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REMARKS ON PUNCTUAL LOCAL CONNECTEDNESS

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Abstract. We study the condition, on a connected and locally connected geometric morphism $p\colon \mathcal{E} \to \mathcal{S}$, that the canonical natural transformation $p_* \to p_!$ should be (pointwise) epimorphic — a condition which F.W. Lawvere [11] called the 'Nullstellensatz', but which we prefer to call 'punctual local connectedness'. We show that this condition implies that $p_!$ preserves finite products, and that, for bounded morphisms between toposes with natural number objects, it is equivalent to being both local and hyperconnected.

Introduction

In his search for an axiomatic theory of cohesion [11], Bill Lawvere has emphasized the importance of two conditions that may be satisfied by a locally connected geometric morphism $p\colon \mathcal{E} \to \mathcal{S}$ between toposes: (a) that the left adjoint $p_!$ of the inverse image functor p^* should preserve finite products (including the empty product 1; thus a geometric morphism satisfying this condition is connected as well as locally connected), and (b) that the canonical natural transformation $p_* \to p_!$, which exists when p is connected as well as locally connected, should be pointwise epimorphic. Lawvere calls this second condition the 'Nullstellensatz'; we have chosen to name it 'punctual local connectedness', since (as we shall see) it is the expression in the internal logic of the base topos $\mathcal{S}$ of the idea that 'every connected object of $\mathcal{E}$ has a point'. The present paper is a contribution to clarifying the status of these two conditions: we show that the second implies the first, and that (at least for bounded morphisms between toposes with natural number objects) the second is equivalent to the conjunction of two more familiar conditions, namely that p should be hyperconnected and local.

In the paper, we provide site characterizations of the Grothendieck toposes $\mathcal{E}$ which satisfy (a) or (b) (that is, such that the unique morphism $p\colon \mathcal{E} \to \mathbf{Set}$ satisfies (a) or (b)), which make it clear why Grothendieck toposes satisfying (b) are local and hyperconnected, and also satisfy (a). However, we also provide 'site-free' proofs of the results stated above, most of which require no more than elementary facts about multiple adjunctions. The layout of the paper is as follows: section 1 contains the site characterization of stable and punctual local connectedness, section 2 contains the elementary results on multiple adjunctions, and section 3 applies these to give the site-free proofs of our results. All

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topos-theoretic terminology and notation used in the paper, other than that specifically introduced here, is taken from the author's book [8].

Before concluding this introduction, I should record my indebtedness to two of my Ph.D. students. The train of thought that led to this paper was started by reading a first draft of Nick Duncan's thesis [6], and the development of my thoughts was aided by stimulating e-mail exchanges with Olivia Caramello.

1. Site Characterizations of SLC and PLC Morphisms

We recall that a geometric morphism $p\colon \mathcal{E} \to \mathcal{S}$ is said to be *connected* if the inverse image functor p^* is full and faithful (equivalently, the unit $1_{\mathcal{S}} \to p_* p^*$ is an isomorphism), and *locally connected* if p^* has an $\mathcal{S}$-indexed left adjoint (commonly denoted $p_!$). Locally connected morphisms were first studied in [2]. If p is locally connected, then p^* preserves $\mathcal{S}$-indexed products; in particular it is a cartesian closed functor (i.e., preserves exponentials), cf. [8], C3.3.1. A locally connected morphism is connected iff $p_!$ preserves the terminal object ([8], C3.3.3). If the codomain topos is **Set**, local connectedness amounts to saying that each object A of $\mathcal{E}$ can be decomposed as a coproduct of *connected objects*, i.e. objects with no nontrivial coproduct decompositions; moreover, $p_!A$ is simply the set of 'connected components' of A, i.e. the set indexing this coproduct decomposition. (The same interpretation is valid over an arbitrary base $\mathcal{S}$, provided we interpret 'connected' in a suitable constructive sense.) Similarly, p_*A may be interpreted as the set of points of A (i.e. morphisms $1 \to A$). If p is connected as well as locally connected, then we have a natural transformation $\theta\colon p_* \to p_!$, which may be obtained by applying fullness and faithfulness of p^* to the composite $p^*p_* \to 1_{\mathcal{E}} \to p^*p_!$ of the counit of $(p^* \dashv p_*)$ and the unit of $(p_! \dashv p^*)$; interpreted in the internal logic of $\mathcal{S}$, it may be interpreted as the mapping which sends a point of A, considered as a connected subobject of A, to the connected component which contains it. Thus the assertion that θ_A is surjective says that 'every connected component of A contains a point'. For this reason, we shall call an arbitrary geometric morphism p *punctually locally connected* (or PLC, for short) if it is connected and locally connected, and $\theta\colon p_* \to p_!$ is (pointwise) epic.

