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FLOWS: COCYCLIC AND ALMOST COCYCLIC

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ABSTRACT. A flow on a compact Hausdorff space is an automorphism. Using the closed structure on the category of uniform spaces, a flow gives rise, by iteration, to an action of the integers on the topological group of automorphisms of the object. We study special classes of flows: periodic, cocyclic, and almost cocyclic, mainly in term of the possibility of extending this action continuously to various compactifications of the integers.

1. Introduction

In this paper we denote by $\mathcal{Flow}$ the category whose objects are pairs (X, t) where X is a compact Hausdorff space and $t : X \longrightarrow X$ is an automorphism. The notion of **flow** is obviously much broader (any category, any endomorphism), but we are restricting to this special case here. If (X, t) and (Y, u) are flows, a continuous map $f : X \longrightarrow Y$ is a **flow morphism** if $uf = ft$. A flow (X, t) is called **periodic** if there is an integer $n > 0$ such that $t^n = \mathrm{id}$.

We let $C(X)$ denote the lattice-ordered ring of all continuous functions from X to $\mathbf{R}$ and, by abuse of notation, also denote by $t : C(X) \longrightarrow C(X)$ the automorphism induced by t (so that $t(f)$ is the composite ft). Since t is an automorphism, it follows that $t^n : X \longrightarrow X$ and $t^n : C(X) \longrightarrow C(X)$ are defined for all $n \in \mathbf{Z}$.

The ring $C(X)$ has a norm $\| \ \|$, defined by $\|f\| = \sup_{x\in X} |f(x)|$. It satisfies $\|ft\| = \|f\|$.

Section 2 surveys some mostly well-known definitions and facts on uniform spaces. Section 3.1 is a review of uniform completions of the group $\mathbf{Z}$ of integers. In section 4, we define the notion of a t-periodic element of $C(X)$. We show that a flow is periodic if and only if every element of $C(X)$ is t-periodic (Theorem 4.5). We define a flow to be cocyclic if the t-periodic elements of $C(X)$ are dense. We then show that a flow is cocyclic if and only if it is a filtered inverse limit of periodic flows (Theorem 4.9). We also show that cocyclic flows are characterized by the fact that the action $\mathbf{Z} \times X \longrightarrow X$, defined by $(n, x) \mapsto t^n x$ extends to a continuous action $\mathrm{pf}(\mathbf{Z}) \times X \longrightarrow X$, where $\mathrm{pf}(\mathbf{Z})$ is the profinite completion of $\mathbf{Z}$ (Theorem 4.10). In Section 5, we define the notion of

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an almost periodic element of $C(X)$. In Section 6, we define and study almost cocyclic flows. These are similarly characterized by the possibility of extending the action of $\mathbf{Z}$ to a continuous action by $b(\mathbf{Z})$, the Bohr compactification of $\mathbf{Z}$, from which it is immediate that a cocyclic flow is almost cocyclic.

2. On uniform spaces

We refer to [Isbell, 1964] for an introduction to uniform spaces and uniform maps. Every completely regular space has at least one uniformity that induces the same topology, but there are usually more than one. A uniform map is continuous, but the converse is false in general. An easy counter-example is given by the homeomorphism between the space of positive integers and the space of their reciprocals, each given the subspace metric of $\mathbf{R}$.

However, there are two cases, each relevant here, in which the uniformity is unique and continuous functions are uniform. The first is a compact Hausdorff space, which has a unique uniformity, in which all covers are uniform; equivalently every neighbourhood of the the diagonal is an entourage. Every continuous function from a compact space to a uniform space that is continuous in the uniform topologies, is also uniform. The second case is that of an abelian topological group. Such a group does not, as a space, have a unique uniformity, but it does as a group. A basic uniform cover is the set of all translates of a neighbourhood of 0 by all the group elements. Should the neighbourhood be a subgroup, this is just the cover by its cosets. All continuous homomorphisms between abelian topological groups are uniform. Non-abelian groups have two uniformities, one in which the left translates of neighbourhoods of the identity are the uniform covers and one using the right translates. A continuous homomorphism between topological groups is uniform so long as you use left translates for both or right translates for both. If one or both groups are abelian (or if the domain is compact) then such a homomorphism is uniform regardless. One point that should be made is that a non-abelian topological group is not necessarily a group in the category of uniform spaces since neither the inverse map nor the multiplication need be uniform.

On these two classes of objects, we will not distinguish between topological and uniform structures or between continuous and uniform morphisms.

If X and Y are uniform spaces, then $\hom(X, Y)$ denotes the set of all uniform functions $X \longrightarrow Y$ equipped with the uniformity of uniform convergence on all of X. This means that if $V \subseteq Y \times Y$ is an entourage on Y, then the set

$$\{(f, g) \mid (fx, gx) \in V \text{ for all } x \in X\ \}$$

is an entourage for the uniform structure on $\hom(X, Y)$. This is an obvious generalization of the sup norm when Y is metric. As usual, we denote by $\mathrm{Hom}(X, Y)$ the set of all uniform functions $X \longrightarrow Y$. It is, of course, the underlying set of $\hom(X, Y)$.

For later use, we record the following property of compact spaces. It follows immediately from [Isbell, 1964, Theorem 31].

2.1. Proposition. *Suppose X and Y are compact Hausdorff spaces. Then* $\hom(X, Y)$ *is complete.* ■

2.2. Definition. *A uniform space is called* **totally bounded** *if it can be uniformly embedded in a compact space. This is equivalent to the fact that every uniform cover has a finite subcover, see [Isbell, 1964].*

The following result is implicit in [Isbell, 1964, Theorem III.24] but we find the proof somewhat opaque and therefore we include our own proof.

2.3. Theorem. *When the space Z is totally bounded, then for any uniform spaces X and Y,* $\mathrm{Hom}(Z \times X, Y) \cong \mathrm{Hom}(Z, \hom(X, Y))$.

This is true also for the internal hom, but we have no need of it. The proof is based on the following lemma in which all entourages used in both the statement and proof will be understood to be symmetric.

2.4. Lemma. *Suppose Z, X, Y are uniform spaces with Z totally bounded and suppose $\theta : Z \longrightarrow \hom(X, Y)$ is a uniform map. Then for each entourage $V \subseteq Y \times Y$ there are entourages $U \subseteq X \times X$ and $W \subseteq Z \times Z$ such that for all $(z, z') \in W$ and $(x, x') \in U$, we have $(\theta z(x), \theta z'(x')) \in V$.*

Proof. Let $\widehat{V} \subseteq Y \times Y$ be an entourage such that $\widehat{V} \circ \widehat{V} \circ \widehat{V} \circ \widehat{V} \subseteq V$. Let

$$H = \{(f, g) \in \mathrm{Hom}(X, Y) \times \mathrm{Hom}(X, Y) \mid (fx, gx) \in \widehat{V} \text{ for all } x \in X\ \}$$

Then H is an entourage in $\hom(X, Y)$. Since θ is a uniform map, there is an entourage $\widehat{W} \subseteq Z \times Z$ such that $(\theta \times \theta)(\widehat{W}) \subseteq H$. Let $W \subseteq \widehat{W}$ be an entourage such that $W \circ W \circ W \subseteq \widehat{W}$.

Since Z is totally bounded, there is a finite subset $z_1, z_2, \ldots, z_n \in Z$ such that $\{W[z_i]\}$ covers Z. For each $i = 1, \ldots, n$, we have that θz_i is a uniform map $X \longrightarrow Y$ and so there is an entourage $U_i \subseteq X \times X$ such that whenever $(x, x') \in U_i$, we have $(\theta z_i(x), \theta z_i(x')) \in \widehat{V}$. If $U = \bigcap U_i$, then we have that for all $i = 1, \ldots, n$ and all $(x, x') \in U$, $(\theta z_i(x), \theta z_i(x')) \in \widehat{V}$.

