'. We aim to *extend* the Cauchy completion functor along the symmetrisation functor. To wit, we define an obvious 'symmetric completion' $(-)_{\mathsf{sc}}\colon\mathsf{SymCat}(\mathcal{Q})\longrightarrow\mathsf{SymCat}(\mathcal{Q})$ (Proposition 3.3) which, by construction, comes with a natural transformation

$$\begin{tikzcd}
\mathsf{Cat}(\mathcal{Q}) \arrow[r, "(-)_{\mathsf{cc}}"] & \mathsf{Cat}(\mathcal{Q}) \\
\mathsf{SymCat}(\mathcal{Q}) \arrow[u, "\text{incl.}"] \arrow[r, "(-)_{\mathsf{sc}}"'] \arrow[ur, phantom, "\Uparrow K"] & \mathsf{SymCat}(\mathcal{Q}) \arrow[u, "\text{incl.}"']
\end{tikzcd}$$

whose components $K_{\mathbb{A}}\colon\mathbb{A}_{\mathsf{sc}}\longrightarrow\mathbb{A}_{\mathsf{cc}}$ are full embeddings. Considering its mate

$$\begin{tikzcd}
\mathsf{Cat}(\mathcal{Q}) \arrow[r, "(-)_{\mathsf{cc}}"] \arrow[d, "(-)_{\mathsf{s}}"'] \arrow[dr, phantom, "\Nearrow L"] & \mathsf{Cat}(\mathcal{Q}) \arrow[d, "(-)_{\mathsf{s}}"] \\
\mathsf{SymCat}(\mathcal{Q}) \arrow[r, "(-)_{\mathsf{sc}}"'] & \mathsf{SymCat}(\mathcal{Q})
\end{tikzcd}$$

we formulate an elementary necessary-and-sufficient condition on $\mathcal{Q}$ under which L is a natural isomorphism (Theorem 3.7); in that case, we say that $\mathcal{Q}$ is a *Cauchy-bilateral* quantaloid (Definition 3.8). If $\mathcal{Q}$ is Cauchy-bilateral, then K is in fact the identity transformation, thus in particular is the Cauchy completion of any symmetric $\mathcal{Q}$-category again symmetric. And moreover, as a corollary, we obtain a distributive law of the Cauchy completion monad over the symmetrisation comonad on $\mathsf{Cat}(\mathcal{Q})$ (Corollary 3.9). In a separate section we point out a number of examples, including Walters' small quantaloids of closed cribles and Lawvere's quantale of non-negative real numbers.

For an overview of the theory of quantaloid-enriched categories, and a list of appropriate historical references, we refer to [Stubbe, 2005], whose notations we adopt.

2. Symmetric quantaloid-enriched categories

In this section, after quickly recalling the notion of involutive quantaloid $\mathcal{Q}$, we give the obvious definition of symmetric $\mathcal{Q}$-category and explain how any $\mathcal{Q}$-category can be symmetrised. Examples are postponed to Section 4.

2.1. Definition. *A* quantaloid *is a* Sup*-enriched category. An* involution *on a quantaloid* $\mathcal{Q}$ *is a* Sup*-functor* $(-)^{\mathsf{o}}\colon \mathcal{Q}^{\mathsf{op}} \longrightarrow \mathcal{Q}$ *which is the identity on objects and satisfies* $f^{\mathsf{oo}} = f$ *for any morphism* f *in* $\mathcal{Q}$*. The pair* $(\mathcal{Q}, (-)^{\mathsf{o}})$ *is then said to form an* involutive quantaloid.

We shall often simply speak of "an involutive quantaloid $\mathcal{Q}$", leaving the notation for the involution understood. Note that the above definition is equivalent to an apparently weaker condition: in fact, any function $f \mapsto f^{\mathsf{o}}$ on the morphisms of a quantaloid $\mathcal{Q}$ such that $f \leq g$ implies $f^{\mathsf{o}} \leq g^{\mathsf{o}}$, $(g \circ f)^{\mathsf{o}} = f^{\mathsf{o}} \circ g^{\mathsf{o}}$, and $f^{\mathsf{oo}} = f$, is an involution. It is furthermore clear that an involution is an isomorphism between $\mathcal{Q}$ and $\mathcal{Q}^{\mathsf{op}}$.

Whenever a morphism $f\colon A \longrightarrow B$ in a quantaloid (or in a locally ordered category, for that matter) is supposed to be a left adjoint, we write f^* for its right adjoint. In many examples there is a big difference between the involute f^{o} and the adjoint f^* of a given morphism f, so morphisms for which involute and adjoint coincide, deserve a name:

2.2. Definition. *In a quantaloid* $\mathcal{Q}$ *with involution* $f \mapsto f^{\mathsf{o}}$*, an* o-symmetric left adjoint *(or simply* symmetric left adjoint *if the context makes the involution clear) is a left adjoint whose right adjoint is its involute.*

Precisely as we write $\mathsf{Map}(\mathcal{Q})$ for the category of left adjoints in $\mathcal{Q}$ (this notation being motivated by the widespread use of the word "map" synonymously with "left adjoint"), we shall write $\mathsf{SymMap}(\mathcal{Q})$ for the category of symmetric left adjoints.

Recall that a category $\mathbb{A}$ enriched in a quantaloid $\mathcal{Q}$ consists of a set $\mathbb{A}_0$ of objects, each $x \in \mathbb{A}_0$ having a type $ta \in \mathcal{Q}_0$, and for any $x, y \in \mathbb{A}_0$ there is a hom-arrow $\mathbb{A}(y, x)\colon tx \longrightarrow ty$ in $\mathcal{Q}$, subject to associativity and unit requirements: $\mathbb{A}(z, y) \circ \mathbb{A}(y, x) \leq \mathbb{A}(z, x)$ and $1_{tx} \leq \mathbb{A}(x, x)$ for all $x, y, z \in \mathbb{A}_0$. A functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ between such $\mathcal{Q}$-categories is an object-map $x \mapsto Fx$ such that $tx = t(Fx)$ and $\mathbb{A}(y, x) \leq \mathbb{B}(Fy, Fx)$ for all $x, y \in \mathbb{A}$. Such a functor is smaller than a functor $G\colon \mathbb{A} \longrightarrow \mathbb{B}$ if $1_{tx} \leq \mathbb{B}(Fx, Gx)$ for every $x \in \mathbb{A}$. With obvious composition one gets a locally ordered 2-category $\mathsf{Cat}(\mathcal{Q})$ of $\mathcal{Q}$-categories and functors.

For two objects $x, y \in \mathbb{A}$, the hom-arrows $\mathbb{A}(y, x)$ and $\mathbb{A}(x, y)$ thus go in opposite directions. Hence, to formulate a notion of "symmetry" for $\mathcal{Q}$-categories, it is far too strong to require $\mathbb{A}(y, x) = \mathbb{A}(x, y)$. Instead, at least for involutive quantaloids, we better do as follows [Betti and Walters, 1982]:

2.3. Definition. *Let* $\mathcal{Q}$ *be a small involutive quantaloid, with involution* $f \mapsto f^{\mathsf{o}}$*. A* $\mathcal{Q}$*-category* $\mathbb{A}$ *is* o-symmetric *(or* symmetric *if there is no confusion about the involved involution) when* $\mathbb{A}(x, y) = \mathbb{A}(y, x)^{\mathsf{o}}$ *for every two objects* $x, y \in \mathbb{A}$.

We shall write $\mathsf{SymCat}(\mathcal{Q})$ for the full sub-2-category of $\mathsf{Cat}(\mathcal{Q})$ determined by the symmetric $\mathcal{Q}$-categories; it is easy to see that the local order in $\mathsf{SymCat}(\mathcal{Q})$ is in fact symmetric (but not anti-symmetric). The full embedding $\mathsf{SymCat}(\mathcal{Q}) \hookrightarrow \mathsf{Cat}(\mathcal{Q})$ has a right adjoint functor[1]:

$$\begin{tikzcd} \mathsf{SymCat}(\mathcal{Q}) \arrow[r, bend left, "\text{incl.}"] \arrow[r, phantom, "\perp"] & \mathsf{Cat}(\mathcal{Q}). \arrow[l, bend left, "(-)_{\mathsf{s}}"] \end{tikzcd} \tag{1}$$

This 'symmetrisation' sends a $\mathcal{Q}$-category $\mathbb{A}$ to the symmetric $\mathcal{Q}$-category $\mathbb{A}_{\mathsf{s}}$ whose objects (and types) are those of $\mathbb{A}$, but for any two objects x, y the hom-arrow is

$$\mathbb{A}_{\mathsf{s}}(y,x) := \mathbb{A}(y,x) \wedge \mathbb{A}(x,y)^{\circ}.$$

A functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ is sent to $F_{\mathsf{s}}\colon \mathbb{A}_{\mathsf{s}} \longrightarrow \mathbb{B}_{\mathsf{s}}\colon a \mapsto Fa$. Quite obviously, the counit of this adjunction has components $S_{\mathbb{A}}\colon \mathbb{A}_{\mathsf{s}} \longrightarrow \mathbb{A}\colon a \mapsto a$.