We shall also consider the condition that the left adjoint $p_!$ preserves finite products (including the empty product 1); we shall say that a locally connected morphism p is *stably locally connected* (or SLC) if this condition holds. A special case of stable local connectedness is *total connectedness*, studied by M.C. Bunge and J. Funk [5] (and also in [8], section C3.6): p is said to be totally connected if it is locally connected and $p_!$ preserves all finite limits (so that it is the inverse image of a geometric morphism, right adjoint to p in the 2-category of toposes and geometric morphisms). For localic morphisms, there is no difference between stable local connectedness and total connectedness: we shall prove this only when the base topos is **Set**, but the argument is constructive, and so can be generalized to an arbitrary base.

1.1. Lemma. *Let X be a locale. Then the topos* $\mathbf{Sh}(X)$ *is stably locally connected iff it is totally connected.*

Proof. One direction is trivial. For the converse, note that stable local connectedness says that the product of any two connected objects is connected. But if U and V are two connected open sublocales of X, considered as subterminal objects of $\mathbf{Sh}(X)$, then their product is their intersection; so this implies that the intersection of any two connected opens of X is connected. Hence the opens which contain a connected open (equivalently, since X is locally connected, the inhabited opens of X) form a filter, which is easily seen to be completely prime; and the corresponding point of X is clearly dense. So by [8], C3.6.17(a), $\mathbf{Sh}(X)$ is totally connected. ■

We recall the site characterization of locally connected Grothendieck toposes. We say that a site $(\mathcal{C}, J)$ is *locally connected* if each J-covering sieve (on an object U of $\mathcal{C}$, say) is connected when regarded as a full subcategory of $\mathcal{C}/U$; if in addition $\mathcal{C}$ has a terminal object, then we say $(\mathcal{C}, J)$ is *connected and locally connected*. Then we have

1.2. Proposition. *Let $p\colon \mathcal{E} \to \mathbf{Set}$ be a bounded geometric morphism. Then p is (connected and) locally connected iff $\mathcal{E}$ has a site of definition which is (connected and) locally connected.*

Proof. See [8], C3.3.10. ■

The construction of the site is straightforward: we take $\mathcal{C}$ to be a generating full subcategory of $\mathcal{E}$ consisting of connected objects (and containing the terminal object if $\mathcal{E}$ itself is connected), and equip it with the coverage J induced by the canonical coverage on $\mathcal{E}$. Clearly, if p is stably locally connected, we may choose $\mathcal{C}$ to be closed under finite products in $\mathcal{E}$; so we arrive at the following characterization of stably locally connected Grothendieck toposes. (Once again, we shall prove it only when the base topos is $\mathbf{Set}$, but by interpreting the proof in the internal logic of $\mathcal{S}$ it can be shown to hold for an arbitrary bounded morphism $\mathcal{E} \to \mathcal{S}$, at least provided $\mathcal{S}$ has a natural number object.) We recall that a category $\mathcal{C}$ is said to be *sifted* [1] (a translation of the French *tamisante* [10]) if it is nonempty and the diagonal functor $\Delta\colon \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ is final. This is equivalent to saying that the colimit functor $[\mathcal{C}, \mathbf{Set}] \to \mathbf{Set}$ preserves finite products.

1.3. Proposition. *For a bounded geometric morphism $p\colon \mathcal{E} \to \mathbf{Set}$, the following are equivalent:*

(i) *p is stably locally connected.*

(ii) *$\mathcal{E}$ has a locally connected site of definition $(\mathcal{C}, J)$ such that $\mathcal{C}$ has finite products.*

(iii) *$\mathcal{E}$ has a locally connected site of definition $(\mathcal{C}, J)$ such that $\mathcal{C}^{\mathrm{op}}$ is sifted.*

Proof. (i) ⇒ (ii) follows from the discussion above. (ii) ⇒ (iii) because a category with finite products is cosifted: for each pair of objects (U, V), the category $(\Delta \downarrow (U, V))$ is connected because it has a terminal object. Finally, if (iii) holds, then the colimit functor $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{Set}$ preserves finite products; but local connectedness of the site implies that constant presheaves on $\mathcal{C}$ are J-sheaves, and hence that the left adjoint $p_!$ of p^* is simply the restriction of the colimit functor to $\mathbf{Sh}(\mathcal{C}, J)$. So it too preserves finite products. ■

We may similarly obtain a site characterization of punctual local connectedness. Clearly, if p is punctually locally connected, and $\mathcal{C}$ is a generating full subcategory of connected objects of $\mathcal{E}$, then $\mathcal{C}$ will have the property that every object U has a point (i.e. a morphism $1 \to U$).

