Theory and Applications of Categories, Vol. 25, No. 4, 2011, pp. 64–117.

COMPARATIVE SMOOTHEOLOGY

ANDREW STACEY

Abstract. We compare various different definitions of "the category of smooth objects". The definitions compared are due to Chen, Frölicher, Sikorski, Smith, and Souriau. The method of comparison is to construct functors between the categories that enable us to see how the categories relate to each other. Our method of study involves finding a general context into which these categories can be placed. This involves considering categories wherein objects are considered in relation to a certain collection of standard test objects. This therefore applies beyond the question of categories of smooth spaces.

1. Introduction

The purpose of this paper is to compare various definitions of categories of smooth objects. The definitions compared are due to [Chen (1977)], [Frölicher (1982)], [Sikorski (1971)] (see also [Mostow (1979)]), [Smith (1966)], and [Souriau (1980)]. Each has the same underlying principle: we know what it means for a map to be smooth between certain subsets of Euclidean spaces and so in general we declare a function to be smooth if whenever we examine it using these subsets, it is smooth. This is a rather vague statement—what do we mean by "examine"?—and the various definitions can all be seen as different ways of making this precise.

Each definition of a category of smooth objects was introduced because someone found a specific defect in the category of manifolds with smooth maps and tried to find a way of correcting it. Often the defect was in the shape of a "missing object". That is, there was some space which, it was felt, was amenable to being studied using the tools of differential topology but which was not, technically, a smooth manifold. These definitions, therefore, were usually motivated by a specific problem and this influenced the choice of category. A definition was considered suitable if it helped solve the problem. This has led to a proliferation of possible definitions for "smooth objects". Such comparison as is already in the literature tends to focus on applications, see for example [Mostow (1979), §4].

With so many different definitions of "the category of smooth objects" two questions naturally arise:

1. Which, if any, truly captures the essence of "smoothness"?

2. How are the various categories related to each other?

Received by the editors 2010-07-07 and, in revised form, 2011-02-07.
Transmitted by Richard Blute. Published on 2011-02-23.
2000 Mathematics Subject Classification: Primary: 58A40, Secondary: 18F15.
Key words and phrases: generalised smooth spaces.

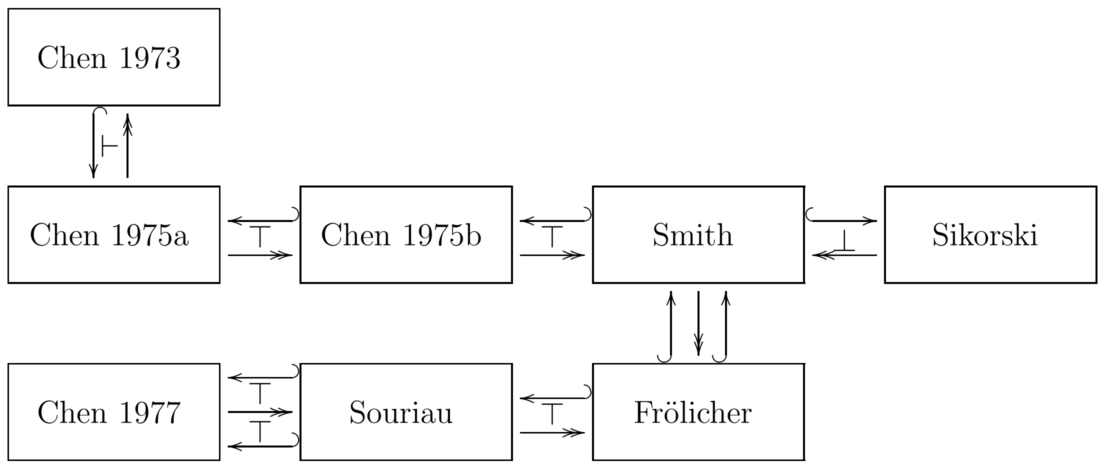

Figure 1: The relationships between the categories

The first is, ultimately, subjective. The answer depends on what one decides the essence of smoothness to be. The second can be either subjective or (reasonably) objective. It is subjective if one considers the question as an attempt to order the definitions by deciding that one is better than another. It is (reasonably) objective if one looks for natural relationships between the categories and considers the nature of those relationships. The qualifier "reasonably" is there because the word "natural" in the previous sentence is being used in its traditional English meaning of "not contrived" rather than its mathematical meaning.

This paper attempts to answer the objective version of the second question.

To look for relationships between categories means finding functors between them. Thus our first goal is to find functors between the various categories. Initially we want to consider uncontrived functors. As each of the categories is an extension of that of smooth manifolds, we start by looking for functors that preserve this subcategory within each.

Once we have found such functors, the next step is to classify these functors (as with the earlier use of the word "natural", "classify" has its traditional English meaning here). The ideal situation here is to be able to say that one category is a reflective or co-reflective subcategory of another (using the given functors as the inclusion and (co-)reflector). We therefore look for adjunctions between the functors and for embeddings.

Our answer at this stage can be summarised in Figure 1.

Once we have examined all the uncontrived functors, we consider the question of whether or not there are any contrived functors. To limit this question slightly, we consider the question as to whether or not any of the various categories of smooth spaces are equivalent, where we allow contrived functors; that is to say, we allow any functors irrespective of how they behave on manifolds.

Our answer to this is satisfying: there are none.

Let us now give a tour of this study, ensuring that we point out the main attractions.

In Section 2 we recall the definitions of five of the various categories of smooth spaces that have appeared in the literature. All of these proposals have the same basic shape and in Section 3 we extract that shape and put it into a general setting.

To explain this basic shape we need to consider how the various categories of smooth spaces might have been devised. Let us start with the definition of a smooth manifold.

1.1. DEFINITION. *A* smooth manifold *consists of a topological manifold together with a smooth structure. A* smooth structure *consists of a maximal smooth atlas. A* smooth atlas *is a family of charts, which are continuous maps to or from open subsets of Euclidean spaces (satisfying certain other conditions).*

We say "to or from" because charts, being homeomorphisms, are invertible and it is a matter of taste as to whether the term "chart" means the map to the manifold or off it.

Let us compare this with one of the definitions for a category of smooth objects.

1.2. DEFINITION. [Souriau (1980)] *A* Souriau space *or* diffeological space *is a pair* $(X, \mathcal{D})$ *where* X *is a set and* $\mathcal{D}$ *is a* diffeology *on* X*. That is to say,* $\mathcal{D}$ *is a family of maps (also called* plots*) into* X *with domains open subsets of Euclidean spaces which satisfy the following conditions.*

1. *Every constant map is a plot.*
2. *If* $\phi \colon U \to X$ *is a plot and* $\theta \colon U' \to U$ *is a* C^∞*-map between open subsets of Euclidean spaces then* $\phi\theta$ *is a plot.*
3. *If* $\phi \colon U \to X$ *is a map which is everywhere locally a plot then it is a plot. That is, if there is an open cover* $\mathcal{V}$ *of* U *such that for each* $V \in \mathcal{V}$*, the restriction* $\phi|_V \colon V \to X$ *is a plot then* ϕ *is a plot.*

By comparing these two definitions, we can see certain common themes.

A. In each there is an underlying category of objects to which one might wish to give a smooth structure. For manifolds, this is the category of topological manifolds; for Souriau spaces, this is the category of sets.

B. In each there is a category of test spaces and a smooth structure consists of a family of morphisms to or from these test spaces and the object in question. For both, this is the category of open subsets of Euclidean spaces.

C. In each these families are not completely arbitrary. There is a forcing condition which must be met. For manifolds, this is that the maps from test spaces are diffeomorphisms on the overlaps. For Souriau spaces, this is the condition that a map which is locally a plot is again a plot.

In Section 3 we generalise this. Starting with an underlying category, $\mathcal{U}$, a test category, $\mathcal{T}$, and a forcing condition we define the category of forced virtual $\mathcal{T}$–objects in $\mathcal{U}$, $\mathcal{U}\mathcal{T}_{\mathrm{f}}^{\mathrm{v}}$. A forced virtual $\mathcal{T}$–object in $\mathcal{U}$ consists of an object in $\mathcal{U}$ together with families of $\mathcal{U}$-morphisms to and from the test objects with the property that the appropriate forcing condition is satisfied. A forcing condition is a way of encoding the idea that if certain maps are smooth then some other map ought to be smooth as well. The latter map is then considered to be "forced" to be smooth by the others.

At this point it is worth making a comment about the direction of the test morphisms. In the various categories of smooth spaces there is almost always a preferred direction of test morphism, either to the smooth space or from it. Only one, that of Frölicher spaces, directly uses both. However, it is possible to recast all of the definitions to use both without changing the actual category (up to canonical isomorphism). Informally, we do this by taking as many test morphisms in the other direction as possible: a process we refer to as "saturation". In the example of Souriau spaces, the test morphisms from a Souriau space, say $(X, \mathcal{D})$, to a test space, say U, can be described in one of two ways:

1. They are those set maps $\psi \colon X \to U$ for which $\psi \circ \phi \in C^\infty(U', U)$ for all $\phi \in \mathcal{D}$.
2. They are the set maps $\psi \colon X \to U$ which underlie morphisms of diffeological spaces $(X, \mathcal{D}) \to (U, \mathcal{D})$ where $(U, \mathcal{D})$ is U with its "standard" diffeology.

These two descriptions make sense in any of the categories.

This recasting makes the comparison simpler.

Having laid out the general recipe, we proceed in Section 4 to the search for functors. As we are initially looking for uncontrived functors, we begin by looking at functors that are induced by one of three obvious operations: changing the forcing condition, changing the category of test spaces, and changing the underlying category. As we have a specific situation to which we wish to apply this theory, we do not work in full generality but rather seek out that midpoint where there is sufficient generality to make matters clear but not so much that we lose sight of our goal. In particular, when looking at what happens when we change the underlying category we consider only certain simple changes.

Having set up this general theory, we apply it in Section 5 to the case at hand. This is where we produce all the functors used in Figure 1. We also give simple descriptions of the functors so that someone who is not interested in the detailed construction can still work out what the functors actually are.

The adjunctions between these functors all follow from the general work in Section 4. However, whilst we can use this general theory to say that a particular functor has an adjoint (and to identify that adjoint), it cannot be used to say that it doesn't have an adjoint. Therefore in Sections 6 and 7 we show that all the adjunctions that didn't follow from the work of Section 4 do not hold. Thus the adjunctions indicated in Figure 1 are all the adjoints that exist that involve any of the functors in that diagram.

Section 8 concludes the main part of this paper by considering arbitrary relationships between the categories. To make this a more specific question, we look for equivalences of categories. Our strategy here is to look for *invariants* of the categories to enable us to say that there are no such equivalences.

The required invariants are extremely simple. We look at terminal objects, the two element set, and the real line. The first two essentially say that if two of these categories are equivalent then there is an equivalence that preserves the underlying category. The third, the real line, is the one that characterises the smooth structures. One highlight of this section is the result that in these categories of smooth spaces it is possible to categorically find the real line. That is, purely using categorical tools, one can say "This is $\mathbb{R}$.".

The final two sections deal with issues that are to one side of the main purpose of this paper but which are nonetheless consistent with its general theme. Section 9 considers the rôle of topology in defining a smooth space. Several of the definitions use a subcategory of the category of topological spaces as the underlying category but others just use Set, the category of sets. Section 9 considers how to remove the topology from those that start with it.

Section 10 is concerned with non-set–based theories. In all of the categories considered in this paper the objects have underlying sets. It is an interesting question as to how to remove this requirement, but the answer is not obvious. Thus in Section 10 we do no more than raise this question.

Because the purpose of this paper is to examine the word "smooth", the term "C^∞-map" will be used when referring to a map between locally convex subsets of Euclidean spaces which is smooth in the standard sense; that is, all definable directional derivatives exist and are continuous on their domain of definition.

As many people defining an extension of smooth manifolds have used the term "differentiable space", we adopt the convention whereby we refer to each type of structure by the name of its original author. We do this even for diffeological spaces though that name is unambiguous and has a nice ring to it.

Finally, I am grateful to the various people who commented on preliminary versions of this article, several of whom did so via discussions at the n–Category Café. In particular I wish to acknowledge the helpful comments of Bruce Bartlett, Urs Schreiber, and the anonymous referee. I am especially grateful to Urs Schreiber for suggesting the title.

2. Categories of Smooth Spaces

In this section we shall describe five of the various categories of smooth spaces that have appeared in the literature. As remarked in the introduction, these categories were initially introduced to correct some defect in the category of smooth manifolds. For several of the categories, in particular Chen's and Souriau's, the motivation for the definition was to apply the tools of differential topology to some space that does not quite fit the definition of a (finite dimensional) smooth manifold. Examples of such spaces include loop spaces and diffeomorphism groups. Note that whilst loop spaces can be treated as infinite dimensional manifolds, the closely associated path space (with domain $\mathbb{R}$) cannot be so described. A closely related idea is that of seeing how far it is possible to push a particular concept in differential topology. Smith's category of smooth spaces was introduced to see how far the de Rham theorem can be extended. Another motivation is to apply the tools of another category, for example the category of rings, to smooth manifolds. To do this, one wants to associate a ring to every smooth manifold and characterise those rings that can be obtained in this fashion. Inevitably in this situation one has to balance precision with usability and allow for things that are sufficiently similar to the type of ring that comes from a smooth manifold. This was the motivation for the category of Sikorski.

Let us now consider the definitions themselves.

2.1. Definition. [Chen (1977)] *A* Chen space *is a pair* $(X, \mathcal{P})$ *where* X *is a set and* $\mathcal{P}$ *is a family of maps (called* plots*) into* X *with domains convex subsets of Euclidean spaces. These have to satisfy the following conditions.*

1. *Every constant map is a plot.*
2. *If* $\phi\colon C \to X$ *is a plot and* $\theta\colon C' \to C$ *is a* C^∞*-map between convex regions then* $\phi\theta$ *is a plot.*
3. *If* $\phi\colon C \to X$ *is a map which is everywhere locally a plot then it is a plot. That is, if there is an open cover* $\mathcal{V}$ *of* C *consisting of convex sets such that for each* $V \in \mathcal{V}$*, the restriction* $\phi|_V\colon V \to X$ *is a plot then* ϕ *is a plot.*

A morphism of Chen spaces $g\colon (X, \mathcal{P}_X) \to (Y, \mathcal{P}_Y)$ *is a map* $g\colon X \to Y$ *on the underlying sets with the property that* $g\phi \in \mathcal{P}_Y$ *for all* $\phi \in \mathcal{P}_X$*.*

2.2. Definition. [Souriau (1980)] *A* Souriau space *is a pair* $(X, \mathcal{D})$ *where* X *is a set and* $\mathcal{D}$ *is a diffeology on* X*. That is,* $\mathcal{D}$ *is a family of maps (also called* plots*) into* X *with domains open subsets of Euclidean spaces. These have to satisfy the following conditions.*

1. *Every constant map is a plot.*
2. *If* $\phi\colon U \to X$ *is a plot and* $\theta\colon U' \to U$ *is a* C^∞*-map between open subsets of Euclidean spaces then* $\phi\theta$ *is a plot.*
3. *If* $\phi\colon U \to X$ *is a map which is everywhere locally a plot then it is a plot.*

A morphism of Souriau spaces $g\colon (X, \mathcal{D}_X) \to (Y, \mathcal{D}_Y)$ *is a map* $g\colon X \to Y$ *on the underlying sets with the property that* $g\phi \in \mathcal{D}_Y$ *for all* $\phi \in \mathcal{D}_X$*.*

A comprehensive treatment of diffeological spaces is in [Iglesias-Zemmour].

2.3. Definition. [Sikorski (1971)] *A* Sikorski space *is a triple* $(X, \mathcal{T}, \mathcal{F})$ *where* X *is a set,* $\mathcal{T}$ *a topology on* X*, and* $\mathcal{F}$ *is a subalgebra of the algebra of continuous functionals on* X *satisfying the following conditions.*

1. *Functionals in* $\mathcal{F}$ *are* locally detectable *in that* $f\colon X \to \mathbb{R}$ *is in* $\mathcal{F}$ *if each point* $x \in X$ *has a neighborhood, say* V*, for which there is a functional* $g \in \mathcal{F}$ *with* $f|_V = g|_V$*.*
2. *If* $f_1, \dots, f_k \in \mathcal{F}$ *and* $g \in C^\infty(\mathbb{R}^k, \mathbb{R})$ *then* $g(f_1, \dots, f_k) \in \mathcal{F}$*.*

A morphism of Sikorski spaces $g\colon (X, \mathcal{T}_X, \mathcal{F}_X) \to (Y, \mathcal{T}_Y, \mathcal{F}_Y)$ *is a continuous map on the underlying topological spaces,* $g\colon (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$*, such that* $fg \in \mathcal{F}_X$ *for all* $f \in \mathcal{F}_Y$*.*

2.4. Definition. [Smith (1966)] *A* Smith space *is a triple* $(X, \mathcal{T}, \mathcal{F})$ *where* X *is a set,* $\mathcal{T}$ *a topology on* X*, and* $\mathcal{F}$ *a set of continuous real-valued functions on* X*. The set* $\mathcal{F}$ *has to satisfy a certain closure condition. For an open set* $U \subseteq \mathbb{R}^n$*, let* $\mathcal{F}(U)$ *denote the set of continuous maps* $\phi\colon U \to X$ *with the property that* $f\phi \in C^\infty(U, \mathbb{R})$ *for all* $f \in \mathcal{F}$*. The closure condition is that* $\mathcal{F}$ *contains all continuous functions* $g\colon X \to \mathbb{R}$ *with the property that for all open sets* $U \subseteq \mathbb{R}^n$ *(*n *arbitrary) and* $\phi \in \mathcal{F}(U)$*,* $g\phi \in C^\infty(U, \mathbb{R})$*.*

A morphism of Smith spaces $g\colon (X, \mathcal{T}_X, \mathcal{F}_X) \to (Y, \mathcal{T}_Y, \mathcal{F}_Y)$ *is a continuous map on the underlying topological spaces,* $g\colon (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$*, such that* $fg \in \mathcal{F}_X$ *for all* $f \in \mathcal{F}_Y$*.*

2.5. Definition. [Frölicher (1982)] *A* Frölicher space *is a triple* $(X, \mathcal{C}, \mathcal{F})$ *where* X *is a set,* $\mathcal{C}$ *is a family of curves in* X*, i.e. a subset of* $\operatorname{Map}(\mathbb{R}, X)$*, and* $\mathcal{F}$ *is a family of functionals on* X*, i.e. a subset of* $\operatorname{Map}(X, \mathbb{R})$*. The sets* $\mathcal{C}$ *and* $\mathcal{F}$ *have to satisfy the following compatibility condition: a curve* $c\colon \mathbb{R} \to X$ *is in* $\mathcal{C}$ *if and only if* $fc \in C^\infty(\mathbb{R}, \mathbb{R})$ *for all functionals* $f \in \mathcal{F}$*, and similarly a functional* $f\colon X \to \mathbb{R}$ *is in* $\mathcal{F}$ *if and only if* $fc \in C^\infty(\mathbb{R}, \mathbb{R})$ *for all curves* $c\colon \mathbb{R} \to X$*.*

A morphism of Frölicher spaces $g\colon (X, \mathcal{C}_X, \mathcal{F}_X) \to (Y, \mathcal{C}_Y, \mathcal{F}_Y)$ *is a map on the underlying sets,* $g\colon X \to Y$*, satisfying the following (equivalent) conditions.*

1. $gc \in \mathcal{C}_Y$ *for all* $c \in \mathcal{C}_X$*,*
2. $fg \in \mathcal{F}_X$ *for all* $f \in \mathcal{F}_Y$*,*
3. $fgc \in C^\infty(\mathbb{R}, \mathbb{R})$ *for all* $c \in \mathcal{C}_X$ *and* $f \in \mathcal{F}_Y$*.*

2.6. Remarks.

1. As we have done in these definitions, we shall use the word "functional" as shorthand for "function with codomain $\mathbb{R}$".
2. Chen modified his definition considerably as he worked with it. The earliest definition seems to be from [Chen (1973)] and the latest from [Chen (1977)]. In between these two lies [Chen (1975)] which includes two further definitions (one, it should

be said, is clearly an incorrect recollection of the definition from [Chen (1973)]). Although all his definitions are based on the theme of detecting smoothness by testing it using convex spaces, there is considerable variation from the first definition to the last. We shall comment a little on this in the next section. We shall use the term "Chen space" to refer only to the definition listed in this section which corresponds to that given in [Chen (1977)].