Recall that a distributor $\Phi\colon \mathbb{A} \longrightarrow\!\!\!\!\!\!\circ\;\;\, \mathbb{B}$ between $\mathcal{Q}$-categories consists of a family arrows $\Phi(y,x)\colon tx \longrightarrow ty$ in $\mathcal{Q}$, one for each $(x,y) \in \mathbb{A}_0 \times \mathbb{B}_0$, subject to $\mathbb{B}(y',y) \circ \Phi(y,x) \leq \Phi(y',x)$ and $\Phi(y,x) \circ \mathbb{A}(x,x') \leq \Phi(y,x')$ for all $y, y' \in \mathbb{B}_0$ and $x, x' \in \mathbb{A}_0$. The composite of such a distributor with another $\Psi\colon \mathbb{B} \longrightarrow\!\!\!\!\!\!\circ\;\;\, \mathbb{C}$ is written as $\Psi \otimes \Phi\colon \mathbb{A} \longrightarrow\!\!\!\!\!\!\circ\;\;\, \mathbb{C}$, and its elements are computed with a "matrix formula": for $x \in \mathbb{A}_0$ and $z \in \mathbb{C}_0$,

$$(\Psi \otimes \Phi)(z,x) = \bigvee_{y \in \mathbb{B}_0} \Psi(z,y) \circ \Phi(y,x).$$

Parallel distributors can be compared elementwise, and in fact one gets a (large) quantaloid $\mathsf{Dist}(\mathcal{Q})$ of $\mathcal{Q}$-categories and distributors. Each functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ determines an adjoint pair of distributors: $\mathbb{B}(-, F-)\colon \mathbb{A} \longrightarrow\!\!\!\!\!\!\circ\;\;\, \mathbb{B}$, with elements $\mathbb{B}(y, Fx)$ for $(x,y) \in \mathbb{A}_0 \times \mathbb{B}_0$, is left adjoint to $\mathbb{B}(F-, -)\colon \mathbb{B} \longrightarrow\!\!\!\!\!\!\circ\;\;\, \mathbb{A}$ in the quantaloid $\mathsf{Dist}(\mathcal{Q})$. These distributors are said to be 'represented by F'. (More generally, a (necessarily left adjoint) distributor $\Phi\colon \mathbb{A} \longrightarrow\!\!\!\!\!\!\circ\;\;\, \mathbb{B}$ is 'representable' if there exists a (necessarily essentially unique) functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ such that $\Phi = \mathbb{B}(-, F-)$.) This amounts to a 2-functor

$$\mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{Map}(\mathsf{Dist}(\mathcal{Q}))\colon \Big(F\colon \mathbb{A} \longrightarrow \mathbb{B}\Big) \mapsto \Big(\mathbb{B}(-, F-)\colon \mathbb{A} \longrightarrow\!\!\!\!\!\!\circ\;\;\, \mathbb{B}\Big). \tag{2}$$

We shall write $\mathsf{SymDist}(\mathcal{Q})$ for the full subquantaloid of $\mathsf{Dist}(\mathcal{Q})$ determined by the symmetric $\mathcal{Q}$-categories. It is easily verified that the involution $f \mapsto f^{\circ}$ on the base quantaloid $\mathcal{Q}$ extends to the quantaloid $\mathsf{SymDist}(\mathcal{Q})$: explicitly, if $\Phi\colon \mathbb{A} \longrightarrow\!\!\!\!\!\!\circ\;\;\, \mathbb{B}$ is a distributor between symmetric $\mathcal{Q}$-categories, then so is $\Phi^{\circ}\colon \mathbb{B} \longrightarrow\!\!\!\!\!\!\circ\;\;\, \mathbb{A}$, with elements $\Phi^{\circ}(a,b) := \Phi(b,a)^{\circ}$. And if $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ is a functor between symmetric $\mathcal{Q}$-categories, then the left adjoint distributor represented by F has the particular feature that it is a symmetric left adjoint in $\mathsf{SymDist}(\mathcal{Q})$ (in the sense of Definition 2.2). That is to say, the functor in (2) restricts to the symmetric situation as

$$\mathsf{SymCat}(\mathcal{Q}) \longrightarrow \mathsf{SymMap}(\mathsf{SymDist}(\mathcal{Q}))\colon \Big(F\colon \mathbb{A} \longrightarrow \mathbb{B}\Big) \mapsto \Big(\mathbb{B}(-, F-)\colon \mathbb{A} \longrightarrow\!\!\!\!\!\!\circ\;\;\, \mathbb{B}\Big), \tag{3}$$

[1] But the right adjoint is not a 2-functor, for this would imply the local order in $\mathsf{Cat}(\mathcal{Q})$ to be symmetric, so this is not a 2-adjunction.

obviously giving a commutative diagram

$$\begin{tikzcd} \mathsf{Cat}(\mathcal{Q}) \arrow[r] & \mathsf{Map}(\mathsf{Dist}(\mathcal{Q})) \\ \mathsf{SymCat}(\mathcal{Q}) \arrow[r] \arrow[u, "\text{incl.}"] & \mathsf{SymMap}(\mathsf{SymDist}(\mathcal{Q})) \arrow[u, "\text{incl.}"'] \end{tikzcd} \tag{4}$$

3. Cauchy completion and symmetric completion

A $\mathcal{Q}$-category $\mathbb{A}$ is said to be 'Cauchy complete' when each left adjoint distributor with codomain $\mathbb{A}$ is represented by a functor [Lawvere, 1973], that is, when for each $\mathcal{Q}$-category $\mathbb{B}$ the functor in (2) determines an equivalence

$$\mathsf{Cat}(\mathcal{Q})(\mathbb{B},\mathbb{A}) \simeq \mathsf{Map}(\mathsf{Dist}(\mathcal{Q}))(\mathbb{B},\mathbb{A}).$$

It is equivalent to require this only for left adjoint presheaves[2] on $\mathbb{A}$. The full inclusion of the Cauchy complete $\mathcal{Q}$-categories in $\mathsf{Cat}(\mathcal{Q})$ admits a left adjoint:

$$\begin{tikzcd} \mathsf{Cat}_{\mathsf{cc}}(\mathcal{Q}) \arrow[r, bend right, "\text{full incl.}"'] \arrow[r, phantom, "\perp"] & \mathsf{Cat}(\mathcal{Q}). \arrow[l, bend right, "(-)_{\mathsf{cc}}"'] \end{tikzcd} \tag{5}$$

That is to say, each $\mathcal{Q}$-category $\mathbb{A}$ has a Cauchy completion $\mathbb{A}_{\mathsf{cc}}$, which can be computed explicitly as follows: objects are the left adjoint presheaves on $\mathbb{A}$, the type of such a left adjoint $\phi\colon *_X \to \mathbb{A}$ is $X \in \mathcal{Q}$, and for another such $\psi\colon *_Y \to \mathbb{A}$ the hom-arrow $\mathbb{A}_{\mathsf{cc}}(\psi,\phi)\colon X \longrightarrow Y$ in $\mathcal{Q}$ is the single element of the composite distributor $\psi^* \otimes \phi$ (where $\psi \dashv \psi^*$). The component at $\mathbb{A} \in \mathsf{Cat}(\mathcal{Q})$ of the unit of this adjunction is a suitable corestriction of the Yoneda embedding: $Y_{\mathbb{A}}\colon \mathbb{A} \longrightarrow \mathbb{A}_{\mathsf{cc}}\colon x \mapsto \mathbb{A}(-,x)$. It is straightforward that $(-)_{\mathsf{cc}}\colon \mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q})$ sends a functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ to $F_{\mathsf{cc}}\colon \mathbb{A}_{\mathsf{cc}} \longrightarrow \mathbb{B}_{\mathsf{cc}}\colon \phi \mapsto \mathbb{B}(-,F-) \otimes \phi$. (For details, see e.g. [Stubbe, 2005, Section 7].)

The Cauchy completion can of course be applied to a symmetric $\mathcal{Q}$-category, but the resulting Cauchy complete category need not be symmetric anymore (see Example 4.7)! That is to say, the functor $(-)_{\mathsf{cc}}\colon \mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q})$ does not restrict to $\mathsf{SymCat}(\mathcal{Q})$ in general. However, its very definition suggests the following modification:

3.1. Definition. *Let $\mathcal{Q}$ be a small involutive quantaloid. A symmetric $\mathcal{Q}$-category $\mathbb{A}$ is* symmetrically complete *if, for any symmetric $\mathcal{Q}$-category $\mathbb{B}$, the functor in (3) determines an equivalence*

$$\mathsf{SymCat}(\mathcal{Q})(\mathbb{B},\mathbb{A}) \simeq \mathsf{SymMap}(\mathsf{SymDist}(\mathcal{Q}))(\mathbb{B},\mathbb{A}).$$

[2] A 'presheaf' on $\mathbb{A}$ is a distributor into $\mathbb{A}$ whose domain is a one-object category with an identity hom-arrow. Writing $*_X$ for the one-object $\mathcal{Q}$-category whose single object $*$ has type $X \in \mathcal{Q}_0$ and whose single hom-arrow is the identity 1_X, a presheaf is then typically written as $\phi\colon *_X \to \mathbb{A}$. (These are really the *contravariant* presheaves on $\mathbb{A}$; the *covariant* presheaves are the distributors from $\mathbb{A}$ to $*_X$. In this paper, however, we shall only consider contravariant presheaves.)

In analogy with the notion of Cauchy completeness of a $\mathcal{Q}$-category, it is straightforward to check the following equivalent expressions:

3.2. PROPOSITION. *Let $\mathcal{Q}$ be a small involutive quantaloid. For a symmetric $\mathcal{Q}$-category $\mathbb{A}$, the following conditions are equivalent:*

1. *$\mathbb{A}$ is symmetrically complete,*
2. *for any symmetric $\mathcal{Q}$-category $\mathbb{B}$, every symmetric left adjoint distributor $\Phi: \mathbb{B} \to\!\!\!\!\!\!\circ\;\; \mathbb{A}$ is representable,*
3. *for any $X \in \mathcal{Q}$, every symmetric left adjoint presheaf $\phi: *_X \to\!\!\!\!\!\!\circ\;\; \mathbb{A}$ is representable.*

And precisely as the Cauchy completion of a $\mathcal{Q}$-category can be computed explicitly with left adjoint presheaves, we can do as follows for the symmetric completion of a symmetric $\mathcal{Q}$-category:

3.3. PROPOSITION. *Let $\mathcal{Q}$ be a small involutive quantaloid. The full embedding of the symmetrically complete symmetric $\mathcal{Q}$-categories in $\mathsf{SymCat}(\mathcal{Q})$ admits a left adjoint:*

$$\begin{tikzcd} \mathsf{SymCat}_{\mathsf{sc}}(\mathcal{Q}) \arrow[r, bend right, "\text{full incl.}"'] \arrow[r, phantom, "\perp"] & \mathsf{SymCat}(\mathcal{Q}). \arrow[l, bend right, "(-)_{\mathsf{sc}}"'] \end{tikzcd} \tag{6}$$

Explicitly, for a symmetric $\mathcal{Q}$-category $\mathbb{A}$, its symmetric completion $\mathbb{A}_{\mathsf{sc}}$ is the full subcategory of $\mathbb{A}_{\mathsf{cc}}$ determined by the symmetric *left adjoint presheaves. The component at $\mathbb{A} \in \mathsf{SymCat}(\mathcal{Q})$ of the unit of this adjunction is a corestriction of the Yoneda embedding: $Y_{\mathbb{A}}: \mathbb{A} \longrightarrow \mathbb{A}_{\mathsf{sc}}: x \mapsto \mathbb{A}(-, x)$.*

Note that $(-)_{\mathsf{sc}}: \mathsf{SymCat}(\mathcal{Q}) \longrightarrow \mathsf{SymCat}(\mathcal{Q})$ sends a functor $F: \mathbb{A} \longrightarrow \mathbb{B}$ between symmetric $\mathcal{Q}$-categories to $F_{\mathsf{sc}}: \mathbb{A}_{\mathsf{sc}} \longrightarrow \mathbb{B}_{\mathsf{sc}}: \phi \mapsto \mathbb{B}(-, F-) \otimes \phi$. Indeed, because $\mathbb{B}$ is symmetric, the distributor $\mathbb{B}(-, F-)$ is a symmetric left adjoint, hence its composition with $\phi \in \mathbb{A}_{\mathsf{sc}}$ gives an object of $\mathbb{B}_{\mathsf{sc}}$.

