Theory and Applications of Categories, Vol. 25, No. 16, 2011, pp. 426–435.

SEMIDIRECT PRODUCTS AND CROSSED MODULES IN VARIETIES OF RIGHT Ω-LOOPS

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ABSTRACT. We present a new explicit construction of categorical semidirect products in an arbitrary variety $\mathbf{V}$ of right Ω-loops and use it to obtain simplified descriptions of internal precrossed and crossed modules in $\mathbf{V}$.

1. Introduction

Categorical semidirect products were introduced by D. Bourn and G. Janelidze in [6], and, as follows from the results of [6], they exist in all semi-abelian categories in the sense of G. Janelidze, L. Márki and W. Tholen [13], and in particular in all semi-abelian varieties of universal algebras. They have also been studied in several contexts by various authors; see e.g. F. Borceux, G. Janelidze, and G.M. Kelly [4], S. Mantovani and G. Metere [14], G. Metere and A. Montoli [15], and references therein. This paper is devoted to their construction in an arbitrary variety $\mathbf{V}$ of right Ω-loops, and to an accordingly simplified description of internal precrossed and crossed modules in $\mathbf{V}$ in the sense of G. Janelidze [12].

While Ω-groups, also called groups with multiple operators are well known from P.J. Higgins [9], the Ω-loops are defined similarly, just replacing the group structure with a loop structure. As it was observed already in [9], they share many basic properties of Ω-groups; it is also known that some of such properties, and in particular Bourn protomodularity, hold for right (and left) Ω-loops too. The following definition of *right Ω-loop* should be considered as well known; according to the definition of *left closed magma* in [3], the term *right closed Ω-magma* would also be appropriate.

1.1. DEFINITION. *A variety $\mathbf{V}$ of right Ω-loops is a pointed variety of universal algebras that has, among its terms, a binary $+$ and a binary $-$ satisfying the identities*

$$x + 0 = x, \tag{1}$$

$$0 + x = x, \tag{2}$$

$$(x - y) + y = x, \tag{3}$$

$$(x + y) - y = x, \tag{4}$$

Received by the editors 2010-02-18 and, in revised form, 2011-07-05.
Transmitted by Walter Tholen. Published on 2011-10-30.
2000 Mathematics Subject Classification: 08C05, 18D35, 18G50, 18C10.
Key words and phrases: semidirect products, variety of right loops, crossed module, precrossed module.

where 0 is the unique constant of $\mathbf{V}$.

1.2. Remark. The identities above easily imply that $x - x = 0$.
Indeed, $x - x = (0 + x) - x = 0$.

Briefly, our descriptions of semidirect products and (pre)crossed modules are obtained using the fact that the right Ω-loops are, in some sense, exactly the algebraic structures whose semidirect products have the corresponding cartesian products as their underlying sets.

2. Preliminaries

A pointed finitely cocomplete category $\mathbf{C}$ is said to be semi-abelian (in the sense of [13]) if it is exact in the sense of M. Barr [1] and protomodular in the sense of D. Bourn [5] (see also F. Borceux [2] and F. Borceux and D. Bourn [3]). Since every variety of universal algebras is (small-)cocomplete and exact, it is semi-abelian if and only if it is pointed and protomodular. As shown by D. Bourn and G. Janelidze [7], a variety of universal algebras is protomodular if and only if it admits nullary terms $e_1, ..., e_n$, binary terms $t_1, ..., t_n$, and $(n+1)$-ary term t, satisfying the identities

$$t(t_1(x,y), ..., t_n(x,y), y) = x \quad \textit{and} \quad t_i(x,x) = e_i \ (i = 1, ..., n).$$

In particular every variety of right Ω-loops is protomodular, and therefore semi-abelian: just take $n = 1$, $e_1 = 0$, $t_1(x,y) = x - y$, *and* $t(x,y) = x + y$. Moreover, the pointed case is well known in universal algebra from the work of A. Ursini and his collaborators (see [8], [18], [19]).