3. A Souriau space is very similar to a Chen space except that the domains of the test functions are different. This is related to a very interesting fact. All of the above definitions consist of a set together with certain "test functions" between the set and certain "test spaces" with the functions either in to the set or out of it. What is worth noting is that for models consisting of maps to the set, several choices of test spaces have been proposed. However, for models consisting of maps out of the set, all have used only $\mathbb{R}$ for the test space. One could speculate that the reason for this is that it is well-known that a map into a subset of a Euclidean space is smooth if and only if all the coordinate projections are smooth; the corresponding result for maps out of a (suitable) subset of a Euclidean space—namely, Boman's theorem [Boman (1967)] (see also [Kriegl and Michor (1997), 3.4]) and Kriegl and Michor's extension [Kriegl and Michor (1997), 24.5]—is much less well-known.

3. A General Recipe

All of the heretofore proposed categories of "smooth objects" can be put into a *standard form*. This standard form is determined by certain choices. In short, these choices are of an underlying category, of test spaces, and of forcing conditions. The underlying category should be thought of as "those objects to which one might wish to give a smooth structure". The test spaces should be thought of as "those objects for which there is an indisputable smooth structure". The forcing conditions should be thought of as ensuring that "morphisms which ought to be smooth actually are smooth".

A close relative of this structure is of sheaves on a site. The site is the category of test spaces and the sheaf condition is the forcing condition. To see where the underlying category fits in, one should consider *concrete* sheaves on a *concrete* site, see [Baez and Hoffnung] and the references therein.

Another close relative of this is the *Isbell envelope* of an essentially small category. The Isbell envelope of $\mathcal{A}$, written $E(\mathcal{A})$, is a category whose objects consist of a contravariant functor $\mathfrak{I}\colon \mathcal{A}^{\mathrm{op}} \to \mathrm{Set}$, a covariant functor $\mathfrak{O}\colon \mathcal{A} \to \mathrm{Set}$, and a natural transformation $\mathfrak{I} \times \mathfrak{O} \to \mathcal{A}(-,-)$. Morphisms in this category are pairs of natural transformations between the functors. One can think of an object in $E(\mathcal{A})$ as a *virtual* object of $\mathcal{A}$ in that it can be experimented on using objects in $\mathcal{A}$. This notion can also be encoded using *profunctors*. We shall comment on this relationship later in Section 10.

3.1. Virtual Objects Let us now describe our structure precisely. We are not aiming for the most general approach here; rather we wish to find a setting that helps with our study of the actual examples already posited. Therefore, we wish to keep our recipe as close as possible to the definitions in Section 2.

The easy part is the two categories. We fix these: the underlying category, $\mathcal{U}$, and the test category, $\mathcal{T}$. We also choose a faithful functor $\mathfrak{U}\colon \mathcal{T} \to \mathcal{U}$ from the test category to the underlying category (although not necessary, it will usually be the case that the test category will be a subcategory of the underlying category). This allows us to define the first stage.

3.2. Definition. *Let $\mathcal{U}$ and $\mathcal{T}$ be categories—the underlying category and the test category respectively. Let $\mathfrak{U}\colon \mathcal{T} \to \mathcal{U}$ be a faithful functor.*

The category of virtual $\mathcal{T}$–objects in $\mathcal{U}$, $\mathcal{U}\mathcal{T}^{\mathrm{v}}$, is the following category. A virtual $\mathcal{T}$–object in $\mathcal{U}$ *consists of a triple $X := (U, \mathfrak{I}, \mathfrak{O})$ where*

- *U is an object in $\mathcal{U}$,*
- *$\mathfrak{I}\colon \mathcal{T} \to \mathrm{Set}$ is a contravariant functor,*
- *$\mathfrak{O}\colon \mathcal{T} \to \mathrm{Set}$ is a covariant functor.*

These have to satisfy the following conditions.

- *$\mathfrak{I}$ is a subfunctor of the functor $T \mapsto \mathcal{U}(\mathfrak{U}(T), U)$.*
- *$\mathfrak{O}$ is a subfunctor of the functor $T \mapsto \mathcal{U}(U, \mathfrak{U}(T))$.*
- *Consider the functors $\mathcal{T} \times \mathcal{T}^{op} \to \mathrm{Set}$,*

$$(T, T') \to \mathcal{U}(\mathfrak{U}(T), U) \times \mathcal{U}(U, \mathfrak{U}(T')), \qquad (T, T') \to \mathcal{U}(\mathfrak{U}(T), \mathfrak{U}(T')).$$

  Composition defines a natural transformation from the first to the second. This natural transformation must induce a natural transformation from $\mathfrak{I} \times \mathfrak{O}$ to $\mathcal{T}(-,-)$ (with the latter viewed as a subfunctor of $\mathcal{U}(\mathfrak{U}(-), \mathfrak{U}(-))$.

A morphism in $\mathcal{U}\mathcal{T}^{\mathrm{v}}$, say $(U_1, \mathfrak{I}_1, \mathfrak{O}_1)$ to $(U_2, \mathfrak{I}_2, \mathfrak{O}_2)$, is a $\mathcal{U}$-morphism $u\colon U_1 \to U_2$ with the properties that $u\psi \in \mathfrak{I}_2$ for all $\psi \in \mathfrak{I}_1$ and $\phi u \in \mathfrak{O}_1$ for all $\phi \in \mathfrak{O}_2$.

We shall denote the obvious forgetful functor $\mathcal{U}\mathcal{T}^{\mathrm{v}} \to \mathcal{U}$ by $X \mapsto |X|$.

3.3. Remarks.

1. When we say "for all $\psi \in \mathfrak{I}$" we mean "for all test objects, T, and all $\psi \in \mathfrak{I}(T)$". This is a shorthand that we shall frequently use to avoid having to introduce unnecessary dummies. For example, given a $\mathcal{U}$-morphism, u, we write "$u \in \mathrm{im}\mathfrak{U}$" to mean that there is a $\mathcal{T}$-morphism, t, such that $u = \mathfrak{U}(t)$. Implicit in this is that the domain and codomain of u come from test objects via $\mathfrak{U}$.

2. In our definition we have allowed for test maps both into and out of our test spaces. This seems a little at variance with the definitions that we are generalising. We shall see that, in the presence of certain forcing conditions, one of these families can be effectively removed. Putting both in at the start allows us to consider both situations at the same time.

3. When we say "subfunctor" we mean this in the strictest sense: that $\mathfrak{I}(T)$ is a subset of $\mathcal{U}(\mathfrak{U}(T), U)$. This is not technically necessary but makes the notation and exposition considerably simpler. This means that $\mathcal{U}\mathcal{T}^{\mathrm{v}}$ is an *amnestic construct* over $\mathcal{U}$. It is obvious that it is *(uniquely) transportable* as well. (The terms *amnestic*, *uniquely transportable*, and *construct* refer to types of *concrete category*, for short definitions see Section 8; for a detailed consideration, see [Adámek et al. (2006)]). Moreover, if $\mathcal{T}$ has a small skeleton then the fibres of the forgetful functor $|-|\colon \mathcal{U}\mathcal{T}^{\mathrm{v}} \to \mathcal{U}$ are small.

Now let us consider the forcing conditions. As we have said before, we are more interested in finding a context into which we can place all known examples than in finding the most general setting. A forcing condition will consist of two parts: an input forcing condition and an output forcing condition. The two are formally similar, related by an obvious "flip", so to ease the exposition we shall focus on one type. We choose, for no good reason, the input forcing condition.

As the definition is a little intricate, we shall take a moment before we begin to explain how it works. We take a virtual $\mathcal{T}$–object in $\mathcal{U}$, a test object, and a $\mathcal{U}$-morphism $u\colon \mathfrak{U}(T) \to |X|$. The question we wish to answer is: should u be a "smooth map"? By this we mean, should u be in $\mathfrak{I}_X(T)$? As we shall show in a moment, there is a functor $\mathfrak{S}\colon \mathcal{T} \to \mathcal{U}\mathcal{T}^{\mathrm{v}}$ which allows us to rephrase this question as the following: should there be a $\mathcal{U}\mathcal{T}^{\mathrm{v}}$-morphism $\mathfrak{S}(T) \to X$ which is taken to u by the forgetful functor $\mathcal{U}\mathcal{T}^{\mathrm{v}} \to \mathcal{U}$? To answer this question, we test it. The procedure for testing it is to compose it on the right by a $\mathcal{U}\mathcal{T}^{\mathrm{v}}$-morphism and on the left by a $\mathcal{T}$-morphism, say x and t. That is to say, we consider the composition $|x|u\mathfrak{U}(t)$ and ask whether or not this is smooth (that is, whether or not it lifts to a $\mathcal{U}\mathcal{T}^{\mathrm{v}}$-morphism). If this happens for "enough" pairs (t, x) then we conclude that u should be in $\mathfrak{I}_X(T)$. To decide this, we look at the family of all the pairs (t, x) for which $|x|u\mathfrak{U}(t)$ lifts. If that family has "enough" pairs, then we answer "yes" to our original question and if not, we answer "no". So to decide what the word "enough" means, we need to choose an assignment of 0 or 1 to each family of pairs (t, x). We cannot choose these completely arbitrarily: there are two obvious conditions that need to be satisfied. Firstly, if u_1 and u_2 should both be smooth (according to this test) then the composition $u_1 \circ u_2$ should be smooth (assuming that the composition makes sense). Secondly, since test objects are those objects which are considered to have an indisputable smooth structure, when we consider them as virtual $\mathcal{T}$–objects in $\mathcal{U}$ under the functor $\mathfrak{S}\colon \mathcal{T} \to \mathcal{U}\mathcal{T}^{\mathrm{v}}$ (to be defined in a moment), then testing smoothness in $\mathcal{T}$ and $\mathcal{U}\mathcal{T}^{\mathrm{v}}$ should give us the same answer. That is to say, this method of testing should not produce any more smooth maps between test objects in the category of virtual $\mathcal{T}$–objects in $\mathcal{U}$ than were already there in the test category.

With this plan of action in mind, let us proceed with defining an input forcing condition. This will take us several steps. The first is to record an obvious lemma that says that the test category naturally embeds in the category of virtual $\mathcal{T}$–objects in $\mathcal{U}$.

3.4. Lemma. *There is a functor $\mathfrak{S} \colon \mathcal{T} \to \mathcal{U}\mathcal{T}^{\mathrm{v}}$ which embeds $\mathcal{T}$ as a full subcategory of $\mathcal{U}\mathcal{T}^{\mathrm{v}}$. Let T be a test object. The virtual $\mathcal{T}$–object in $\mathcal{U}$, $\mathfrak{S}(T)$, has underlying object, $\mathfrak{U}(T)$, in $\mathcal{U}$, input test functor $\mathfrak{I}(T') = \mathcal{T}(T', T)$, and output test functor $\mathfrak{O}(T') = \mathcal{T}(T, T')$. On morphisms, $\mathfrak{S}(t)$ is determined by the requirement that $|\mathfrak{S}(t)| = \mathfrak{U}(t)$.*

This functor has several important properties. As functors $\mathcal{T} \to \mathcal{U}$, we have $|\mathfrak{S}| = \mathfrak{U}$. For X a virtual $\mathcal{T}$–object in $\mathcal{U}$ and T a test object we have $\mathfrak{I}(T) = \mathcal{U}\mathcal{T}^{\mathrm{v}}(\mathfrak{S}(T), X)$ as subsets of $\mathcal{U}(\mathfrak{U}(T), |X|)$; similarly for the output test functions. ∎

The next stage is to define the notion of a *trial* for a pair (T, X) where T is a test object and X a virtual $\mathcal{T}$–object in $\mathcal{U}$. This provides a way to test whether a $\mathcal{U}$-morphism $u \colon \mathfrak{U}(T) \to |X|$ "ought" to be in $\mathfrak{I}_X(T)$. The idea being that if such a morphism succeeds at sufficiently many trials, it is "forced". In reality, rather than trying to impose the forcing condition, we restrict our attention to those objects that already satisfy it.

3.5. Definition. *Let X be a virtual $\mathcal{T}$–object in $\mathcal{U}$. Let T be a test object. A* trial *from T to X is defined to be a pair (t, x) where t is a $\mathcal{T}$-morphism with target T and x is a $\mathcal{U}\mathcal{T}^{\mathrm{v}}$-morphism with source X.*

We can illustrate a trial, (t, x), diagrammatically as follows.

$$\xrightarrow{t} T \dashrightarrow X \xrightarrow{x}$$

A trial is a way of testing a $\mathcal{U}$-morphism to see if it "looks smooth". The idea is to see if a particular $\mathcal{U}$-morphism fills in the dotted arrow. As stated, this does not make sense as the categories on the left and right are not the same. The following definition makes this more precise.

3.6. Definition. *Let X be a virtual $\mathcal{T}$–object in $\mathcal{U}$. Let T be a test object. Let (t, x) be a trial from T to X. A $\mathcal{U}$-morphism $u \colon \mathfrak{U}(T) \to |X|$* succeeds at the trial *if the $\mathcal{U}$-morphism $|x| u \mathfrak{U}(t)$ underlies a $\mathcal{U}\mathcal{T}^{\mathrm{v}}$-morphism.*

Let us write T' for the source of t and X' for the target of x. Then a $\mathcal{U}$-morphism, $u \colon \mathfrak{U}(T) \to |X|$, succeeds at this trial if the $\mathcal{U}$-morphism

$$\mathfrak{U}(T') \xrightarrow{\mathfrak{U}(t)} \mathfrak{U}(T) \xrightarrow{u} |X| \xrightarrow{|x|} |X'|$$

lifts to a $\mathcal{U}\mathcal{T}^{\mathrm{v}}$-morphism $\mathfrak{S}(T') \to X'$. Equivalently, if the above $\mathcal{U}$-morphism is in $\mathfrak{I}_{X'}(T')$.

Given a $\mathcal{U}$-morphism we want to know which trials it succeeds at.

3.7. Definition. *For a fixed $\mathcal{U}$-morphism, $u\colon \mathfrak{U}(T) \to |X|$, we define* $\mathrm{Tri}(u)$ *to be the class of trials from T to X at which u succeeds.*

The idea here being that if a $\mathcal{U}$-morphism succeeds at enough trials, then it ought to be in $\mathfrak{I}_X(T)$. In order to keep $\mathfrak{I}_X(-)$ a functor, if one $\mathcal{U}$-morphism is forced to be in it then many others will also be forced. We can keep track of these dependencies by making the trials into a category. We start by defining what it means for trials between different pairs of objects to be compatible.

3.8. Definition. *For $i = 1, 2$, let T_i be a test object and X_i be a virtual $\mathcal{T}$–object in $\mathcal{U}$. Let (t_i, x_i) be a trial from T_i to X_i. Let $t\colon T_2 \to T_1$ be a $\mathcal{T}$-morphism and $x\colon X_1 \to X_2$ be a $\mathcal{U}\mathcal{T}^{\mathrm{v}}$-morphism. We say that (t_2, x_2)* is compatible with (t_1, x_1) along t and x *if there exist a $\mathcal{T}$-morphism t' and $\mathcal{U}\mathcal{T}^{\mathrm{v}}$-morphism x' such that $tt_2 = t_1 t'$ and $x_2 x = x' x_1$.*

For a family of trials, $\mathcal{F}$, from T_1 to X_1 we define $x\mathcal{F}t$ to be the family of trials from T_2 to X_2 with the property that each trial in $x\mathcal{F}t$ is compatible with a trial in $\mathcal{F}$.

This is illustrated by the following diagram.

$$\begin{tikzcd}
{} \arrow[r, "t_1"] & T_1 \arrow[r, dotted] & X_1 \arrow[r, "x_1"] \arrow[d, "x"] & {} \arrow[d, "x'"] \\
{} \arrow[u, "t'"] \arrow[r, "t_2"] & T_2 \arrow[u, "t"'] \arrow[r, dotted] & X_2 \arrow[r, "x_2"] & {}
\end{tikzcd}$$

The point being that if a $\mathcal{U}$-morphism, $u\colon \mathfrak{U}(T_1) \to |X_1|$, succeeds at the first trial then $|x|u\mathfrak{U}(t)$ will succeed at the second.

We think of compatibility as like a function that moves individual trials from the first pair (T_1, X_1) to the second pair (T_2, X_2). It is not an honest function as it may be one-to-many. However, when we apply it to families of trials we obtain a well-defined function and this is the function $\mathcal{F} \mapsto x\mathcal{F}t$.

Now we can define the category of trials.

3.9. Definition. *We define the category of trials,* Tri*, to be the category whose objects are triples $(T, X, \mathcal{F})$ with T a test object, X a virtual $\mathcal{T}$–object in $\mathcal{U}$, and $\mathcal{F}$ a family of trials from T to X. The morphisms from $(T_1, X_2, \mathcal{F}_1)$ to $(T_2, X_2, \mathcal{F}_2)$ are pairs (t, x) where $t\colon T_2 \to T_1$ is a $\mathcal{T}$-morphism and $x\colon X_1 \to X_2$ is a $\mathcal{U}\mathcal{T}^{\mathrm{v}}$-morphism such that $x\mathcal{F}_1 t \subseteq \mathcal{F}_2$.*

The category of trials has an obvious functor to $\mathcal{T}^{\mathrm{op}} \times \mathcal{U}\mathcal{T}^{\mathrm{v}}$. The fibre of this functor at (T, X) is the partially ordered class of subclasses of trials from T to X. Observe that every triple (T, X, u) with u a $\mathcal{U}$-morphism from $\mathfrak{U}(T)$ to $|X|$ defines an object in Tri, $(T, X, \mathrm{Tri}(u))$. Given a $\mathcal{T}$-morphism, $t\colon T' \to T$, and a $\mathcal{U}\mathcal{T}^{\mathrm{v}}$-morphism, $x\colon X \to X'$, we have that $x\,\mathrm{Tri}(u)t \subseteq \mathrm{Tri}(|x|u\mathfrak{U}(t))$ whence there is a Tri-morphism from $(T, X, \mathrm{Tri}(u))$ to $(T', X', \mathrm{Tri}(|x|u\mathfrak{U}(t)))$.

3.10. Definition. *An* input forcing condition *for virtual $\mathcal{T}$–objects in $\mathcal{U}$ is a functor $\mathfrak{F}_i\colon \mathrm{Tri} \to \{0 \to 1\}$ with the property that for test objects, T_1 and T_2, $\mathfrak{F}_i(T_1, \mathfrak{S}(T_2), \mathcal{F}) = 0$ if $(1, 1) \notin \mathcal{F}$.*

For a test object, T, and a virtual $\mathcal{T}$–object in $\mathcal{U}$, X, we shall say that a family of trials, $\mathcal{F}$, from T to X is sufficient *if* $\mathfrak{F}_i(T, X, \mathcal{F}) = 1$.

For a test object, T, and a virtual $\mathcal{T}$–object in $\mathcal{U}$, X, we shall say that a $\mathcal{U}$-morphism $u \colon \mathfrak{U}(T) \to |X|$ *is* forced *if* $\mathfrak{F}_i(T, X, \mathrm{Tri}(u)) = 1$.

3.11. Remarks.

1. There is an obvious generalisation to output forcing conditions by "flipping" all the arrows. We shall write $\mathfrak{F}_o$ for the corresponding functor.
2. From examining the definition of morphisms in the category of trials we see that if $u \colon \mathfrak{U}(T) \to |X|$ is forced and $t \colon T' \to T$, $x \colon X \to X'$ are suitable morphisms, then $|x| u \mathfrak{U}(t)$ is also forced.
3. Similarly, if $\mathcal{F}$ is a sufficient family of trials from T to X and $\mathcal{F} \subseteq \mathcal{F}'$ then $\mathcal{F}'$ is also sufficient.
4. The one restraint on the functor translates into the statement: "Nothing obviously non-smooth should ever be forced to be smooth".