1.4. PROPOSITION. *A bounded geometric morphism* $p\colon \mathcal{E} \to \mathbf{Set}$ *is punctually locally connected iff* $\mathcal{E}$ *has a connected and locally connected site of definition* $\mathbf{Sh}(\mathcal{C}, J)$ *such that every object of* $\mathcal{C}$ *has a point.*

PROOF. One direction follows from the remarks above. For the converse, we have to show that if $(\mathcal{C}, J)$ satisfies the given conditions, then every connected object of the topos $\mathbf{Sh}(\mathcal{C}, J)$ has a point. But in fact every nonzero object has a point, since if $F(U)$ is nonempty for any U, then $F(1)$ must be nonempty. ∎

Examples of sites satisfying the conditions of 1.4 are not hard to find:

1.5. EXAMPLE. Consider the 'topological gros topos' $\mathcal{E} = \mathbf{Sh}(\mathcal{C}, J)$, where $\mathcal{C}$ is a small full subcategory of topological spaces, closed under passage to open subspaces, and a sieve is J-covering iff it contains a jointly surjective family of open inclusions. As noted in [8], C3.3.7, if we allow $\mathcal{C}$ to contain spaces which are not locally connected, then $\mathcal{E}$ is not a locally connected topos. However, if we restrict $\mathcal{C}$ to contain only locally connected spaces, then the full subcategory $\mathcal{D} \subseteq \mathcal{C}$ whose objects are the connected spaces is easily seen to satisfy the hypotheses of the Comparison Lemma ([8], C2.2.3), and the induced coverage J' on $\mathcal{D}$ has all covers connected. Moreover, since connected spaces are by definition nonempty, every object of $\mathcal{D}$ has a point. So we deduce that in this case $\mathbf{Sh}(\mathcal{C}, J) \simeq \mathbf{Sh}(\mathcal{D}, J')$ is punctually locally connected.

Similar examples can be given on replacing $\mathcal{C}$ by the category of smooth manifolds (note that manifolds are automatically locally connected), or by the category of affine schemes of finite type over an algebraically closed field K (that is, the dual of the category of finitely-presented K-algebras), equipped with the Zariski topology. (Algebraic closedness is needed to ensure that every connected object has a point.) On the other hand, we cannot replace spaces by locales in this example: there are examples of connected and locally connected locales with no points, and if we allow such locales to be objects of $\mathcal{C}$ our topos will be (connected and) locally connected, but not punctually so.

From the site characterization of 1.4, we may immediately deduce some further properties of punctually locally connected Grothendieck toposes.

1.6. PROPOSITION. *Let* $p\colon \mathcal{E} \to \mathbf{Set}$ *be a bounded, punctually locally connected* $\mathbf{Set}$*-topos. Then*

(i) *p is local.*

(ii) *p is hyperconnected.*

(iii) *p is stably locally connected.*

Proof. (i) We recall the site characterization of local Grothendieck toposes in [9], 1.7: $\mathcal{E}$ is local iff it can be generated by a site $(\mathcal{C}, J)$ such that $\mathcal{C}$ has a terminal object which is J-irreducible (that is, the only J-covering sieve on 1 is the maximal one). It is easy to see that if $(\mathcal{C}, J)$ is a site satisfying the conditions of 1.4, then the terminal object of $\mathcal{C}$ is J-irreducible: for every J-covering sieve is (connected, and hence) inhabited, and every morphism with codomain 1 is split epic, and so generates the maximal sieve.

(ii) Similarly, the underlying category of a site as in 1.4 is strongly connected (i.e., given any two objects, there are morphisms between them in both directions), from which it follows easily that its topos of sheaves is hyperconnected (cf. [8], A4.6.9).