Now suppose that $(z, z') \in W$. There are indices i and j such that $(z_i, z) \in W$ and $(z', z_j) \in W$. This implies that $(z_i, z_j) \in W \circ W \circ W \subseteq \widehat{W}$. We can now infer that all of $(\theta z, \theta z_i)$, $(\theta z_i, \theta z_j)$ and $(\theta z_j, \theta z')$ belong to H. Using the fact that $(\theta z_j(x), \theta z_j(x')) \in \widehat{V}$, we see that all of $(\theta z(x), \theta z_i(x))$, $(\theta z_i(x), \theta z_j(x))$, $(\theta z_j(x), \theta z_j(x'))$, and $(\theta z_j(x'), \theta z(x'))$ belong to $\widehat{V}$, whence $(\theta z(x), \theta z'(x')) \in V$. ■

Proof of the theorem. The lemma implies immediately that if $\theta : Z \longrightarrow \hom(X, Y)$ is uniform, so is the transpose $Z \times X \longrightarrow Y$. For the converse, simply observe that if we have a uniform map $Z \times X \longrightarrow Y$, the conclusion of the lemma is valid from which it is immediate that the transpose $Z \longrightarrow \hom(X, Y)$ is uniform. ■

In general, the closed structure on the category of uniform spaces is not symmetric, so this condition is specific to Z and it is not sufficient that X be totally bounded. It is sufficient that X be discrete, but that will not be the case in our applications.

2.5. PROPOSITION. *The supremum of a family of totally bounded uniformities on a set is totally bounded.*

PROOF. Let X be a set and let $\{U_\alpha\}$ be a family of totally bounded uniformities on X. For each α let $f_\alpha : X \longrightarrow C_\alpha$ be a uniform embedding of X, equipped with the U_α uniformity, to a compact space C_α. Then the map $X \longrightarrow \prod C_\alpha$ whose α coordinate is f_α is an injection of X into a compact space for which the induced uniformity on X is the supremum of the U_α. ■

3. Completions and compactifications of $\mathbf{Z}$

A **group compactification of $\mathbf{Z}$** is a compact group $\overline{\mathbf{Z}}$ and a homomorphism $\mathbf{Z} \longrightarrow \overline{\mathbf{Z}}$ whose image is dense. We do not require the map to be injective, so that the canonical map $\mathbf{Z} \longrightarrow \mathbf{Z}/n\mathbf{Z}$, for any positive integer n, is a compactification. It is not hard to see that these are the only finite compactifications of $\mathbf{Z}$ and that in all other cases, the map $\mathbf{Z} \longrightarrow \overline{\mathbf{Z}}$ is injective. If G is a topological group, a map $\mathbf{Z} \longrightarrow G$ will be said to **lift to $\overline{\mathbf{Z}}$** when there is a commutative diagram

$$\begin{tikzcd} & \mathbf{Z} \arrow[dl] \arrow[dr] & \\ \overline{\mathbf{Z}} \arrow[rr, dashed] & & G \end{tikzcd}$$

Although this terminology is not entirely reasonable when $\mathbf{Z} \longrightarrow \overline{\mathbf{Z}}$ fails to be injective, we will use it anyway. *When we talk of compactifications of $\mathbf{Z}$ we will always mean group compactifications. Similarly, when we talk of topologies on $\mathbf{Z}$, we always mean group topologies.*

3.1. UNIFORM COMPLETIONS OF $\mathbf{Z}$. We will denote by $\mathbf{Z}_{\mathrm{pf}}$ the topological group whose underlying group is the integers, but whose topology has the non-zero subgroups as its basic neighbourhoods at 0. In the discrete topology, the integers are complete, but in this topology they are not. We denote by $\mathrm{pf}(\mathbf{Z})$ the completion in the uniformity defined by the topology just described. There are at least two good ways of describing it. One is as the inverse limit of all $\mathbf{Z}/n\mathbf{Z}$ as n ranges over the positive integers. When $n|m$, we have a commutative diagram

$$\begin{tikzcd} & \mathbf{Z} \arrow[dl] \arrow[dr] & \\ \mathbf{Z}/m\mathbf{Z} \arrow[rr] & & \mathbf{Z}/n\mathbf{Z} \end{tikzcd}$$

More generally, if m and n are arbitrary and l their least common multiple, we have the commutative diagram

$$\begin{tikzcd} & \mathbf{Z} \arrow[dl] \arrow[d] \arrow[dr] & \\ \mathbf{Z}/m\mathbf{Z} & \mathbf{Z}/l\mathbf{Z} \arrow[l] \arrow[r] & \mathbf{Z}/n\mathbf{Z} \end{tikzcd}$$

which shows that the diagram is cofiltered.

Since for non-zero $n \in \mathbf{Z}$ a finite number of translates of the subgoup $n\mathbf{Z}$ covers $\mathbf{Z}$, it follows that the topology is totally bounded and the completion is compact.

Here is the second way to construct $\mathrm{pf}(\mathbf{Z})$. If p is a prime, let $\mathrm{pf}(\mathbf{Z})_p$ denote p-adic integers, which is the inverse limit of the sequence

$$\mathbf{Z}/p\mathbf{Z} \longleftarrow \mathbf{Z}/p^2\mathbf{Z} \longleftarrow \mathbf{Z}/p^3\mathbf{Z} \longleftarrow \cdots$$

Then $\mathrm{pf}(\mathbf{Z}) = \prod \mathrm{pf}(\mathbf{Z})_p$, the product over all primes. An element of $\mathrm{pf}(\mathbf{Z})_p$ is an infinite series $\alpha = a_0 + a_1 p + \cdots + a_n p^n + \cdots$ with coefficients between 0 and $p-1$. Note, however, that addition and multiplication are not mod p, but involve carries into higher powers of p. The usual argument shows that such an element is invertible if and only if $a_0 \neq 0$. We say that the element above has **order** n if $a_0 = \cdots = a_{n-1} = 0$ and $a_n \neq 0$. It will simplify the exposition below if we let p^∞ denote the 0 element of $\mathrm{pf}(\mathbf{Z})_p$ and write $\mathrm{ord}_p 0 = \infty$.

We let $\rho_n : \mathrm{pf}(\mathbf{Z}) \longrightarrow \mathbf{Z}/n\mathbf{Z}$ and $\sigma_p : \mathrm{pf}(\mathbf{Z}) \longrightarrow \mathrm{pf}(\mathbf{Z})_p$ be the canonical projections. Let us denote by $\mathrm{ord}_p\lambda$ the order of $\sigma_p\lambda$. If $\lambda, \mu \in \mathrm{pf}(\mathbf{Z})$, it is clear that $\lambda|\mu$ if and only if, for each prime p, $\mathrm{ord}_p\lambda \le \mathrm{ord}_p\mu$.

3.2. PROPOSITION. *A closed ideal in* $\mathrm{pf}(\mathbf{Z})$ *has the form* $I = \prod I_p$, *where* I_p *is an ideal in* $\mathrm{pf}(\mathbf{Z})_p$. *Every non-zero ideal in* $\mathrm{pf}(\mathbf{Z})_p$ *is principal, generated by a power of* p.

PROOF. Let I be a closed ideal in $\mathrm{pf}(\mathbf{Z})$. Let e_p denote the element of $\prod \mathrm{pf}(\mathbf{Z})_q$ whose pth coordinate is 1 and all other coordinates are 0. Let $I_p = e_p I$. Clearly $\sum I_p \subseteq I \subseteq \prod I_p$. But since the sum is dense in the product and I is closed, it is clear that $\prod I_p = I$.