3.12. Definition. *A* forcing condition *is a choice of input forcing condition, $\mathfrak{F}_i$, and output forcing condition, $\mathfrak{F}_o$.*

A virtual $\mathcal{T}$–object in $\mathcal{U}$, X, satisfies *the forcing condition $(\mathfrak{F}_i, \mathfrak{F}_o)$ if, whenever T is a test object and $u \colon \mathfrak{U}(T) \to |X|$ is forced then $u \in \mathfrak{I}_X(T)$ and, similarly, whenever $u \colon |X| \to \mathfrak{U}(T)$ is forced then $u \in \mathfrak{O}_X(T)$.*

Given a forcing condition, we write $\mathcal{U}\mathcal{T}_{\mathrm{f}}^{\mathrm{v}}$ for the full subcategory of $\mathcal{U}\mathcal{T}^{\mathrm{v}}$ consisting of virtual $\mathcal{T}$–objects in $\mathcal{U}$ satisfying this forcing condition.

Since forcing conditions take values in $\{0 \to 1\}$ they form a lattice and can thus be combined using logical connectors.

3.13. Smooth Objects in the Wild Let us now specify to the matter in hand. The above describes an extremely general set-up, far broader than we shall need. The first step to reducing to our examples is to find a minimal setting containing all of them. For this, we limit our choices for test category, underlying category, and forcing condition.

As the forcing conditions are the most unfamiliar part, we start by introducing a list of examples. This list contains all that are needed to specify the given categories of smooth spaces. We shall then translate the various categories of smooth spaces into this language. We conclude with some remarks as to the characteristics of the test categories and underlying categories.

The way that we define, say, an input forcing condition is to list certain sufficient families of trials from a generic test object to a generic virtual $\mathcal{T}$–object in $\mathcal{U}$. As the input forcing condition is a functor to $\{0 \to 1\}$, this will force many other families to also be sufficient—any family that is the target of a morphism from one of the specified ones. All the others are insufficient.

To check that these are well-defined, the only thing to check is the "non-stupid" condition, namely that nothing is forced that really shouldn't be forced. We shall not do this here.

3.14. Definition. *In the following, X will be a virtual $\mathcal{T}$–object in $\mathcal{U}$ and T a test object. We list the determining families of trials.*

The input saturation condition

$$\{(1_T, \phi) : \phi \in \mathfrak{O}_X\}$$

Here, we use the fact that $\mathfrak{O}_X(T') = \mathcal{U}\mathcal{T}^{\mathrm{v}}(X, \mathfrak{S}(T'))$ to regard ϕ as a $\mathcal{U}\mathcal{T}^{\mathrm{v}}$-morphism.

The output saturation condition

$$\{(\psi, 1_T) : \psi \in \mathfrak{I}_X\}$$

The input determined condition

$$\{(t_\lambda, 1_X) : \lambda \in \Lambda\}$$

where $\{t_\lambda \colon T_\lambda \to T : \lambda \in \Lambda\}$ is a family of $\mathcal{T}$-morphisms with the property that a $\mathcal{U}$-morphism $u \colon \mathfrak{U}(T) \to \mathfrak{U}(T')$ is in the image of $\mathfrak{U}$ if $u\mathfrak{U}(t_\lambda)$ is in the image of $\mathfrak{U}$ for all λ.

The output determined condition

$$\{(1_X, t_\lambda) : \lambda \in \Lambda\}$$

where $\{t_\lambda \colon T \to T_\lambda : \lambda \in \Lambda\}$ is a family of $\mathcal{T}$-morphisms with the property that a $\mathcal{U}$-morphism $u \colon \mathfrak{U}(T') \to \mathfrak{U}(T)$ is in the image of $\mathfrak{U}$ if $\mathfrak{U}(t_\lambda)u$ is in the image of $\mathfrak{U}$ for all λ.

The input specifically–determined condition *This is the same as the input determined condition except that the families $\{t_\lambda\}$ must be drawn from a pre-specified list.*

The output specifically–determined condition *This is to the output determined condition as the input specifically–determined condition is to the input determined condition.*

The input sheaf condition *To define this, we need to assume that the test category is a site.*

$$\{(t_\lambda, 1_X) : \lambda \in \Lambda\}$$

where $\{t_\lambda \colon T_\lambda \to T : \lambda \in \Lambda\}$ is a covering of T.

The output sheaf condition *To define this, we require the underlying category to be a subcategory of the category of topological spaces. Also given a subset $U \subseteq |X|$ we define the* subspace *virtual $\mathcal{T}$–object structure on U to be given by*

$$\mathfrak{I}(T') = \{\psi \colon \mathfrak{U}(T') \to U : |\iota_\lambda|\psi \in \mathfrak{I}_X(T')\},$$
$$\mathfrak{O}(T') = \{\phi|\iota_\lambda| \colon U \to \mathfrak{U}(T') : \phi \in \mathfrak{O}_X(T')\}.$$

With this we take families of trials of the form

$$\{(\iota_\lambda, 1_T) : \lambda \in \Lambda\}$$

where $\iota_\lambda \colon X_\lambda \to X$ is a family of $\mathcal{U}\mathcal{T}^{\mathrm{v}}$-morphisms such that $|\iota_\lambda| \colon |X_\lambda| \to |X|$ is the inclusion of an open subset of $|X|$, the $|X_\lambda|$ cover $|X|$, and the virtual $\mathcal{T}$–object structure on X_λ is the subspace virtual $\mathcal{T}$–object.

The input terminal condition *Assume that the test category has a terminal object, say $*_\mathcal{T}$. The empty family of trials from $*_\mathcal{T}$ to X is sufficient.*

The output terminal condition *Assume that the test category has a terminal object, say $*_\mathcal{T}$. The empty family of trials from X to $*_\mathcal{T}$ is sufficient.*

The empty input condition *No family of trials is sufficient.*

The empty output condition *No family of trials is sufficient.*

3.15. Remarks.

1. One caveat of this method of specifying forcing conditions is that, due to the functorial nature of a forcing condition, there may be "unexpected" sufficient families. In the list above we gave, for most of the conditions, some sufficient families of trials for any pair (T, X) (thinking of input forcing conditions). It is tempting to think that a $\mathcal{U}$-morphism $u \colon \mathfrak{U}(T) \to |X|$ is forced if *and only if* $\mathrm{Tri}(u)$ contains one of these generating families. This may not be true; for example, if u is forced then $u\mathfrak{U}(t)$ must also be forced even if $\mathrm{Tri}(u\mathfrak{U}(t))$ doesn't contain a generating family.

   However, most of the conditions do have this property: that if a $\mathcal{U}$-morphism, u, is forced then $\mathrm{Tri}(u)$ contains one of the given generating families. The terminal conditions do not have this property, and if the families are not chosen wisely then the specifically–determined conditions may not.

2. In the input saturation condition, a $\mathcal{U}$-morphism $u \colon \mathfrak{U}(T) \to |X|$ is forced if $\phi u \colon \mathfrak{U}(T) \to \mathfrak{U}(T')$ lifts to a $\mathcal{U}\mathcal{T}^{\mathrm{v}}$-morphism $\mathfrak{S}(T) \to \mathfrak{S}(T')$ for all T'; equivalently, if it comes from a $\mathcal{T}$-morphism $T \to T'$.

3. The difference between the determined and specifically–determined conditions is that in the former *any* family of morphisms satisfying the requirement may be used

in the test. In the latter, the family of morphisms has to come from a list which is drawn up at the start. The reason that one might prefer to use this is that the full list of families of morphisms that satisfy the requirement may be difficult to write down. It can therefore be some work to decide whether or not a given morphism is forced. With a fixed list, the task becomes much easier. However, as remarked above, unless these lists are chosen carefully there may still be some morphisms that are forced which cannot be tested merely by the families on the stated lists.

4. The input sheaf condition says that the forced input test morphisms to a specific virtual $\mathcal{T}$–object in $\mathcal{U}$ form a sheaf on the test category. This is often phrased as saying that a morphism that is locally smooth is itself smooth.

5. Let us consider the output sheaf condition. Consider a trial

$$U \xrightarrow{\iota} X \dashrightarrow T \xrightarrow{=} T$$

The statement that $u \colon |X| \to \mathfrak{U}(T)$ succeeds this trial means that $u|_U \colon U \to \mathfrak{U}(T)$ is an output test morphism for U. By definition, therefore, there is an output test morphism $u' \colon |X| \to \mathfrak{U}(T)$ such that $u'|_U = u|_U$.

Thus as for the input sheaf condition, an output morphism is smooth if it is locally smooth. As the word "locally" here refers to the object in $\mathcal{U}$, we need to know that objects in $\mathcal{U}$ are topological spaces.

6. The saturation conditions allow us to effectively ignore the corresponding family of test functions when looking at whether a morphism on the underlying objects in $\mathcal{U}$ lifts to a morphism in $\mathcal{U}\mathcal{T}_{\mathrm{f}}^{\mathrm{v}}$. For example, suppose that the output test functions are saturated. Then if $u \colon |X_1| \to |X_2|$ is such that $u\psi \in \mathfrak{I}_{X_2}$ for all $\psi \in \mathfrak{I}_{X_1}$ then for $\phi \in \mathfrak{O}_{X_2}$ and $\psi \in \mathfrak{I}_{X_1}$, $\phi u \psi$ came from a morphism in $\mathcal{T}$. Whence ϕu is forced and hence $\phi u \in \mathfrak{O}_{X_1}$. Thus u underlies a $\mathcal{U}\mathcal{T}_{\mathrm{f}}^{\mathrm{v}}$-morphism. Note that we can only ignore one family of test functions by this method.

7. The empty forcing conditions translate to the fact that no morphisms are forced.

8. The terminal forcing conditions translate to the fact that any morphism which factors through the terminal object of the test category is forced.

9. The saturation condition is the "top" of the family of forcing conditions whilst the empty condition is the "bottom". The forcing conditions thus form a complete (possibly large) lattice.

Let us now translate the examples already "in the wild" into our formalism. We shall also include all of Chen's definitions to provide a wider scope for comparisons.

Frölicher spaces

1. The underlying category is Set.
2. There is one test space, $\mathbb{R}$.
3. The input forcing condition is saturation.
4. The output forcing condition is saturation.

Chen spaces Although we have only given Chen's last definition in Section 2, we shall give the standard form of all four of his definitions.

[Chen (1973)]
1. The underlying category is that of all Hausdorff topological spaces.
2. The test spaces are all closed, convex subsets of Euclidean spaces.
3. The input forcing condition is the terminal condition.
4. The output forcing condition is saturation.

[Chen (1975)]
1. The underlying category is that of all topological spaces.
2. The test spaces are all closed, convex subsets of Euclidean spaces.
3. The input forcing condition is the terminal condition.
4. The output forcing condition is saturation.

[Chen (1975)]
1. The underlying category is that of all topological spaces.
2. The test spaces are all closed, convex subsets of Euclidean spaces.
3. The input forcing condition is the determined condition with the terminal condition.
4. The output forcing condition is saturation.

[Chen (1977)]
1. The underlying category is Set.
2. The test spaces are all convex subsets of Euclidean spaces.
3. The input forcing condition is the sheaf condition with the terminal condition.
4. The output forcing condition is saturation.

Souriau Spaces

1. The underlying category is Set.
2. The test spaces are all open subsets of Euclidean spaces.
3. The input forcing condition is the sheaf condition with the terminal condition.

4. The output forcing condition is saturation.

Sikorski Spaces

1. The underlying category is that of topological spaces.
2. The test spaces are the Euclidean spaces.
3. The input forcing condition is saturation.
4. The output forcing condition is the sheaf condition, with the specifically–determined condition and the terminal condition.

These spaces are, perhaps, the hardest to see how to make them fit our standard form. There appears to be one test space, $\mathbb{R}$, and three conditions that need to be satisfied: that of being an algebra, the locally detectable condition, and the last condition involving k–tuples of maps. The locally detectable condition is clearly the output sheaf condition. The last condition leads us to expand our set of test spaces. This condition appears to be saying that the composition of a test morphism to $\mathbb{R}^k$ and a C^∞-map $\mathbb{R}^k \to \mathbb{R}$ is again a test morphism. Thus if we expand the test spaces to all Euclidean spaces, we appear to get this condition for free. The catch is that we need to force the condition that if $f_1, \ldots, f_k$ are test maps to $\mathbb{R}$ then $(f_1, \ldots, f_k)$ is a test map to $\mathbb{R}^k$. This is the *specifically–determined* condition with the coordinate projections as the determining family. The condition that the functions be an algebra is then almost vacuous. For providing that $\mathfrak{O}(\mathbb{R})$ is not empty we obtain the structure of an algebra using the smooth maps $1 \in C^\infty(\mathbb{R}, \mathbb{R})$ and $\alpha, \mu \in C^\infty(\mathbb{R}^2, \mathbb{R})$ defined by $1(t) = 1$, $\alpha(s,t) = s + t$, and $\mu(s,t) = s \cdot t$. To ensure that $\mathfrak{O}(\mathbb{R})$ is not empty, we impose the output terminal condition.

Smith Spaces

1. The underlying category is that of topological spaces.
2. There is one test space, $\mathbb{R}$.
3. The input forcing condition is saturation.
4. The output forcing condition is saturation.

From the definition, it would appear that Smith requires different families for the input and output test spaces. However, due to Boman's theorem, [Boman (1967)], it is clear that in the closure condition it is sufficient to consider only those maps from $\mathbb{R}$. It is interesting to note that the definition of a Smith space preceded Boman's result.

Let us comment now on the characteristics of the categories appearing in the above. Firstly, let us consider the test categories. It is possible to embed these categories as full subcategories of one "maximal" category.

3.16. Definition. The maximal test category *is the amnestic, transportable construct generated by the following category. The objects of this category are those subsets $M \subseteq \mathbb{R}^n$ with the property that each $m \in M$ has an open neighborhood, say V, in $\mathbb{R}^n$ and a diffeomorphism $\psi \colon V \cong U$ with U also an open subset of $\mathbb{R}^n$ such that $\psi(V \cap M)$ is convex. The morphisms in this category are the C^∞-maps.*

The maximal test category contains all convex subsets of finite dimensional affine spaces, all open subsets of affine spaces, all smooth manifolds, and all smooth manifolds with boundary, as well as a good deal else.

Our reason for doing this embedding is that when comparing the categories of smooth objects we shall want to consider modifications of the test category. By embedding our categories in this way, we can always factor our modifications as passing from a category to a full subcategory or vice versa. This will simplify the exposition without diminishing its relevance.

Now let us consider the underlying category. In the examples given, it is either Set or some category of topological spaces; either all topological spaces or Hausdorff topological spaces. One might conceivably wish to restrict ones attention further to, say, regular, normal, paracompact, or metrisable spaces.

When considering how to define a category of smooth spaces, the main distinction is between Set and the others; the question being as to whether "smoothness" is a property that is built on continuity or whether it stands in its own right.

For the purpose of comparing the categories, the important factor is in how the categories relate to each other. The category of Hausdorff spaces is a reflective full subcategory of that of all topological spaces, whilst topological spaces forms a topological category over Set. Thus these are the features that we shall consider.

We shall return briefly to the issue of topology in Section 9.

4. Functors

Our purpose now is to define certain functors between our categories. We shall start by defining the "obvious" functors. Later, we shall show that these contain all the "interesting" functors.

The functors fall into three types: change of underlying category, change of family of test spaces, and change of forcing condition. By composing these functors, we obtain functors between any two of our categories. In addition, we can restrict our attention to functors where the change is particularly simple. The test category is specified by a subclass of the class of objects of the maximal test category. We can therefore order the test categories by inclusion and need only consider the case where one is contained in the other. Similarly, we can order forcing conditions. Informally, we say that one forcing condition is less than another if every forced morphism for the first is forced for the second. Formally, we say that $(\mathfrak{F}_i^1, \mathfrak{F}_o^1) \preceq (\mathfrak{F}_i^2, \mathfrak{F}_o^2)$ if $\mathfrak{F}_i^1 \le \mathfrak{F}_i^2$ and $\mathfrak{F}_o^1 \le \mathfrak{F}_o^2$ (using the ordering on $\{0 \to 1\}$). Since we can "or" and "and" forcing conditions, we need only consider the case where one forcing condition is less than another.

The functor in one direction is usually straightforward. For functors in the opposite direction we use a little lattice theory on the fibres of $\mathcal{U}\mathcal{T}_{\mathrm{f}}^{\mathrm{v}}$ over $\mathcal{U}$.

In the following we shall use standard lattice notation. That is, in a complete lattice L, $\top$ and $\bot$ refer to the maximum and minimum respectively, $\bigwedge$ is the meet (intersection), $\bigvee$ is the join (union), and we use the standard interval notation, so for $a, b \in L$ with $a \preceq b$ we write $[a, b]$ for $\{c \in L : a \preceq c \preceq b\}$. Recall that if $f \colon L_1 \to L_2$ is an order-preserving map then $a \mapsto \bigwedge f^{-1}[a, \top]$ and $a \mapsto \bigvee f^{-1}[\bot, a]$ are also order-preserving.

4.1. The Fibre Categories Let us fix the underlying category, $\mathcal{U}$, the test category, $\mathcal{T}$, and a forcing condition $(\mathfrak{F}_i, \mathfrak{F}_o)$. Let $\mathcal{U}\mathcal{T}_{\mathrm{f}}^{\mathrm{v}}$ be the category of forced virtual $\mathcal{T}$–objects in $\mathcal{U}$. Let U be an object in $\mathcal{U}$. Let ${\mathcal{U}\mathcal{T}_{\mathrm{f}}^{\mathrm{v}}}_U$ be the fibre at U of the forgetful functor $\mathcal{U}\mathcal{T}_{\mathrm{f}}^{\mathrm{v}} \to \mathcal{U}$. We shall show that this is a complete lattice. It may be a large lattice (that is, it may be a proper class rather than a set) but will be a set if $\mathcal{T}$ has a small skeleton. The ordering is defined by saying that $X_1 \preceq X_2$ if there is a $\mathcal{U}\mathcal{T}_{\mathrm{f}}^{\mathrm{v}}$-morphism $x \colon X_1 \to X_2$ which maps to the identity on U under the forgetful functor. *A priori* this defines a quasi-ordering on ${\mathcal{U}\mathcal{T}_{\mathrm{f}}^{\mathrm{v}}}_U$ but in fact it is a partial ordering since we insisted in the definition of a virtual $\mathcal{T}$–object in $\mathcal{U}$ that the input and output test functors be strict subfunctors of the hom-functor.

4.2. Proposition. *${\mathcal{U}\mathcal{T}_{\mathrm{f}}^{\mathrm{v}}}_U$ is a complete lattice.*

Proof. Let $\mathfrak{s}$ be a family in ${\mathcal{U}\mathcal{T}_{\mathrm{f}}^{\mathrm{v}}}_U$. Let $\mathfrak{s}^c$ be the (possibly empty) family

$$\mathfrak{s}^c := \{X \in {\mathcal{U}\mathcal{T}_{\mathrm{f}}^{\mathrm{v}}}_U : X \preceq X' \text{ for all } X' \in \mathfrak{s}\}.$$

For a test object, T, define

$$\begin{aligned}
\mathfrak{I}(T) &:= \bigcap_{X \in \mathfrak{s}} \mathfrak{I}_X(T), \\
\mathfrak{O}(T) &:= \bigcap_{X \in \mathfrak{s}^c} \mathfrak{O}_X(T), \\
\mathfrak{I}'(T) &:= \{\psi \colon \mathfrak{U}(T) \to U : \phi\psi \in \operatorname{im}\mathfrak{U} \text{ for all } \phi \in \mathfrak{O}\}, \\
\mathfrak{O}'(T) &:= \{\phi \colon U \to \mathfrak{U}(T) : \phi\psi \in \operatorname{im}\mathfrak{U} \text{ for all } \psi \in \mathfrak{I}\}.
\end{aligned}$$

Although the families may be large, the intersections are all taking place within the *sets* $\mathcal{U}(\mathfrak{U}(T), U)$ and $\mathcal{U}(U, \mathfrak{U}(T))$. In particular, if $\mathfrak{s}^c$ is empty, $\mathfrak{O}(T) = \mathcal{U}(U, \mathfrak{U}(T))$.