(iii) After 1.3 and 1.4, it suffices to show that if $\mathcal{C}$ has a terminal object and every object of $\mathcal{C}$ has a point, then $\mathcal{C}^{\mathrm{op}}$ is sifted. But, for any pair of objects (U, V), the category $(\Delta \downarrow (U, V))$ is nonempty because both U and V have points; and, if $(f\colon W \to U, g\colon W \to V)$ and $(h\colon X \to U, k\colon X \to V)$ are any two objects of this category, we may connect them by choosing points w, x of W, X and forming the diagram

$$(f,g) \xleftarrow{w} (fw, gw) \xrightarrow{fw} (1_U, y) \xleftarrow{hx} (hx, gw) \xrightarrow{gw} (z, 1_B) \xleftarrow{kx} (hx, kx) \xrightarrow{x} (h,k)$$

where y and z are respectively the constant morphism $U \to 1 \overset{gw}{\to} V$, and the constant morphism $V \to 1 \overset{hx}{\to} U$. ■

It is at first sight rather surprising that the seemingly innocuous assumption '$\theta\colon p_* \to p_!$ is epic' should have such varied consequences. The genesis of the present paper was the author's attempt to find 'site-independent' reasons why these implications should hold.

2. Generalities on multiple adjunctions

When we study a geometric morphism $p\colon \mathcal{E} \to \mathcal{S}$ which is both local and locally connected, we are dealing with a chain of four adjoint functors $(p_! \dashv p^* \dashv p_* \dashv p^{\#})$. There is a certain amount of 'folklore' concerning chains of adjoint functors which does not seem to be written down anywhere in the literature; we devote this section to studying some of it.

We recall that an adjunction $(F \dashv G)$ with unit η and counit ϵ, is said to be *idempotent* if the natural transformations $F\eta\colon F \to FGF$ and $\epsilon_F\colon FGF \to F$ are two-sided inverses to each other (rather than merely one-sided inverses, as asserted by one of the 'triangular identities'). The apparent asymmetry of this condition is only apparent; it is equivalent to the assertion that η_G and $G\epsilon$ are two-sided inverses. Clearly, an adjunction which is either a reflection or a coreflection (i.e., such that either ϵ or η is an isomorphism) is idempotent. Also, given a chain of three adjoint functors $(L \dashv F \dashv R)$, the adjunction $(L \dashv F)$ is idempotent iff $(F \dashv R)$ is; we do not need this result, but we mention it in connection with the following lemma.

2.1. Lemma. *Suppose given functors $L, R\colon \mathcal{D} \rightrightarrows \mathcal{C}$ and $F\colon \mathcal{C} \to \mathcal{D}$ such that $(L \dashv F \dashv R)$; let α and β denote the unit and counit of $(L \dashv F)$, and η and ϵ those of $(F \dashv R)$. Suppose further that the two adjunctions are idempotent. Then the composite natural transformations $R\alpha.\beta_R\colon LFR \to R \to RFL$ and $\eta_L.L\epsilon\colon LFR \to L \to RFL$ are equal.*

Proof. Consider the diagram

$$\begin{tikzcd}
LFR \arrow[r, "\beta_R"] \arrow[dd, "L\epsilon"'] \arrow[dr, "LFR\alpha"] & R \arrow[r, "R\alpha"] & RFL \\
& LFRFL \arrow[ur, "\beta_{RFL}"] \arrow[d, shift right, "L\epsilon_{FL}"'] & \\
L \arrow[r, "L\alpha"] & LFL \arrow[u, shift right, "LF\eta_L"'] \arrow[r, "\beta_L"] & L \arrow[uu, "\eta_L"']
\end{tikzcd}$$

in which the three quadrilateral cells commute by naturality, the two vertical morphisms in the middle are inverse to each other by idempotency of $(F \dashv R)$, and the bottom composite is the identity by one of the triangular identities for $(L \dashv F)$. ■

2.2. Corollary. *Suppose given a chain of three adjoint functors* $(L \dashv F \dashv R)$.

(i) *If F is full and faithful, then there is a canonical natural transformation $R \to L$, which may be defined either as the composite $L\epsilon.(\beta_R)^{-1}\colon R \to LFR \to L$ or as $(\eta_L)^{-1}.R\alpha\colon R \to RFL \to L$.*

(ii) *If L and R are full and faithful, then there is a canonical natural transformation $L \to R$, which may be defined either as $\beta_R.(L\epsilon)^{-1}$ or as $(R\alpha)^{-1}.\eta_L$.*

Proof. (i) If F is full and faithful, then the adjunctions are idempotent and η and β are natural isomorphisms, so this follows immediately from 2.1.

(ii) Similarly, if L and R are full and faithful then the adjunctions are idempotent and α and ϵ are natural isomorphisms. ■

Now suppose we have a chain of four adjoint functors $(L \dashv F \dashv G \dashv R)$. If we suppose that F (and hence also R) is full and faithful, then by the above we have canonical natural transformations $\theta\colon G \to L$ and $\phi\colon F \to R$.