For a non-zero ideal $I_p \subseteq \mathrm{pf}(\mathbf{Z})_p$, let $p^n(a_0 + a_1 p + \cdots)$, $a_0 \neq 0$, be an element of I of least degree. Since the element in parentheses is invertible, we see that $p^n \in I_p$. Since n was chosen as small as possible, it is clear the I_p is generated by p^n. ∎

THE BOHR COMPACTIFICATION. The **Bohr compactifcation** of an abelian group G can be defined as the value at G of the left adjoint to the underlying functor from compact abelian groups to topological abelian groups. It can be directly constructed, as Bohr did, using the construction from the adjoint functor theorem (it is a matter of historical fact that Bohr's construction was one of the motivations behind Freyd's construction). Another method constructs it as $|G^*|^*$ where * is the Pontrjagin dual. A third is as the completion under the finest totally bounded topology on G.

Let $\mathrm{b}(\mathbf{Z})$ denote the Bohr compactification of $\mathbf{Z}$. We say that the topology on $\mathbf{Z}$ induced by the embedding $\mathbf{Z} \longrightarrow \mathrm{b}(\mathbf{Z})$ is the **Bohr topology** on $\mathbf{Z}$ and it will be denoted by $\mathbf{Z}_b$. Although this topology is not well understood, we can give some idea of the open neighbourhoods of 0. Like all compact abelian groups, $b(\mathbf{Z})$ can be embedded algebraically and topologically into a power of the circle group $\mathbf{T} = \mathbf{R}/\mathbf{Z}$. Thus, for some set S, we have an embedding $\mathbf{Z}_b \hookrightarrow \mathbf{T}^S$. A standard argument shows that we can take $S = \mathrm{Hom}(\mathbf{Z}_b, \mathbf{T}) = \mathrm{Hom}(\mathbf{Z}, \mathbf{T})$ since every homomorphism from $\mathbf{Z}$ to a compact group extends to $b(\mathbf{Z})$ and thus is continuous on $\mathbf{Z}_b$. A homomorphism $f : \mathbf{Z} \longrightarrow \mathbf{T}$ is entirely determined by $f(1)$ which can be any element of $\gamma \in \mathbf{T}$ and thus is multiplication mod $\mathbf{Z}$ by some real number γ, which can be taken to lie in the interval $[0, 1)$ although any real number could be used; two reals that differ by an integer give the same homomorphism. For each $\epsilon > 0$, let

$$U(\gamma, \epsilon) = \{n \in \mathbf{Z} \mid n\gamma \text{ is within } \epsilon \text{ of an integer}\}$$

Then the $U(\gamma, \epsilon)$ form a subbase for the neighbourhood system at 0 in $\mathbf{Z}_b$. Informally, these sets look like “almost subgroups” or “almost periodic sets”, although we will not attempt to make this notion precise. For example, when $\gamma = \pi$, and $\epsilon = 0.1$, the numbers in $U(\pi, \epsilon)$ between 0 and 113 consist of the multiples of 7 up to 77 together with the numbers congruent to 1 mod 7 between 36 and 113. Since 355/113 is such a good approximation for π, this sequence of elements of $U(\pi, \epsilon)$ will continue to repeat as follows. Let A denote the set consisting of the 24 numbers $\{0, 7, 14, \ldots, 77, 36, 43, 50, \ldots, 13\}$. Then $U(\pi, 0.1)$ continues with the numbers of the form $a + 113k$, for $a \in A$ and k up to $k = 87$. After that, determined by a better continued fraction approximation for π, the sequence begins to veer off.

4. Periodic and cocyclic flows

Recall that in this paper, all spaces are assumed compact Hausdorff. Also recall that a flow (X, t) is called periodic if there is an $n > 0$ such that $t^n = \mathrm{id}$. We say that n is **a period** of the flow. The least such positive n will be called **the minimal period** of the flow.

4.1. Action of $\mathbf{Z}$ on a flow. If (X, t) is a flow, then there is an action of $\mathbf{Z}$ on X given by $(n, x) \mapsto t^n(x)$. This gives rise to a group homomorphism $\varphi : \mathbf{Z} \longrightarrow \mathrm{Aut}(X)$ given by $n \mapsto t^n$. When n is the minimal period of t, then the kernel of $\mathbf{Z} \longrightarrow \mathrm{Aut}(X)$ is exactly $n\mathbf{Z}$.

4.2. Proposition. *The group* $\mathrm{Aut}(X)$ *becomes a topological group if we equip it with the uniformity it inherits from* $\hom(X, X)$ *(see Section 2).*

Proof. A neighbourhood of an automorphism s is determined by an entourage E as

$$E(s) = \{r \mid (rx, sx) \in E \text{ for all } x \in X\}$$

For the remainder of this computation, treat all set builders as indexed over all the elements of X. Then we have

$$\begin{aligned} E(su) &= \{r \mid (rx, sux) \in E\} \\ &= \{r \mid (ru^{-1}y, sy) \in E\} \text{ (replace } x \text{ by } u^{-1}y) \\ &= \{vu \mid (vy, sy) \in E\} \text{ (replace } r \text{ by } vu) \\ &= E(s)u \end{aligned}$$

In particular, $E(s) = E(1)s$ so that the topology is given by right translates of neighbourhoods of the identity. Assume now that E is a symmetric entourage. Then

$$\begin{aligned} E(1) = \{r \mid (rx, x) \in E\} &= \{r \mid (x, rx) \in E\} = \{s^{-1} \mid (x, s^{-1}x) \in E\} \\ &= \{s^{-1} \mid (sy, y) \in E\} \text{ (replace } x \text{ by } sy) \\ &= E(1)^{-1} \end{aligned}$$

Finally, let F be an entourage such that $F \circ F^{\mathrm{op}} \subseteq E$. Then for $r, s \in F(1)$, we have $(rx, x), (sx, x) \in F$ so that $(rx, sx) \in F \circ F^{\mathrm{op}} \subseteq E$. But then for $y = sx$, $(rs^{-1}y, y) \in E$ and so we see that $F(1)F(1)^{-1} \subseteq E(1)$ and the conditions for being a topological group are satisfied. ∎

Notation. We will denote by $\mathrm{aut}(X)$ the group $\mathrm{Aut}(X)$ of automorphisms of X equipped with this topology.

Since X is compact Hausdorff it is a subspace of $\mathbf{R}^{C(X)}$. Thus for all $f \in C(X)$, we define $\varphi_f : \mathbf{Z} \longrightarrow C(X)$ as the composite:

$$\mathbf{Z} \longrightarrow \mathrm{aut}(X) \longrightarrow \mathrm{Hom}(X, X) \longrightarrow \mathrm{Hom}(X, \mathbf{R}^{C(X)}) \xrightarrow{p_f} \mathrm{Hom}(X, \mathbf{R}) = C(X)$$

Then $\varphi_f(n) = ft^n$.

We will say that an $f \in C(X)$ is **t-periodic** if there is an $n > 0$ for which $ft^n = f$.

4.3. Theorem. *Let (X, t) be a flow and $f \in C(X)$. Of the following conditions, the first three are equivalent and imply the fourth:*

(1) $\{ft^n \mid n \in \mathbf{Z}\}$ *is finite;*

(2) f *is t-periodic;*

(3) $\varphi_f : \mathbf{Z} \longrightarrow C(X)$ *extends to a finite compactification of* $\mathbf{Z}$*;*

(4) $\varphi_f : \mathbf{Z} \longrightarrow C(X)$ *extends continuously to* $\mathrm{pf}(\mathbf{Z})$.

Proof. The proofs that (1) $\implies$ (2) $\implies$ (3) $\implies$ (1) are trivial and left to the reader. The proof that (3) $\implies$ (4) follows from the fact that $\mathbf{Z}/n\mathbf{Z}$ is a quotient of $\mathrm{pf}(\mathbf{Z})$. ∎

4.4. Example. Here is an example to show that (4) does not imply the other three. Let $X = \mathrm{pf}(\mathbf{Z})$ and let $t : X \longrightarrow X$ be given by $t(x) = x+1$. Then for all $n \in \mathbf{Z}$, $t^n(x) = x+n$ and we can evidently extend this to all $z \in \mathrm{pf}(\mathbf{Z})$ by $t^z(x) = x + z$. Clearly t is not periodic and it follows from Theorem 4.5 below that some $f \in C(X)$ is not t-periodic.