For an object B in a semi-abelian category $\mathbf{C}$ consider the diagram

$$\begin{tikzcd} & Pt_{\mathbf{C}}(B) \arrow[dl, shift left, "U"] \arrow[dr, shift left, "K"] & \\ \mathbf{C} \arrow[ur, shift left, "F"] \arrow[rr, shift right, "F^T"'] & & \mathbf{C}^T \arrow[ul, shift left, "L"] \arrow[ll, shift right, "U^T"'] \end{tikzcd} \tag{5}$$

in which:

- $Pt_{\mathbf{C}}(B)$ is the category of points over B as used in various above-mentioned papers (originally [5]). That is, the objects in $Pt_{\mathbf{C}}(B)$ are triples (A, α, β), where $\alpha : A \longrightarrow B$ and $\beta : B \longrightarrow A$ are morphisms in $\mathbf{C}$ in with $\alpha\beta = 1_B$. A morphism $f : (A, \alpha, \beta) \longrightarrow (A', \alpha', \beta')$ in $Pt_{\mathbf{C}}(B)$ is a morphism $f : A \longrightarrow A'$ in $\mathbf{C}$ with $\alpha' f = \alpha$ and $f\beta = \beta'$.

- U is a functor defined by $U(A, \alpha, \beta) = Ker(\alpha)$, and F is its left adjoint, defined therefore by $F(X) = (B + X, [1, 0], \iota_1)$, in the obvious notation.

- T is the monad on $\mathbf{C}$ determined by the adjoint pair (F, U), $\mathbf{C}^T$ the category of T^B-algebras, and U^T, F^T, and K are the corresponding forgetful functor, free functor, and comparison functor respectively. L is the left adjoint of K. Recall that, for X in $\mathbf{C}$, we have $T(X) = B\flat X = \ker([1,0] : B + X \longrightarrow B)$.

As shown in [6], the functor U is monadic; and, according to [6], for a T^B-algebra (X, ξ) (or a B-action (X, ξ) in the sense of [4]), the semidirect product $(B \ltimes (X, \xi), \pi_\xi, \iota_\xi)$ is defined as the object in $Pt_{\mathbf{C}}(B)$ corresponding to (X, ξ) under the equivalence (K, L). Equivalently,

$$(B \ltimes (X, \xi), \pi_\xi, \iota_\xi) = L(X, \xi). \tag{6}$$

As usually, by the semidirect product of B and (X, ξ) we will sometimes mean (just) the object $B \ltimes (X, \xi)$. When $\mathbf{C}$ is a pointed protomodular variety of universal algebras, we have

$$B\flat X = \{t(b_1, ..., b_p, x_1, ..., x_q) \in B + X \mid t(b_1, ..., b_p, 0, ..., 0) = 0\}, \tag{7}$$

and $B \ltimes (X, \xi)$ can be presented as $B \ltimes (X, \xi) = (B + X)/E$, where E is the congruence on the coproduct $B + X$ generated by $\{(t, \xi(t)) | t \in B\flat X\}$, considering both $B\flat X$ and X as subalgebras of $B + X$.

Let us also recall the explicit description of the comparison functor K. For (A, α, β) in $Pt_{\mathbf{C}}(B)$, consider the diagram

$$\begin{tikzcd}
B\flat X \arrow[r, "\kappa_{B,X}"] \arrow[d, "\xi"] & B + X \arrow[r, "{[1,0]}"'] \arrow[d, "{[\beta,\kappa]}"] & B \arrow[l, bend right, "\iota_1"'] \arrow[d, equal] \\
X \arrow[r, "\kappa"'] & A \arrow[r, "\alpha"] & B \arrow[l, bend left, "\beta"]
\end{tikzcd} \tag{8}$$

where $(B\flat X, \kappa_{B,X})$ is the kernel of $[1, 0]$, (X, κ) is the kernel of α, and ξ is the induced morphism between these kernels. We can write $K(A, \alpha, \beta) = (X, \xi)$.

3. Simplified description of semidirect products

The main result of this paper is the following:

3.1. Theorem. *Let $\mathbf{V}$ variety of right Ω-loops. Given an object B and a T^B-algebra (X, ξ), the semidirect product $B \ltimes (X, \xi)$ is the set-theoretical product $B \times X$ equipped with the Ω-algebra structure defined by:*

$$\omega((b_1, x_1), ..., (b_n, x_n)) = (\omega(b_1, ..., b_n), \xi(\omega(x_1 + b_1, ..., x_n + b_n) - \omega(b_1, ..., b_n))), \tag{9}$$

for each n-ary operation $\omega \in \Omega$ *and for all* $b_1, ..., b_n \in B$, $x_1, ..., x_n \in X$. *The corresponding* $\pi_\xi : B \times X \longrightarrow B$ *and* $\iota_\xi : B \longrightarrow B \times X$ *are given by* $\pi_\xi(b, x) = b$ *and* $\iota_\xi(b) = (b, 0)$ *respectively.*