2.3. Lemma. *Given functors $(L \dashv F \dashv G \dashv R)$ as above, the following conditions are equivalent:*

(i) *θ is pointwise epic.*

(ii) *G is faithful on morphisms whose codomain is in the image of F.*

(iii) *ϕ is pointwise monic.*

Proof. From the definition of θ as $(\eta_L)^{-1}.G\alpha$, it is easy to see that the effect of composing with θ_A on morphisms $LA \to B$ may be described as follows: first transpose across the adjunction $(L \dashv F)$, then apply G, then compose with the isomorphism $(\eta_B)^{-1}\colon GFB \cong B$. The first and third steps are bijective, so the composite is injective iff the second step is; i.e., iff (ii) holds. Similarly, the effect of composing with ϕ_B on morphisms $A \to FB$ is given by first applying G, then composing with $GFB \cong B$, and then transposing across $(G \dashv R)$, so it is injective iff (ii) holds. ∎

The equivalence of (i) and (iii) in 2.3 is stated, but not proved, in [11]; the proof above is due to the present author. Next, we consider these two conditions in isolation.

2.4. Lemma.

(i) *Given functors $(F \dashv G \dashv R)$ with F and R full and faithful, the condition that $\phi\colon F \to R$ is monic is equivalent to saying that the counit of $(F \dashv G)$ is monic.*

(ii) *Given functors $(L \dashv F \dashv G)$ with F full and faithful, the condition that $\theta\colon G \to L$ is epic implies that the unit of $(L \dashv F)$ is epic; and the converse holds if G preserves epimorphisms.*

Proof. (i) Clearly, if ϵ is monic, so is ϵ_R; but ϕ is the composite $\epsilon_R.(F\delta)^{-1}\colon F \cong FGR \to R$, where δ is the counit of $(G \dashv R)$, so it too is monic. Conversely, if ϕ is monic, so is ϕ_G; but ϕ_G is the composite $\epsilon_{RG}(F\delta_G)^{-1}$, which is equal to $\epsilon_{RG}(GF\gamma)$ where γ is the unit of $(G \dashv R)$, and hence to $\gamma\epsilon$ by naturality of ϵ. So this implies that ϵ is monic.

(ii) The arguments involving θ are essentially dual to those involving ϕ, the only difference being that we have to apply functors to the left rather than the right of natural transformations. Since F, being a left adjoint, preserves epimorphisms, the assumption that θ is epic implies that $F\theta$ is epic, from which we may deduce exactly as in (i) that the unit α of $(L \dashv F)$ is epic; but for the converse we need to know that $G\alpha$ is epic. (Of course, if the additional right adjoint R exists, then G will preserve epimorphisms.) ∎

It is well known (see [3], 3.6.2) that if the unit of a reflection $(L \dashv F)$ is epic, then the reflective subcategory which is the repletion of the image of F is closed under strong subobjects; and the converse holds if every morphism of the ambient category factors as an epimorphism followed by a strong monomorphism. Similarly, the conditions in 2.4(i) are equivalent to saying that the repletion of the image of F is closed under strong quotients, provided every morphism factors as a strong epi followed by a mono.

Up to this point, we have not needed to make any assumptions about the categories which appear as the domain and codomain of our functors. For the next result, we do need such an assumption — but it is one which holds in any topos.

2.5. Corollary. *Suppose given functors $(L \dashv F \dashv G)$ with F full and faithful, and suppose further that the domain ($\mathcal{D}$, say) of G has pullbacks and that epimorphisms are stable under pullback in $\mathcal{D}$. Then the natural transformation $\theta\colon G \to L$ is epic iff the unit of $(L \dashv F)$ is epic and G preserves epis.*

PROOF. After 2.4(ii), it remains to prove that if θ is epic then G preserves epis. So let $e\colon A \twoheadrightarrow B$ be an epimorphism in $\mathcal{D}$, and form the pullback

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "g"] & A \arrow[d, "e"] \\ FGB \arrow[r, "\epsilon_B"] & B \end{tikzcd} .$$