Clearly, if (X,t) is periodic, then every $f \in C(X)$ is t-periodic. For the converse we have the following theorem. *Note that the argument makes no use of the compactness of the space. Also note that all the functions used in the argument are bounded.*

4.5. Theorem. *Let (X,t) be a flow. If every $f \in C^*(X)$ is t-periodic, then t is periodic on X.*

Proof. If t is not periodic, then there must be, for any $m > 0$, an infinite set $S_m \subseteq X$ such that $t^m x \neq x$ for all $x \in S_m$. The set S_m might consist of a single infinite orbit, if there is one, or a set of periodic points whose orbit size does not divide m. An obvious argument using LCM would show that if $t^m x = x$ for all but a finite set of periodic points, then t would be periodic.

Suppose $x_1 \in S_1$. Let $f_1 : X \longrightarrow [0,1]$ be a continuous function such that $f_1(x_1) = 1$ and $f_1(tx_1) = 0$. Let $g_1 = f_1/2$. Clearly $g_1(x_1) \neq g_1(tx_1)$. Next choose $x_2 \in S_2$ such that $\{x_2, t^2x_2\}$ is disjoint from $\{x_1, tx_1\}$. The fact that S_2 is infinite and t a bijection allows such a choice to be made. Let $f_2 : X \longrightarrow [0,1]$ be a continuous function such that $f_2(x_1) = f_2(tx_1) = 0$ and

$$\begin{cases} f_2(x_2) = 1 \text{ and } f_2(t^2x_2) = 0 & \text{if } g_1(x_2) > g_1(t^2x_2) \\ f_2(x_2) = 0 \text{ and } f_2(t^2x_2) = 1 & \text{if } g_1(x_2) \leq g_1(t^2x_2) \end{cases}$$

Let $g_2 = g_1 + f_2/4$. Clearly $g_2(x_1) = 1/2$ while $g_2(tx_1) = 0$. If $g_1(x_2) > g_1(t^2x_2)$, we have that $g_2(x_2) = g_1(x_2) + 1/4 > g_1(t^2x_2) = g_2(t^2x_2)$, while if $g_1(x_2) \leq g_1(t^2x_2)$, we have $g_2(x_2) = g_1(x_2) < g_1(t^2x_2) + 1/4 = g_2(t^2x_2)$. Thus we see that $g_2(x_i) \neq g_2(t^ix_i)$ for $i = 1, 2$.

Suppose that points $x_1 \in S_1$, $x_2 \in S_2$, $\dots$, $x_k \in S_k$ and functions $f_1, f_2, \dots, f_k : X \longrightarrow [0,1]$ have been chosen so that for all $j < i$, $f_i(x_j) = f_i(t^jx_j) = 0$ and such that the function $g_k = \sum_{i=1}^k 2^{-i} f_i$ satisfies $g_k(x_i) \neq g_k(t^ix_i)$ for all $i \leq k$. As above, we can choose $x_{k+1} \in S_{k+1}$ so that $\{x_{k+1}, t^{k+1}x_{k+1}\}$ is disjoint from $\{x_1, x_2, \dots, x_k, tx_1, t^2x_2, \dots t^kx_k\}$. Now let $f_{k+1} : X \longrightarrow [0,1]$ be a continuous function that vanishes on all the elements of $\{x_1, x_2, \dots, x_k, tx_1, t^2x_2, \dots t^kx_k\}$ and such that

$$\begin{cases} f_{k+1}(x_{k+1}) = 1 \text{ and } f_{k+1}(t^{k+1}x_{k+1}) = 0 & \text{if } g_k(x_{k+1}) > g_k(t^{k+1}x_{k+1}) \\ f_{k+1}(x_{k+1}) = 0 \text{ and } f_{k+1}(t^{k+1}x_{k+1}) = 1 & \text{if } g_k(x_{k+1}) \leq g_k(t^{k+1}x_{k+1}) \end{cases}$$

Let $g_{k+1} = g_k + f_{k+1}/2^{k+1}$. We have that for $i < k+1$, $g_{k+1}(x_i) = g_k(x_i) \neq g_k(t^ix_i) = g_{k+1}(t^ix_i)$. If $g_k(x_{k+1}) > g_k(t^{k+1}x_{k+1})$, we have that $g_{k+1}(x_{k+1}) = g_k(x_{k+1}) + 1/2^{k+1} > g_k(t^{k+1}x_{k+1}) = g_{k+1}(t^{k+1}x_{k+1})$, while if $g_k(x_{k+1}) \leq g_k(t^{k+1}x_{k+1})$, we have that $g_{k+1}(x_{k+1}) = g_k(x_{k+1}) < g_k(t^{k+1}x_{k+1}) + 1/2^{k+1} = g_{k+1}(t^{k+1}x_{k+1})$, which completes the induction step. We conclude by letting $g = \sum_{i=1}^\infty 2^{-i} f_i = \lim g_i$. We readily see that $g = g_{k+1}$ on the set $\{x_1, x_2, \dots, x_k, tx_1, t^2x_2, \dots t^kx_k\}$ so that $g(x_k) \neq g(t^kx_k)$, so that $g \neq gt^k$ for all k, which contradicts the assumption that every bounded function on X is t-periodic. ∎

4.6. Definition. *We say that a flow (X, t) is* **cocyclic** *if the set of t-periodic elements of $C(X)$ is dense.*

4.7. Notation. We denote by $C_n(X)$ the subset of $C(X)$ consisting of all f such that $ft^n = f$. In other words it consists of all the t-periodic elements whose minimal period divides n.

4.8. Proposition. *Let (X, t) be a flow. Then $C_n(X)$ is a complete t-invariant lattice-ordered subalgebra.*

Proof. All but the completeness is completely obvious. If $f_1, f_2, \ldots, f_k, \ldots$ is a sequence of functions that converges to f, then the sequence $f_1t^n, f_2t^n, \ldots, f_kt^n$ converges to ft^n. But that sequence is the original one. ∎

4.9. Theorem. *A flow is cocyclic if and only if it is a filtered inverse limit of periodic flows.*

Proof. Assume (X, t) is cocyclic. It follows from Gelfand duality that $C_n(X) \cong C(X_n)$ for a quotient space X_n of X. The family of these quotients X_n is filtered, which together with compactness, implies that the induced map $X \longrightarrow \lim X_n$ is surjective. To see that it is injective, let $x \neq y \in X$. If $f(x) = f(y)$ for all t-periodic f, the same would be true for all functions in the closure of the algebra of the t-periodic functions, which is all of $C(X)$.

Conversely, suppose $\{(X_\alpha, t_\alpha)\}$ is a filtered diagram of periodic flows, such that $X = \lim X_\alpha$ and t is the flow on X uniquely determined by $\{t_\alpha\}$. By Gelfand duality, $C(X) = \operatorname{colim} C(X_\alpha)$. Since the colimit is filtered, it is just the closure of the union of the corresponding subalgebras of $C(X)$ so that the t-periodic functions are dense in the union, since they are dense in each $C(X_\alpha)$. ∎

4.10. Theorem. *Let (X, t) be a flow. Then the following are equivalent:*

(1) *(X, t) is cocyclic;*

(2) *The action $\widetilde{\varphi} : \mathbf{Z} \times X \longrightarrow X$ given by $\widetilde{\varphi}(n, x) = t^n x$ is continuous when $\mathbf{Z}$ is replaced by $\mathbf{Z}_{\mathrm{pf}}$;*

(3) *The action $\widetilde{\varphi} : \mathbf{Z} \times X \longrightarrow X$ extends to a continuous action by $\mathrm{pf}(\mathbf{Z})$.*

Proof. (1) $\Longrightarrow$ (2): Assume that (X, t) is cocyclic. Then $X = \lim X_n$, as seen in the proof of Theorem 4.9. We claim that $\mathbf{Z}_{\mathrm{pf}} \times X \longrightarrow X$ (see 3.1) is continuous. Since $X = \lim X_n$, it is sufficient to show that the composite $\mathbf{Z}_{\mathrm{pf}} \times X \longrightarrow X \longrightarrow X_n$ is continuous for each $n \in \mathbf{Z}$. Clearly the composite factors through $\mathbf{Z}/n\mathbf{Z} \times X \longrightarrow X_n$. That map is clearly continuous since $\mathbf{Z}/n\mathbf{Z}$ is discrete and each power of t is continuous.