All this now raises a natural question: given any $\mathcal{Q}$-category $\mathbb{A}$, how does the symmetrisation of its Cauchy completion relate to the symmetric completion of its symmetrisation? It is clear that there is a natural transformation

$$\begin{tikzcd} \mathsf{Cat}(\mathcal{Q}) \arrow[r, "(-)_{\mathsf{cc}}"] & \mathsf{Cat}(\mathcal{Q}) \\ \mathsf{SymCat}(\mathcal{Q}) \arrow[u, "\text{incl.}"] \arrow[r, "(-)_{\mathsf{sc}}"'] \arrow[ur, phantom, "\Nwarrow K"] & \mathsf{SymCat}(\mathcal{Q}) \arrow[u, "\text{incl.}"'] \end{tikzcd} \tag{7}$$

whose components are the full embeddings $K_{\mathbb{A}}: \mathbb{A}_{\mathsf{sc}} \longrightarrow \mathbb{A}_{\mathsf{cc}}: \phi \mapsto \phi$ of which the construction, in Proposition 3.3, of the symmetric completion speaks. From the calculus of mates [Kelly and Street, 1974] at least part of the following statement is then straightforward:

3.4. Proposition. *Let $\mathcal{Q}$ be an involutive quantaloid. There is a natural transformation*

$$\begin{tikzcd}
\mathsf{Cat}(\mathcal{Q}) \arrow[r, "(-)_{\mathsf{cc}}"] \arrow[d, "(-)_{\mathsf{s}}"'] \arrow[dr, phantom, "\overset{L}{\Nearrow}"] & \mathsf{Cat}(\mathcal{Q}) \arrow[d, "(-)_{\mathsf{s}}"] \\
\mathsf{SymCat}(\mathcal{Q}) \arrow[r, "(-)_{\mathsf{sc}}"'] & \mathsf{SymCat}(\mathcal{Q})
\end{tikzcd} \tag{8}$$

whose component at $\mathbb{A}$ in $\mathsf{Cat}(\mathcal{Q})$ is the full embedding[3]

$$L_{\mathbb{A}}\colon (\mathbb{A}_{\mathsf{s}})_{\mathsf{sc}} \longrightarrow (\mathbb{A}_{\mathsf{cc}})_{\mathsf{s}}\colon \phi \mapsto \mathbb{A}(-, S_{\mathbb{A}}-) \otimes \phi.$$

Moreover, each $(\mathbb{A}_{\mathsf{cc}})_{\mathsf{s}}$ is symmetrically complete.

Proof. It is straightforward to check that the explicit definition of L is indeed obtained as the mate of the natural transformation in diagram (7).

Given $\phi, \psi \in (\mathbb{A}_{\mathsf{s}})_{\mathsf{sc}}$, consider the distributors

$$\begin{tikzcd}
*_X \arrow[r, "\phi", "\circ" marking] & \mathbb{A}_{\mathsf{s}} \arrow[r, "\psi^{\circ}", "\circ" marking] \arrow[loop above, "\mathbb{A}_{\mathsf{s}}", "\circ" marking] \arrow[d, shift right, "\mathbb{A}(-, S_{\mathbb{A}}-)"', "\circ" marking] & *_Y \\
& \mathbb{A} \arrow[u, shift right, "\mathbb{A}(S_{\mathbb{A}}-, -)"', "\circ" marking] &
\end{tikzcd}$$

of which we know that $\mathbb{A}_{\mathsf{s}} = \mathbb{A}(S_{\mathbb{A}}-, S_{\mathbb{A}}-) \wedge \mathbb{A}(S_{\mathbb{A}}-, S_{\mathbb{A}}-)^{\circ}$, and compute that

$$\begin{aligned}
\psi^{\circ} \otimes \phi &= \psi^{\circ} \otimes \mathbb{A}_{\mathsf{s}} \otimes \phi \\
&= \psi^{\circ} \otimes (\mathbb{A}(S_{\mathbb{A}}-, S_{\mathbb{A}}-) \wedge \mathbb{A}(S_{\mathbb{A}}-, S_{\mathbb{A}}-)^{\circ}) \otimes \phi \\
&= (\psi^{\circ} \otimes \mathbb{A}(S_{\mathbb{A}}-, S_{\mathbb{A}}-) \otimes \phi) \wedge (\psi^{\circ} \otimes \mathbb{A}(S_{\mathbb{A}}-, S_{\mathbb{A}}-)^{\circ} \otimes \phi) \\
&= (\psi^{\circ} \otimes \mathbb{A}(S_{\mathbb{A}}-, S_{\mathbb{A}}-) \otimes \phi) \wedge (\phi^{\circ} \otimes \mathbb{A}(S_{\mathbb{A}}-, S_{\mathbb{A}}-) \otimes \psi)^{\circ} \\
&= (\psi^{\circ} \otimes \mathbb{A}(S_{\mathbb{A}}-, -) \otimes \mathbb{A}(-, S_{\mathbb{A}}-) \otimes \phi) \wedge (\phi^{\circ} \otimes \mathbb{A}(S_{\mathbb{A}}-, -) \otimes \mathbb{A}(-, S_{\mathbb{A}}-) \otimes \psi)^{\circ} \\
&= (L_{\mathbb{A}}(\psi)^{*} \otimes L_{\mathbb{A}}(\phi)) \wedge (L_{\mathbb{A}}(\phi)^{*} \otimes L_{\mathbb{A}}(\psi))^{\circ}
\end{aligned}$$

which asserts precisely the fully faithfulness of $L_{\mathbb{A}}$. (To pass from the second to the third line, we use that $\psi^{\circ} \otimes - \otimes \phi$ preserves infima, due to $\psi \dashv \psi^{\circ}$ and $\phi \dashv \phi^{\circ}$. From the third to the fourth line we use the involution on $\mathsf{SymDist}(\mathcal{Q})$ provided by the involution on $\mathcal{Q}$. And from line four to line five we use that $\mathbb{A}(S_{\mathbb{A}}-, S_{\mathbb{A}}-) = \mathbb{A}(S_{\mathbb{A}}-, -) \otimes \mathbb{A}(-, S_{\mathbb{A}}-)$.)

[3]Recall that $S_{\mathbb{A}}\colon \mathbb{A}_{\mathsf{s}} \longrightarrow \mathbb{A}\colon a \mapsto a$ is the counit of the adjunction in diagram (1). Trivial as it may seem, it plays a crucial role throughout this section.

For any $a \in (\mathbb{A}_{\mathsf{s}})_0$ it is straightforward that $L_{\mathbb{A}}(\mathbb{A}_{\mathsf{s}}(-,a)) = \mathbb{A}(-,S_{\mathbb{A}}a)$. Putting $\psi = \mathbb{A}_{\mathsf{s}}(-,a)$ in the previous calculation, we thus find for any $\phi \in (\mathbb{A}_{\mathsf{s}})_{\mathsf{sc}}$ that

$$\mathbb{A}_{\mathsf{s}}(a,-)\otimes\phi = \Big(\mathbb{A}(S_{\mathbb{A}}a,-)\otimes L_{\mathbb{A}}(\phi)\Big)\wedge\Big(L_{\mathbb{A}}(\phi)^*\otimes\mathbb{A}(-,S_{\mathbb{A}}a)\Big)^{\circ}.$$

Letting a vary in $\mathbb{A}_{\mathsf{s}}$, this shows that

$$\phi = \Big(\mathbb{A}(S_{\mathbb{A}}-,-)\otimes L_{\mathbb{A}}(\phi)\Big)\wedge\Big(L_{\mathbb{A}}(\phi)^*\otimes\mathbb{A}(-,S_{\mathbb{A}}-)\Big)^{\circ} \tag{9}$$

which implies that $L_{\mathbb{A}}$ is injective on objects.

As for the final part of the proposition, suppose that $\mathbb{C}$ is a Cauchy complete $\mathcal{Q}$-category and that $\phi\colon *_X \mathrel{\circ\!\!\rightarrow} \mathbb{C}_{\mathsf{s}}$ is a symmetric left adjoint. Then there exists a $c \in \mathbb{C}_0$ such that $L_{\mathbb{C}}(\phi) = \mathbb{C}(-,c)$ and we use the formula in (9) to compute that

$$\phi = \Big(\mathbb{C}(S_{\mathbb{C}}-,-)\otimes L_{\mathbb{C}}(\phi)\Big)\wedge\Big(L_{\mathbb{C}}(\phi)^*\otimes\mathbb{C}(-,S_{\mathbb{C}}-)\Big)^{\circ} = \mathbb{C}(S_{\mathbb{C}}-,c)\wedge\mathbb{C}(c,S_{\mathbb{C}}-)^{\circ} = \mathbb{C}_{\mathsf{s}}(-,c).$$

Therefore $\mathbb{C}_{\mathsf{s}}$ is symmetrically complete. This of course applies to $\mathbb{A}_{\mathsf{cc}}$. $\square$

Whereas the previous proposition establishes a *comparison* between $(\mathbb{A}_{\mathsf{s}})_{\mathsf{sc}}$ and $(\mathbb{A}_{\mathsf{cc}})_{\mathsf{s}}$, we shall now study when these two constructions *coincide*. This is related with the symmetrisation not only of $\mathcal{Q}$-categories and functors, but also of left adjoint distributors. We start by putting the formula in (9) in a broader context:

3.5. Lemma. *If $\Psi\colon\mathbb{A}\mathrel{\circ\!\!\rightarrow}\mathbb{B}$ is a left adjoint distributor between categories enriched in a small involutive quantaloid $\mathcal{Q}$, then the distributor*

$$\Psi_{\mathsf{s}} := \Big(\mathbb{B}(S_{\mathbb{B}}-,-)\otimes\Psi\otimes\mathbb{A}(-,S_{\mathbb{A}}-)\Big)\wedge\Big(\mathbb{A}(S_{\mathbb{A}}-,-)\otimes\Psi^*\otimes\mathbb{B}(-,S_{\mathbb{B}}-)\Big)^{\circ}\colon\mathbb{A}_{\mathsf{s}}\mathrel{\circ\!\!\rightarrow}\mathbb{B}_{\mathsf{s}}$$

satisfies $\Psi_{\mathsf{s}}\otimes(\Psi_{\mathsf{s}})^{\circ}\leq\mathbb{B}_{\mathsf{s}}$. Thus Ψ_{s} is a symmetric left adjoint if and only if $\mathbb{A}_{\mathsf{s}}\leq(\Psi_{\mathsf{s}})^{\circ}\otimes\Psi_{\mathsf{s}}$, and if this is the case then it follows that $\Psi = \mathbb{B}(-,S_{\mathbb{B}}-)\otimes\Psi_{\mathsf{s}}\otimes\mathbb{A}(S_{\mathbb{A}}-,-)$.