Now Lg is epic (since L is a left adjoint), and hence so is the composite

$$GC \xrightarrow{\theta_C} LC \xrightarrow{Lg} LFGB \xrightarrow{\beta_{GB}} GB$$

since the counit β of $(L \dashv F)$ is an isomorphism. But we have

$$\begin{aligned} \beta_{GB}.Lg.\theta_C &= \beta_{GB}.\theta_{FGB}.Gg \\ &= \beta_{GB}.L\epsilon_{FGB}.(\beta_{GFGB})^{-1}.Gg \\ &= \beta_{GB}.LFG\epsilon_B.(\beta_{GFGB})^{-1}.Gg \\ &= G\epsilon_B.Gg \\ &= Ge.Gf \end{aligned}$$

where we have used the fact that $L\epsilon_{FGB} = LFG\epsilon_B$ since both morphisms are two-sided inverses for $LF\eta_{GB}$. So Ge is epic. ∎

For the final result in this section, we need to assume that the categories with which we are dealing are cartesian closed. We recall two standard results about adjunctions between cartesian closed categories:

2.6. LEMMA. *Let $\mathcal{C}$ and $\mathcal{D}$ be cartesian closed categories, and $F\colon \mathcal{C} \to \mathcal{D}$ a functor having a left adjoint L. Then*

(i) *F is cartesian closed (that is, the canonical morphism $F(B^A) \to FB^{FA}$ is an isomorphism for all A and B) iff F and L satisfy the 'Frobenius reciprocity' condition that*

$$L(C \times FA) \xrightarrow{(L\pi_1, \beta_A L\pi_2)} LC \times A$$

is an isomorphism for all A and C. (Here β denotes the counit of $(L \dashv F)$.)

(ii) *If F is full and faithful, then L preserves binary products iff the repletion of the image of F is an exponential ideal in $\mathcal{D}$; more specifically, iff the canonical morphism $F(A^{LC}) \to FA^C$ (corresponding to the composite*

$$L(F(A^{LC} \times C) \xrightarrow{(L\pi_1, L\pi_2)} LF(A^{LC}) \times LC \xrightarrow{\beta_{A^{LC}} \times 1} A^{LC} \times LC \xrightarrow{\mathrm{ev}} A\ )$$

is an isomorphism for all A and C.

PROOF. See [8], A1.5.8 and A4.3.1. ∎

If F is cartesian closed and has a right adjoint G as well as a left adjoint L, then Lemma 2.6(i) combined with the counit map $FGD \to D$ yields a natural morphism $\psi_{C,D}\colon LC \times GD \cong L(C \times FGD) \to L(C \times D)$ for arbitrary objects C, D of $\mathcal{D}$. (Note that if L preserves 1, then $\psi_{1,D}$ is just the morphism θ_D defined earlier.) Using this, we may obtain a natural morphism $FA^D \to F(A^{GD})$ by transposing the composite

$$L(FA^D) \times GD \xrightarrow{\psi_{FA^D,D}} L(FA^D \times D) \xrightarrow{L(\mathrm{ev})} LFA \xrightarrow{\beta_A} A\ .$$

If we also assume that F is full and faithful, and that G has a further right adjoint R, then it is straightforward to verify that this morphism makes the diagram

$$\begin{tikzcd}
FA^D \arrow[r] \arrow[d, "(\phi_A)^D"] & F(A^{GD}) \arrow[d, "\phi_{A^{GD}}"] \\
RA^D & R(A^{GD}) \arrow[l]
\end{tikzcd}$$

commute, where ϕ is the natural transformation of 2.2(ii) and the bottom edge of the square is the isomorphism arising as in 2.6(ii) from the fact that G preserves binary products. We may thus conclude:

2.7. Proposition. *Let $\mathcal{C}$ and $\mathcal{D}$ be cartesian closed categories, and suppose we have a string of four adjoint functors $(L \dashv F \dashv G \dashv R)$ between $\mathcal{C}$ and $\mathcal{D}$ satisfying the equivalent conditions of* 2.3. *Suppose further that F is a cartesian closed functor, and that all monomorphisms in $\mathcal{D}$ are strong. Then L preserves binary products.*

Proof. Since ϕ_A is monic, it follows from the commutative diagram above that $FA^D \to F(A^{GD})$ is monic, and hence strong monic, for any A and D. But since the unit of $(L \dashv F)$ is epic by 2.4(ii), the repletion of the image of F is closed under strong subobjects, and hence FA^D belongs to it. So the image of F is an exponential ideal in $\mathcal{D}$, and we can apply 2.6(ii). ∎

3. Why PLC morphisms are local and SLC

Our first result in this section is an immediate translation of 2.4 and 2.5 into topos-theoretic terms:

3.1. Lemma.

(i) *If $p\colon \mathcal{E} \to \mathcal{S}$ is a local geometric morphism, then it is hyperconnected iff the canonical natural transformation $\phi\colon p^* \to p^\#$ is monic. (Here $p^\#$ denotes the right adjoint of p_*.)*

(ii) *If $p\colon \mathcal{E} \to \mathcal{S}$ is connected and locally connected, then it is punctually locally connected iff it is hyperconnected and p_* preserves epimorphisms.*

PROOF. We recall that a connected geometric morphism p is hyperconnected iff (the repletion of) the image of p^* is closed under subobjects, iff it is closed under quotients ([8], A4.6.6). Since all monos and epis in a topos are strong, and all morphisms factor as an epi followed by a mono, the first of these conditions is equivalent to saying that the unit of $(p_! \dashv p^*)$ is epic (provided $p_!$ exists), and the second is equivalent to saying that the counit of $(p^* \dashv p_*)$ is monic. So these are direct translations of the statements of 2.4(i) and of 2.5. ∎

We note in passing that if we modify the site of Example 1.5 by allowing a (connected and locally connected) pointless locale X to be an object of $\mathcal{C}$, then the resulting topos is not hyperconnected over **Set** (though it remains connected, locally connected and local). For the support of the sheaf $\mathcal{C}(-, X)$ is a subterminal object of this topos which is neither 0 nor 1.

The extra hypothesis in 3.1(ii) that p_* preserves epimorphisms will of course be satisfied if p is local (though, in that case, the conclusion that p is punctually locally connected could also be deduced from 3.1(i) and 2.3). To obtain a converse result, we need to do a little more work. The key to it is the following result, which appears as C3.4.14 in [8]:

3.2. PROPOSITION. *A geometric morphism $p \colon \mathcal{E} \to \mathcal{S}$ is connected iff p_* preserves $\mathcal{S}$-indexed coproducts.* ∎

By considering coproducts indexed by a finite copower of 1 in $\mathcal{S}$, we deduce that if p is connected then p_* preserves finite coproducts. Also, if $\mathcal{S}$ has a natural number object then p_* preserves the natural number object, since p^* does and the unit of $(p^* \dashv p_*)$ is an isomorphism. We thus obtain a result which should have appeared in section C3.6 of [8], but did not:

3.3. COROLLARY. *Let $p \colon \mathcal{E} \to \mathcal{S}$ be a connected geometric morphism, and consider the following conditions:*

(i) *p_* preserves epimorphisms.*

(ii) *p_* preserves coequalizers of equivalence relations.*

(iii) *p_* preserves all coequalizers.*

(iv) *p_* preserves all $\mathcal{S}$-indexed colimits.*

(v) *p_* has a right adjoint, i.e. p is local.*

Then we have (v) $\Rightarrow$ (iv) $\Leftrightarrow$ (iii) $\Rightarrow$ (ii) $\Leftrightarrow$ (i). *Moreover,* (ii) $\Rightarrow$ (iii) *holds if $\mathcal{S}$ has a natural number object, and* (iv) $\Rightarrow$ (v) *holds if p is bounded.*

Proof. The 'upward' implications are all trivial (for (v) $\Rightarrow$ (iv) we need the fact that the adjunction $(p_* \dashv p^\#)$, if it exists, is necessarily $\mathcal{S}$-indexed). (i) $\Rightarrow$ (ii) holds because p_* preserves kernel-pairs, and a morphism in a topos is epic iff it is the coequalizer of its kernel-pair. For (ii) $\Rightarrow$ (iii), we use the fact that since p_* preserves the natural number object (as noted above) and also finite limits and images, it preserves the construction of the equivalence relation generated by an arbitrary parallel pair (cf. [8], A2.5.7). (iii) $\Rightarrow$ (iv) follows from 3.2 and the remarks following it, since by definition a functor preserves $\mathcal{S}$-indexed colimits iff it preserves finite colimits (in each fibre) and $\mathcal{S}$-indexed coproducts. Finally, (iv) $\Rightarrow$ (v) follows from the $\mathcal{S}$-indexed version of the Adjoint Functor Theorem (as was already noted in [8], C3.6.2). ■

Combining 3.1(ii) and 3.3, we have

3.4. Theorem. *Let $p\colon \mathcal{E} \to \mathcal{S}$ be a bounded geometric morphism, and suppose $\mathcal{S}$ has a natural number object. Then p is punctually locally connected iff it is locally connected, hyperconnected and local.* ■

The requirement, in 3.4, that $\mathcal{S}$ should have a natural number object could probably be dispensed with. Whether the boundedness assumption could also be dispensed with seems more problematic.