Proof. Consider the diagram in the category of sets

$$\begin{tikzcd}
X \arrow[r, "\kappa"] \arrow[d, equal] & A \arrow[r, "\alpha"] \arrow[d, bend left, "\psi"] & B \arrow[l, bend right, "\beta"'] \arrow[d, equal] \\
X \arrow[r, "\langle 0,1\rangle"'] & B \times X \arrow[u, bend left, "\varphi"] \arrow[r, "\pi_1"] & B \arrow[l, bend left, "\langle 1,0\rangle"]
\end{tikzcd} \tag{10}$$

where $\alpha\beta = 1_B$, $\kappa = ker(\alpha)$ and the maps φ and ψ are defined as follows:

$$\varphi : B \times X \longrightarrow A, \quad (b, x) \longmapsto \kappa(x) + \beta(b)$$
$$\psi : A \longrightarrow B \times X, \quad a \longmapsto \left(\alpha(a), \kappa^{-1}(a - \beta\alpha(a))\right).$$

We then have

$$\psi\varphi(b, x) = \psi(\kappa(x) + \beta(b)) = (b, \kappa^{-1}((\kappa(x) + \beta(b)) - \beta(b)) = (b, x)$$
$$\varphi\psi(a) = \varphi(\alpha(a), \kappa^{-1}(a - \beta\alpha(a))) = (a - \beta\alpha(a)) + \beta\alpha(a) = a.$$

Therefore φ and ψ are bijections, inverse to each other. Together with (6) and (8) this allows us to construct the semidirect product $B \ltimes (X, \xi)$ as the cartesian product $B \times X$ (in the category of sets) equipped with the unique algebraic structure making φ and ψ isomorphisms. For this, structure we have

$$\begin{aligned}
&\omega((b_1, x_1), ...(b_n, x_n)) = \psi(\omega(\varphi(b_1, x_1), ..., \varphi(b_n, x_n))) \quad (using\ \psi\varphi = 1_{B\times X}) \\
&= \psi(\omega(\kappa(x_1) + \beta(b_1), ..., \kappa(x_n) + \beta(b_n))) = (\alpha(\omega(\kappa(x_1) + \beta(b_1), ..., \kappa(x_n) + \beta(b_n))), \\
&\kappa^{-1}[\omega(\kappa(x_1) + \beta(b_1), ..., \kappa(x_n) + \beta(b_n)) - \beta\alpha\omega(\kappa(x_1) + \beta(b_1), ..., \kappa(x_n) + \beta(b_n))]) \\
&= (\omega(b_1, ..., b_n), \kappa^{-1}[\omega(\kappa(x_1) + \beta(b_1), ..., \kappa(x_n) + \beta(b_n)) - \omega(\beta(b_1), ..., \beta(b_n))]) \\
&= (\omega(b_1, ..., b_n), \kappa^{-1}[\beta, \kappa](\omega(x_1 + b_1, ..., x_n + b_n) - \omega(b_1, ..., b_n))) \\
&= (\omega(b_1, ..., b_n), \xi(\omega(x_1 + b_1, ..., x_n + b_n) - \omega(b_1, ..., b_n)))
\end{aligned} \tag{11}$$

where the last equality follows from the commutativity of the left-hand square in diagram (8). Next, we have $\alpha\varphi = \pi_1$, $\varphi\langle 1, 0\rangle = \beta$ and $\varphi\langle 0, 1\rangle = \kappa$ in (10). Indeed

$$\alpha\varphi(b, x) = \alpha\kappa(x) + \alpha\beta(x) = 0 + x,$$
$$\varphi\langle 0, 1\rangle(x) = \varphi(0, x) = \kappa(x) + 0 = \kappa(x),$$
$$\varphi\langle 1, 0\rangle(b) = \varphi(b, 0) = 0 + \beta(b) = \beta(b).$$

This allows us to identify π_ξ with $\pi_1 : B \times X \longrightarrow B$ and ι_ξ with $\langle 1, 0\rangle : B \longrightarrow B \times X$. Therefore

$$\pi_\xi(b, x) = \pi_1(b, x) = b, \quad \iota_\xi(b) = \langle 1, 0\rangle(b) = (b, 0).$$