(2) $\Longrightarrow$ (3): Assume that $\widetilde{\varphi}$ is continuous on $\mathbf{Z}_{\mathrm{pf}}$. It follows from Theorem 2.3 that $\mathrm{Hom}(\mathbf{Z}_{\mathrm{pf}} \times X, X) \cong \mathrm{Hom}(\mathbf{Z}_{\mathrm{pf}}, \mathrm{hom}(X, X))$ so the transposed map $\varphi : \mathbf{Z}_{\mathrm{pf}} \longrightarrow \mathrm{hom}(X, X)$ is continuous. But by Proposition 2.1, $\mathrm{hom}(X, X)$ is complete and so we can extend $\widetilde{\varphi}$ to

a map $\mathrm{pf}(\mathbf{Z}) \longrightarrow \hom(X, X)$ which, by another application of 2.3 transposes to a uniform morphism $\mathrm{pf}(\mathbf{Z}) \times X \longrightarrow X$. The fact that $\widetilde{\varphi}(n+m, x) = \widetilde{\varphi}(n, \widetilde{\varphi}(m, x))$ for all $n, m \in \mathbf{Z}$, together with continuity and the fact that $\mathbf{Z}_{\mathrm{pf}}$ is dense in $\mathrm{pf}(\mathbf{Z})$, implies that the extension remains a group action.

(3) $\implies$ (1): Assume that the action of $\mathbf{Z}$ on X extends to an action of $\mathrm{pf}(\mathbf{Z})$ on X. We denote $\widetilde{\varphi}(\lambda, x)$ by $t^\lambda x$. For $n > 0$ in $\mathbf{Z}$, let

$$E_n = \{(x, y) \in X \times X \mid y = t^{n\ell} x \text{ for some } \ell \in \mathbf{Z}\}$$

$$F_n = \{(x, y) \in X \times X \mid y = t^{n\lambda} x \text{ for some } \lambda \in \mathrm{pf}(\mathbf{Z})\}$$

At this point, we insert:

4.11. Claim. $F_n = \mathrm{cl}(E_n)$ in $X \times X$.

Proof. We will show that E_n is dense in F_n and that F_n is closed. For the first, suppose $\lambda \in \mathrm{pf}(\mathbf{Z})$. Let $\{\ell_\alpha\}$ denote a net in $\mathbf{Z}$ that converges to λ. Then continuity of the action implies that for all $x \in X$, the net $\{t^{n\ell_\alpha} x\}$ converges to $t^{n\lambda} x$. This shows that E_n is dense in F_n. To see that F_n is closed, suppose that $\{(x_\alpha, t^{n\lambda_\alpha} x_\alpha)\}$ is a net in F_n that converges to (x, y). Since $n\mathrm{pf}(\mathbf{Z})$ is the continuous image of the compact space $\mathrm{pf}(\mathbf{Z})$ it is compact and hence the net $\{n\lambda_\alpha\}$ has a subnet that converges to an element $n\lambda \in \mathrm{pf}(\mathbf{Z})$. If we restrict to this subnet, the original net converges to $(x, t^{n\lambda} x)$. But the original net converges and the space is Hausdorff, so it can only converge to $(x, t^{n\lambda} x)$ and hence we conclude that $y = t^{n\lambda} x$ which shows that $(x, y) \in F_n$. ∎

We now return to the proof of the theorem. As before, let $C_n(X)$ denote the subalgebra of $C(X)$ consisting of all f such that $f = ft^n$. As in 4.9, there is quotient $\pi_n : X \longrightarrow X_n$ such that $C(X_n) = C_n(X) \subseteq C(X)$. Let $K_n \rightrightarrows X$ be the kernel pair of π_n. It is immediate that $E_n \subseteq K_n$ from which we conclude that $F_n \subseteq K_n$. But then X/F_n maps surjectively onto $X/K_n = X_n$, whence $C(X_n)$ is a subalgebra of $C(X/F_n)$. But for every $(x \in X)$, $(x, t^n x) \in F_n$, from which we see that every $f \in C(X/F_n)$ is t-periodic of period n. Thus $C(X/F_n) = C(X_n)$ so that $X_n \cong X/F_n$ and therefore $F_n = K_n$.

Now suppose that x and y are distinct elements of X. We want to show that there is an $n \in \mathbf{Z}$ such that $\pi_n x \neq \pi_n y$. Assume that $\pi_n x = \pi_n y$ for all $n > 0$. Then, since $F_n = K_n$ for each $n > 0$, there is a $\lambda \in \mathrm{pf}(\mathbf{Z})$ for which $y = t^\lambda x$ and n divides λ. Choose such a λ. For each prime p let λ_p be the projection of λ in $\mathrm{pf}(\mathbf{Z})_p$. Let $\ell_p = \mathrm{ord}_p \lambda_p$ (see 3.2 and the discussion that precedes it.) Let $I[x] = \{\zeta \in \mathrm{pf}(\mathbf{Z}) \mid t^\zeta x = x\}$. It is readily shown that $I[x]$ is a closed ideal, so by Proposition 3.2, we can write $I[x] = \prod I_p$, where I_p is an ideal of $\mathrm{pf}(\mathbf{Z})_p$ generated by a power, p^{k_p}. If $\ell_p \geq k_p$ for each prime p, then $\lambda \in I[x]$, which implies that $y = t^\lambda x = x$. Hence there is a prime p for which $\ell_p < k_p$ (which includes the possibility that $k_p = \infty$). Let $n = p^{\ell_p}$. Suppose that $\pi_n x = \pi_n y$. Then there is a $\mu \in \mathrm{pf}(\mathbf{Z})$ such that $n | \mu$, whence $\mathrm{ord}_p \mu \geq \ell_p$, and also $t^\mu x = y$. But then $\mu - \lambda \in I[x]$ which means that $\mathrm{ord}_p(\mu - \lambda) \geq k_p > \ell_p$ which is impossible since $\mathrm{ord}_p \mu > \mathrm{ord}_p \lambda$ implies that $\mathrm{ord}_p(\mu - \lambda) = \mathrm{ord}_p \lambda = \ell_p$. We conclude from this that the canonical map $X \longrightarrow \lim X_n$ is a monomorphism and as seen in the proof of 4.9 that it is surjective and therefore a homeomorphism. ∎

4.12. THEOREM. *Let (X,t) be a flow. Then (X,t) is cocyclic if and only if the map $\varphi : \mathbf{Z} \longrightarrow \mathrm{aut}(X)$ is uniform on the topology of $\mathbf{Z}_{\mathrm{pf}}$.*

PROOF. Suppose that (X,t) is cocyclic. Then $\widetilde{\varphi} : \mathbf{Z} \times X \longrightarrow X$ extends continuously to $\mathrm{pf}(\mathbf{Z}) \times X \longrightarrow X$ and thus restricts to $\mathbf{Z}_{\mathrm{pf}} \times X \longrightarrow X$.