Proof. It is clear that $\Psi_{\mathsf{s}}\colon\mathbb{A}_{\mathsf{s}}\mathrel{\circ\!\!\rightarrow}\mathbb{B}_{\mathsf{s}}$ is a distributor: it is the infimum of two distributors, the first term of which is a composite of three distributors, and the second term is the involute of a composite of three distributors (which makes sense because domain and codomain of this composite are symmetric $\mathcal{Q}$-categories). Precisely because Ψ_{s} is a distributor between symmetric $\mathcal{Q}$-categories, it makes sense to speak of its involute $(\Psi_{\mathsf{s}})^{\circ}$, and it is straightforward to compute that

$$\begin{aligned}
\Psi_{\mathsf{s}}\otimes(\Psi_{\mathsf{s}})^{\circ} &\leq \Big(\mathbb{B}(S_{\mathbb{B}}-,-)\otimes\Psi\otimes\mathbb{A}(-,S_{\mathbb{A}}-)\Big)\otimes\Big(\mathbb{A}(S_{\mathbb{A}}-,-)\otimes\Psi^*\otimes\mathbb{B}(-,S_{\mathbb{B}}-)\Big)^{\circ\circ}\\
&\leq \mathbb{B}(S_{\mathbb{B}}-,-)\otimes\Psi\otimes\mathbb{A}(-,-)\otimes\Psi^*\otimes\mathbb{B}(-,S_{\mathbb{B}}-)\\
&\leq \mathbb{B}(S_{\mathbb{B}}-,-)\otimes\mathbb{B}(-,-)\otimes\mathbb{B}(-,S_{\mathbb{B}}-)\\
&= \mathbb{B}(S_{\mathbb{B}}-,S_{\mathbb{B}}-)
\end{aligned}$$

and therefore, by involution, also $\Psi_{\mathsf{s}}\otimes(\Psi_{\mathsf{s}})^{\circ}\leq\mathbb{B}(S_{\mathbb{B}}-,S_{\mathbb{B}}-)^{\circ}$ holds, from which we can conclude that $\Psi_{\mathsf{s}}\otimes(\Psi_{\mathsf{s}})^{\circ}\leq\mathbb{B}(S_{\mathbb{B}}-,S_{\mathbb{B}}-)\wedge\mathbb{B}(S_{\mathbb{B}}-,S_{\mathbb{B}}-)^{\circ} = \mathbb{B}_{\mathsf{s}}(-,-)$ as claimed.

Now Ψ_{s} is a symmetric left adjoint if and only if $\Psi_{\mathsf{s}} \dashv (\Psi_{\mathsf{s}})^{\circ}$, and because the counit inequality of this adjunction always holds, this adjunction is equivalent to the truth of the unit inequality $\mathbb{A}_{\mathsf{s}} \leq (\Psi_{\mathsf{s}})^{\circ} \otimes \Psi_{\mathsf{s}}$. Suppose that this is indeed the case, then we can compute that

$$
\begin{array}{rcl}
\Psi & = & \Psi \otimes \mathbb{A}(-, S_{\mathbb{A}}-) \otimes \mathbb{A}_{\mathsf{s}}(-,-) \otimes \mathbb{A}(S_{\mathbb{A}}-,-) \\
& \leq & \Psi \otimes \mathbb{A}(-, S_{\mathbb{A}}-) \otimes (\Psi_{\mathsf{s}})^{\circ} \otimes \Psi_{\mathsf{s}} \otimes \mathbb{A}(S_{\mathbb{A}}-,-) \\
& \leq & \Psi \otimes \mathbb{A}(-, S_{\mathbb{A}}-) \otimes \Big(\mathbb{A}(S_{\mathbb{A}}-,-) \otimes \Psi^{*} \otimes \mathbb{B}(-, S_{\mathbb{B}}-)\Big)^{\circ\circ} \otimes \Psi_{\mathsf{s}} \otimes \mathbb{A}(S_{\mathbb{A}}-,-) \\
& \leq & \Psi \otimes \mathbb{A}(-,-) \otimes \Psi^{*} \otimes \mathbb{B}(-, S_{\mathbb{B}}-) \otimes \Psi_{\mathsf{s}} \otimes \mathbb{A}(S_{\mathbb{A}}-,-) \\
& \leq & \mathbb{B} \otimes \mathbb{B}(-, S_{\mathbb{B}}-) \otimes \Psi_{\mathsf{s}} \otimes \mathbb{A}(S_{\mathbb{A}}-,-) \\
& \leq & \mathbb{B}(-, S_{\mathbb{B}}-) \otimes \Big(\mathbb{B}(S_{\mathbb{B}}-,-) \otimes \Psi \otimes \mathbb{A}(-, S_{\mathbb{A}}-)\Big) \otimes \mathbb{A}(S_{\mathbb{A}}-,-) \\
& \leq & \mathbb{B}(-,-) \otimes \Psi \otimes \mathbb{A}(-,-) \\
& = & \Psi
\end{array}
$$

which means that $\Psi = \mathbb{B}(-, S_{\mathbb{B}}-) \otimes \Psi_{\mathsf{s}} \otimes \mathbb{A}(S_{\mathbb{A}}-,-)$ as claimed. $\square$

The notation introduced in the previous lemma will be used in the remainder of this section. In particular shall we use it in the next proposition.

3.6. Proposition. *For a category $\mathbb{A}$ enriched in a small involutive quantaloid $\mathcal{Q}$, the following conditions are equivalent:*

1. *the functor $L_{\mathbb{A}}\colon (\mathbb{A}_{\mathsf{s}})_{\mathsf{sc}} \longrightarrow (\mathbb{A}_{\mathsf{cc}})_{\mathsf{s}}$ from Proposition 3.4 is surjective on objects (and therefore an isomorphism, with inverse $\psi \mapsto \psi_{\mathsf{s}}$),*
2. *for every left adjoint presheaf $\psi\colon *_X \to \mathbb{A}$, the presheaf $\psi_{\mathsf{s}}\colon *_X \to \mathbb{A}_{\mathsf{s}}$ is a symmetric left adjoint,*
3. *for every left adjoint distributor $\Psi\colon \mathbb{X} \to \mathbb{A}$, the distributor $\Psi_{\mathsf{s}}\colon \mathbb{X}_{\mathsf{s}} \to \mathbb{A}_{\mathsf{s}}$ is a symmetric left adjoint.*

Proof. (1 $\Leftrightarrow$ 2) From (the proof of) Proposition 3.4 we know that $L_{\mathbb{A}}$ is injective on objects, and that $\phi = (L_{\mathbb{A}}(\phi))_{\mathsf{s}}$ for any symmetric left adjoint $\phi\colon *_X \to \mathbb{A}_{\mathsf{s}}$ (this is the formula in (9) rewritten with the notation introduced in Lemma 3.5, taking into account that the domain of ϕ is the symmetric $\mathcal{Q}$-category $*_X$, so that S_{*_X} is the identity on $*_X$). To say that $L_{\mathbb{A}}$ is surjective on objects thus means that for any left adjoint $\psi\colon *_X \to \mathbb{A}$ there exists a (necessarily unique) symmetric left adjoint $\phi\colon *_X \to \mathbb{A}_{\mathsf{s}}$ such that $L_{\mathbb{A}}(\phi) = \psi$. Thus indeed $\psi_{\mathsf{s}} = \phi$ is a symmetric left adjoint. Conversely, if we assume that for every left adjoint presheaf $\psi\colon *_X \to \mathbb{A}$ the presheaf $\psi_{\mathsf{s}}\colon *_X \to \mathbb{A}_{\mathsf{s}}$ is a symmetric left adjoint, then Lemma 3.5 implies $L(\psi_{\mathsf{s}}) = \psi$ so that $L_{\mathbb{A}}$ is surjective on objects.

(3 $\Leftrightarrow$ 2) One implication is trivial. For the other, by Lemma 3.5 we only need to prove that

$$\mathbb{X}_{\mathsf{s}}(y,x) \leq (\Psi_{\mathsf{s}})^{\circ}(y,-) \otimes \Psi_{\mathsf{s}}(-,x)$$

for every left adjoint $\Psi\colon \mathbb{X} \to \mathbb{A}$ and every $x, y \in (\mathbb{X}_{\mathsf{s}})_0$. But for every $x \in \mathbb{X}_0$ we have a left adjoint presheaf $\Psi(-,x)\colon *_{tx} \to \mathbb{A}$ and by hypothesis thus also a symmetric left adjoint presheaf $\Psi(-,x)_{\mathsf{s}}\colon *_{tx} \to \mathbb{A}_{\mathsf{s}}$. Because $\Psi(-,x)_{\mathsf{s}} = \Psi_{\mathsf{s}}(-,x)$ and $(\Psi_{\mathsf{s}})^{\circ}(y,-) = (\Psi_{\mathsf{s}}(-,y))^{\circ} = (\Psi(-,y)_{\mathsf{s}})^{\circ}$, the sought-after inequation is equivalent to

$$\mathbb{X}_{\mathsf{s}}(y,x) \leq (\Psi(-,y)_{\mathsf{s}})^{\circ} \otimes \Psi(-,x)_{\mathsf{s}}.$$

Using the adjunction $\Psi(-,x)_{\mathsf{s}} \dashv (\Psi(-,x)_{\mathsf{s}})^{\circ}$ this is in turn equivalent to

$$\mathbb{X}_{\mathsf{s}}(y,x) \otimes (\Psi(-,x)_{\mathsf{s}})^{\circ} \leq (\Psi(-,y)_{\mathsf{s}})^{\circ}$$

which is an instance of the action inequality $\mathbb{X}_{\mathsf{s}} \otimes (\Psi_{\mathsf{s}})^{\circ} \leq (\Psi_{\mathsf{s}})^{\circ}$ for $(\Psi_{\mathsf{s}})^{\circ}\colon \mathbb{A}_{\mathsf{s}} \to \mathbb{X}_{\mathsf{s}}$. $\Box$

Now we have everything in place to prove our main theorem, establishing in particular an elementary necessary-and-sufficient condition on the base quantaloid $\mathcal{Q}$ under which $(\mathbb{A}_{\mathsf{s}})_{\mathsf{sc}} \cong (\mathbb{A}_{\mathsf{cc}})_{\mathsf{s}}$ holds for *every* $\mathcal{Q}$-category $\mathbb{A}$.