No two of the three conditions of 3.4 imply the third. As we have already remarked, the locale-based version of the gros topos is locally connected and local, but not hyperconnected. On the other hand, if we build the gros topos based on a category of spaces which includes non-locally-connected spaces, then we obtain a topos which is hyperconnected and local, but not locally connected. Finally, if $\mathcal{C}$ is any small category which is strongly connected but whose idempotent-completion does not have a terminal object (for example, if $\mathcal{C}$ is a nontrivial group), then the functor category $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ is locally connected and hyperconnected, but not local (cf. [8], C3.6.3(b)).

To complete the story, we need a 'site-free' substitute for part (iii) of 1.6. Fortunately, Proposition 2.7 provides this:

3.5. Proposition. *Let $p\colon \mathcal{E} \to \mathcal{S}$ be a geometric morphism which is locally connected, hyperconnected and local. Then p is stably locally connected.*

Proof. We have a string of four adjoint functors $(p_! \dashv p^* \dashv p_* \dashv p^\#)$ between $\mathcal{E}$ and $\mathcal{S}$, and (by either part of 3.1) they satisfy the conditions of 2.3. Moreover, p^* is a cartesian closed functor because p is locally connected, and all monomorphisms in $\mathcal{E}$ are strong. So the hypotheses of 2.7 are satisfied, and hence $p_!$ preserves binary products; but it also preserves the terminal object since p is connected. ■

3.6. Remark. For completeness, we note that an alternative proof of the implication 'PLC $\Rightarrow$ SLC' may also be given by translating the construction in the proof of 1.6(iii) from the site to the ambient topos. That is, we may prove in the internal logic of $\mathcal{S}$ that if every connected object of $\mathcal{E}$ has a point, then the product of two connected objects is connected, as follows.

Let A and B be two connected objects. To show that $A \times B$ is connected, it suffices to show that the equivalence relation, on the set of connected subobjects of $A \times B$, which is the transitive closure of the relation of having inhabited intersection, is the total relation (i.e. that any two connected subobjects may be linked by a finite chain of connected subobjects in which adjacent members have inhabited intersection), since by definition the connected components of $A \times B$ are the unions of the equivalence classes of this relation (cf. [8], C1.5.8). So let C and C' be two such subobjects; by assumption each contains a point, so suppose $(a, b)\colon 1 \to A \times B$ factors through $C \rightarrowtail A \times B$, and (a', b') factors through C'. Then the subobject $\{a\} \times B$ is connected, since it is isomorphic to B, and meets C in the point (a, b), and similarly $A \times \{b'\}$ is connected and has inhabited intersection with C'. But these two subobjects meet in the point (a, b'), so the result is proved.

It is worth noting that the essence of the above argument may be found in some textbooks on general topology (for example, [4]) as a proof that the product of two connected spaces is connected — though most such books give non-constructive proofs of this result. In a similar vein to 3.6, we note that the natural morphism $\psi_{A,B}\colon p_! A \times p_* B \to p_!(A \times B)$, which we constructed for an arbitrary locally connected morphism p in the course of proving 2.7, may be described in the internal logic of the base topos as the map which sends a pair consisting of a connected component $A' \rightarrowtail A$ and a point $b\colon 1 \to B$ to the component of $A \times B$ containing the connected subobject $A' \times \{b\}$.

Also for completeness, we record

3.7. Proposition. *Suppose $p\colon \mathcal{E} \to \mathcal{S}$ is punctually locally connected. Then the following are equivalent:*

(i) *p is totally connected.*

(ii) *$\theta\colon p_* \to p_!$ is an isomorphism.*

Proof. Clearly (ii) implies (i) since p_* preserves finite limits. But if (i) holds, then for any connected object A of $\mathcal{E}$ the equalizer of any two morphisms $1 \rightrightarrows A$ must be connected, and hence equal to 1: that is, any connected object of $\mathcal{E}$ has a unique point. Interpreting this argument in the internal logic of $\mathcal{S}$, we see that θ is an isomorphism. ∎

Geometric morphisms satisfying the condition in 3.7(ii) were studied in [7], under the name 'quintessential localizations'. In particular, we showed there that such morphisms with domain $\mathcal{E}$ correspond bijectively to idempotent natural endomorphisms of the identity functor on $\mathcal{E}$. (In [11], Lawvere refers to quintessential localizations of $\mathcal{E}$ as 'abstract qualities'.)

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