Let us show these are functors. Let t be a $\mathcal{T}$-morphism and $\psi \in \mathfrak{I}$. Then for $X \in \mathfrak{s}$ we have $\psi \in \mathfrak{I}_X$ and so $\psi\mathfrak{U}(t) \in \mathfrak{I}_X$. As this is for all $X \in \mathfrak{s}$, we have $\psi\mathfrak{U}(t) \in \mathfrak{I}$. Hence $\mathfrak{I}'$ is a functor $\mathcal{T} \to \mathrm{Set}$. For $\mathfrak{I}'$, let $\psi \in \mathfrak{I}'(T)$ and let $t \colon T' \to T$ be a $\mathcal{T}$-morphism. Let $\phi \in \mathfrak{O}$. By definition, $\phi\psi \in \operatorname{im}\mathfrak{U}$ whence $\phi\psi\mathfrak{U}(t) \in \operatorname{im}\mathfrak{U}$. As this holds for all ϕ, $\psi\mathfrak{U}(t) \in \mathfrak{I}'(T')$. Hence $\mathfrak{I}$ is a functor $\mathcal{T} \to \mathrm{Set}$. Both are clearly subfunctors of the requisite hom-functor. A similar argument works for $\mathfrak{O}$ and $\mathfrak{O}'$.

By construction, $\mathfrak{I}$ and $\mathfrak{O}'$ are compatible, as are $\mathfrak{O}$ and $\mathfrak{I}'$. Hence $(U, \mathfrak{I}, \mathfrak{O}')$ and $(U, \mathfrak{I}', \mathfrak{O})$ are virtual $\mathcal{T}$–objects in $\mathcal{U}$. We observe that, for any $X \in \mathfrak{s}$, $\mathfrak{I} \subseteq \mathfrak{I}_X$ and—by the compatibility condition—$\mathfrak{O}' \supseteq \mathfrak{O}_X$. Hence the identity on U lifts to a $\mathcal{U}\mathcal{T}^{\mathrm{v}}$-morphism $(U, \mathfrak{I}, \mathfrak{O}') \to X$ for any $X \in \mathfrak{s}$. Similarly, the identity on U lifts to a $\mathcal{U}\mathcal{T}^{\mathrm{v}}$-morphism $X \to (U, \mathfrak{I}', \mathfrak{O})$ for any $X \in \mathfrak{s}^c$.

To show that both are forced virtual $\mathcal{T}$–objects in $\mathcal{U}$, we need to show that they satisfy the forcing condition. Let us consider $(U, \mathfrak{I}, \mathfrak{O}')$. Let T be a test object and let $u \colon \mathfrak{U}(T) \to U$ be a $\mathcal{U}$-morphism that is forced for the pair $(T, (U, \mathfrak{I}, \mathfrak{O}'))$. Let $X \in \mathfrak{s}$. As the identity on U lifts to a $\mathcal{U}\mathcal{T}^{\mathrm{v}}$-morphism $(U, \mathfrak{I}, \mathfrak{O}') \to X$, u is also forced for the pair (T, X). Hence $u \in \mathfrak{I}_X$. As this holds for all $X \in \mathfrak{s}$, $u \in \mathfrak{I}$. Now let $u \colon U \to \mathfrak{U}(T)$ be forced for the pair $((U, \mathfrak{I}, \mathfrak{O}'), T)$. Let $\psi \in \mathfrak{I}(T')$. By Lemma 3.4, ψ lifts to a $\mathcal{U}\mathcal{T}^{\mathrm{v}}$-morphism $\mathfrak{S}(T') \to (U, \mathfrak{I}, \mathfrak{O}')$. Hence $u\psi \colon \mathfrak{U}(T') \to \mathfrak{U}(T)$ is forced. From Definition 3.10, $u\psi \in \operatorname{im}\mathfrak{U}$. Hence $u \in \mathfrak{O}'(T)$. Thus $(U, \mathfrak{I}, \mathfrak{O}')$ is a forced virtual $\mathcal{T}$–object in $\mathcal{U}$.

Similarly, $(U, \mathfrak{I}', \mathfrak{O})$ is a forced virtual $\mathcal{T}$–object in $\mathcal{U}$. Clearly, $(U, \mathfrak{I}, \mathfrak{O}') \in \mathfrak{s}^c$. Hence $(U, \mathfrak{I}, \mathfrak{O}') \preceq (U, \mathfrak{I}', \mathfrak{O})$. Thus $\mathfrak{O} \subseteq \mathfrak{O}'$ and $\mathfrak{I} \subseteq \mathfrak{I}'$; hence $(U, \mathfrak{I}, \mathfrak{O})$ is a virtual $\mathcal{T}$–object in $\mathcal{U}$.

For any $X \in \mathfrak{s}$, clearly $\mathfrak{I} \subseteq \mathfrak{I}_X$. Then for $X' \in \mathfrak{s}^c$, $\mathfrak{O}_X \subseteq \mathfrak{O}_{X'}$ so $\mathfrak{O}_X \subseteq \mathfrak{O}$. Hence the identity on U lifts to a $\mathcal{U}\mathcal{T}^{\mathrm{v}}$-morphism $(U, \mathfrak{I}, \mathfrak{O}) \to X$. Similarly, for $X \in \mathfrak{s}^c$, the identity on U lifts to a $\mathcal{U}\mathcal{T}^{\mathrm{v}}$-morphism $X \to (U, \mathfrak{I}, \mathfrak{O})$. We therefore have our candidate for the meet of $\mathfrak{s}$. It remains to show that it is a forced virtual $\mathcal{T}$–object in $\mathcal{U}$.

This is similar to the arguments above for $(U, \mathfrak{I}, \mathfrak{O}')$ and $(U, \mathfrak{I}', \mathfrak{O})$. If $u \colon \mathfrak{U}(T) \to U$ is forced for $(T, (U, \mathfrak{I}, \mathfrak{O}))$ then u is forced for (T, X) for all $X \in \mathfrak{s}$. Hence $u \in \mathfrak{I}_X$ for all $X \in \mathfrak{s}$, whence $u \in \mathfrak{I}$. Similarly, $\mathfrak{O}$ satisfies the forcing condition. Thus $(U, \mathfrak{I}, \mathfrak{O})$ is the meet of $\mathfrak{s}$.

It is obvious how to adapt this to define the join of $\mathfrak{s}$. ■

It is interesting to see what is the maximum of ${\mathcal{U}\mathcal{T}_{\mathrm{f}}^{\mathrm{v}}}_U$. From the above proof, it will have test functors

$$\mathfrak{I}(T) = \mathcal{U}(\mathfrak{U}(T), U),$$
$$\mathfrak{O}(T) = \bigcap_{X \in {\mathcal{U}\mathcal{T}_{\mathrm{f}}^{\mathrm{v}}}_U} \mathfrak{O}_X(T).$$

By sending an object in $\mathcal{U}$ to the maximum and minimum of its fibre categories, we obtain functors from $\mathcal{U}$ to $\mathcal{U}\mathcal{T}_{\mathrm{f}}^{\mathrm{v}}$.

4.3. Definition. *Let* $\mathfrak{Ind} \colon \mathcal{U} \to \mathcal{U}\mathcal{T}_{\mathrm{f}}^{\mathrm{v}}$ *and* $\mathfrak{Dis} \colon \mathcal{U} \to \mathcal{U}\mathcal{T}_{\mathrm{f}}^{\mathrm{v}}$ *be the functors*

$$\mathfrak{Ind} \colon U \mapsto \bigwedge {\mathcal{U}\mathcal{T}_{\mathrm{f}}^{\mathrm{v}}}_U,$$
$$\mathfrak{Dis} \colon U \mapsto \bigvee {\mathcal{U}\mathcal{T}_{\mathrm{f}}^{\mathrm{v}}}_U.$$

We refer to these as, respectively, the indiscrete *and* discrete $\mathcal{U}\mathcal{T}_{\mathrm{f}}^{\mathrm{v}}$*–functors.*

4.4. Forcing Functors For this section, we fix the underlying category, $\mathcal{U}$, and the test category, $\mathcal{T}$. We choose two forcing conditions, $(\mathfrak{F}_{i1}, \mathfrak{F}_{o1})$ and $(\mathfrak{F}_{i2}, \mathfrak{F}_{o2})$, with $(\mathfrak{F}_{i1}, \mathfrak{F}_{o1}) \preceq (\mathfrak{F}_{i2}, \mathfrak{F}_{o2})$. These define two categories of smooth objects, ${\mathcal{U}\mathcal{T}_{\mathrm{f}}^{\mathrm{v}}}_1$ and ${\mathcal{U}\mathcal{T}_{\mathrm{f}}^{\mathrm{v}}}_2$, which are full subcategories of the category of virtual $\mathcal{T}$–objects in $\mathcal{U}$.

4.5. Proposition. *The inclusion functor* $\mathcal{UT}_{\mathrm{f}}^{\mathrm{v}}{}_{2} \to \mathcal{UT}^{\mathrm{v}}$ *factors through* $\mathcal{UT}_{\mathrm{f}}^{\mathrm{v}}{}_{1}$.

Proof. Let X be an object in $\mathcal{UT}_{\mathrm{f}}^{\mathrm{v}}{}_{2}$. Let T be a test object. Let $u \colon \mathfrak{U}(T) \to |X|$ be a $\mathcal{U}$-morphism. Suppose that u is forced by $(\mathfrak{F}_{i1}, \mathfrak{F}_{o1})$. Then $\mathfrak{F}_{i1}(\mathrm{Tri}(u)) = 1$. Since $\mathfrak{F}_{i2} \geq \mathfrak{F}_{i1}$, $\mathfrak{F}_{i2}(\mathrm{Tri}(u)) = 1$ also. Hence as X is an object in $\mathcal{UT}_{\mathrm{f}}^{\mathrm{v}}{}_{2}$, $u \in \mathfrak{I}_X(T)$. The same holds for $\mathcal{U}$-morphisms out of X, whence X is an object in $\mathcal{UT}_{\mathrm{f}}^{\mathrm{v}}{}_{1}$. ∎

4.6. Definition. *We write* $\mathfrak{Incl} \colon \mathcal{UT}_{\mathrm{f}}^{\mathrm{v}}{}_{2} \to \mathcal{UT}_{\mathrm{f}}^{\mathrm{v}}{}_{1}$ *for the inclusion functor.*

Now we shall construct a functor in the opposite direction. Given an object in $\mathcal{UT}_{\mathrm{f}}^{\mathrm{v}}{}_{1}$, we wish to define a "nearest" object in $\mathcal{UT}_{\mathrm{f}}^{\mathrm{v}}{}_{2}$. It is obvious that there are—usually—two choices.

4.7. Definition. *Define two* forcing functors $\mathcal{UT}_{\mathrm{f}}^{\mathrm{v}}{}_{1} \to \mathcal{UT}_{\mathrm{f}}^{\mathrm{v}}{}_{2}$ *by*

$$\begin{aligned} \mathfrak{For}^{-} &\colon X \mapsto \bigvee \mathfrak{Incl}^{-1}\left[\perp, X\right], \\ \mathfrak{For}^{+} &\colon X \mapsto \bigwedge \mathfrak{Incl}^{-1}\left[X, \top\right]. \end{aligned}$$

The idea, if not the fact, of these functors is that $\mathfrak{For}^{-}(X)$ should be the nearest object in $\mathcal{UT}_{\mathrm{f}}^{\mathrm{v}}{}_{2}$ below X and $\mathfrak{For}^{+}(X)$ should be the nearest object in $\mathcal{UT}_{\mathrm{f}}^{\mathrm{v}}{}_{2}$ above X (comparisons actually happening in $\mathcal{UT}_{\mathrm{f}}^{\mathrm{v}}{}_{1}$). However, these naïve expectations may not be met as it is entirely possible that, for example, $\mathfrak{Incl}\mathfrak{For}^{-}(X)$ is actually above X.

The restriction of $\mathfrak{Incl}$ to a fibre is the inclusion of one lattice in another and the order on the first is that induced from the second. From this we can deduce some elementary properties of $\mathfrak{For}^{-}$ and $\mathfrak{For}^{+}$.

4.8. Lemma. *The compositions* $\mathfrak{For}^{-}\mathfrak{Incl}$ *and* $\mathfrak{For}^{+}\mathfrak{Incl}$ *are the identity on* $\mathcal{UT}_{\mathrm{f}}^{\mathrm{v}}{}_{2}$. *For all objects, X, in* $\mathcal{UT}_{\mathrm{f}}^{\mathrm{v}}{}_{1}$, $\mathfrak{For}^{-}(X) \preceq \mathfrak{For}^{+}(X)$.

Proof. The first comes from the fact that, as $\mathfrak{Incl}$ is an inclusion, for X an object in $\mathcal{UT}_{\mathrm{f}}^{\mathrm{v}}{}_{2}$, $\mathfrak{Incl}^{-1}\left[\perp, \mathfrak{Incl}(X)\right] = [\perp, X]$. Thus $\mathfrak{For}^{-}\mathfrak{Incl}(X) = X$. The case of $\mathfrak{For}^{+}$ is similar.

For the second, observe that if $X_1 \in \mathfrak{Incl}^{-1}[\perp, X]$ and $X_2 \in \mathfrak{Incl}^{-1}[X, \perp]$ then $\mathfrak{Incl}(X_1) \preceq \mathfrak{Incl}(X_2)$. Hence $X_1 \preceq X_2$. Thus $\mathfrak{For}^{-}(X) \preceq \mathfrak{For}^{+}(X)$. ∎

We shall be particularly interested in the question of when it is the case that $X \preceq \mathfrak{Incl}\mathfrak{For}^{+}(X)$ and $\mathfrak{Incl}\mathfrak{For}^{-}(X) \preceq X$. To answer this, we need explicit descriptions of $\mathfrak{Incl}\mathfrak{For}^{+}(X)$ and $\mathfrak{Incl}\mathfrak{For}^{-}(X)$.

Let X be an object in $\mathcal{UT}_{\mathrm{f}}^{\mathrm{v}}{}_{1}$. Let us write X^{+} for $\mathfrak{Incl}\mathfrak{For}^{+}(X)$. From the proof of Proposition 4.2, we see that

$$\mathfrak{I}_{X^{+}} = \bigcap \mathfrak{I}_{X'}$$

where the indexing family is over objects, X', in $\mathcal{UT}_{\mathrm{f}}^{\mathrm{v}}{}_{2}$ such that $X \preceq \mathfrak{Incl}(X')$. Thus $\mathfrak{I}_X \subseteq \mathfrak{I}_{X'}$ and so $\mathfrak{I}_X \subseteq \mathfrak{I}_{X^{+}}$. Thus to test whether or not $X \preceq X^{+}$ it is sufficient to test whether or not $\mathfrak{O}_{X^{+}} \subseteq \mathfrak{O}_X$. This is not guaranteed—we shall see some examples later—but we can give some conditions for when it does hold.

4.9. PROPOSITION.

1. *In the first case, we assume a condition on the output forcing condition,* $\mathfrak{F}_{o2}$. *This condition is that if* X_1 *and* X_2 *are virtual* $\mathcal{T}$*–objects in* $\mathcal{U}$ *with the same underlying object in* $\mathcal{U}$ *and the same input test functor then* $\mathfrak{F}_{o2}(X_1) = \mathfrak{F}_{o2}(X_2)$. *When this condition holds then* $X \preceq X^+$ *if and only if* $X^+ = (|X|, \mathfrak{I}_{X^+}, \mathfrak{O}_X)$.

2. *If the output forcing conditions* $\mathfrak{F}_{o1}$ *and* $\mathfrak{F}_{o2}$ *are the same then* $X^+ = (|X|, \mathfrak{I}_{X^+}, \mathfrak{O}_X)$.

PROOF. The key to both parts is the same: showing that $(|X|, \mathfrak{I}_{X^+}, \mathfrak{O}_X)$ is an object in $\mathcal{U}\mathcal{T}_{\mathrm{f}}^{\mathrm{v}}{}_2$. Once this is shown, it is obviously X^+. In particular, in the first condition the reverse implication is obvious. Let us consider the conditions in turn.

1. If $X \preceq X^+$ then X^+ is an object in $\mathcal{U}\mathcal{T}_{\mathrm{f}}^{\mathrm{v}}{}_2$ with the property that $\mathfrak{O}_{X^+}(T) \subseteq \mathfrak{O}_X(T)$ for all test objects, T,. Suppose that $u \colon |X| \to T$ is forced (via $\mathfrak{F}_{o2}$) for $(|X|, \mathfrak{I}_{X^+}, \mathfrak{O}_X)$. By assumption, it is therefore also forced for $(|X|, \mathfrak{I}_{X^+}, \mathfrak{O}_{X^+}) = X^+$. Hence $u \in \mathfrak{O}_{X^+}(T)$ whence $u \in \mathfrak{O}_X(T)$.

   Now consider the input side. Suppose that $u \colon T \to |X|$ is forced (via $\mathfrak{F}_{i2}$) for $(|X|, \mathfrak{I}_{X^+}, \mathfrak{O}_X)$. Then as the identity on $|X|$ lifts to a $\mathcal{U}\mathcal{T}^{\mathrm{v}}$-morphism $x \colon (|X|, \mathfrak{I}_{X^+}, \mathfrak{O}_X) \to X^+$, $u = |x|u$ is forced for X^+. Thus as X^+ is an object in $\mathcal{U}\mathcal{T}_{\mathrm{f}}^{\mathrm{v}}{}_2$, $u \in \mathfrak{I}_{X^+}(T)$.

   Hence $(|X|, \mathfrak{I}_{X^+}, \mathfrak{O}_X)$ is an object in $\mathcal{U}\mathcal{T}_{\mathrm{f}}^{\mathrm{v}}{}_2$.

2. Suppose that $u \colon |X| \to T$ is forced (via $\mathfrak{F}_{o2}$) for $(|X|, \mathfrak{I}_{X^+}, \mathfrak{O}_X)$. By assumption, it is therefore also forced via $\mathfrak{F}_{o1}$. Since the identity on $|X|$ lifts to a $\mathcal{U}\mathcal{T}^{\mathrm{v}}$-morphism $x \colon X \to (|X|, \mathfrak{I}_{X^+}, \mathfrak{O}_X)$, $u = u|x|$ is forced for X. Thus as X^+ is an object in $\mathcal{U}\mathcal{T}_{\mathrm{f}}^{\mathrm{v}}{}_1$, $u \in \mathfrak{O}_X(T)$.

   Now consider the input side. Observe that if X' is an object in $\mathcal{U}\mathcal{T}_{\mathrm{f}}^{\mathrm{v}}{}_2$ such that $X \preceq X'$ then $(|X|, \mathfrak{I}_{X^+}, \mathfrak{O}_X) \preceq X'$ (comparisons in $\mathcal{U}\mathcal{T}^{\mathrm{v}}$ for simplicity). Therefore if $u \colon T \to |X|$ is forced for $(|X|, \mathfrak{I}_{X^+}, \mathfrak{O}_X)$ then it is forced for X'. Hence it is in $\mathfrak{I}_{X^+}(T)$ as this is the intersection of the corresponding $\mathfrak{I}_{X'}(T)$.

   Hence $(|X|, \mathfrak{I}_{X^+}, \mathfrak{O}_X)$ is an object in $\mathcal{U}\mathcal{T}_{\mathrm{f}}^{\mathrm{v}}{}_2$. ■

A particular example of when the first condition holds is when the output forcing condition is saturation. The second case shows that if we change the forcing conditions one component at a time then we get good control over how the changes occur.

There are obvious analogues for the input forcing condition.

Let us conclude this section with an example of when $X^- = \mathfrak{For}^-(X) \preceq X$ fails. In this example, the underlying category is Set and the test category is the category of open subsets of $\mathbb{R}$ with C^∞-maps between them. Both forcing conditions have saturation as output forcing condition. The weaker forcing condition has no input forcing condition whilst the stronger has the sheaf condition. As the output forcing conditions are saturation, both an object in $\mathcal{U}\mathcal{T}_{\mathrm{f}}^{\mathrm{v}}{}_1$ and an object in $\mathcal{U}\mathcal{T}_{\mathrm{f}}^{\mathrm{v}}{}_2$ are determined by their input test functions.