3.7. THEOREM. *For a small involutive quantaloid $\mathcal{Q}$, the following conditions are equivalent:*

1. *each functor $L_{\mathbb{A}}\colon (\mathbb{A}_{\mathsf{s}})_{\mathsf{sc}} \longrightarrow (\mathbb{A}_{\mathsf{cc}})_{\mathsf{s}}$ as in Proposition 3.4 is an isomorphism (making diagram (8) commute up to isomorphism),*
2. *for every left adjoint presheaf $\psi\colon *_X \to \mathbb{A}$, the presheaf $\psi_{\mathsf{s}}\colon *_X \to \mathbb{A}_{\mathsf{s}}$ is a symmetric left adjoint,*
3. *for each left adjoint distributor $\Psi\colon \mathbb{A} \to \mathbb{B}$, the distributor $\Psi_{\mathsf{s}}\colon \mathbb{A}_{\mathsf{s}} \to \mathbb{B}_{\mathsf{s}}$ is a symmetric left adjoint,*
4. *the inclusion $\mathsf{SymMap}(\mathsf{SymDist}(\mathcal{Q})) \longrightarrow \mathsf{Map}(\mathsf{Dist}(\mathcal{Q}))$ admits a right adjoint making the following square commute:*

$$\begin{tikzcd} \mathsf{Cat}(\mathcal{Q}) \arrow[r] \arrow[d, "(-)_{\mathsf{s}}"'] & \mathsf{Map}(\mathsf{Dist}(\mathcal{Q})) \arrow[d, dotted] \\ \mathsf{SymCat}(\mathcal{Q}) \arrow[r] & \mathsf{SymMap}(\mathsf{SymDist}(\mathcal{Q})) \end{tikzcd}$$

5. *for each family $(f_i\colon X \longrightarrow X_i, g_i\colon X_i \longrightarrow X)_{i \in I}$ of morphisms in $\mathcal{Q}$,*

$$\left.\begin{array}{c} \forall j,k \in I:\ f_k \circ g_j \circ f_j \leq f_k \\ \forall j,k \in I:\ g_j \circ f_j \circ g_k \leq g_k \\ 1_X \leq \bigvee_{i \in I} g_i \circ f_i \end{array}\right\} \Longrightarrow\ 1_X \leq \bigvee_{i \in I} (g_i \wedge f_i^{\circ}) \circ (g_i^{\circ} \wedge f_i).$$

In fact, the right adjoint of which the fourth statement speaks, is

$$(-)_{\mathsf{s}}\colon \mathsf{Map}(\mathsf{Dist}(\mathcal{Q})) \longrightarrow \mathsf{SymMap}(\mathsf{SymDist}(\mathcal{Q}))\colon \Big(\Psi\colon \mathbb{A} \to \mathbb{B}\Big) \mapsto \Big(\Psi_{\mathsf{s}}\colon \mathbb{A}_{\mathsf{s}} \to \mathbb{B}_{\mathsf{s}}\Big). \quad (10)$$

PROOF. ($1 \Leftrightarrow 2 \Leftrightarrow 3$) Are taken care of in Proposition 3.6.

($3 \Rightarrow 4$) With the help of Lemma 3.5, it can be checked that the left adjoint distributor $\mathbb{C}(-, S_{\mathbb{C}}-)\colon \mathbb{C}_{\mathsf{s}} \multimap \mathbb{C}$ displays $\mathbb{C}_{\mathsf{s}}$ as the coreflection of a $\mathcal{Q}$-category $\mathbb{C}$ along the inclusion $\mathsf{SymMap}(\mathsf{SymDist}(\mathcal{Q})) \longrightarrow \mathsf{Map}(\mathsf{Dist}(\mathcal{Q}))$: if $\mathbb{A}$ is a symmetric $\mathcal{Q}$-category and $\Psi\colon \mathbb{A} \multimap \mathbb{C}$ is a left adjoint distributor, then by assumption we have that $\Psi_{\mathsf{s}}\colon \mathbb{A} \multimap \mathbb{C}_{\mathsf{s}}$ is a symmetric left adjoint distributor such that $\Psi = \mathbb{C}(-, S_{\mathbb{C}}-) \otimes \Psi_{\mathsf{s}}$; and if $\Phi\colon \mathbb{A} \multimap \mathbb{C}_{\mathsf{s}}$ would be another symmetric left adjoint distributor such that $\Psi = \mathbb{C}(-, S_{\mathbb{C}}-) \otimes \Phi$ (and therefore also $\Psi^* = \Phi^{\circ} \otimes \mathbb{C}(S_{\mathbb{C}}-, -)$), then necessarily

$$
\begin{aligned}
\Psi_{\mathsf{s}} &= \Big(\mathbb{C}(S_{\mathbb{C}}-, -) \otimes \Psi\Big) \wedge \Big(\Psi^* \otimes \mathbb{C}(-, S_{\mathbb{C}}-)\Big)^{\circ} \\
&= \Big(\mathbb{C}(S_{\mathbb{C}}-, -) \otimes \mathbb{C}(-, S_{\mathbb{C}}-) \otimes \Phi\Big) \wedge \Big(\Phi^{\circ} \otimes \mathbb{C}(S_{\mathbb{C}}-, -) \otimes \mathbb{C}(-, S_{\mathbb{C}}-)\Big)^{\circ} \\
&= \Big(\mathbb{C}(S_{\mathbb{C}}-, S_{\mathbb{C}}-) \otimes \Phi\Big) \wedge \Big(\mathbb{C}(S_{\mathbb{C}}-, S_{\mathbb{C}}-)^{\circ} \otimes \Phi\Big) \\
&= \Big(\mathbb{C}(S_{\mathbb{C}}-, S_{\mathbb{C}}-) \wedge \mathbb{C}(S_{\mathbb{C}}-, S_{\mathbb{C}}-)^{\circ}\Big) \otimes \Phi \\
&= \mathbb{C}_{\mathsf{s}} \otimes \Phi \\
&= \Phi
\end{aligned}
$$

(To pass from the third to the fourth line we use that $- \otimes \Phi$ preserves infima because Φ is a left adjoint.) By general theory for adjoint functors, these coreflections $\mathbb{C}(-, S_{\mathbb{C}}-)\colon \mathbb{C}_{\mathsf{s}} \multimap \mathbb{C}$ determine the right adjoint, which is the functor given in (10).

($4 \Rightarrow 1$) Suppose that $G\colon \mathsf{SymMap}(\mathsf{Dist}(\mathcal{Q})) \longrightarrow \mathsf{Map}(\mathsf{Dist}(\mathcal{Q}))$ is right adjoint to the inclusion and makes the square commute; thus G necessarily acts on objects as $\mathbb{C} \mapsto \mathbb{C}_{\mathsf{s}}$. Writing $\varepsilon_{\mathbb{C}}\colon \mathbb{C}_{\mathsf{s}} \multimap \mathbb{C}$ for the counit of the adjunction, $G(\varepsilon_{\mathbb{C}})$ is necessarily the identity distributor on $\mathbb{C}_{\mathsf{s}}$. But on the other hand, by commutativity of the diagram, G also sends $\mathbb{C}(-, S_{\mathbb{C}}-)\colon \mathbb{C}_{\mathsf{s}} \multimap \mathbb{C}$ to the identity distributor on $\mathbb{C}_{\mathsf{s}}$. By general theory for adjoint functors, $G(\varepsilon_{\mathbb{C}}) = G(\mathbb{C}(-, S_{\mathbb{C}}-))$ implies $\varepsilon_{\mathbb{C}} = \mathbb{C}(-, S_{\mathbb{C}}-)$. Thus $\mathbb{C}(-, S_{\mathbb{C}}-)\colon \mathbb{C}_{\mathsf{s}} \multimap \mathbb{C}$ enjoys a universal property, saying in particular that: for every left adjoint $\phi\colon *_X \multimap \mathbb{C}$ there is a unique symmetric left adjoint $\psi\colon *_X \multimap \mathbb{C}_{\mathsf{s}}$ such that $\mathbb{C}(-, S_{\mathbb{C}}-) \otimes \psi = \phi$. In other words, $L_{\mathbb{C}}\colon (\mathbb{C}_{\mathsf{s}})_{\mathsf{sc}} \longrightarrow (\mathbb{C}_{\mathsf{cc}})_{\mathsf{s}}\colon \psi \mapsto \mathbb{C}(-, S_{\mathbb{C}}-) \otimes \psi$ is surjective on objects, and therefore an isomorphism.

($2 \Rightarrow 5$) Putting $\mathbb{A}_0 = I$, $ti = X_i$ and $\mathbb{A}(j, i) = f_j \circ g_j \vee \delta_{ij}$ defines a $\mathcal{Q}$-category $\mathbb{A}$ (the "Kronecker delta" $\delta_{ij}\colon X_i \longrightarrow X_j$ denotes the identity when $i = j$ and the zero morphism otherwise), and putting $\psi(i) = f_i$ defines a presheaf $\psi\colon *_X \multimap \mathbb{A}$ with a right adjoint $\psi^*\colon \mathbb{A} \multimap *_X$ which is given by $\psi^*(i) = g_i$. By hypothesis we infer that $\psi_{\mathsf{s}}\colon *_X \multimap \mathbb{A}_{\mathsf{s}}$ is a symmetric left adjoint. This means in particular that $1_X \leq (\psi_{\mathsf{s}})^{\circ} \otimes \psi_{\mathsf{s}}$, or in other terms $1_X \leq \bigvee_i (f_i^{\circ} \wedge g_i) \circ (f_i \wedge g_i^{\circ})$, as wanted.