■

The identities (1), (2), (3) and (4) of Definition 1.1 are actually not only sufficient but also necessary for our definition of φ and ψ to determine bijections inverse to each other and to make the relevant parts of diagram (10) commutative. To show this consider the following three instances of diagram (10):

$$\begin{tikzcd}
X \arrow[r, "\kappa=1_X"] \arrow[d, equal] & X \arrow[r, "\alpha=0"] \arrow[d, bend left, "\psi"] & 0 \arrow[l, bend right, "\beta=0"'] \arrow[d, equal] \\
X \arrow[r, "\langle 0,1\rangle"'] & 0 \times X \arrow[r, "\pi_1"] \arrow[u, bend left, "\varphi"] & 0 \arrow[l, bend left, "\langle 0,0\rangle"]
\end{tikzcd} \tag{12}$$

$$\begin{tikzcd}
0 \arrow[r, "\kappa=0"] \arrow[d, equal] & X \arrow[r, "\alpha=1_X"] \arrow[d, bend left, "\psi"] & X \arrow[l, bend right, "\beta=1_X"'] \arrow[d, equal] \\
0 \arrow[r, "\langle 0,0\rangle"'] & X \times 0 \arrow[r, "\pi_1"] \arrow[u, bend left, "\varphi"] & X \arrow[l, bend left, "\langle 1,0\rangle"]
\end{tikzcd} \tag{13}$$

$$\begin{tikzcd}
X \arrow[r, "\kappa=\langle 0,1\rangle"] \arrow[d, equal] & X \times X \arrow[r, "\alpha=\pi_1"] \arrow[d, bend left, "\psi"] & X \arrow[l, bend right, "\beta=\langle 1,1\rangle"'] \arrow[d, equal] \\
X \arrow[r, "\langle 0,1\rangle"'] & X \times X \arrow[r, "\pi_1"] \arrow[u, bend left, "\varphi"] & X \arrow[l, bend left, "\langle 1,0\rangle"]
\end{tikzcd} \tag{14}$$

In (12), we have $\varphi\langle 0, 1\rangle(x) = \varphi(0, x) = x + 0$, $\kappa(x) = x$, and since the left-hand square commutes, we have $x + 0 = x$.

In (13), we have $\alpha\varphi(x, 0) = 0 + x$, $\pi_1(x, 0) = 0$ and since $\alpha\varphi = \pi_1$, we obtain $0 + x = x$.

In (14), we have

$$\begin{aligned}\varphi\psi(y,x) &= \varphi(y,\kappa^{-1}((y,x)-(y,y)))\\ &= ((y,x)-(y,y))+(y,y)\\ &= ((y-y)+y,(x-y)+y),\end{aligned}$$

$$\begin{aligned}\psi\varphi(y,x) &= \psi((0,x)+(y,y))\\ &= \psi(0+y,x+y)=\psi(y,x+y)\\ &= (y,\kappa^{-1}((y,x+y)-(y,y)))\\ &= (y,\kappa^{-1}(y-y,(x+y)-y))\end{aligned}$$

and since φ and ψ must be inverse to each other, we obtain $(x-y)+y=x$ and $(x+y)-y=x$.

3.2. Remarks. (a) As follows from (9), we have

$$(b_1,x_1)+(b_2,x_2)=(b_1+b_2,\xi(((x_1+b_1)+(x_2+b_2))-(b_1+b_2))).$$

(b) For any variety of right Ω-loops, we have

$$(0,x)+(b,0)=(0+b,\xi(((x+0)+(0+b))-(0+b)))=(b,\xi((x+b)-b))=(b,\xi(x))=(b,x),$$

and in particular for Ω-groups this gives $(b_1,x_1)+(b_2,x_2)=(b_1+b_2,x_1+\xi(b_1+x_2-b_1))$. for all $b_1,b_2\in B$ and $x_1,x_2\in X$.

4. Crossed modules in a variety of right Ω-loops

In this section we apply the construction of the semidirect product to describe precrossed and crossed modules in varieties of right Ω-loops. Let us recall the following definitions, in which $\iota_2: X\longrightarrow B+X$ and $\kappa_{B,X}: B\flat X\longrightarrow B+X$ denote the second coproduct injection and the kernel of $[1,0]: B+X\longrightarrow B$ respectively.

4.1. Definition. *[12] An internal precrossed module in a semi-abelian category* $\mathbf{C}$ *is a 4-tuple* (B,X,ξ,δ) *in which* (X,ξ) *is a* B*-action and* $\delta: X\longrightarrow B$ *a morphism in* $\mathbf{C}$ *such that the diagram*

$$\begin{tikzcd} B\flat X \arrow[r, "\kappa_{B,X}"] \arrow[d, "\xi"'] & B+X \arrow[d, "{[1,\delta]}"] \\ X \arrow[r, "\delta"] & B \end{tikzcd} \tag{15}$$

commutes.