Conversely, assume that $\varphi : \mathbf{Z} \longrightarrow \mathrm{aut}(X)$ is uniform on $\mathbf{Z}_{\mathrm{pf}}$. We claim that $\mathrm{aut}(X)$ is complete, since it is evidently closed in $\hom(X,X)$ and the latter is complete by Proposition 2.1. Thus $\varphi : \mathbf{Z}_{\mathrm{pf}} \longrightarrow \mathrm{aut}(X)$ extends continuously to $\mathrm{pf}(\mathbf{Z}) \longrightarrow \mathrm{aut}(X)$ and then, by Theorem 2.3 (which is also [Isbell, 1964, Theorem 24]) it transposes to $\mathrm{pf}(\mathbf{Z}) \times X \longrightarrow X$. The conclusion follows from the preceding theorem. ∎

4.13. THEOREM. *A quotient of a cocyclic flow is cocyclic.*

PROOF. Let $q : (X,t) \longrightarrow (Y,s)$ be a surjective flow map in compact Hausdorff spaces and assume that (X,t) is cocyclic. The continuous action $\mathrm{pf}(\mathbf{Z}) \times X \xrightarrow{\widetilde{\varphi}} X \xrightarrow{q} Y$ gives, by Theorem 2.3, a map $\mathrm{pf}(\mathbf{Z}) \longrightarrow \hom(X,Y)$ that takes $z \in \mathrm{pf}(\mathbf{Z})$ to the function $x \mapsto q(t^z x)$. Since the category of compact Hausdorff spaces is equational, q is a regular epimorphism. If E is the kernel pair of q, the coequalizer diagram $E \rightrightarrows X \longrightarrow Y$ gives an equalizer $\hom(Y,Y) \longrightarrow \hom(X,Y) \rightrightarrows \hom(E,Y)$, since $\hom(-,Y)$ has a left adjoint. In particular, $\hom(Y,Y)$ is closed in $\hom(X,Y)$. We then have a commutative square

$$\begin{tikzcd} \mathbf{Z} \arrow[r] \arrow[d] & \mathrm{pf}(\mathbf{Z}) \arrow[d] \\ \hom(Y,Y) \arrow[r] & \hom(X,Y) \end{tikzcd}$$

in which the top arrow is dense and the bottom one is a closed inclusion, so that the diagonal fill-in gives the map $\mathrm{pf}(\mathbf{Z}) \longrightarrow \hom(Y,Y)$. The conclusion follows from Theorem 4.10. ∎

4.14. THEOREM. *The full subcategory of cocyclic flows is a reflective subcategory of the category of flows.*

PROOF. Let (X,t) be a flow. Let $P(X)$ be the full subalgebra of $C(X)$ consisting of the t-periodic elements. The sum and product of t-periodic elements as well as all constants are t-periodic so that $P(X)$ is a subalgebra, evidently t-invariant (that is, invariant under the actions of both t and t^{-1} on $C(X)$). The topological closure $\mathrm{cl}(P(X))$ has by [Gillman & Jerison (1960), Theorem 5.14] the form $C(Y)$ for some quotient Y of X. Moreover, by Gelfand duality, there is an action of t on Y that is compatible with the quotient mapping $X \longrightarrow Y$. If $(X,t) \longrightarrow (Z,t)$ is a flow map to a cocyclic flow, then it is clear that the image of $P(Z) \longrightarrow C(X)$ lies in $P(X) = P(Y)$ and hence the image of $C(Z)$ lies in $C(Y)$ so that the map $X \longrightarrow Z$ factors through Y. ∎

4.15. COROLLARY. *An arbitrary limit of cocyclic flows is cocyclic.* ∎

4.16. Proposition. *A finite limit or finite colimit of periodic flows is periodic. A finite colimit of cocyclic flows is cocyclic.*

Proof. Periodic flows are obviously closed under finite products, finite sums, t-invariant subobjects, and quotients. As for cocyclic flows, closure under finite sums is evident from Gelfand duality: if $X = \sum X_\alpha$ is a finite sum, then $C(X) = \prod C(X_\alpha)$ and if the periodic elements are dense in each factor, they are dense in the product. Closure under quotients is given by Theorem 4.13.

4.17. Examples: Two flows that are cocyclic, but not periodic. Let $[n]$ denote the finite set $\{0, 1, \ldots, n-1\}$ with the discrete topology and the action that takes i to $i+1 \pmod{n}$. The first example is the one point compactification of $\sum[n]$. We extend the action to fix the point at infinity. The second example is $\prod[n]$. It is easy to see that neither flow is periodic because the periods get too large, while each is the inverse limit of a chain of periodic flows and hence cocyclic.

5. Almost t-periodic functions

5.1. Definition. *Let (X, t) be a flow. Suppose that $f \in C(X)$. For each $\epsilon > 0$, let $U(f, \epsilon)$ denote $\{n \in \mathbf{Z} \mid \|ft^n - f\| < \epsilon\}$.*

Recall that when (X, t) is a flow and $f \in C(X)$, we have defined $\varphi_f : \mathbf{Z} \longrightarrow C(X)$ by $\varphi_f(n) = ft^n$.

The following observation is routine and left to the reader, (The **weak topology** is the coarsest topology on the domain that renders a function continuous.)

5.2. Proposition. *The topology generated by the $U(f, \epsilon)$ for all $\epsilon > 0$ is just the weak topology from φ_f.* ∎

5.3. Proposition. *For any $f \in C(X)$, the weak topology $\mathbf{Z}$ gets from φ_f is a group topology on $\mathbf{Z}$.*

Proof. For any $m \in \mathbf{Z}$, we have $\|ft^{m+n} - ft^m\| = \|ft^n - f\|$ since as x ranges over all of X, so does $t^m x$. Clearly $U(f, \epsilon/2) - U(f, \epsilon/2) \subseteq U(f, \epsilon)$. ∎

We will call this the **topology on $\mathbf{Z}$ induced by** f.

5.4. Remark. This topology is Hausdorff if and only if f is *not* t-periodic.

5.5. Definition. *We say that $f \in C(X)$ is* **almost t-periodic** *if the topology on $\mathbf{Z}$ induced by f is totally bounded.*

This means that for every $\epsilon > 0$, a finite number of translates of $U(f, \epsilon)$ covers $\mathbf{Z}$. If we unwind that, it turns out to mean that for all $\epsilon > 0$, there is a finite set $S \subseteq \mathbf{Z}$ such that for all $n \in \mathbf{Z}$ there is an $s \in S$ with $\|ft^n - ft^s\| < \epsilon$. We will call such a finite set S an **ϵ-span for** $U(f, \epsilon)$. Although the phrase "almost periodic" has been used in many not entirely compatible ways, our definition seems to capture their spirit. See the first item in the following theorem.

5.6. Theorem. *Let (X, t) be a flow and $f \in C(X)$. Then the following are equivalent:*

(1) *The closure of $\{ft^n \mid n \in \mathbf{Z}\}$ is compact in the sup norm on $C(X)$;*

(2) *f is almost t-periodic;*

(3) *The topology on $\mathbf{Z}$ induced by $\varphi_f : \mathbf{Z} \longrightarrow C(X)$ is totally bounded and hence the uniform completion is compact;*

(4) *$\varphi_f : \mathbf{Z} \longrightarrow C(X)$ extends continuously to the Bohr compactification $\mathrm{b}(\mathbf{Z})$ of $\mathbf{Z}$.*

Proof. (1) $\implies$ (2): Let $A = \{ft^n \mid n \in \mathbf{Z}\}$. For any $\epsilon > 0$, every element of $\mathrm{cl}(A)$ is within ϵ of an element of A. Thus the cover of A by ϵ-spheres around each element of A also covers $\mathrm{cl}(A)$ and hence has a finite refinement, say the ϵ-spheres around the elements $ft^{s_1}, ft^{s_2}, \ldots, ft^{s_m}$. Clearly $S = \{s_1, s_2, \ldots s_m\}$ is an ϵ-span.