($5 \Rightarrow 2$) If $\psi\colon *_X \multimap \mathbb{A}$ is a left adjoint, the family $(\psi(a)\colon X \longrightarrow ta, \psi^*(a)\colon ta \longrightarrow X)_{a \in \mathbb{A}_0}$ of morphisms in $\mathcal{Q}$ is easily seen to satisfy the conditions in the hypothesis, thus

$$1_X \leq \bigvee_{a \in \mathbb{A}_0} \Big(\psi(a)^{\circ} \wedge \psi^*(a)\Big) \circ \Big(\psi(a) \wedge \psi^*(a)\Big)^{\circ}.$$

The right hand side is exactly $\psi_{\mathsf{s}}^{\circ} \otimes \psi_{\mathsf{s}}$ so this is equivalent to $\psi_{\mathsf{s}} \dashv (\psi_{\mathsf{s}})^{\circ}$ (cf. Lemma 3.5). $\square$

For further reference, we give a name to those quantaloids (small or large) that satisfy the fifth condition in the above Theorem 3.7:

3.8. Definition. *A quantaloid $\mathcal{Q}$ is* Cauchy-bilateral *if it is involutive (with involution $f \mapsto f^{\circ}$) and for each family $(f_i\colon X \longrightarrow X_i, g_i\colon X_i \longrightarrow X)_{i \in I}$ of morphisms in $\mathcal{Q}$,*

$$\left.\begin{array}{c} \forall j,k \in I:\ \ f_k \circ g_j \circ f_j \le f_k \\ \forall j,k \in I:\ \ g_j \circ f_j \circ g_k \le g_k \\ 1_X \le \bigvee_{i \in I} g_i \circ f_i \end{array}\right\} \Longrightarrow\ 1_X \le \bigvee_{i \in I} (g_i \wedge f_i^{\circ}) \circ (g_i^{\circ} \wedge f_i).$$

Thus, a *small* Cauchy-bilateral quantaloid $\mathcal{Q}$ is precisely one that satisfies the equivalent conditions in Theorem 3.7.

To finish this section we explain an important consequence of Theorem 3.7, containing an answer to R. Betti and R.F.C. Walters' [1982] question about the symmetry of the Cauchy completion of a symmetric category:

3.9. Corollary. *If $\mathcal{Q}$ is a small Cauchy-bilateral quantaloid, then the following diagrams commute:*

$$\begin{tikzcd} \mathsf{Cat}(\mathcal{Q}) \arrow[r, "(-)_{\mathsf{cc}}"] & \mathsf{Cat}(\mathcal{Q}) \\ \mathsf{SymCat}(\mathcal{Q}) \arrow[u, "\text{incl.}"] \arrow[r, "(-)_{\mathsf{cc}}"'] & \mathsf{SymCat}(\mathcal{Q}) \arrow[u, "\text{incl.}"'] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{Cat}(\mathcal{Q}) \arrow[r, "(-)_{\mathsf{s}}"] & \mathsf{Cat}(\mathcal{Q}) \\ \mathsf{Cat}_{\mathsf{cc}}(\mathcal{Q}) \arrow[u, "\text{incl.}"] \arrow[r, "(-)_{\mathsf{s}}"'] & \mathsf{Cat}_{\mathsf{cc}}(\mathcal{Q}) \arrow[u, "\text{incl.}"'] \end{tikzcd}$$

Proof. Suppose that the equivalent conditions in Theorem 3.7 hold. If $\psi\colon *_X \mathrel{-\!\!\circ\!\!\rightarrow} \mathbb{A}$ is a left adjoint presheaf on a symmetric $\mathcal{Q}$-category, then $\psi_{\mathsf{s}}\colon *_X \mathrel{-\!\!\circ\!\!\rightarrow} \mathbb{A}$ is a symmetric left adjoint presheaf which satisfies $\psi = \mathbb{A}(-, S_{\mathbb{A}}-) \otimes \psi_{\mathsf{s}} = \psi_{\mathsf{s}}$ (by Lemma 3.5 and symmetry of $\mathbb{A}$). So ψ is necessarily a symmetric left adjoint. Hence the full embedding $\mathbb{A}_{\mathsf{sc}} \hookrightarrow \mathbb{A}_{\mathsf{cc}}$ is surjective-on-objects, or in other words, $\mathbb{A}_{\mathsf{sc}} = \mathbb{A}_{\mathsf{cc}}$. Therefore the Cauchy completion of a symmetric $\mathcal{Q}$-category is symmetric, making the first square commute. Now suppose that $\mathbb{C}$ is a Cauchy complete category. In (the proof of) Proposition 3.4 it was stipulated that $\mathbb{C}_{\mathsf{s}}$ is symmetrically complete, so – knowing now that the symmetric completion and the Cauchy completion of any symmetric $\mathcal{Q}$-category coincide – it follows that $\mathbb{C}_{\mathsf{s}}$ is Cauchy complete too, making the second square commute. $\square$

This corollary implies that, whenever a small $\mathcal{Q}$ is Cauchy-bilateral, there is a *distributive law* [Beck, 1969; Street, 1972; Power and Watanabe, 2002] of the Cauchy completion monad over the symmetrisation comonad on the category $\mathsf{Cat}(\mathcal{Q})$. More precisely, the reflective subcategory $\mathsf{Cat}_{\mathsf{cc}}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q})$ is the category of algebras of a monad $(\mathcal{T}, \mu, \eta)$ on $\mathsf{Cat}(\mathcal{Q})$; and similarly, the coreflective subcategory $\mathsf{SymCat}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q})$ is the category of coalgebras of a comonad $(\mathcal{D}, \delta, \varepsilon)$ on $\mathsf{Cat}(\mathcal{Q})$. (The functors $\mathcal{T}$ and $\mathcal{D}$ are precisely $(-)_{\mathsf{cc}}$ and $(-)_{\mathsf{s}}$, of course.) As shown in [Power and Watanabe, 2002, Theorems 3.10 and 5.10], the commutativity of the squares in Corollary 3.9 is *equivalent* to the existence of a natural transformation $\lambda\colon \mathcal{T} \circ \mathcal{D} \Longrightarrow \mathcal{D} \circ \mathcal{T}$ making the following diagrams commute:

$$\begin{tikzcd}
\mathcal{T}\mathcal{T}\mathcal{D}\mathbb{C} \arrow[r, "\mu_{\mathcal{D}\mathbb{C}}"] \arrow[d, "\mathcal{T}\lambda_{\mathbb{C}}"'] & \mathcal{T}\mathcal{D}\mathbb{C} \arrow[dd, "\lambda_{\mathbb{C}}"] & \\
\mathcal{T}\mathcal{D}\mathcal{T}\mathbb{C} \arrow[d, "\lambda_{\mathcal{T}\mathbb{C}}"'] & & \mathcal{D}\mathbb{C} \arrow[ul, "\eta_{\mathcal{D}\mathbb{C}}"'] \arrow[dl, "\mathcal{D}\eta_{\mathbb{C}}"] \\
\mathcal{D}\mathcal{T}\mathcal{T}\mathbb{C} \arrow[r, "\mathcal{D}\mu_{\mathbb{C}}"'] & \mathcal{D}\mathcal{T}\mathbb{C} &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathcal{T}\mathcal{D}\mathbb{C} \arrow[r, "\mathcal{T}\delta_{\mathbb{C}}"] \arrow[dl, "\mathcal{T}\varepsilon_{\mathbb{C}}"'] \arrow[dd, "\lambda_{\mathbb{C}}"'] & \mathcal{T}\mathcal{D}\mathcal{D}\mathbb{C} \arrow[d, "\lambda_{\mathcal{D}\mathbb{C}}"] \\
\mathcal{T}\mathbb{C} & & \mathcal{D}\mathcal{T}\mathcal{D}\mathbb{C} \arrow[d, "\mathcal{D}\lambda_{\mathbb{C}}"] \\
& \mathcal{D}\mathcal{T}\mathbb{C} \arrow[ul, "\varepsilon_{\mathcal{T}\mathbb{C}}"] \arrow[r, "\delta_{\mathcal{T}\mathbb{C}}"'] & \mathcal{D}\mathcal{D}\mathcal{T}\mathbb{C}
\end{tikzcd}$$

This, in turn, says exactly that λ is a *distributive law of the monad $\mathcal{T}$ over the comonad $\mathcal{D}$* [Power and Watanabe, 2002, Definition 6.1]. Because $\mathcal{T}$ and $\mathcal{D}$ arise from (co)reflective subcategories, there is *at most one* such distributive law; its components are necessarily

$$\lambda_{\mathbb{C}}\colon (\mathbb{C}_{\mathsf{s}})_{\mathsf{cc}} \longrightarrow (\mathbb{C}_{\mathsf{cc}})_{\mathsf{s}}\colon \phi \mapsto \mathbb{C}(-, S_{\mathbb{C}}-) \otimes \phi.$$

(Thus $\lambda_{\mathbb{C}}$ is precisely the functor $L_{\mathbb{C}}$ of Proposition 3.4, reckoning that – under the conditions of Theorem 3.7 – the symmetric completion of a symmetric $\mathcal{Q}$-category coincides with its Cauchy completion.) It is a consequence of the general theory of distributive laws that the monad $\mathcal{T}$ restricts to the category of $\mathcal{D}$-coalgebras, that the comonad $\mathcal{D}$ restricts to the category of $\mathcal{T}$-algebras, and that the categories of (co)algebras for these restricted (co)monads are equivalent to each other and are further equivalent to the category of so-called λ-*bialgebras* [Power and Watanabe, 2002, Corollary 6.8]. In the case at hand, a λ-bialgebra is simply a $\mathcal{Q}$-category which is both symmetric and Cauchy-complete (the "λ-compatibility" between algebra and coalgebra structure is trivially satisfied), and a morphism between λ-bialgebras is simply a functor between such $\mathcal{Q}$-categories.

4. Examples

4.1. Example. [Commutative quantales] A quantale is, by definition, a one-object quantaloid. (For some authors, a quantale need not be unital, so for them it is *not* a one-object quantaloid; but for us, a quantale is always unital.) Put differently, a quantale is a monoid in the monoidal category Sup (whereas a quantaloid is a Sup-enriched category). Obviously, a quantale Q is commutative if and only if the identity function $1_Q\colon Q \longrightarrow Q$ is an involution.