Consider the object, X, in $\mathcal{UT}_{\mathrm{f}}^{\mathrm{v}}{}_1$ with $|X| = \mathbb{R}$ and $\mathfrak{I}_X(T)$ those C^∞-maps $\psi\colon T \to \mathbb{R}$ which are bounded. To find X^- we take the join of all objects in $\mathcal{UT}_{\mathrm{f}}^{\mathrm{v}}{}_2$ below X with the same object in $\mathcal{U}$. For any bounded open interval, $I \subseteq \mathbb{R}$, we can define a object, X_I, in $\mathcal{UT}_{\mathrm{f}}^{\mathrm{v}}{}_2$ with $|X_I| = \mathbb{R}$ and $\mathfrak{I}_{X_I}(T)$ those C^∞-maps $\psi\colon T \to \mathbb{R}$ which factor through I. This is an object in $\mathcal{UT}_{\mathrm{f}}^{\mathrm{v}}{}_2$. (Note that we have not assumed the constant forcing condition on inputs, if we had we would have to include constant maps but this makes no substantial difference to the example.) Moreover, this object in $\mathcal{UT}_{\mathrm{f}}^{\mathrm{v}}{}_2$ is below X. However, the identity on $\mathbb{R}$ is locally in some X_I and so, because of the sheaf condition, is in the join of the X_I. Hence X^- is at least the object in $\mathcal{UT}_{\mathrm{f}}^{\mathrm{v}}{}_2$ with input test functor $\mathfrak{I}(T) = C^\infty(T, \mathbb{R})$. In fact, it is exactly that.

Thus $X^- \not\leq X$.

4.10. Change of Test Spaces In this section we wish to examine what happens when we change the test spaces. The underlying category remains the same and we choose two categories of test spaces, $\mathcal{T}_1$ and $\mathcal{T}_2$. We therefore obtain the category of virtual $\mathcal{T}_1$–objects in $\mathcal{U}$, $\mathcal{UT}_1{}^{\mathrm{v}}$, and the category of virtual $\mathcal{T}_2$–objects in $\mathcal{U}$, $\mathcal{UT}_2{}^{\mathrm{v}}$.

We restrict ourselves to the case where $\mathcal{T}_1$ is a full subcategory of $\mathcal{T}_2$ since, as remarked earlier (in the second paragraph of Section 4), we are viewing our test categories as being full subcategories of the maximal test category. Restriction of the test functors defines an obvious functor $\mathfrak{Res}\colon \mathcal{UT}_2{}^{\mathrm{v}} \to \mathcal{UT}_1{}^{\mathrm{v}}$.

As the forcing condition depends slightly on the test category, we must consider how to relate the two. Clearly, we wish to meddle with this as little as possible since we can apply a forcing functor afterwards.

Consider the categories of trials, Tri_1 and Tri_2. There is an obvious inclusion functor $\mathrm{Tri}_1 \to \mathrm{Tri}_2$ induced from the inclusion $\mathcal{T}_1 \to \mathcal{T}_2$. We shall not give this a symbol, leaving to context the rôle of distinguishing. This has the property that if T is a test object in $\mathcal{T}_1$, X a virtual $\mathcal{T}$–object in $\mathcal{U}$, and $u\colon \mathfrak{U}(T) \to |X|$ a $\mathcal{U}$-morphism, then $\mathrm{Tri}_1(u) \subseteq \mathrm{Tri}_2(u)$.

Let $(\mathfrak{F}_{i2}, \mathfrak{F}_{o2})$ be a forcing condition with respect to $\mathcal{T}_2$. Then we can restrict $\mathfrak{F}_{i2}$ and $\mathfrak{F}_{o2}$ to Tri_1. Let us call the resulting functors $\mathfrak{F}_{i1}$ and $\mathfrak{F}_{o1}$. If, for T_1 and T_2 in $\mathcal{T}_1$, $u\colon \mathfrak{U}(T_1) \to \mathfrak{U}(T_2)$ is not in the image of $\mathfrak{U}$ then $\mathfrak{F}_{i1}(\mathrm{Tri}_1(u)) = \mathfrak{F}_{i2}(\mathrm{Tri}_1(u)) \leq \mathfrak{F}_{i2}(\mathrm{Tri}_2(u)) = 0$, and similarly for $\mathfrak{F}_{o1}$. Hence $(\mathfrak{F}_{i1}, \mathfrak{F}_{o1})$ is a forcing condition.

Thus we have $\mathcal{UT}_{1\mathrm{f}}^{\mathrm{v}}$, and $\mathcal{UT}_{2\mathrm{f}}^{\mathrm{v}}$.

4.11. Proposition. *The restriction of a forced virtual $\mathcal{T}_2$–object in $\mathcal{U}$ is a forced virtual $\mathcal{T}_1$–object in $\mathcal{U}$.*

Proof. Let X be a forced virtual $\mathcal{T}_2$–object in $\mathcal{U}$. Let us consider the input test functor. Let T be a test object in $\mathcal{T}_1$. Let $u\colon \mathfrak{U}(T) \to |X|$ be a $\mathcal{U}$-morphism which is forced via $\mathfrak{F}_{i1}$. Then $\mathfrak{F}_{i1}(\mathrm{Tri}_1(u)) = 1$. By definition, therefore, $\mathfrak{F}_{i2}(\mathrm{Tri}_1(u)) = 1$. Since $\mathrm{Tri}_1(u) \subseteq \mathrm{Tri}_2(u)$, u is forced via $\mathfrak{F}_{i2}$. Hence $u \in \mathfrak{I}_X$. Since the source u is a test object from $\mathcal{T}_1$, it persists in the restriction. Hence the restriction of X is again a forced virtual $\mathcal{T}_1$–object in $\mathcal{U}$. ∎

Since $\mathfrak{Res}$ preserves the underlying objects in $\mathcal{U}$, for each object, U, in $\mathcal{U}$ it restricts to a functor, i.e. an order-preserving map, $\mathcal{U}\mathcal{T}_{2\mathrm{f}}^{\mathrm{v}}{}_U \to \mathcal{U}\mathcal{T}_{1\mathrm{f}}^{\mathrm{v}}{}_U$. We can therefore define two reverse functors using the lattice structure of $\mathcal{U}\mathcal{T}_{2\mathrm{f}}^{\mathrm{v}}{}_U$ in a similar fashion to the change of forcing condition functors.

4.12. Definition. *Define two* extension functors $\mathcal{U}\mathcal{T}_{1\mathrm{f}}^{\mathrm{v}} \to \mathcal{U}\mathcal{T}_{2\mathrm{f}}^{\mathrm{v}}$ *by*

$$\mathfrak{Ext}^+ \colon X \mapsto \bigvee \mathfrak{Res}^{-1}\left[\bot, X\right],$$
$$\mathfrak{Ext}^- \colon X \mapsto \bigwedge \mathfrak{Res}^{-1}\left[X, \top\right].$$

In studying functors from $\mathcal{U}\mathcal{T}_{1\mathrm{f}}^{\mathrm{v}}$ to $\mathcal{U}\mathcal{T}_{2\mathrm{f}}^{\mathrm{v}}$ one encounters an obvious question: if T is a test object in $\mathcal{T}_2$ that is *not* in $\mathcal{T}_1$, which $\mathcal{U}$-morphisms $\mathfrak{U}(T) \to |X|$ should be included? The functors $\mathfrak{Ext}^-$ and $\mathfrak{Ext}^+$ are intended to give, respectively, the minimum and maximum answers to this question. However, as with the forcing functors, these intentions are not always carried out.

Under a mild assumption on the relationship between $\mathcal{T}_1$ and $\mathcal{T}_2$ we can factor $\mathfrak{Ext}^+$ and $\mathfrak{Ext}^-$ through $\mathfrak{For}^-$ and $\mathfrak{For}^+$ respectively. This assumption is closely related to the concept of an *adequate* subcategory as studied in [Isbell (1960)]. In essence, it says that when looking in $\mathcal{U}$, $\mathcal{T}_2$ can be determined by looking at morphisms to and from objects in $\mathcal{T}_1$.

4.13. Definition. *We say that* $\mathcal{T}_1$ *is* $\mathcal{U}$–adequate *in* $\mathcal{T}_2$ *if, in* $\mathcal{U}$*, it determines* $\mathrm{im}\mathfrak{U}$*. That is to say, if* T_1 *and* T_2 *are objects in* $\mathcal{T}_2$ *and* $u \colon \mathfrak{U}(T_1) \to \mathfrak{U}(T_2)$ *is a* $\mathcal{U}$*-morphism which is* not *in the image of* $\mathfrak{U}$ *then there are objects in* $\mathcal{T}_1$*, say* T_1' *and* T_2'*, and morphisms* $t_1 \colon T_1' \to T_1$*,* $t_2 \colon T_2 \to T_2'$ *such that* $\mathfrak{U}(t_2)u\mathfrak{U}(t_1)$ *is not in the image of* $\mathfrak{U}$*.*

In the following we shall assume that this condition holds. This allows us to extend the embedding of $\mathcal{T}_1$ in $\mathcal{U}\mathcal{T}_{1\mathrm{f}}^{\mathrm{v}}$ to the whole of $\mathcal{T}_2$. As this is an extension of $\mathfrak{S} \colon \mathcal{T}_1 \to \mathcal{U}\mathcal{T}_{1\mathrm{f}}^{\mathrm{v}}$ we shall use the same symbol.

4.14. Lemma. *There is a functor* $\mathfrak{S} \colon \mathcal{T}_2 \to \mathcal{U}\mathcal{T}_1{}^{\mathrm{v}}$ *which embeds* $\mathcal{T}_2$ *as a full subcategory of* $\mathcal{U}\mathcal{T}_1{}^{\mathrm{v}}$*. The object,* $\mathfrak{S}(T)$*, in* $\mathcal{U}\mathcal{T}_1{}^{\mathrm{v}}$ *has underlying object,* $\mathfrak{U}(T)$*, in* $\mathcal{U}$*, input test functor* $\mathfrak{I}(T') = \mathcal{T}_2(T', T)$*, and output test functor* $\mathfrak{O}(T') = \mathcal{T}_2(T, T')$*. On morphisms,* $\mathfrak{S}(t)$ *is determined by the requirement that* $|\mathfrak{S}(t)| = \mathfrak{U}(t)$*.*

The restriction of this functor to $\mathcal{T}_1$ *is the functor from Lemma 3.4.*

Proof. The only part we need to worry about is showing that $\mathcal{T}_2$ embeds as a *full* subcategory of $\mathcal{U}\mathcal{T}_1{}^{\mathrm{v}}$. This is where we need our assumption.

We know that it is true when restricted to $\mathcal{T}_1$. Let T_1 and T_2 be objects in $\mathcal{T}_2$. Let $u \colon |T_1| \to |T_2|$ be a $\mathcal{U}$-morphism which is not in the image of $\mathfrak{U}$. Then, by assumption, there are $\mathcal{T}_2$-morphisms $t_1 \colon T_1' \to T_1$ and $t_2 \colon T_2 \to T_2'$ such that $\mathfrak{U}(t_2)u\mathfrak{U}(t_1)$ is not in the image of $\mathfrak{U}$. However from Lemma 3.4, t_1 underlies a morphism of objects in $\mathcal{U}\mathcal{T}_2{}^{\mathrm{v}}$, $\mathfrak{S}(T_1') \to \mathfrak{S}(T_1)$; similarly for t_2. Thus if u underlay a $\mathcal{U}\mathcal{T}_2{}^{\mathrm{v}}$-morphism, we would have that $\mathfrak{U}(t_2)u\mathfrak{U}(t_1)$ underlay a $\mathcal{U}\mathcal{T}_2{}^{\mathrm{v}}$-morphism from $\mathfrak{S}(T_1')$ to $\mathfrak{S}(T_2')$. As these come from $\mathcal{T}_1$, we know that this would mean that $\mathfrak{U}(t_2)u\mathfrak{U}(t_1)$ lay in the image of $\mathfrak{U}$, a contradiction. Hence $\mathfrak{S}$ is full. ■

Let us write $\mathfrak{Ext}^+_{\mathrm{pre}}$ and $\mathfrak{Ext}^-_{\mathrm{pre}}$ for the extension functors $\mathcal{U}\mathcal{T}_1{}^{\mathrm{v}} \to \mathcal{U}\mathcal{T}_2{}^{\mathrm{v}}$; i.e. when there are no forcing conditions. Under our assumption on the relationship of $\mathcal{T}_1$ to $\mathcal{T}_2$ we can give explicit descriptions of $\mathfrak{Ext}^+_{\mathrm{pre}}(X)$ and $\mathfrak{Ext}^-_{\mathrm{pre}}(X)$.

4.15. Proposition. *Let X be a virtual $\mathcal{T}_1$–object in $\mathcal{U}$. For T an object in $\mathcal{T}_2$, define*

$$\begin{aligned}
\mathfrak{I}_a(T) &:= \mathcal{U}\mathcal{T}_1{}^{\mathrm{v}}(\mathfrak{S}(T), X),\\
\mathfrak{O}_a(T) &:= \{\phi\colon |X| \to \mathfrak{U}(T) : \phi = \mathfrak{U}(t)\phi' \text{ for some } T' \in \mathcal{T}_1, \phi' \in \mathfrak{O}_X(T'), t\colon T \to T'\},\\
\mathfrak{I}_b(T) &:= \{\psi\colon \mathfrak{U}(T) \to |X| : \psi = \psi'\mathfrak{U}(t) \text{ for some } T' \in \mathcal{T}_1, \psi' \in \mathfrak{I}_X(T'), t\colon T' \to T\},\\
\mathfrak{O}_b(T) &:= \mathcal{U}\mathcal{T}_1{}^{\mathrm{v}}(X, \mathfrak{S}(T)).
\end{aligned}$$

Then

$$\begin{aligned}
\mathfrak{Ext}^+_{pre}(X) &= (|X|, \mathfrak{I}_a, \mathfrak{O}_a), \text{ and}\\
\mathfrak{Ext}^-_{pre}(X) &= (|X|, \mathfrak{I}_b, \mathfrak{O}_b).
\end{aligned}$$

Proof. There is an obvious symmetry here so we shall concentrate on $(|X|, \mathfrak{I}_a, \mathfrak{O}_a)$. Let us start by showing that this is a virtual $\mathcal{T}_2$–object in $\mathcal{U}$. It is clear that the test functors are subfunctors of the requisite hom-functors. Therefore we just need to check the compatibility condition. Let $\psi \in \mathfrak{I}_a(T)$ and $\phi \in \mathfrak{O}_a(T')$. Then $\phi = \mathfrak{U}(t)\phi'$ for some $T'' \in \mathcal{T}_1$, $\phi' \in \mathfrak{O}_X(T'')$, and $t\colon T' \to T''$. As $\phi' \in \mathfrak{O}_X(T'')$ it underlies a $\mathcal{U}\mathcal{T}_2{}^{\mathrm{v}}$-morphism $X \to \mathfrak{S}(T'')$. Hence $\phi'\psi$ underlies a $\mathcal{U}\mathcal{T}_2{}^{\mathrm{v}}$-morphism $\mathfrak{S}(T) \to \mathfrak{S}(T'')$ which, as $\mathfrak{S}$ is full, comes from a morphism in $\mathcal{T}_2$. Hence $\phi\psi = \mathfrak{U}(t)\phi'\psi \in \mathrm{im}\mathfrak{U}$.

For T an object in $\mathcal{T}_1$,

$$\mathfrak{I}_a(T) = \mathcal{U}\mathcal{T}_1{}^{\mathrm{v}}(\mathfrak{S}(T), X) = \mathfrak{I}_X(T),$$

whilst in $\mathfrak{O}_a(T)$ we can take 1_T as the auxiliary morphism whence $\mathfrak{O}_a(T) = \mathfrak{O}_X(T)$. Hence

$$\mathfrak{Res}(|X|, \mathfrak{I}_a, \mathfrak{O}_a) = X.$$

We therefore have $(|X|, \mathfrak{I}_a, \mathfrak{O}_a) \preceq \mathfrak{Ext}^+_{\mathrm{pre}}(X)$.

Let X' be an object in $\mathcal{U}\mathcal{T}_1{}^{\mathrm{v}}$ with $|X'| = |X|$ such that $\mathfrak{Res}(X') \preceq X$. Then for T an object in $\mathcal{T}_1$, $\mathfrak{I}_{X'}(T) \subseteq \mathfrak{I}_X(T)$ and $\mathfrak{O}_{X'}(T) \supseteq \mathfrak{O}_X(T)$. Let T' be an object in $\mathcal{T}_2$. Let $\phi \in \mathfrak{O}_X(T)$ and $t\colon T \to T'$ a $\mathcal{T}_2$-morphism. Then $\phi \in \mathfrak{O}_{X'}(T)$ so, by functorality, $\mathfrak{U}(t)\phi \in \mathfrak{O}_{X'}(T)$. Hence $\mathfrak{O}_{X'}(T') \supseteq \mathfrak{O}_a(T')$. Let $\psi \in \mathfrak{I}_{X'}(T')$. Let $t\colon T \to T'$ be a $\mathcal{T}_2$-morphism. Then $\psi\mathfrak{U}(t) \in \mathfrak{I}_{X'}(T) \subseteq \mathfrak{I}_X(T)$. Hence $\psi\colon \mathfrak{U}(T') \to |X|$ underlies a $\mathcal{U}\mathcal{T}_2{}^{\mathrm{v}}$-morphism $\mathfrak{S}(T') \to X$, whence $\psi \in \mathfrak{I}_a(T')$. Thus $\mathfrak{I}_{X'}(T') \subseteq \mathfrak{I}_a(T')$. Hence $X' \preceq (|X|, \mathfrak{I}_a, \mathfrak{O}_a)$.

Thus $(|X|, \mathfrak{I}_a, \mathfrak{O}_a) = \bigvee \mathfrak{Res}^{-1}[\bot, X] = \mathfrak{Ext}^+_{\mathrm{pre}}(X)$. ∎

From this we can deduce a useful factorisation for when we do have a forcing condition.

4.16. Corollary. $\mathfrak{Ext}^+ = \mathfrak{For}^-\mathfrak{Ext}^+_{pre}$ *and* $\mathfrak{Ext}^- = \mathfrak{For}^+\mathfrak{Ext}^-_{pre}$.

Proof. As part of the previous proof we showed that $\mathfrak{Res}\mathfrak{Ext}^+_{\text{pre}}(X) = X$. Thus

$$\mathfrak{Res}^{-1}\left[\perp, X\right] = \left[\perp, \mathfrak{Ext}^+_{\text{pre}}(X)\right].$$

If we are extremely careful on the functors, we ought to write

$$\mathfrak{Ext}^+(X) = \bigvee (\mathfrak{Res}\mathfrak{Incl})^{-1}\left[\perp, X\right].$$

From which we deduce that

$$\begin{aligned} \mathfrak{Ext}^+(X) &= \bigvee \mathfrak{Incl}^{-1}\mathfrak{Res}^{-1}\left[\perp, X\right] \\ &= \bigvee \mathfrak{Incl}^{-1}\left[\perp, \mathfrak{Ext}^+_{\text{pre}}(X)\right] \\ &= \mathfrak{For}^{-}\mathfrak{Ext}^+_{\text{pre}}(X). \end{aligned}$$

Equality on morphisms is a formality. ∎

The idea here is that $\mathfrak{Ext}^+_{\text{pre}}(X)$ ought to be the maximum extension of X by a virtual $\mathcal{T}_2$–object in $\mathcal{U}$. Therefore any extension of X to a forced virtual $\mathcal{T}_2$–object in $\mathcal{U}$ must lie below it. Since $\mathfrak{Ext}^+(X)$ is meant to be the maximum such forced virtual $\mathcal{T}_2$–object in $\mathcal{U}$, we can find it by looking at the "nearest" forced virtual $\mathcal{T}_2$–object in $\mathcal{U}$ below $\mathfrak{Ext}^+_{\text{pre}}(X)$. In other words, by applying $\mathfrak{For}^{-}$.

Since we know that it is not always true that $\mathfrak{For}^{-}(X) \preceq X$, the obvious question is whether or not $\mathfrak{Ext}^+(X) \preceq \mathfrak{Ext}^+_{\text{pre}}(X)$.

Before proving this we observe that from Proposition 4.15 we obtain another extension functor which will help us establish the relationships between the other various extension functors.