(2) $\implies$ (3): This is just the definition.

(3) $\implies$ (4): Assuming the uniform completion $\mathbf{Z}^\sharp$ is compact, the map $\mathbf{Z} \longrightarrow C(X)$ extends, since $C(X)$ is complete, to a continuous map $\mathbf{Z}^\sharp \longrightarrow C(X)$. But the topology on $\mathbf{Z}$ is a group topology and hence $\mathbf{Z}^\sharp$ is a group. Since every group compactification of $\mathbf{Z}$ is a quotient of $\mathrm{b}(\mathbf{Z})$, we have $\mathrm{b}(\mathbf{Z}) \longrightarrow \mathbf{Z}^\sharp \longrightarrow C(X)$.

(4) $\implies$ (1): Trivial since the image of $\mathrm{b}(\mathbf{Z}) \longrightarrow C(X)$, which is compact, includes the image of $\varphi_f : \mathbf{Z} \longrightarrow C(X)$. ∎

Recall from 4.1 that, given a flow (X, t), we get a homomorphism $\varphi : \mathbf{Z} \longrightarrow \mathrm{aut}(X)$ defined by $\varphi(n) = t^n$.

5.7. Definition. *Let (X, t) be a flow. The* **t-induced topology** *on $\mathbf{Z}$ is the weak topology induced by φ. This is the coarsest topology on $\mathbf{Z}$ for which φ is continuous and is automatically a group topology since $\mathrm{aut}(X)$ is a topological group and φ is a group homomorphism.*

5.8. Proposition. *A topology on $\mathbf{Z}$ makes φ continuous (and therefore uniform) if and only if for each $f \in C(X)$, and each $\epsilon > 0$, the set $U(f, \epsilon)$ is a neighbourhood of 0.*

Proof. Using the fact that every compact Hausdorff space is homeomorphic to a subspace of a power of the unit interval, indexed by its maps to the interval, we have a sequence of embeddings

$$\mathrm{aut}(X) \subseteq \hom(X, X) \hookrightarrow \hom(X, \mathbf{R}^{C(X)}) \cong \hom(X, \mathbf{R})^{C(X)}$$

so that φ is continuous if and only if, for each $f \in C(X)$, the composite

$$\mathbf{Z} \longrightarrow \mathrm{aut}(X) \subseteq \hom(X, X) \hookrightarrow \hom(X, \mathbf{R}^{C(X)}) \cong \hom(X, \mathbf{R})^{C(X)} \xrightarrow{p_f} C(X)$$

is continuous, where p_f is the projection on the f coordinate. When the identifications are sorted out, the composite map takes $n \in \mathbf{Z}$ to ft^n. In particular the inverse image of an ϵ-neighbourhood in $\mathbf{R}$ is

$$\{m \in \mathbf{Z} \mid \|ft^n - ft^m\| < \epsilon\} = \{m \in \mathbf{Z} \mid \left\|ft^{n-m} - f\right\| < \epsilon\}$$

and it follows that φ is continuous if and only if for every $f \in C(X)$ and every $\epsilon > 0$, $U(f, \epsilon)$ is a neighbourhood of 0. ∎

This result has two interesting consequences:

5.9. Corollary. *The t-induced topology is the sup of the topologies induced by all the $f \in C(X)$ (see 5.3).* ∎

5.10. Corollary. *Suppose (X, t) is a flow and suppose that $U(f, \epsilon)$ contains a non-zero subgroup of $\mathbf{Z}$ for each $f \in C(X)$ and each $\epsilon > 0$. Then (X, t) is cocyclic.*

Proof. In that case $\varphi : \mathbf{Z} \longrightarrow \mathrm{aut}(X)$ will be continuous in the topology generated by those subgroups. But that topology is coarser than that of the $\mathbf{Z}_{\mathrm{pf}}$ and hence φ is also continuous on $\mathbf{Z}_{\mathrm{pf}}$. Now Theorems 4.10 and 2.3 give the desired conclusion. ∎

Simultaneous almost t-periodicity. Recall that an $f \in C(X)$ is almost t-periodic if for all $\epsilon > 0$ a finite number of translates of the set $U(f, \epsilon)$ covers $\mathbf{Z}$. A set F of functions is **simultaneously almost t-periodic** if a finite number of translates of $\bigcap_{f \in F} U(f, \epsilon)$ covers $\mathbf{Z}$.

5.11. Proposition. *A finite set of almost t-periodic functions is simultaneously almost t-periodic.*

Proof. Suppose $F = \{f_1, \ldots, f_k\}$. The set $\bigcap_{i=1}^k U(f_i, \epsilon)$ is open in the supremum of the topologies induced by the $f_i \in F$. Since each of those topologies is totally bounded it follows from 2.5 that the supremum is also and hence a finite number of translates covers $\mathbf{Z}$. ∎

5.12. Proposition. *Let F be any set of almost t-periodic functions. Then every element of the complete t-invariant subalgebra generated by F is almost t-periodic.*

Proof. If $f, g \in F$, Then $U(f \pm g, \epsilon) \supseteq U(f, \epsilon/2) \cap U(g, \epsilon/2)$ and the preceding argument implies that $\{f, g\}$ is simultaneously almost t-periodic so that a finite number of translates of the the right-hand side covers $\mathbf{Z}$. A similar argument works for $f \wedge g$ and $f \vee g$. For the product, it is sufficient to show that fg is almost t-periodic when $\|f\| = \|g\| = 1$. But in that case we also get that $U(fg, \epsilon) \supseteq U(f, \epsilon/2) \cap U(g, \epsilon/2)$. Since $U(f, \epsilon) = U(ft, \epsilon)$ we conclude that every element of the smallest t-invariant subalgebra generated by F is almost t-periodic. Finally, suppose that f is in the closure of that subalgebra. Given $\epsilon > 0$, there is a g in the subalgebra such that $\|f - g\| \leq \epsilon/3$. But then a standard argument shows that $U(f, \epsilon) \supseteq U(g, \epsilon/3)$. ∎

6. Almost cocyclic flows

We say that a flow is **almost cocyclic** if every $f \in C(X)$ is almost t-periodic. We might have defined this to mean that the almost t-periodic functions were dense, in parallel with the definition of cocyclic, but 5.12 shows that the conditions are equivalent.

6.1. Example of an almost cocyclic flow that is not cocyclic. Let $X = \mathbf{R}/\mathbf{Z}$, the circle group. Define $t : X \longrightarrow X$ as addition mod $\mathbf{Z}$ of an irrational number γ. It is well known that the orbit of any point is dense. Fix an $x \in X$. For any $n > 0$, the orbit of x under t^n is dense. If $ft^n = f$, then for any $y \in X$, the set $\{t^{kn}x\}$ comes arbitrarily close to y so that for some $k \in \mathbf{Z}$, we have $f(x) = ft^{kn}(x)$, which is arbitrarily close to $f(y)$. Thus $f(x) = f(y)$ and we conclude that the only t-periodic functions in $C(X)$ are the constants. Hence the flow is not cocyclic.

Now let $f \in C(X)$ and let $\epsilon > 0$ be given. Since X is compact, there is a $\delta > 0$ such that $|x_1 - x_2| < \delta$ implies $|f(x_1) - f(x_2)| < \epsilon$. In the following, all arithmetic is carried out mod $\mathbf{Z}$. Since the integer multiples of γ are dense, the intervals $\{(n\gamma - \delta, n\gamma + \delta)\}$ cover X and hence a finite set of them, say $\{(s\gamma - \delta, s\gamma + \delta) \mid s \in S\}$, S finite, also covers X. Now given $n \in \mathbf{Z}$, let $s \in S$ so that $|n\gamma - s\gamma| < \delta$, whence for all $x \in X$,

$$|t^s x - t^n x| = |(s\gamma + x) - (n\gamma + x)| = |s\gamma - n\gamma| < \delta$$

so that $|ft^s x - ft^n x| < \epsilon$. Since this is independent of x, it follows that $\|ft^s - ft^n\| < \epsilon$ so that S is an ϵ-span.