As we shall point out below, many an interesting involutive quantaloid $\mathcal{Q}$ satisfies the following condition:

4.2. DEFINITION. *A quantaloid $\mathcal{Q}$ is* strongly Cauchy-bilateral *when it is involutive (with involution $f \mapsto f^{\circ}$) and for any family $(f_i\colon X \longrightarrow X_i, g_i\colon X_i \longrightarrow X)_{i\in I}$ of morphisms in $\mathcal{Q}$,*

$$1_X \leq \bigvee_i g_i \circ f_i \implies 1_X \leq \bigvee_i (f_i^{\circ} \wedge g_i) \circ (f_i \wedge g_i^{\circ}).$$

Obviously, a strongly Cauchy-bilateral $\mathcal{Q}$ is Cauchy-bilateral in the sense of Definition 3.8. For a so-called integral quantaloid – that is, when the top element of each $\mathcal{Q}(X, X)$ is 1_X – Cauchy-bilaterality and strong Cauchy-bilaterality are equivalent notions, but in general the latter is *strictly stronger* than the former:

4.3. EXAMPLE. We take an example of an involutive quantale from [Resende, 2007, Example 3.18]: the complete lattice with Hesse diagram

$$\begin{tikzcd}[column sep=small, row sep=small] & \top \arrow[dl, dash] \arrow[ddr, dash] & \\ 1 \arrow[dd, dash] & & \\ & & a \arrow[ddl, dash] \\ b \arrow[dr, dash] & & \\ & 0 & \end{tikzcd}$$

together with the commutative multiplication defined by $a \circ \top = \top$, $a \circ a = b$ and $a \circ b = a$. This gives a quantale which we can equip with the identity involution. It is straightforward to check that this quantale is Cauchy-bilateral, but not strongly so.

4.4. EXAMPLE. [Generalised metric spaces] The condition for strong Cauchy-bilaterality is satisfied by the integral and commutative quantale $Q = ([0, \infty], \bigwedge, +, 0)$ with its trivial involution: for any family $(a_i, b_i)_{i\in I}$ of pairs of elements of $[0, \infty]$, if $\bigwedge_i (a_i + b_i) \leq 0$ is assumed then

$$\bigwedge_i (\max\{a_i, b_i\} + \max\{a_i, b_i\}) = 2 \cdot \bigwedge_i \max\{a_i, b_i\} \leq 2 \cdot \bigwedge_i (a_i + b_i) \leq 0.$$

This "explains" the well known fact that the Cauchy completion of a symmetric generalised metric space [Lawvere, 1973] is again symmetric.

4.5. EXAMPLE. [Locales] Any locale $(L, \bigvee, \wedge, \top)$ is a commutative (hence trivially involutive) and integral quantale; it is easily checked that L is strongly Cauchy-bilateral. Splitting the idempotents of the Sup-monoid $(L, \wedge, \top)$ gives an integral quantaloid with an obvious involution; it too is strongly Cauchy-bilateral.

4.6. EXAMPLE. [Groupoid-quantaloids with canonical involution] For a category $\mathcal{C}$, let $\mathcal{Q}(\mathcal{C})$ be the quantaloid with the same objects as $\mathcal{C}$ but where $\mathcal{Q}(\mathcal{C})(X, Y)$ is the complete lattice of subsets of $\mathcal{C}(X, Y)$, composition is done "pointwise" (for $S \subseteq \mathcal{C}(X, Y)$ and $T \subseteq \mathcal{C}(Y, Z)$ let $T \circ S := \{t \circ s \mid t \in T, s \in S\}$) and the identity on an object X is the singleton $\{1_X\}$. With local suprema in $\mathcal{Q}(\mathcal{C})$ given by union, it is straightforward to check that $\mathcal{Q}(\mathcal{C})$ is a quantaloid; it is the free quantaloid on the category $\mathcal{C}$.

If $\mathcal{G}$ is a groupoid, then $S \mapsto S^\circ := \{s^{-1} \mid s \in S\}$ is the *canonical involution* on $\mathcal{Q}(\mathcal{G})$. For any family $(T_i \subseteq \mathcal{G}(X, X_i), S_i \subseteq \mathcal{G}(X_i, X))_{i \in I}$ we can prove that

$$1_X \in \bigcup_i S_i \circ T_i \implies 1_X \in \bigcup_i (S_i^\circ \cap T_i) \circ (S_i \cap T_i^\circ).$$

Indeed, the premise says that there is a $i_0 \in I$ for which we have $x \in S_{i_0}$ and $y \in T_{i_0}$ such that $1_X = x \circ y$ in $\mathcal{G}$. But then $y \in S_{i_0}^\circ \cap T_{i_0}$ and $x \in S_{i_0} \cap T_{i_0}^\circ$, so that the conclusion follows. That is to say, $\mathcal{Q}(\mathcal{G})$ is strongly Cauchy-bilateral.

4.7. Example. [Commutative group-quantales with trivial involution] For a commutative group $(G, \cdot, 1)$, also the group-quantale $\mathcal{Q}(G)$ is commutative, and – in contrast with the above example – it can therefore be equipped with the *trivial involution* $S \mapsto S^\circ := S$. Betti and Walters [1982] gave a simple example of such a commutative group-quantale with trivial involution for which the Cauchy completion of a symmetric enriched category is not necessarily symmetric. We repeat it here: Let $G = \{1, a, b\}$ be the commutative group defined by $a \cdot a = b$, $b \cdot b = a$ and $a \cdot b = 1$; then Betti and Walters showed that the Cauchy completion of the (symmetric) singleton $\mathcal{Q}(G)$-category whose hom is $\{1\}$, is not symmetric. In fact, the pair $(\{a\}, \{b\})$ of elements of $\mathcal{Q}(G)$ does satisfy the premise but not the conclusion of the condition in Definition 3.8: thus, in retrospect, Theorem 3.7 predicts that there must exist a symmetric category whose Cauchy completion is no longer symmetric.

There is a common generalisation of Examples 4.5 and 4.6, due to [Walters, 1982]: given a small site $(\mathcal{C}, J)$, there is an involutive quantaloid $\mathcal{R}(\mathcal{C}, J)$ such that the category of symmetric and Cauchy complete $\mathcal{R}(\mathcal{C}, J)$-categories is equivalent to $\mathsf{Sh}(\mathcal{C}, J)$. We shall spell out this important example, and show that it is strongly Cauchy-bilateral (and thus also satisfies the equivalent conditions in Theorem 3.7). In retrospect, this proves that the *symmetric and Cauchy complete* $\mathcal{R}(\mathcal{C}, J)$*-categories* can be computed as the *Cauchy completions of the symmetric* $\mathcal{R}(\mathcal{C}, J)$*-categories*.

4.8. Example. [Quantaloids determined by small sites] If $\mathcal{C}$ is a small category, then the small quantaloid $\mathcal{R}(\mathcal{C})$ of *cribles* in $\mathcal{C}$ is the full sub-quantaloid of $\mathsf{Rel}(\mathsf{Set}^{\mathcal{C}^{\mathsf{op}}})$ whose objects are the representable presheaves. It is useful to have an explicit description. We write a *span* in $\mathcal{C}$ as $(f, g)\colon D \to\!\!\!\mapsto C$, and intend it to be a pair of arrows with $\mathsf{dom}(f) = \mathsf{dom}(g)$, $\mathsf{cod}(f) = C$ and $\mathsf{cod}(g) = D$. A crible $R\colon D \to\!\!\!\!\!\!\mapsto\!\!\mapsto C$ is then a set of spans $D \to\!\!\!\mapsto C$ such that for any $(f, g) \in R$ and any $h \in \mathcal{C}$ with $\mathsf{cod}(h) = \mathsf{dom}(f)$, also $(f \circ h, g \circ h) \in R$. Composition in $\mathcal{R}(\mathcal{C})$ is obvious: for $R\colon D \to\!\!\!\!\!\!\mapsto\!\!\mapsto C$ and $S\colon E \to\!\!\!\!\!\!\mapsto\!\!\mapsto D$ the elements of $R \circ S\colon E \to\!\!\!\!\!\!\mapsto\!\!\mapsto C$ are the spans $(f, g) : E \to\!\!\!\mapsto C$ for which there exists a morphism $h \in \mathcal{C}$ such that $(f, h) \in R$ and $(h, g) \in S$. The identity crible $\mathsf{id}_C\colon C \to\!\!\!\!\!\!\mapsto\!\!\mapsto C$ is the set $\{(f, f) \mid \mathsf{cod}(f) = C\}$, and the supremum of a set of cribles is their union. In fact, $\mathcal{R}(\mathcal{C})$ is an involutive quantaloid: the involute $R^\circ\colon C \to\!\!\!\!\!\!\mapsto\!\!\mapsto D$ of a crible $R\colon D \to\!\!\!\!\!\!\mapsto\!\!\mapsto C$ is obtained by reversing the spans in R.

If J is a Grothendieck topology on the category $\mathcal{C}$, then there is a locally left exact nucleus[4] $j\colon \mathcal{R}(\mathcal{C}) \longrightarrow \mathcal{R}(\mathcal{C})$ sending a crible $R\colon D \twoheadrightarrow C$ to

$$j(R) := \{(f,g)\colon D \twoheadrightarrow C \mid \exists S \in J(\mathsf{dom}(f)) : \forall s \in S,\ (g \circ s, f \circ s) \in R\}.$$

Conversely, if $j\colon \mathcal{R}(\mathcal{C}) \longrightarrow \mathcal{R}(\mathcal{C})$ is a locally left exact nucleus, then

$$J(C) := \{S \text{ is a sieve on } C \mid \mathsf{id}_C \leq j(\{(s,s) \mid, s \in S\})\}$$

defines a Grothendieck topology on $\mathcal{C}$. These procedures are each other's inverse [Betti and Carboni, 1983; Rosenthal, 1996]. For a small site $(\mathcal{C}, J)$ we write, following [Walters, 1982], $\mathcal{R}(\mathcal{C}, J)$ for the quotient quantaloid $\mathcal{R}(\mathcal{C})_j$ where j is the nucleus determined by the Grothendieck topology J; it is the *small quantaloid of closed cribles* determined by the site $(\mathcal{C}, J)$. Because every locally left exact nucleus on $\mathcal{R}(\mathcal{C})$ preserves the involution, $\mathcal{R}(\mathcal{C}, J)$ too is involutive. (Walters [1982] originally called $\mathcal{R}(\mathcal{C}, J)$ a 'bicategory of relations', wrote it as $\mathsf{Rel}(\mathcal{C}, J)$, and called its arrows 'relations'. To avoid confusion with other constructions that have been called 'bicategories of relations' since then, we prefer to speak of 'small quantaloids of closed cribles'. For an axiomatic study of these, we refer to [Heymans and Stubbe, 2011].)