4.17. Lemma. *The assignment $X \mapsto (|X|, \mathfrak{I}_a, \mathfrak{O}_b)$ defines another functor $\mathfrak{Ext}\colon \mathcal{U}\mathcal{T}_{1\text{f}}^{\text{v}} \to \mathcal{U}\mathcal{T}_{2\text{f}}^{\text{v}}$. This functor has the following properties:*

1. *$\mathfrak{Res}\mathfrak{Ext}$ is the identity on $\mathcal{U}\mathcal{T}_{1\text{f}}^{\text{v}}$,*
2. *$\mathfrak{Ext}^{-}(X) \preceq \mathfrak{Ext}(X) \preceq \mathfrak{Ext}^+(X)$ for all forced virtual $\mathcal{T}_1$–objects, X, in $\mathcal{U}$, and*
3. *$\mathfrak{Ext}\mathfrak{S} = \mathfrak{S}\colon \mathcal{T}_2 \to \mathcal{U}\mathcal{T}_{2\text{f}}^{\text{v}}$.*

Proof. That this is a virtual $\mathcal{T}_2$–object in $\mathcal{U}$ comes from the fullness of $\mathfrak{S}\colon \mathcal{T}_2 \to \mathcal{U}\mathcal{T}_1{}^{\text{v}}$. Composition defines a map

$$\mathcal{U}\mathcal{T}_1{}^{\text{v}}(\mathfrak{S}(T'), X) \times \mathcal{U}\mathcal{T}_1{}^{\text{v}}(X, \mathfrak{S}(T)) \to \mathcal{U}\mathcal{T}_1{}^{\text{v}}(\mathfrak{S}(T'), \mathfrak{S}(T)) \cong \mathcal{T}_2(T', T).$$

It is clear that the restriction of this to $\mathcal{T}_1$ is X.

To show that it satisfies the forcing conditions, let $\psi\colon \mathfrak{U}(T) \to |X|$ be forced for this object in $\mathcal{U}\mathcal{T}_2{}^{\text{v}}$. Then for any $\mathcal{T}_2$-morphism $t\colon T' \to T$ for a test object, T', the composition ψt is forced. Since X satisfies the forcing conditions with respect to $\mathcal{T}_1$, we therefore have that $\psi t \in \mathfrak{I}_X(T')$. As this holds for all such t, ψ underlies a $\mathcal{U}\mathcal{T}_1{}^{\text{v}}$-morphism $\mathfrak{S}(T) \to X$. It is therefore in $\mathfrak{I}_a(T)$. The case for the output forcing condition is similar, and hence we have an object in $\mathcal{U}\mathcal{T}_{2\text{f}}^{\text{v}}$.

The properties are obvious. ∎

4.18. Lemma. *For all forced virtual $\mathcal{T}_1$–objects, X, in $\mathcal{U}$,*

$$\mathfrak{Ext}^-_{pre}(X) \preceq \mathfrak{Ext}^-(X) \preceq \mathfrak{Ext}(X) \preceq \mathfrak{Ext}^+(X) \preceq \mathfrak{Ext}^+_{pre}(X).$$

Proof. The relationship of $\mathfrak{Ext}$ to the others is straightforward. From its description we have

$$\mathfrak{Ext}^-_{\mathrm{pre}}(X) \preceq \mathfrak{Ext}(X) \preceq \mathfrak{Ext}^+_{\mathrm{pre}}(X).$$

Hence as $\mathfrak{Ext}^- = \mathfrak{For}^+\mathfrak{Ext}^-_{\mathrm{pre}}$ and $\mathfrak{Ext}^+ = \mathfrak{For}^-\mathfrak{Ext}^+_{\mathrm{pre}}$ we see that

$$\mathfrak{Ext}^-(X) \preceq \mathfrak{Ext}(X) \preceq \mathfrak{Ext}^+(X).$$

The two outer relations are proved in the same fashion as each other so let us take the right-hand one. From Section 4.4 we see that this depends on the input test functors. Let us write X^+ for $\mathfrak{Ext}^+(X)$. Consider the virtual $\mathcal{T}_2$–object, $\widehat{X}$, in $\mathcal{U}$ with object in $\mathcal{U}$ $|X|$, input test functor $\mathfrak{I}_a$, and output test functor $\mathfrak{I}_{X^+}$. Note that $\mathfrak{I}_a$ is the input test functor for $\mathfrak{Ext}^+_{\mathrm{pre}}(X)$ and so $\widehat{X} \preceq \mathfrak{Ext}^+_{\mathrm{pre}}(X)$.

Suppose that $\psi \colon \mathfrak{U}(T) \to |X|$ is forced for $\widehat{X}$. Then for every test object, T', and $\mathcal{T}$-morphism $t \colon T' \to T$, $\psi\mathfrak{U}(t) \colon \mathfrak{U}(T') \to |X|$ is forced for $\widehat{X}$ and hence for $\mathfrak{Res}(\widehat{X})$.

It is possible to show that $\mathfrak{Res}(\widehat{X}) = X$ but for this step it is sufficient to note that as $\widehat{X} \preceq \mathfrak{Ext}^+_{\mathrm{pre}}(X)$, the identity on $|X|$ lifts to a $\mathcal{U}\mathcal{T}_{1\mathrm{f}}^{\mathrm{v}}$-morphism $\mathfrak{Res}(\widehat{X}) \to X$. Thus $\psi\mathfrak{U}(t) \colon \mathfrak{U}(T') \to |X|$ is forced for X and thus in $\mathfrak{I}_X(T')$. This is sufficient to show that $\psi \colon \mathfrak{U}(T) \to |X|$ lifts to a $\mathcal{U}\mathcal{T}_1{}^{\mathrm{v}}$-morphism $\mathfrak{S}(T) \to X$ since for $\phi \in \mathfrak{O}_X(T'')$ and $t \colon T' \to T$, $\phi\psi t \in \mathcal{T}_2(T', T'')$ and hence $\phi\psi \in \mathfrak{O}_{\mathfrak{S}(T)}(T'')$ by the assumption that $\mathcal{T}_1$ is $\mathcal{U}$–adequate in $\mathcal{T}_2$.

Thus $\psi \in \mathfrak{I}_a(T)$ and so $\widehat{X}$ satisfies the input forcing condition.

Now suppose that X' is a forced virtual $\mathcal{T}_2$–object in $\mathcal{U}$ with underlying object in $\mathcal{U}$ $|X|$ such that the identity on $|X|$ lifts to a $\mathcal{U}\mathcal{T}_2{}^{\mathrm{v}}$-morphism $X' \to \mathfrak{Ext}^+_{\mathrm{pre}}(X)$. Then $\mathfrak{I}_{X'} \subseteq \mathfrak{I}_a$ and $\mathfrak{O}_{X'} \supseteq \mathfrak{O}_{X^+}$. Hence the identity on $|X|$ lifts to a $\mathcal{U}\mathcal{T}_2{}^{\mathrm{v}}$-morphism $X' \to \widehat{X}$. Thus if $\phi \colon |X| \to \mathfrak{U}(T)$ is forced for $\widehat{X}$, ϕ is forced for X' as well. Hence as X' is a forced virtual $\mathcal{T}_2$–object in $\mathcal{U}$, $\phi \in \mathfrak{O}_{X'}(T)$. As this holds for all such X', $\phi \in \mathfrak{O}_{\widehat{X}}(T)$ and thus $\widehat{X}$ satisfies the output forcing condition.

Hence $\widehat{X}$ is a forced virtual $\mathcal{T}_2$–object in $\mathcal{U}$ and so is equal to $\mathfrak{Ext}^+(X)$. Thus $\mathfrak{Ext}^+(X) \preceq \mathfrak{Ext}^+_{\mathrm{pre}}(X)$. ∎

4.19. Corollary. *$\mathfrak{Res}\mathfrak{Ext}^+$ and $\mathfrak{Res}\mathfrak{Ext}^-$ are the identity on $\mathcal{U}\mathcal{T}_{1\mathrm{f}}^{\mathrm{v}}$ and $\mathfrak{Ext}^-\mathfrak{Res}(X) \preceq X \preceq \mathfrak{Ext}^+\mathfrak{Res}(X)$ for all objects, X, in $\mathcal{U}\mathcal{T}_{2\mathrm{f}}^{\mathrm{v}}$.* ∎

One further interesting question is as to when $\mathfrak{Ext}^+$ and $\mathfrak{Ext}^-$ agree, or if either is the same as $\mathfrak{Ext}$. It transpires that it is sufficient to check this on the image of $\mathfrak{S} \colon \mathcal{T}_2 \to \mathcal{U}\mathcal{T}_{1\mathrm{f}}^{\mathrm{v}}$.

4.20. Proposition. *If any of $\mathfrak{Ext}^+$, $\mathfrak{Ext}^-$, and $\mathfrak{Ext}$ agree on the image of $\mathfrak{S} \colon \mathcal{T}_2 \to \mathcal{U}\mathcal{T}_{1\mathrm{f}}^{\mathrm{v}}$ then they agree on the whole of $\mathcal{U}\mathcal{T}_{1\mathrm{f}}^{\mathrm{v}}$.*

Proof. Suppose that, say, $\mathfrak{Ext}^+$ and $\mathfrak{Ext}$ agree on the image of $\mathfrak{S}$. Since, for a forced virtual $\mathcal{T}_1$–object, X, in $\mathcal{U}$, $\mathfrak{Ext}(X) \preceq \mathfrak{Ext}^+(X) \preceq \mathfrak{Ext}^+_{\text{pre}}(X)$, and $\mathfrak{Ext}(X)$ and $\mathfrak{Ext}^+_{\text{pre}}(X)$ have the same input test functor, to show that $\mathfrak{Ext}^+$ and $\mathfrak{Ext}$ agree it is sufficient to show that whenever $\phi\colon |X| \to \mathfrak{U}(T)$ is in $\mathfrak{O}_{\mathfrak{Ext}(X)}(T)$ then it is in $\mathfrak{O}_{\mathfrak{Ext}^+(X)}(T)$.

Let $\phi \in \mathfrak{O}_{\mathfrak{Ext}(X)}(T)$. Then for every $t\colon T \to T'$ with T' an object in $\mathcal{T}_1$, $t\phi \in \mathfrak{O}_X(T')$. Hence ϕ underlies a $\mathcal{U}\mathcal{T}_1{}^{\mathrm{v}}$-morphism $X \to \mathfrak{S}(T)$. Applying $\mathfrak{Ext}^+$ we obtain a $\mathcal{U}\mathcal{T}_2{}^{\mathrm{v}}$-morphism $\mathfrak{Ext}^+(X) \to \mathfrak{Ext}^+\mathfrak{S}(T)$. By assumption, $\mathfrak{Ext}^+\mathfrak{S}(T) = \mathfrak{Ext}\mathfrak{S}(T)$. Since $\mathfrak{Ext}\mathfrak{S}(T) = \mathfrak{S}(T)$, we deduce that ϕ defines a $\mathcal{U}\mathcal{T}_2{}^{\mathrm{v}}$-morphism $\mathfrak{Ext}^+(X) \to \mathfrak{S}(T)$ and hence $\phi \in \mathfrak{O}_{\mathfrak{Ext}^+(X)}$.

Thus $\mathfrak{Ext}^+ = \mathfrak{Ext}$ as required.

The case for $\mathfrak{Ext}^-$ and $\mathfrak{Ext}$ is similar. Then if $\mathfrak{Ext}^-$ and $\mathfrak{Ext}^+$ agree on the image of $\mathfrak{S}$, they both agree with $\mathfrak{Ext}$ and hence they agree on the whole of $\mathcal{U}\mathcal{T}_{1\mathrm{f}}^{\mathrm{v}}$. ∎

As a concluding remark for this section, let us consider a subtly different question on extensions. We have assumed in this section that we have a forcing condition on the larger test category which we restrict to the smaller. It is possible to consider the situation from the opposite angle: suppose we have a forcing condition on the smaller test category which we wish to extend to the larger. Can this be done, and in how many ways?

The work of this section shows that it can be done, and that the two extremes are straightforward to describe. The minimal extension of the forcing conditions is, say for the input forcing condition, to say that a morphism $u\colon \mathfrak{U}(T) \to |X|$ is forced if there is a factorisation of u as $u'\mathfrak{U}(t)$ where $t\colon T \to T'$ is such that T' is an object in $\mathcal{T}_1$ and $u'\colon \mathfrak{U}(T') \to |X|$ is forced. This is the minimum that can be done to preserve the fact that the forcing conditions are defined by functors. With this extension of the forcing conditions, the extension functors are $\mathfrak{Ext}^+ = \mathfrak{Ext}^+_{\text{pre}}$ and $\mathfrak{Ext}^- = \mathfrak{Ext}^-_{\text{pre}}$. The maximal extension of the forcing conditions is, again for the input, to say that a morphism $u\colon \mathfrak{U}(T) \to |X|$ is forced if whenever $t\colon T' \to T$ is a $\mathcal{T}_2$-morphism with T' an object in $\mathcal{T}_1$ then $u\mathfrak{U}(t)$ is forced. With this extension of the forcing conditions, the extension functors are $\mathfrak{Ext}^+ = \mathfrak{Ext} = \mathfrak{Ext}^-$ and, in fact, the categories $\mathcal{U}\mathcal{T}_{1\mathrm{f}}^{\mathrm{v}}$ and $\mathcal{U}\mathcal{T}_{2\mathrm{f}}^{\mathrm{v}}$ are isomorphic via the extension and restriction functors.

4.21. Change of Underlying Category Now we turn to the possibility of changing the underlying category. We wish to consider two ways to do this. The first is when we have a functor from one underlying category to another. This will allow us to consider inclusions of subcategories and reflections and coreflections. The second is when we have a topological category over another category and wish to transfer forced virtual $\mathcal{T}$–objects in $\mathcal{U}$ from the lower category to the higher one.

4.21.1. Simple Transfers The most obvious way to change the underlying category is to have two underlying categories, say $\mathcal{U}_1$ and $\mathcal{U}_2$, together with a covariant functor $\mathfrak{G}\colon \mathcal{U}_1 \to \mathcal{U}_2$. Given that we want to say something about virtual $\mathcal{T}$–objects in $\mathcal{U}$ and forced virtual $\mathcal{T}$–objects in $\mathcal{U}$, we should ensure that this functor "plays nicely" with the structures defining them. Since we are focussing on the underlying categories here, we wish to avoid changing the rest of the structure as much as we can. Therefore we fix a the test category, $\mathcal{T}$, with functors to $\mathcal{U}_1$ and $\mathcal{U}_2$, both of which we shall denote by $\mathfrak{U}$. The compatibility that we require is that $\mathfrak{G}\mathfrak{U} = \mathfrak{U}$.

In the absence of forcing conditions, we get some simple results.

4.22. Proposition. *For X a virtual $\mathcal{T}$–object in $\mathcal{U}_1$, $(\mathfrak{G}(|X|), \mathfrak{G}\mathfrak{I}_X, \mathfrak{G}\mathfrak{O}_X)$ is a virtual $\mathcal{T}$–object in $\mathcal{U}_2$. This assignment is functorial and covers $\mathfrak{G}$.*

Proof. The only thing to check is the compatibility between the input and output test morphisms. Let $\psi \colon \mathfrak{U}(T) \to \mathfrak{G}(|X|)$ be an input test morphism and $\phi \colon \mathfrak{G}(|X|) \to T'$ be an output test morphism. By definition, $\psi = \mathfrak{G}(\psi')$ for some $\psi' \colon \mathfrak{U}(T) \to |X|$ and $\phi = \mathfrak{G}(\phi')$ for some $\phi' \colon \mathfrak{U}(T') \to |X|$. Hence $\phi\psi = \mathfrak{G}(\phi'\psi')$. As X is a virtual $\mathcal{T}$–object in $\mathcal{U}$, $\phi'\psi' \in \operatorname{im}\mathfrak{U}$ and so $\phi\psi \in \operatorname{im}\mathfrak{G}\mathfrak{U} = \operatorname{im}\mathfrak{U}$. ∎

We shall denote this functor $\mathfrak{G}^{\mathrm{v}}$. Let us write $\operatorname{Fun}(\mathcal{U}_1, \mathcal{U}_2; \mathfrak{U})$ for the category of functors $\mathcal{U}_1 \to \mathcal{U}_2$ which intertwine $\mathfrak{U}$, with natural transformations that are the identity on $\operatorname{im}\mathfrak{U}$.

4.23. Corollary. *The assignment $\mathfrak{G} \mapsto \mathfrak{G}^{\mathrm{v}}$ defines a functor*

$$\operatorname{Fun}(\mathcal{U}_1, \mathcal{U}_2; \mathfrak{U}) \to \operatorname{Fun}(\mathcal{U}_1\mathcal{T}^{\mathrm{v}}, \mathcal{U}_2\mathcal{T}^{\mathrm{v}}).$$

In particular, adjoint pairs map to adjoint pairs. ∎

Now let us consider forcing functors. Here we run into difficulties. What we want to be able to say is that if X is a forced virtual $\mathcal{T}$–object in $\mathcal{U}_1$ then $\mathfrak{G}^{\mathrm{v}}(X)$ is a forced virtual $\mathcal{T}$–object in $\mathcal{U}_2$. To do this we need to be able to assert that, say for input morphisms, if X is a forced virtual $\mathcal{T}$–object in $\mathcal{U}_1$ and $u \colon \mathfrak{U}(T) \to |\mathfrak{G}^{\mathrm{v}}(X)|$ is a forced $\mathcal{U}_2$-morphism then there is a forced $\mathcal{U}_1$-morphism $u' \colon \mathfrak{U}(T) \to |X|$ such that $\mathfrak{G}(u') = u$. This is not a simple condition. Firstly, we observe that we can only force $\mathcal{U}_1$-morphisms that lie in the image of $\mathfrak{G}$. Secondly, there is a potential problem if more than one forced virtual $\mathcal{T}$–object in $\mathcal{U}_1$ maps to the same virtual $\mathcal{T}$–object in $\mathcal{U}_2$: we need to know that a given forced $\mathcal{U}_2$-morphism has a forced preimage for each of the preimages of the virtual $\mathcal{T}$–object in $\mathcal{U}_2$. This is particularly tricky when the virtual $\mathcal{T}$–object in $\mathcal{U}_2$ in question is the natural image of a test object since there we have additional restrictions on which $\mathcal{U}_2$-morphisms can be forced. Thirdly, when testing whether a $\mathcal{U}_2$-morphism is forced, we may need to use trials that have target (or source) outside the image of $\mathfrak{G}$. Thus it may be that a $\mathcal{U}_2$-morphism would not be forced if we only considered those trials in the image of $\mathfrak{G}$ but would be forced if we allowed all trials.

At the loss of some generality we can simplify matters a little. Let $(\mathfrak{F}_{i2}, \mathfrak{F}_{o2})$ be a forcing condition on the category of virtual $\mathcal{T}$–objects in $\mathcal{U}_2$. We apply $\mathfrak{G}$ to the category of trials; this is defined by sending (t, x) to $(t, \mathfrak{G}^{\mathrm{v}}(x))$. The temptation at this point would be to define $(\mathfrak{F}_{i1}, \mathfrak{F}_{o1})$ by composition. However, this does not help resolve the potential difficulties given above. Rather, we define

$$\mathfrak{F}_{i1}(T, X, \mathcal{F}) = \max\{\mathfrak{F}_{i2}(T, \mathfrak{G}^{\mathrm{v}}(X), \mathcal{F}') : \mathcal{F}' \cap \operatorname{im}\mathfrak{G} = \mathfrak{G}(\mathcal{F})\}.$$

We define $\mathfrak{F}_{o1}$ similarly.

4.24. LEMMA. $(\mathfrak{F}_{i1}, \mathfrak{F}_{o1})$ *is a forcing condition. If* $\mathfrak{G}$ *is injective on* $\mathcal{U}_1$*-morphisms out of* im$\mathfrak{U}$ *then* $\mathfrak{F}_{i1}$ *has the property that for a* $\mathcal{U}_1$*-morphism,* $u \colon \mathfrak{U}(T) \to |X|$*,* u *is forced if* $\mathfrak{G}(u)$ *is forced. Similarly, if* $\mathfrak{G}$ *is injective on* $\mathcal{U}_1$*-morphisms into* im$\mathfrak{U}$ *then* $\mathfrak{F}_{o1}$ *has the property that for a* $\mathcal{U}_1$*-morphism,* $u \colon |X| \to \mathfrak{U}(T)$*,* u *is forced if* $\mathfrak{G}(u)$ *is forced.*

PROOF. It is straightforward to show that $\mathfrak{F}_{i1}$ and $\mathfrak{F}_{o1}$ are functors. Thus we consider the "non-stupid" condition. This is immediate from the fact that for test objects T_1 and T_2, $(1_{T_1}, 1_{\mathfrak{S}(T_2)}) \in \operatorname{im}\mathfrak{G}$ so if $(T_1, \mathfrak{S}(T_2), \mathcal{F})$ is such that $(1,1) \notin \mathcal{F}$ and $(T_1, \mathfrak{S}(T_2), \mathcal{F}')$ is such that $\mathcal{F}' \cap \operatorname{im}\mathfrak{G} = \mathfrak{G}(\mathcal{F})$ then $(1,1) \notin \mathcal{F}'$ and so $\mathfrak{F}_{i2}(T_1, \mathfrak{S}(T_2), \mathcal{F}') = 0$. Thus $\mathfrak{F}_{i1}(T_1, \mathfrak{S}(T_2), \mathcal{F}) = 0$. The case for $\mathfrak{F}_{o1}$ is similar.