4.2. Definition. *[12] An internal crossed module in a semi-abelian category* $\mathbf{C}$ *is an internal precrossed module* (B, X, ξ, δ) *for which the diagram*

$$\begin{tikzcd}
(B+X)\flat X \arrow[r, "{[1_B,\delta]\flat 1_X}"] \arrow[d, "{[1_{B+X},\iota_2]^\sharp}"'] & B\flat X \arrow[d, "\xi"] \\
B\flat X \arrow[r, "\xi"'] & X
\end{tikzcd} \tag{16}$$

commutes. Here, $[1_{B+X}, \iota_2]^\sharp$ *is the unique morphism such that* $\kappa_{B,X}[1_{B+X}, \iota_2]^\sharp = [1_{B+X}, \iota_2]\kappa_{B+X,X}$.

4.3. Theorem. *A precrossed module in a variety* $\mathbf{V}$ *of right* Ω*-loops can equivalently be defined as a quadruple* (B, X, ξ, δ) *in which* (X, ξ) *is a* B*-action and* $\delta : X \longrightarrow B$ *is a morphism such that for an* n*-ary operation* $\omega \in \Omega$,

$$\begin{aligned}
&\omega(\delta(x_1) + b_1, ..., \delta(x_n) + b_n) - \omega(b_1, ..., b_n) \\
&\qquad = \delta(\xi(\omega(x_1 + b_1, ..., x_n + b_n) - \omega(b_1, ..., b_n)))
\end{aligned} \tag{17}$$

for all $x_1, ..., x_n \in X$, $b_1, ..., b_n \in B$.

Proof. As explained in [12], a precrossed module in $\mathbf{V}$ corresponds to a reflexive graph

$$\begin{tikzcd}
B \ltimes X \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\gamma"'] & B \arrow[l, "\beta"']
\end{tikzcd} \tag{18}$$

with $\alpha(b, x) = b$ and $\beta(b) = (b, 0)$.
Since γ is a homomorphism and $(b, x) = (0, x) + (b, 0)$, we have $\gamma(b, x) = \gamma(0, x) + \gamma(b, 0) = \gamma(0, x) + b$. This shows that γ is completely determined by $\gamma(0, x)$, $for\ all$ $x \in X$. We introduce the morphism $\delta : X \longrightarrow B$ defined by $\delta(x) = \gamma(0, x)$ and then $\gamma(b, x) = \delta(x) + b$ for each $b \in B$ and each $x \in X$. This means that a precrossed module in $\mathbf{V}$ can be defined as quadruple (B, X, ξ, δ), in which (X, ξ) is a B-action and $\delta : X \longrightarrow B$ a morphism in $\mathbf{V}$ such that $\gamma : B \ltimes X \longrightarrow B$ defined by $\gamma(b, x) = \delta(x) + b$ is also a morphism in $\mathbf{V}$.
For an n-ary operation $\omega \in \Omega$,

$$\omega(\gamma(b_1, x_1), ..., \gamma(b_n, x_n)) = \omega(\delta(x_1) + b_1, ..., \delta(x_n) + b_n), \tag{19}$$

$$\begin{aligned}
\gamma(\omega((b_1, x_1), ..., (b_n, x_n))) &= \gamma(\omega(b_1, ..., b_n), \xi(\omega(x_1 + b_1, ..., x_n + b_n) - \omega(b_1, ...b_n))) \\
&= \delta(\xi(\omega(x_1 + b_1, ..., x_n + b_n) - \omega(b_1, ..., b_n))) + \omega(b_1, ..., b_n)
\end{aligned} \tag{20}$$

for $x_1, ..., x_n \in X$ and $b_1, ..., b_n \in B$. Therefore, γ is a morphism if and only if (17) holds. ∎

4.4. Theorem. *A crossed module in a variety* $\boldsymbol{V}$ *of* Ω*-loops can be defined as a precrossed module* (B, X, ξ, δ) *with*

$$\xi(\omega(x_1' + (\delta(x_1) + b_1), ..., x_n' + (\delta(x_n) + b_n)) - \omega(\delta(x_1) + b_1, ..., \delta(x_n) + b_n))+ \xi(\omega(x_1 + b_1, ..., x_n + b_n) - \omega(b_1, ..., b_n)) = \xi(\omega((x_1' + x_1) + b_1, ..., (x_n' + x_n) + b_n) - \omega(b_1, ..., b_n)) \quad (21)$$

for all $b_1', ..., b_n' \in B$, $x_1, ..., x_n \in X$ *and* $x_1', ...x_n' \in X$.