6.2. Theorem. *Let (X, t) be a flow. Then the following are equivalent:*

(1) *(X, t) is almost cocyclic;*

(2) *The t-induced topology is totally bounded;*

(3) *The action of $\mathbf{Z}$ extends continuously to* $\mathrm{b}(\mathbf{Z})$*;*

Proof. That (1) holds if and only if (2) does is obvious. That (2) holds if and only if (3) does follows from Proposition 5.8. ∎

6.3. Proposition. *A quotient of an almost cocyclic flow is almost cocyclic.*

Proof. Suppose $(X, t) \longrightarrow (Y, s)$ is a quotient mapping between flows. Then $C(Y)$ is a subalgebra of $C(X)$. If the elements of $C(X)$ are almost t-periodic, the same is true of $C(Y)$. ∎

6.4. Proposition. *Let (X, t) be almost cocyclic. For each $x \in X$, the map $\theta_x : \mathbf{Z} \longrightarrow X$, defined by $\theta_x(n) = t^n(x)$ is uniform when $\mathbf{Z}$ is topologized by all $U(f, \epsilon)$, $f \in C(X)$ and $\epsilon > 0$.*

Proof. The map $\mathbf{Z} \longrightarrow \mathrm{aut}(X)$ is uniform by 5.8 and the map $\mathrm{aut}(X) \longrightarrow X$ that evaluates at x is induced by $x : 1 \longrightarrow X$ and hence is uniform by [Isbell, 1964, III.2] and therefore so is the composite. ∎

Recall that $\mathrm{b}(\mathbf{Z})$ is the Bohr compactification of $\mathbf{Z}$. Define $t : \mathrm{b}(\mathbf{Z}) \longrightarrow \mathrm{b}(\mathbf{Z})$ as the extension to $\mathrm{b}(\mathbf{Z})$ of the map from $\mathbf{Z} \longrightarrow \mathrm{b}(\mathbf{Z})$ defined by $t(\zeta) = \zeta + 1$, which is obviously uniform.

6.5. Proposition. *The flow* $(\mathrm{b}(\mathbf{Z}), t)$ *is almost cocyclic but is not cocyclic.*

Proof. For any $f \in C(\mathrm{b}(\mathbf{Z}))$, the restriction of f to $\mathbf{Z}$ is continuous in the topology on $\mathbf{Z}$ induced by its inclusion in $\mathrm{b}(\mathbf{Z})$. Hence, for any $\epsilon > 0$, $U(f, \epsilon)$ is open in that topology. But $\mathbf{Z}$ is totally bounded in that topology, which comes from the inclusion into a compact group. Hence finitely many translates of $U(f, \epsilon)$ cover $\mathbf{Z}$, which implies that f is almost t-periodic. On the other hand, the orbit of 0 under any power of t is all of $\mathbf{Z}$, which is dense in $\mathrm{b}(\mathbf{Z})$. The same argument used in 6.1 shows that only constant functions are t-periodic and thus the flow is not cocyclic. ∎

6.6. Definition. *Let* $\mathbf{Z}$-Cmp *denote the category of compact Hausdorff spaces equipped with an action of* $\mathbf{Z}$. *This name is chosen by analogy with G-set. We similarly let* $\mathbf{Z}$-$\mathcal{C}$ *denote the category of complete lattice-ordered rings equipped with an action by* $\mathbf{Z}$.

In each of these categories, an object C with a $\mathbf{Z}$-action is determined by an automorphism $t : C \longrightarrow C$, with t being the value of the action at the integer 1. It must be an automorphism since the value of the action at -1 must be t^{-1}. Thus a $\mathbf{Z}$-action is the same thing as an automorphic flow.

6.7. Proposition. *The Gelfand duality between* Cmp *and* $\mathcal{C}$ *extends to a duality between* $\mathbf{Z}$-Cmp *and* $\mathbf{Z}$-$\mathcal{C}$.

Proof. An automorphism $t : X \longrightarrow X$ in Cmp gives a morphism $C(t) : C(X) \longrightarrow C(X)$, which must be an isomorphism since C is a functor and so $C(t^{-1}) = C(t)^{-1}$. Similarly, if R is a complete lattice-ordered ring and $t : R \longrightarrow R$ is an automorphism, then $\mathrm{Max}(t) : \mathrm{Max}(R) \longrightarrow \mathrm{Max}(R)$ is an automorphism of the maximal ideal spaces. ∎

Now suppose that (X, t) is a flow and that R is the ring of almost t-periodic functions with respect to t. Define an equivalence relation E on X by $(x, y) \in E$ if $f(x) = f(y)$ for all $f \in R$. Then it follows from Gelfand duality that $R = C(X/E)$. From the preceding discussion it also follows that X/E has a flow we call t/E and that $(X, t) \longrightarrow (X/E, t/E)$ is a flow morphism. It is trivial that $(X/E, t/E)$ is an almost cocyclic flow.

6.8. Theorem. *The category of almost cocyclic flows is a reflective subcategory of the category of all flows, given by* $(X, t) \longrightarrow (X/E, t/E)$.

Proof. We have done everything except to exhibit the adjunction. Suppose we have a flow map $p : (X, t) \longrightarrow (Y, s)$ in which (Y, s) is almost cocyclic. The category $\mathcal{C}$ of C^*-algebras has as morphisms the "norm-reducing" (which is to say, non-increasing) ring homomorphisms. It follows that if $f \in C(Y)$, $U(f, \epsilon) \subseteq U(fp, \epsilon)$ and if a finite number of translates of $U(f, \epsilon)$ covers $\mathbf{Z}$, then the same must be true of $U(fp, \epsilon)$. It follows that the image of $C(Y)$ in $C(X)$ actually lies in $C(X/E)$. That is, we have a commutative triangle

$$\begin{tikzcd} C(Y) \arrow[rr] \arrow[dr] & & C(X/E) \arrow[dl] \\ & C(X) & \end{tikzcd}$$

which, upon dualizing, gives

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ X/E \arrow[rr] & & Y \end{tikzcd}$$

as required. ■

6.9. Example: A flow that is periodic at each element, but is not cocyclic or even almost cocyclic. Let X be the subset of $\mathbf{R}^2$ consisting of circles centered at the origin of radius $1 - 1/n$ for $n = 2, 3, 4, \ldots, \infty$. Define $t : X \longrightarrow X$ in polar coordinates by $t(r, \theta) = (r, \theta + \pi/n)$ when $r = 1 - 1/n$. This function is continuous on the circles of radius < 1 as well as on the outer circle where it is the identity. But as you get nearer the outer circle the function gets closer and closer to the identity and hence the function is continuous on the union. We use Theorem 6.2 to show the action is not almost cocyclic. Suppose the action $\alpha : \mathbf{Z} \times X \longrightarrow X$ that takes (n, x) to $t^n x$ extended to $\mathrm{b}(\mathbf{Z}) \times X \longrightarrow X$. Let $\zeta \in \mathrm{b}(\mathbf{Z})$ be any accumulation point of the sequence $2, 3, 4, \ldots$. Then $(\zeta, (1, 0))$ is an accumulation point of $(2, (1/2, 0)), (3, (2/3, 0)), (4, (3/4, 0)), \ldots$. Apply α to the sequence to get $(1/2, \pi), (2/3, \pi), (3/4, \pi), \ldots$, which approaches $(1, \pi)$, while clearly $t(\zeta, (1, 0)) = \lim t(n, (1, 0)) = (1, 0)$, which shows that the extended action cannot be continuous.

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