Let $u \colon \mathfrak{U}(T) \to |X|$ be a $\mathcal{U}_1$-morphism. We need to show that, under the stated condition, $\operatorname{Tri}_2(\mathfrak{G}(u)) \cap \operatorname{im}\mathfrak{G} = \operatorname{Tri}_1(\mathfrak{G}(u))$. It is clear that $\mathfrak{G}(\operatorname{Tri}_1(u)) \subseteq \operatorname{Tri}_2(\mathfrak{G}(u))$. Let (t,x) be a trial for u such that $\mathfrak{G}(u)$ succeeds at $(t, \mathfrak{G}^{\mathrm{v}}(x))$. Let X' be the target of x. Then $|\mathfrak{G}^{\mathrm{v}}(x)|\mathfrak{G}(u)\mathfrak{U}(t) \in \mathfrak{I}_{\mathfrak{G}^{\mathrm{v}}(X')}$. By definition of $\mathfrak{G}^{\mathrm{v}}(X)$, this means that there is some $\psi \in \mathfrak{I}_{X'}$ such that $\mathfrak{G}(\psi) = |\mathfrak{G}^{\mathrm{v}}(x)|\mathfrak{G}(u)\mathfrak{U}(t)$. The right-hand side of this is equal to $\mathfrak{G}(|x|u\mathfrak{U}(t))$. Thus, by the assumption, $\psi = |x|u\mathfrak{U}(t)$ so u succeeds at the trial (t,x) as required.

The case for the outputs is similar. ∎

4.25. COROLLARY. *If* $\mathfrak{G}$ *is a full functor and is injective on* $\mathcal{U}_1$*-morphisms in and out of* im$\mathfrak{U}$ *then* $\mathfrak{G}^{\mathrm{v}}$ *takes forced virtual* $\mathcal{T}$*–objects in* $\mathcal{U}_1$ *to forced virtual* $\mathcal{T}$*–objects in* $\mathcal{U}_2$*.* ∎

For example, if $\mathfrak{G}$ is the inclusion of a full subcategory then we have this condition. If we don't have this condition then the best strategy is to use one of the compositions

$$\mathcal{U}_1\mathcal{T}_{\mathrm{f}}^{\mathrm{v}} \to \mathcal{U}_1\mathcal{T}^{\mathrm{v}} \xrightarrow{\mathfrak{G}^{\mathrm{v}}} \mathcal{U}_2\mathcal{T}^{\mathrm{v}} \xrightarrow{\mathfrak{For}^-, \mathfrak{For}^+} \mathcal{U}_2\mathcal{T}_{\mathrm{f}}^{\mathrm{v}}.$$

If we have one of the two conditions then there is an obvious choice for which of the forcing functors to choose.

4.25.1. TOPOLOGICAL TRANSFERS The second way of changing the underlying category that we shall consider concerns the situation when one category is topological over the other category. That is to say, we have two categories $\mathcal{U}_1$ and $\mathcal{U}_2$ with a functor $\mathfrak{G} \colon \mathcal{U}_1 \to \mathcal{U}_2$ such that every $\mathfrak{G}$–structured source has a unique $\mathfrak{G}$–initial lift (see [Adámek et al. (2006), Section 21] for more on topological functors and categories). As at the start of the previous part, we also fix a test category, $\mathcal{T}$, with functors to $\mathcal{U}_1$ and $\mathcal{U}_2$ intertwined by $\mathfrak{G}$. Using the previous part, we obtain a functor $\mathfrak{G}^{\mathrm{v}} \colon \mathcal{U}_1\mathcal{T}^{\mathrm{v}} \to \mathcal{U}_2\mathcal{T}^{\mathrm{v}}$ which may, or may not, restrict to a functor on some subcategories of forced virtual $\mathcal{T}$–objects in $\mathcal{U}$.

4.26. LEMMA. $\mathcal{U}_1\mathcal{T}^{\mathrm{v}}$ *is topological over* $\mathcal{U}_2\mathcal{T}^{\mathrm{v}}$*.*

PROOF. Let X be a virtual $\mathcal{T}$–object in $\mathcal{U}_2$ and let $x_\lambda \colon X \to \mathfrak{G}^{\mathrm{v}}(X_\lambda)$ be a $\mathfrak{G}^{\mathrm{v}}$–structured source.

Let us consider the fibre of $\mathfrak{G}^{\mathrm{v}}$ at X. If X' is a virtual $\mathcal{T}$–object in $\mathcal{U}_1$ with $\mathfrak{G}^{\mathrm{v}}(X') = X$ then the input and output test functors of X' are completely determined by X. This is because we have, for example, $\mathfrak{I}_X = \mathfrak{G}\mathfrak{I}_{X'}$ and $\mathfrak{G}$ is faithful. Thus the only possible variation in X' is in regard to $|X'|$. Here we must have $\mathfrak{G}(|X'|) = |X|$.

Thus we are looking for a lift of $|X|$ with certain properties. Firstly, the $\mathcal{U}_2$-morphisms $\psi\colon \mathfrak{U}(T) \to |X|$ and $\phi\colon |X| \to \mathfrak{U}(T)$ for $\psi \in \mathfrak{I}_X(T)$ and $\phi \in \mathfrak{O}_X(T)$ must lift to $\mathcal{U}_1$-morphisms. Secondly, the $\mathcal{U}_2\mathcal{T}^{\mathrm{v}}$-morphisms $x_\lambda\colon X \to \mathfrak{G}^{\mathrm{v}}(X_\lambda)$ must lift to $\mathcal{U}_1\mathcal{T}^{\mathrm{v}}$-morphisms. However it is straightforward to see that if the first condition holds then the second is equivalent to lifting the $\mathcal{U}_2$-morphisms $|x_\lambda|\colon |X| \to \mathfrak{G}(|X_\lambda|)$ to $\mathcal{U}_1$-morphisms.

If we ignore the input test morphisms for the moment, then we have a family of $\mathcal{U}_2$-morphisms out of $|X|$. Since $\mathcal{U}_1$ is topological over $\mathcal{U}_2$ there is a unique initial lift of this family, say U. Now we see that this also lifts the input test morphisms since the composition of ψ with either ϕ or with $|x_\lambda|$ lifts to an $\mathcal{U}_1$-morphism. Hence $(U, \mathfrak{G}^{-1}\mathfrak{I}_X, \mathfrak{G}^{-1}\mathfrak{O}_X)$ is the desired lift. ∎

4.27. Corollary. *The functor $\mathfrak{G}^{\mathrm{v}}\colon \mathcal{U}_1\mathcal{T}^{\mathrm{v}} \to \mathcal{U}_2\mathcal{T}^{\mathrm{v}}$ has both a left and a right adjoint.* ∎

4.28. Definition. *Let us write $\mathfrak{Fin}$ for the left adjoint and $\mathfrak{Coa}$ for the right adjoint.*

The notation is to suggest "finest" and "coarsest".

Under favourable conditions these adjoints allow us to transfer forcing conditions neatly from $\mathcal{U}_1\mathcal{T}^{\mathrm{v}}$ to $\mathcal{U}_2\mathcal{T}^{\mathrm{v}}$. The idea is that a $\mathcal{U}_2$-morphism $u\colon \mathfrak{U}(T) \to |X|$ should be forced if it is forced for at least one of the preimages of X. But if it is forced for at least one, it will be forced for the maximal one. Thus we need only test $\mathfrak{Coa}(X)$. Similarly, for input test morphisms we need only look at $\mathfrak{Fin}(X)$. The caveat is, as always, that we need to check the "non-stupid" condition.

4.29. Proposition. *Suppose that $\mathfrak{Fin}$ and $\mathfrak{Coa}$ satisfy $\mathfrak{Fin}\mathfrak{S} = \mathfrak{S}$ and $\mathfrak{Coa}\mathfrak{S} = \mathfrak{S}$. Let $(\mathfrak{F}_{i1}, \mathfrak{F}_{o1})$ be a forcing condition on virtual $\mathcal{T}$–objects in $\mathcal{U}_1$. Define*

$$\mathfrak{F}_{i2}(T, X, \mathcal{F}) := \max\{\mathfrak{F}_{i1}(T, \mathfrak{Fin}(X), \mathcal{F}') : \mathcal{F}' \cap \mathrm{im}\mathfrak{Fin} = \mathfrak{Fin}(\mathcal{F})\},$$
$$\mathfrak{F}_{o2}(T, X, \mathcal{F}) := \max\{\mathfrak{F}_{o1}(\mathfrak{Coa}(X), T, \mathcal{F}') : \mathcal{F}' \cap \mathrm{im}\mathfrak{Coa} = \mathfrak{Coa}(\mathcal{F})\}.$$

Then $(\mathfrak{F}_{i2}, \mathfrak{F}_{o2})$ is a forcing condition.

Proof. Functorality is immediate. Let us check the "non-stupid" condition. Observe that the assumption on $\mathfrak{Fin}$ ensures that

$$\mathfrak{F}_{i2}(T, \mathfrak{S}(T'), \mathcal{F}) = \max\{\mathfrak{F}_{i1}(T, \mathfrak{S}(T'), \mathcal{F}') : \mathcal{F}' \cap \mathrm{im}\mathfrak{Fin} = \mathfrak{Fin}(\mathcal{F})\}.$$

Moreover, $(1,1) \in \mathrm{im}\mathfrak{Fin}$ so $(1,1) \in \mathcal{F}'$ if and only if $(1,1) \in \mathcal{F}$. Hence if $(1,1) \notin \mathcal{F}$, $\mathfrak{F}_{i2}(T, \mathfrak{S}(T'), \mathcal{F}) = 0$. The case of $\mathfrak{F}_{o2}$ is similar. ∎

4.30. Proposition. *$\mathfrak{Fin}$ and $\mathfrak{Coa}$ take forced virtual $\mathcal{T}$–objects in $\mathcal{U}_2$ to forced virtual $\mathcal{T}$–objects in $\mathcal{U}_1$.*

Proof. This follows from the argument in Lemma 4.24 since both $\mathfrak{Fin}$ and $\mathfrak{Coa}$ are full and faithful. ∎

5. Functors in the Wild

We can now define functors between the various categories of smooth spaces which we illustrate in Figure 2. To explain this diagram, we need to clarify the notation used. We define the following notation for the categories of test spaces (morphisms are always C^∞-maps):

- $\mathcal{T}\mathcal{C}h$: objects are convex subsets of Euclidean spaces.
- $\mathcal{T}\mathcal{S}o$: objects are open subsets of Euclidean spaces.
- $\mathcal{T}\mathcal{S}i$: objects are Euclidean spaces.
- $\mathcal{T}\mathcal{F}r$: single object, $\mathbb{R}$.

We define the following notation for the forcing conditions:

- $\emptyset$: no condition.
- Tml: the terminal condition.
- Shf: the sheaf condition.
- Dtm: the determined condition.
- SDtm: the specifically determined condition.
- Stn: the saturation condition.

We specify a forcing condition as (input, output) where both are lists of conditions which are to be "anded" together. The underlined categories correspond to the ones in Section 3.13, the others are intermediate ones put in to enable us to define the functors. We denote the category of topological spaces by Top and the functors between Top and Set by the obvious notations $\mathfrak{Ind}, \mathfrak{Dis} \colon \mathrm{Set} \to \mathrm{Top}$ and $|-| \colon \mathrm{Top} \to \mathrm{Set}$.

There are various adjunctions and isomorphisms that can be read off from Figure 2. Firstly we note that, due to Boman's theorem in [Boman (1967)], under the saturation condition then the extension functors always agree. Thus restriction and extension define isomorphisms of categories

$$\big(\mathrm{Set}, \mathcal{T}\mathcal{F}r, (\mathrm{Stn}, \mathrm{Stn})\big) \cong \big(\mathrm{Set}, \mathcal{T}\mathcal{S}o, (\mathrm{Stn}, \mathrm{Stn})\big)$$

and

$$\big(\mathrm{Top}, \mathcal{T}\mathcal{F}r, (\mathrm{Stn}, \mathrm{Stn})\big) \cong \big(\mathrm{Top}, \mathcal{T}\mathcal{S}i, (\mathrm{Stn}, \mathrm{Stn})\big).$$

Furthermore, since any open subset of Euclidean space is locally convex, the sheaf condition is sufficient to ensure that restriction and extension define an isomorphism of categories

$$\big(\mathrm{Set}, \mathcal{T}\mathcal{S}o \cap \mathcal{T}\mathcal{C}h, (\mathrm{Shf}\,\mathrm{Tml}, \mathrm{Stn})\big) \cong \big(\mathrm{Set}, \mathcal{T}\mathcal{S}o, (\mathrm{Shf}\,\mathrm{Tml}, \mathrm{Stn})\big).$$

$$
\begin{tikzcd}[column sep=huge]
\underline{\big(\mathrm{Set}, \mathcal{T}\mathcal{C}h, (\mathrm{Shf}\,\mathrm{Tml}, \mathrm{Stn})\big)} \arrow[d, shift left=4, "\mathfrak{Res}"] & \\
\big(\mathrm{Set}, \mathcal{T}\mathcal{S}o \cap \mathcal{T}\mathcal{C}h, (\mathrm{Shf}\,\mathrm{Tml}, \mathrm{Stn})\big) \arrow[u, "\mathfrak{Ext}^-"] \arrow[u, shift left=8, "\mathfrak{Ext}^+"'] \arrow[d, "\mathfrak{Ext}^-"'] \arrow[d, shift left=8, "\mathfrak{Ext}^+"] & \\
\underline{\big(\mathrm{Set}, \mathcal{T}\mathcal{S}o, (\mathrm{Shf}\,\mathrm{Tml}, \mathrm{Stn})\big)} \arrow[u, shift left=4, "\mathfrak{Res}"] \arrow[d, "\mathfrak{For}^-"'] \arrow[d, shift left=8, "\mathfrak{For}^+"] & \underline{\big(\mathrm{Top}, \mathcal{T}\mathcal{S}i, (\mathrm{Stn}, \mathrm{Shf}\,\mathrm{SDtm}\,\mathrm{Tml})\big)} \arrow[d, "\mathfrak{For}^-"'] \arrow[d, shift left=8, "\mathfrak{For}^+"] \\
\big(\mathrm{Set}, \mathcal{T}\mathcal{S}o, (\mathrm{Stn}, \mathrm{Stn})\big) \arrow[u, shift left=4, "\mathfrak{Incl}"] \arrow[d, shift left=4, "\mathfrak{Res}"] & \big(\mathrm{Top}, \mathcal{T}\mathcal{S}i, (\mathrm{Stn}, \mathrm{Stn})\big) \arrow[u, shift left=4, "\mathfrak{Incl}"] \arrow[d, shift left=4, "\mathfrak{Res}"] \\
\underline{\big(\mathrm{Set}, \mathcal{T}\mathcal{F}r, (\mathrm{Stn}, \mathrm{Stn})\big)} \arrow[u, "\mathfrak{Ext}^-"] \arrow[u, shift left=8, "\mathfrak{Ext}^+"'] \arrow[r, shift left, "\mathfrak{Fin}"] \arrow[r, shift right, "\mathfrak{Coa}"'] \arrow[d, shift left=4, "\mathfrak{Incl}"] & \underline{\big(\mathrm{Top}, \mathcal{T}\mathcal{F}r, (\mathrm{Stn}, \mathrm{Stn})\big)} \arrow[u, "\mathfrak{Ext}^-"] \arrow[u, shift left=8, "\mathfrak{Ext}^+"'] \arrow[d, shift left=4, "\mathfrak{Incl}"] \\
\big(\mathrm{Set}, \mathcal{T}\mathcal{F}r, (\emptyset, \emptyset)\big) \arrow[u, "\mathfrak{For}^-"] \arrow[u, shift left=8, "\mathfrak{For}^+"'] \arrow[r, shift left=3, "\mathfrak{Ind}^{\mathrm{v}}"] \arrow[r, shift right=3, "\mathfrak{Dis}^{\mathrm{v}}"'] & \big(\mathrm{Top}, \mathcal{T}\mathcal{F}r, (\emptyset, \emptyset)\big) \arrow[u, "\mathfrak{For}^-"] \arrow[u, shift left=8, "\mathfrak{For}^+"'] \arrow[l, "|-|^{\mathrm{v}}" description]
\end{tikzcd}
$$

Figure 2: The functors between the categories of smooth spaces.

The extension functors are always adjoint to the restriction functors and thus the functor

$$\big(\mathrm{Set}, \mathcal{T}\mathcal{C}h, (\mathrm{Shf}\,\mathrm{Tml}, \mathrm{Stn})\big) \to \big(\mathrm{Set}, \mathcal{T}\mathcal{S}o, (\mathrm{Shf}\,\mathrm{Tml}, \mathrm{Stn})\big)$$

has both a left and right adjoint.

Saturation is sufficient to ensure that Proposition 4.9 holds, at least for one direction, and thus the functors (effectively given by inclusion)

$$\big(\mathrm{Set}, \mathcal{T}\mathcal{F}r, (\mathrm{Stn}, \mathrm{Stn})\big) \to \big(\mathrm{Set}, \mathcal{T}\mathcal{S}o, (\mathrm{Shf}\,\mathrm{Tml}, \mathrm{Stn})\big)$$
$$\big(\mathrm{Top}, \mathcal{T}\mathcal{F}r, (\mathrm{Stn}, \mathrm{Stn})\big) \to \big(\mathrm{Top}, \mathcal{T}\mathcal{S}i, (\mathrm{Stn}, \mathrm{Shf}\,\mathrm{SDtm}\,\mathrm{Tml})\big)$$

have adjoints. In the first case, the adjoint is a left adjoint given by $\mathfrak{For}^+$. In the second case, the adjoint is a right adjoint given by $\mathfrak{For}^-$.

As the functors are constructed in a fairly abstract manner, let us now give brief descriptions of how they actually work. We shall use the usual descriptions of the spaces which, usually, means only considering one family of test morphisms.

Chen and Souriau. Let $(X, \mathcal{P})$ be a Chen space. For an open set U of some Euclidean space, let $(U, \mathcal{P}_U)$ denote its natural structure as a Chen space; i.e. a map $C \to U$ is a plot if and only if it is a C^∞-map. Let $\mathcal{D}$ be the family of maps $U \to X$ which underlie morphisms of Chen spaces $(U, \mathcal{P}_U) \to (X, \mathcal{P})$. Then $(X, \mathcal{D})$ is a Souriau space.

Conversely, let $(X, \mathcal{D})$ be a Souriau space. For a convex set C of some Euclidean space, let $(C, \mathcal{D}_C)$ denote its natural structure as a Souriau space; i.e. a map $U \to C$ is a plot if and only if it is a C^∞-map. Let $\mathcal{P}^+$ be the family of maps $C \to X$ which underlie morphisms of Souriau spaces $(C, \mathcal{D}_C) \to (X, \mathcal{D})$. Then $(X, \mathcal{P}^+)$ is a Chen space.

Define a second family $\mathcal{P}^-$ on X by taking all the maps $C \to X$ which locally factor through the maps in $\mathcal{D}$; as above, C runs over the family of convex subsets of Euclidean spaces. Then $(X, \mathcal{P}^-)$ is a Chen space.

Souriau and Frölicher. Let $(X, \mathcal{D})$ be a Souriau space. Let $\mathcal{F}$ be the set of morphisms of Souriau spaces from $(X, \mathcal{D})$ to $\mathbb{R}$, viewed as having its standard diffeology. Let $\mathcal{C}$ be the family of curves determined by $\mathcal{F}$. That is to say, $\mathcal{C}$ is the family of curves $c \colon \mathbb{R} \to X$ with the property that $fc \in C^\infty(\mathbb{R}, \mathbb{R})$ for all $f \in \mathcal{F}$. Then $(X, \mathcal{C}, \mathcal{F})$ is a Frölicher space.