Proof. As follows from the results of [11] and [12], a precrossed module module (B, X, ξ, δ) is a crossed module if and only if the map

$$m : (B \ltimes X) \times_B (B \ltimes X) \longrightarrow B \ltimes X$$

defined by $m(u, v) = p(u, \beta\alpha(u), v)$ is a morphism in $\mathbf{V}$; here p is any Mal'tsev term in $\mathbf{V}$ and α and β are as in (18). In our case, using Theorem 3.1, we can simplify the definition of m as follows:

$$m((b', x'), (b, x)) = p((b', x'), (b', 0), (b, x)) = ((b', x') - (b', 0)) + (b, x) = (b, x' + x).$$

We then calculate, for any n-ary ω

$$\begin{aligned} &m(\omega((b_1', x_1'), ..., (b_n', x_n')), \omega((b_1, x_1), ..., (b_n, x_n))) \\ &= m((\omega(b_1', ..., b_n'), \xi(\omega(x_1' + b_1', ..., x_n' + b_n') - \omega(b_1', ..., b_n'))), \\ &\quad (\omega(b_1, ..., b_n), \xi(\omega(x_1 + b_1, ...x_n + b_n) - \omega(b_1, ..., b_n)))) \\ &= (\omega(b_1, ..., b_n), \xi(\omega(x_1' + b_1', ..., x_n' + b_n') - \omega(b_1', ..., b_n'))+ \\ &\quad \xi(\omega(x_1 + b_1, ..., x_n + b_n) - \omega(b_1, ..., b_n))) \quad (22) \end{aligned}$$

and

$$\begin{aligned} &\omega(m((b_1', x_1')(b_1, x_1)), ..., m((b_n', x_n')(b_n, x_n))) \\ &= \omega((b_1, x_1' + x_1), ..., (b_n, x_n' + x_n)) = (\omega(b_1, ..., b_n), \\ &\quad \xi(\omega((x_1' + x_1) + b_1, ..., (x_n' + x_1) + b_n) - \omega(b_1, ..., b_n))) \quad (23) \end{aligned}$$

for $b_1, ..., b_n \in B$, $b_1', ..., b_n' \in B$, $x_1, ..., x_n \in X$, $x_1', ..., x_n' \in X$ with $b_i' = \delta(x_i) + b_i$, $i = 1, ..., n$. Using the fact that

$$((b', x'), (b, x)) \in (B \ltimes X) \times_B (B \ltimes X)$$

if and only if $b' = \delta(x) + b$, we conclude that m is a morphism in $\mathbf{V}$ if and only if (21) is satisfied. ∎

4.5. Remarks and examples. When ω is the same as $+$, and it is associative, the formula $bx = \xi(b+x-b)$ defined by a B-action on X (see [6]), and the equalities (17) and (21) become $\delta(b_1x_2) = b_1+\delta(x_2)-b_1$ and $\delta(x_1)(b_1x_2') = x_1+b_1x_2'-x_1$ respectively. They are obviously equivalent to the classical $\delta(bx) = b + \delta(x) - b$ and $\delta(x)(x') = x + x' - x$, which define crossed modules, originally by J.H.C. Whitehead [17]. More generally, (17) and (21) conveniently apply to the context of G. Orzech [16], and still, more generally, to the context of distributive Ω_2-loops in the sense of S. Mantovani and G. Metere [14]. On the other hand (17) and (21) are special cases of conditions imposed in [12]; they are expressed in [12] as commutativity of diagrams (2.1) and (3.14) respectively. However, we did not obtain (17) and (21) *directly* from those diagrams, and:

(a) We do not know how to deduce the commutativity of diagrams (2.1) and (3.14) in [12] directly from (17) and (21), that is, not using the reflexive graph (18) and the map $m : (B \ltimes X) \times_B (B \ltimes X) \longrightarrow B \ltimes X$ (which in fact means: we do not know how to do it using Theorem 3.1).

(b) Applying (17) and (21) to groups, and to contexts of [14] and [16] (which include not only groups but many classical algebraic structures, e.g. associative and Lie, and general non-associative algebras over rings), gives the known descriptions of (pre)crossed modules much more easily than the results of [12].

Acknowledgements

The results of this paper formed part of the author's PhD thesis [10] at the University of Cape Town. I would like to thank my supervisor Prof. George Janelidze for his encouragement and support. I would also like to thank the referee whose comments and suggestions helped improve the final version of the paper.

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