Any locale L can be thought of as a site $(\mathcal{C}, J)$, where $\mathcal{C}$ is the ordered set L and J is its so-called canonical topology (so $J(u)$ is the set of all covering families of $u \in L$): it is easily verified that $\mathcal{R}(\mathcal{C}, J)$ is then isomorphic (as involutive quantaloid) to the quantaloid obtained by splitting the idempotents in the Sup-monoid L. And if $\mathcal{G}$ is a small groupoid and J is the smallest Grothendieck topology on $\mathcal{G}$, then the quantaloid of relations $\mathcal{R}(\mathcal{G}, J)$ equals the quantaloid of cribles $\mathcal{R}(\mathcal{G})$, which in turn is isomorphic (as involutive quantaloid) to the free quantaloid $\mathcal{Q}(\mathcal{G})$ with its canonical involution. Indeed, any crible $R\colon X \twoheadrightarrow Y$ in $\mathcal{G}$ determines the subset $F(R) := \{h^{-1} \circ g \mid (g,h) \in R\}$ of $\mathcal{G}(X,Y)$. Conversely, for any subset S of $\mathcal{G}(X,Y)$ let $G(S)$ be the smallest crible containing the set of spans $\{(1_X, s) \mid s \in S\}$ in $\mathcal{G}$. Then $R \mapsto F(R)$ and $S \mapsto G(S)$ extend to functors $F\colon \mathcal{R}(\mathcal{G}) \longrightarrow \mathcal{Q}(\mathcal{G})$ and $G\colon \mathcal{Q}(\mathcal{G}) \longrightarrow \mathcal{R}(\mathcal{G})$ which are each other's inverse and which preserve the involution. Hence both Examples 4.5 and 4.6 are covered by the construction of the quantaloid $\mathcal{R}(\mathcal{C}, J)$ from a small site $(\mathcal{C}, J)$.

Now we show that $\mathcal{R}(\mathcal{C}, J)$ is strongly Cauchy-bilateral, as in Definition 4.2. This claim is equivalent to saying that for any family $(F_i\colon X \twoheadrightarrow X_i, G_i\colon X_i \twoheadrightarrow X)_{i \in I}$ of cribles in $\mathcal{C}$ we have

$$j(\mathsf{id}_X) \subseteq j\Big(\bigcup_i j\Big(j(G_i) \circ j(F_i)\Big)\Big) \implies j(\mathsf{id}_X) \subseteq j\Big(\bigcup_i j\Big((j(F_i)^\circ \cap j(G_i) \circ (j(F_i) \cap j(G_i)^\circ)\Big)\Big)$$

in the involutive quantaloid $\mathcal{R}(\mathcal{C})$ with left exact nucleus j constructed from J.

[4]A nucleus j on a quantaloid $\mathcal{Q}$ is a lax functor $j\colon \mathcal{Q} \longrightarrow \mathcal{Q}$ which is the identity on objects and such that each $j\colon \mathcal{Q}(X,Y) \longrightarrow \mathcal{Q}(X,Y)$ is a closure operator; it is locally left exact if it preserves finite infima of arrows. If $j\colon \mathcal{Q} \longrightarrow \mathcal{Q}$ is a nucleus on a quantaloid, then there is a quotient quantaloid $\mathcal{Q}_j$ of j-closed morphisms, that is, those $f \in \mathcal{Q}$ for which $j(f) = f$.

Take the left-hand side: it is equivalent (by general computations with the nucleus j) to $\mathsf{id}_X \subseteq j(\bigcup_i G_i \circ F_i)$ in $\mathcal{R}(\mathcal{C})$. By definition this means that, for any morphism x in $\mathcal{C}$ with $\mathsf{cod}(x) = X$, $(x, x) \in j(\bigcup_i G_i \circ F_i)$, but because $j(\bigcup_i G_i \circ F_i)$ is a crible, this is equivalent to requiring simply that $(1_X, 1_X) \in j(\bigcup_i G_i \circ F_i)$ (and here 1_X denotes the identity on X in $\mathcal{C}$). Spelling out the definition of the nucleus j in terms of the Grothendieck topology J this means that: there exists a covering sieve $S \in J(X)$ such that for all $s \in S$ there exists an $i_s \in I$ satisfying $(s, s) \in G_{i_s} \circ F_{i_s}$. In a similar fashion the right-hand side can be seen to say precisely that: there exists a covering sieve $S \in J(X)$ such that for all $s \in S$ there exists an $i_s \in I$ satisfying $(s, s) \in (F^{\circ}_{i_s} \cap G_{i_s}) \circ (F_{i_s} \cap G^{\circ}_{i_s})$. The implication is now straightforward.

The following example further generalises the previous one.

4.9. Example. [Locally localic and modular quantaloids] Following [Freyd and Scedrov, 1990] we say that a quantaloid $\mathcal{Q}$ is locally localic when each $\mathcal{Q}(X, Y)$ is a locale; and $\mathcal{Q}$ is modular if it is involutive and when for any morphisms $f\colon Z \longrightarrow Y$, $g\colon Y \longrightarrow X$ and $h\colon Z \longrightarrow X$ in $\mathcal{Q}$ we have $gf \wedge h \leq g(f \wedge g^{\circ}h)$ (or equivalently, $gf \wedge h \leq (g \wedge hf^{\circ})f$). (Here we write the composition in $\mathcal{Q}$ by juxtaposition to avoid overly bracketed expressions.) In fact, every locally localic and modular quantaloid $\mathcal{Q}$ is strongly Cauchy-bilateral: suppose that $(f_i\colon X \longrightarrow X_i, g_i\colon X_i \longrightarrow X)_{i \in I}$ is a family of morphisms in $\mathcal{Q}$ such that $1_X \leq \bigvee_i g_i f_i$, then we can compute that:

$$\begin{aligned}
1_X &= 1_X \wedge \bigvee_i g_i f_i \\
&= \bigvee_i (1_X \wedge g_i f_i) \\
&= \bigvee_i (1_X \wedge (1_X \wedge g_i f_i)) \\
&\leq \bigvee_i (1_X \wedge g_i(g_i^{\circ} 1_X \wedge f_i)) \\
&= \bigvee_i (1_X \wedge g_i(g_i^{\circ} \wedge f_i)) \\
&\leq \bigvee_i (1_X (g_i^{\circ} \wedge f_i)^{\circ} \wedge g_i)(g_i^{\circ} \wedge f_i) \\
&= \bigvee_i (g_i \wedge f_i^{\circ})(g_i^{\circ} \wedge f_i).
\end{aligned}$$

To pass from the first to the second line we used that $\mathcal{Q}(X, X)$ is a locale, and both inequalities were introduced by use of the modular law.

Any small quantaloid of relations $\mathcal{R}(\mathcal{C}, J)$ is in fact locally localic and modular, and thus its strong Cauchy-bilaterality follows from the above computation. But to prove that $\mathcal{R}(\mathcal{C}, J)$ is modular, is not simpler than to prove directly that it satisfies the condition in Definition 4.2, as we did in Example 4.8. (The quantale in Example 4.4 is locally localic but not modular, and the quantale in Example 4.3 is neither locally localic nor modular; but both are Cauchy-bilateral.)

4.10. Example. [Sets and relations] The quantaloid Rel of sets and relations is not small, but it is involutive (by taking opposites: $R^{\circ} = \{(y,x) \mid (x,y) \in R\}$) and it is strongly Cauchy-bilateral. In fact, this holds for any quantaloid $\mathsf{Rel}(\mathcal{E})$ of internal relations in a Grothendieck topos $\mathcal{E}$, because it is modular and locally localic [Freyd and Scedrov, 1990].

There is a subtle difference between Examples 4.8 and 4.10: the former deals with the *small* quantaloid $\mathcal{R}(\mathcal{C}, J)$ built from a small site, the latter deals with the *large* quantaloid $\mathsf{Rel}(\mathsf{Sh}(\mathcal{C}, J))$ of relations between the sheaves on that site. However, both constructions give rise to a Cauchy-bilateral quantaloid. We shall further analyse the interplay between these quantaloids in a forthcoming paper.

Finally we mention a difference between "symmetric" and "discrete" $\mathcal{Q}$-categories.

4.11. Example. [Symmetric vs. discrete] In any locally ordered category $\mathcal{K}$, an object D is said to be *discrete* when, for any other object $X \in \mathcal{K}$, the order $\mathcal{K}(X, D)$ is symmetric. It is straightforward to verify that, whenever $\mathcal{Q}$ is a small Cauchy-bilateral quantaloid, every symmetric and Cauchy complete $\mathcal{Q}$-category is a discrete object of $\mathsf{Cat}_{\mathsf{cc}}(\mathcal{Q})$. However, not all discrete objects of $\mathsf{Cat}_{\mathsf{cc}}(\mathcal{Q})$ need to be symmetric, not even when $\mathcal{Q}$ is Cauchy-bilateral! A counterexample can be found in the theory of generalised metric spaces: Suppose that X is a set and $R \subseteq X \times X$. For $x, y \in X$, a *path* from x to y is a sequence $\alpha = (x_0, x_1, \ldots, x_n)$ of elements of X with $x_0 = x$, $x_n = y$ and $(x_i, x_{i+1}) \in R$ for all $i < n$; the *length* $l(\alpha)$ of such a path α is then n. For every $x \in X$, (x) is a path from x to x of length 0. It is easy to verify that

$$d_R(x, y) := \bigwedge \{l(\alpha) \mid \alpha \text{ is a path from } x \text{ to } y\}$$

turns X into a generalised metric space. (This infimum is a minimum, except when there is no path from x to y, in which case $d_R(x, y) = \infty$.) Any such space (X, d_R) is Cauchy complete, as is every generalised metric space with values in $\mathbb{N} \cup \{\infty\}$. And, in fact, it is discrete in the sense given above, because $x \leq y$ if and only if $0 \geq d_R(x, y)$, so there is a path with length 0 from x to y, which means that $x = y$. However, choosing $X = \{0, 1\}$ and $R = \{(0, 1)\}$ gives a non-symmetric example: $d_R(0, 1) = 1 \neq \infty = d_R(1, 0)$.

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Hans Heymans
Department of Mathematics and Computer Science, University of Antwerp
Middelheimlaan 1, 2020 Antwerpen, Belgium

Isar Stubbe
Laboratoire de Mathématiques Pures et Appliquées, Université du Littoral-Côte d'Opale
50 rue Ferdinand Buisson, 62228 Calais, France
Email: `hans.heymans@ua.ac.be`
`isar.stubbe@lmpa.univ-littoral.fr`

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