Let $(X, \mathcal{C}, \mathcal{F})$ be a Frölicher space. Define a family $\mathcal{D}$ on X by taking all maps $U \to X$ which underlie maps of Frölicher spaces with U an open subset of a Euclidean space equipped with its obvious Frölicher space structure. Then $(X, \mathcal{D})$ is a Souriau space.

Smith and Frölicher. The case of Smith spaces and Frölicher spaces is an interesting one because the definitions are so similar. One way to phrase the difference is to say that in a Smith space the smooth functions are a subset of the continuous functions whilst in a Frölicher space the continuous functions are a superset of the smooth functions.

Let $(X, \mathcal{T}, \mathcal{F})$ be a Smith space. Let $\mathcal{C}_X$ be the set of maps $c \colon \mathbb{R} \to X$ for which $fc \in C^\infty(\mathbb{R}, \mathbb{R})$ for all $f \in \mathcal{F}$. Let $\mathcal{F}_X$ be the set of maps $f \colon X \to \mathbb{R}$ for which $fc \in C^\infty(\mathbb{R}, \mathbb{R})$ for all $c \in \mathcal{C}_X$. Then $(X, \mathcal{C}, \mathcal{F})$ is a Frölicher space.

For the reverse direction, let $(X, \mathcal{C}, \mathcal{F})$ be a Frölicher space. To make this a Smith space we need to define a topology on X. There are two obvious choices: the *curvaceous* topology on X and the *functional* topology. The first is the finest topology for which all the smooth curves are continuous. The second is the coarsest topology for which all the smooth functionals are continuous. Let us denote them by $\mathcal{T}_c$ and $\mathcal{T}_f$ respectively. It is immediate that $\mathcal{T}_c$ is finer than $\mathcal{T}_f$. Then $(X, \mathcal{T}_c, \mathcal{F})$ and $(X, \mathcal{T}_f, \mathcal{F})$ are Smith spaces.

Sikorski and Smith. Let $(X, \mathcal{T}, \mathcal{F})$ be a Sikorski space. For $U \subseteq \mathbb{R}^n$ an open subset, let $\mathcal{F}(U)$ be the family of continuous maps $\phi \colon U \to X$ for which $f\phi \in C^\infty(U, \mathbb{R})$ for all $f \in \mathcal{F}$. Let $\overline{\mathcal{F}}$ be the family of continuous functions $f \colon X \to \mathbb{R}$ for which $f\phi \in C^\infty(U, \mathbb{R})$ for all $\phi \in \mathcal{F}(U)$ and for all open $U \subseteq \mathbb{R}^n$. Then $(X, \mathcal{T}, \overline{\mathcal{F}})$ is a Smith space.

Conversely, if $(X, \mathcal{T}, \mathcal{F})$ is a Smith space then it is automatically a Sikorski space.

6. The Differences

We saw in the previous section that the category of Frölicher spaces is (isomorphic to) a full subcategory of that of Souriau spaces, likewise the category of Souriau spaces in that of Chen spaces, Frölicher spaces in Smith spaces, and Smith spaces in Sikorski spaces. An obvious question is whether any of these embeddings is dense.

As an aid to answering that, note that it is easy to characterise Souriau spaces and Chen spaces according to whether or not they have the same underlying Frölicher space: simply examine the set of morphisms to $\mathbb{R}$. In both categories, we regard $\mathbb{R}$ as having its "standard" structure, namely the smallest which contains the identity map.

For Sikorski spaces and Smith spaces, the characterisation is slightly more problematical due to the topology. One cannot simply examine the set of morphisms from $\mathbb{R}$ with its standard Sikorski or Smith structure as these might not be all the smooth curves; the problem being that a smooth curve need not be continuous. Thus one has to simply compute the family of smooth curves irrespective of the topology and this cannot be described in terms of morphism sets. We shall comment more on topology later.

Let us now construct non-isomorphic objects in each of the categories which have the same underlying structure in the "next category down". The most interesting of these examples is probably the one showing that Chen spaces and Souriau spaces are not the same.

We shall use the following notation.

- $\mathcal{F}r$ for the category of Frölicher spaces.
- $\mathcal{C}h$ for the category of Chen spaces.
- $\mathcal{S}o$ for the category of Souriau spaces.
- $\mathcal{S}i$ for the category of Sikorski spaces.
- $\mathcal{S}m$ for the category of Smith spaces.

We shall label the functors between them by the *target* category. For all but two pairs there is a single sensible functor in each direction. There are two functors from the category of Souriau spaces to the category of Chen spaces and from the category of Frölicher spaces to the category of Smith spaces. The latter differ only in a very minor way and can be treated together so we shall not specify which is which. Thus we consider the following list of functors.

- $\mathfrak{So} \colon \mathcal{F}r \to \mathcal{S}o$
- $\mathfrak{Fr} \colon \mathcal{S}o \to \mathcal{F}r$
- $\mathfrak{Ch}^\sharp \colon \mathcal{S}o \to \mathcal{C}h$ (corresponding to $\mathfrak{Ext}^+$)
- $\mathfrak{Ch}^\flat \colon \mathcal{S}o \to \mathcal{C}h$ (corresponding to $\mathfrak{Ext}^-$)
- $\mathfrak{So} \colon \mathcal{C}h \to \mathcal{S}o$
- $\mathfrak{Sm} \colon \mathcal{F}r \to \mathcal{S}m$
- $\mathfrak{Fr} \colon \mathcal{S}m \to \mathcal{F}r$
- $\mathfrak{Si} \colon \mathcal{S}m \to \mathcal{S}i$
- $\mathfrak{Sm} \colon \mathcal{S}i \to \mathcal{S}m$

Smith spaces. For Smith spaces, consider $\mathbb{R}$ with its usual topology and the set of all continuous functions versus $\mathbb{R}$ with the discrete topology and the set of all continuous functions (i.e. continuous for the discrete topology, whence all functions). The smooth curves for the latter are the constant curves. Let us show that the same holds for the former. Firstly, observe that if $c \colon \mathbb{R} \to \mathbb{R}$ is a map such that $fc \in C^\infty(\mathbb{R}, \mathbb{R})$ for all continuous functions $f : \mathbb{R} \to \mathbb{R}$ then c is a C^∞-map as the identity is continuous. Suppose that $c \colon \mathbb{R} \to \mathbb{R}$ is a C^∞-map which is not constant. Then $c'(t) \neq 0$ for some $t \in \mathbb{R}$ and so there is an interval around t on which the restriction of c is a diffeomorphism. Thus if $f : \mathbb{R} \to \mathbb{R}$ is a map for which $fc \in C^\infty(\mathbb{R}, \mathbb{R})$ then f must be a C^∞-map in a neighborhood of $c(t)$. As there is a continuous map which is not a C^∞-map in this neighborhood, c cannot be a smooth curve in the Frölicher structure underlying this Smith space. Thus all the smooth curves in this space are constant. Hence these two different Smith spaces define the same underlying Frölicher space.

Sikorski spaces. In the above, it sufficed that there was a functional that was non-smooth at each point in $\mathbb{R}$. From this we see that if $\mathcal{F}$ is a family of functions $\mathbb{R} \to \mathbb{R}$ which contains the identity, is translation-invariant, and contains a non-smooth function, then the only plots $\phi \colon U \to \mathbb{R}$ are the constant ones. The corresponding Smith space of such a family thus has family of functions all the continuous functions. One can find a family of functions $\mathcal{F}$ satisfying these conditions and the conditions for a Sikorski structure on $\mathbb{R}$ which is not the set of continuous functions. For example, one of the families of piecewise-smooth functions will do (there are various possibilities, any will work here). Thus the inclusion of Smith spaces in Sikorski spaces is not dense.

Souriau spaces. As Souriau spaces, $\mathbb{R}^2$ with its standard diffeology and with its "wire" diffeology (generated by the smooth curves) are not isomorphic. Nonetheless, by Boman's result, [Boman (1967)], they have the same smooth functionals and thus the same underlying Frölicher structure.

Chen spaces. Let us now compare Chen spaces and Souriau spaces. We can show that they are different by exhibiting a Souriau space whose images under the two extension functors are different.

This Souriau space is $[0, 1]$ with its usual diffeology. The Chen space $\mathfrak{Ch}^\sharp([0, 1])$ is easily seen to be the standard Chen space structure on $[0, 1]$. In particular, it contains the identity on $[0, 1]$. On the other hand, every plot in $\mathfrak{Ch}^\flat([0, 1])$ factors through a C^∞-map $U \to [0, 1]$ from an open subset of some Euclidean space into $[0, 1]$. The identity on $[0, 1]$ does not have this property: any factorisation of the inclusion $[0, 1] \to \mathbb{R}$ via some open set U must extend outside $[0, 1]$. Hence $\mathfrak{Ch}^\sharp([0, 1]) \neq \mathfrak{Ch}^\flat([0, 1])$.

One can extend this example to see that the main difference between Chen spaces and Souriau spaces is the ability to "approach boundary points at speed". In the Chen realm, one can approach a point at speed and stop. In the Souriau realm, one must always be able to go a little further. Now suppose that one wishes to declare that a certain point cannot be approached from certain directions. One therefore wishes to consider one-sided derivatives at those points. In the Chen realm, this presents no difficulties: one approaches at a steady speed along those directions that one is allowed to approach along. In the Souriau realm, this is more problematical. One has to approach along an allowed direction and then "bounce back", so ones speed has to approach zero as one nears the point of interest.

However, this difference is illusory. In [Kriegl and Michor (1997), 24.5], Kriegl and Michor show that for convex sets the two ways of examining smoothness on the boundary are equivalent. That is, the partial derivatives of a map from a convex set exist (and are continuous) on the boundary if and only if it takes smooth curves to smooth curves. Thus one can define one-sided derivatives in the Souriau realm using functions that are required to "bounce back".

7. More on Adjunctions

In Section 5 we had several adjunctions. In this section we shall consider whether any other adjunctions are possible. Our conclusion will be that there are none. One thing worth highlighting is that the examples in this section are not overly complicated. All have very simple underlying sets, often $\mathbb{R}$, and reasonably simple structure.

Frölicher and Smith. One surprise is that there is not an adjunction pairing between the categories of Frölicher spaces and Smith spaces. One would expect an adjunction $\mathfrak{Fr} \dashv \mathfrak{Sm}$ since if $(X, \mathcal{T}, \mathcal{F})$ is a Smith space and $(X, \mathcal{T}', \mathcal{F}') = \mathfrak{Fr}\mathfrak{Sm}(X, \mathcal{T}, \mathcal{F})$ then $\mathcal{F} \subseteq \mathcal{F}'$ which suggests that the identity on X should lift to a morphism $(X, \mathcal{T}', \mathcal{F}') \to (X, \mathcal{T}, \mathcal{F})$. The problem is that this need not be continuous. One might suppose that one could fix this by altering the topology used in defining the Smith space from a Frölicher space. For a Frölicher space $(X, \mathcal{C}, \mathcal{F})$ one would need to be able to choose a topology $\mathcal{T}$ on X such that $(X, \mathcal{T}, \mathcal{F})$ was a Smith space and if $(X, \mathcal{T}', \mathcal{F}')$ was another Smith space structure on X with $\mathcal{F}' \subseteq \mathcal{F}$ then the identity on X is continuous as a morphism $(X, \mathcal{T}) \to (X, \mathcal{T}')$.

Let us show by example that this is not possible. Our example will also show that $\mathfrak{Fr}$ cannot be a right adjoint as it does not preserve limits. Note that for any topological space $(X, \mathcal{T})$ the triple $(X, \mathcal{T}, \mathcal{F})$ is a Smith space where $\mathcal{F}$ is the family of all continuous functionals from X to $\mathbb{R}$.

Let $\mathcal{T}_+$ be the topology on $\mathbb{R}$ with basis $\{[a, \infty), a \in \mathbb{R}\}$ and $\mathcal{T}_-$ with basis $\{(-\infty, a], a \in \mathbb{R}\}$. Let $\mathcal{T}_{\mathbb{R}}$ be the standard topology on $\mathbb{R}$. For either of these non-standard topologies, a continuous map $f \colon (\mathbb{R}, \mathcal{T}_\pm) \to (\mathbb{R}, \mathcal{T}_{\mathbb{R}})$ is constant. Let $\mathcal{F}_c$ denote the family of constant functionals on $\mathbb{R}$. As remarked above, each of $(\mathbb{R}, \mathcal{T}_\pm, \mathcal{F}_c)$ is a Smith space. The identity on $\mathbb{R}$ underlies morphisms of Smith spaces $(\mathbb{R}, \mathcal{T}_\pm, \mathcal{F}_c) \to (\mathbb{R}, \mathcal{I}_{\mathbb{R}}, \mathcal{F}_c)$ where $\mathcal{I}_{\mathbb{R}}$ is the indiscrete topology on $\mathbb{R}$. Let us consider the corresponding pullback. It is clear that the underlying set of the pullback is $\mathbb{R}$ and that the underlying set morphisms are the identity. For these to be continuous, the topology on the pullback must be the discrete topology. The family of functionals is then forced, by either the locality condition or the saturation condition, to be *all* functionals. That is to say, the pullback is $(\mathbb{R}, \mathcal{D}_{\mathbb{R}}, \operatorname{Set}(\mathbb{R}, \mathbb{R}))$.

The corresponding Frölicher spaces of $(\mathbb{R}, \mathcal{T}_\pm, \mathcal{F}_c)$ and $(\mathbb{R}, \mathcal{I}, \mathcal{F}_c)$ are both $(\mathbb{R}, \operatorname{Set}(\mathbb{R}, \mathbb{R}), \mathcal{F}_c)$. The pullback in $\mathcal{Fr}$ is thus $(\mathbb{R}, \operatorname{Set}(\mathbb{R}, \mathbb{R}), \mathcal{F}_c)$. However the Frölicher space of

$(\mathbb{R}, \mathcal{D}_{\mathbb{R}}, \mathrm{Set}(\mathbb{R},\mathbb{R}))$ is $(\mathbb{R}, \mathcal{C}_c, \mathrm{Set}(\mathbb{R},\mathbb{R}))$ where $\mathcal{C}_c$ is the set of constant curves. Hence $\mathfrak{Fr}$ cannot be a right adjoint.

To see that this shows that we cannot choose the topology on a Frölicher space in a manner suitable for Smith spaces, note that by this example the topology on $(X, \mathrm{Set}(\mathbb{R},\mathbb{R}), \mathcal{F}_c)$ would have to be finer than both $\mathcal{T}_+$ and $\mathcal{T}_-$. The only such topology is the discrete topology, but by the completion property of Smith spaces the resulting functionals would be *all* functionals.

Let us now show that $\mathfrak{Sm}$ cannot be a left adjoint by examining its behaviour on colimits. Equip $\mathbb{R}$ and $[0,1]$ with their standard structure as Smith spaces and as Frölicher spaces (note that these correspond under $\mathfrak{Sm}$). Define two morphisms $a, b \colon \mathbb{R} \to [0,1]$ by $a(t) = 1$ and $b(t) = 1/(1+t^2)$ (the exact form of b does not matter, just its image). Consider the coequalisers in both categories.

The underlying set of the coequalisers is $\{0,1\}$ in each case. For each, a functional $\{0,1\} \to \mathbb{R}$ is smooth if and only if it pulls back to a smooth functional on $[0,1]$. As a smooth functional on $[0,1]$ is continuous, we see that the smooth functionals on $\{0,1\}$ are the constant functionals. The Frölicher space structure is thus $(\{0,1\}, \mathrm{Set}(\mathbb{R}, \{0,1\}), \mathcal{F}_c)$. For the Smith space structure, we must also work out the topology. As constant functions are always continuous, there are no constraints on the topology from those whence we see that the topology must be the quotient topology. This is $\{\emptyset, \{1\}, \{0,1\}\}$. But the topology on $\mathfrak{Sm}(\{0,1\}, \mathrm{Set}(\mathbb{R}, \{0,1\}), \mathcal{F}_c)$ is the indiscrete topology no matter which method of choosing the topology is used.

We can modify the above example to show that $\mathfrak{Fr}$ also cannot be a left adjoint. In the above, change the Smith structure on $[0,1]$ to be that with all continuous functions. The corresponding Frölicher space is $([0,1], \mathcal{C}_c, \mathrm{Set}([0,1],\mathbb{R}))$. The rest of the structure stays the same as above. Let us calculate the coequalisers in both categories. As before, both have underlying sets $\{0,1\}$. The functionals on each are those functions $\{0,1\} \to \mathbb{R}$ which pull back to "smooth" functionals on $[0,1]$. The Smith structure thus produces all continuous functionals $\{0,1\} \to \mathbb{R}$. As the topology is the same as before, namely $\{\emptyset, \{1\}, \{0,1\}\}$, this is simply the family of constant functionals. The coequaliser in the category of Smith spaces is thus $(\{0,1\}, \mathcal{T}, \mathcal{F}_c)$ where $\mathcal{T}$ is the above topology. The underlying Frölicher space of this is $(\{0,1\}, \mathrm{Set}(\mathbb{R}, \{0,1\}), \mathcal{F}_c)$. On the other hand, the coequaliser in the category of Frölicher spaces is clearly $(\{0,1\}, \mathcal{C}_c, \mathrm{Set}(\{0,1\},\mathbb{R}))$. Hence $\mathfrak{Fr}$ does not preserve colimits.

Finally, for these functors, let us show that $\mathfrak{Sm}$ cannot be a right adjoint by considering its action on limits. For $t \in \mathbb{R}$ define a Frölicher space $(\mathbb{R}, \mathcal{C}_t, \mathcal{F}_t)$ by taking $\mathbb{R}$ and putting a kink in at t. That is, the functionals are those maps $\mathbb{R} \to \mathbb{R}$ which are continuous, are smooth on $\mathbb{R} \smallsetminus \{t\}$, and all left and right derivatives exist at t (but are not necessarily equal). The corresponding curves are characterised by the property that they can only pass through t *infinitely slowly*. As an example, $(\mathbb{R}, \mathcal{C}_0, \mathcal{F}_0)$ can be viewed as the union of the positive x– and y–axes in $\mathbb{R}^2$. There is an obvious morphism from each of these spaces to $\mathbb{R}$ with its usual Frölicher structure. Let us consider the limit of this family.

The underlying set is again $\mathbb{R}$ and the curves in the Frölicher space structure are those maps $c\colon \mathbb{R} \to \mathbb{R}$ which are in each of the $\mathcal{C}_t$. Such a curve must be smooth and everywhere infinitely slow, whence constant. Hence the limiting Frölicher space is $(\mathbb{R}, \mathcal{C}_c, \operatorname{Set}(\mathbb{R}, \mathbb{R}))$.

Now let us transport this to the category of Smith spaces via $\mathfrak{Sm}$. The underlying set of the limit is again $\mathbb{R}$. The topology is the standard topology, which immediately implies that $\mathfrak{Sm}$ does not preserve limits, but let us examine the functionals as well to see that even if one could fix the topology then $\mathfrak{Sm}$ would not preserve limits. As we have seen before, the plots of the corresponding Smith space are constant whence the functionals are all *continuous* functionals. As the topology is the standard one, we see that the limiting Smith space is $(\mathbb{R}, \mathcal{T}_{\mathbb{R}}, C(\mathbb{R}, \mathbb{R}))$. This is not the result of applying $\mathfrak{Sm}$ to $(\mathbb{R}, \mathcal{C}_c, \operatorname{Set}(\mathbb{R}, \mathbb{R}))$.

Smith and Sikorski. From Section 5 we know that $\mathfrak{Si} \dashv \mathfrak{Sm}$. Let us show that $\mathfrak{Si}$ is not a right adjoint and $\mathfrak{Sm}$ is not a left adjoint.

To show that $\mathfrak{Sm}$ is not a left adjoint, let us consider its behaviour on colimits. For $k \in \mathbb{N}$, define a family of functionals $\mathcal{F}_k$ to be the set of functions $f\colon \mathbb{R} \to \mathbb{R}$ which are continuous, are smooth on $\mathbb{R} \setminus \{0\}$, all left and right derivatives exist at 0, and the left and right derivatives agree up to (at least) order k. It is straightforward to show that each of these families satisfies the axioms for a Sikorski space (with the standard topology on $\mathbb{R}$). The colimit of this family obviously has functionals the set of smooth functions $f\colon \mathbb{R} \to \mathbb{R}$.

Let us consider the underlying Smith space of $(\mathbb{R}, \mathcal{T}_{\mathbb{R}}, \mathcal{F}_k)$ ($\mathcal{T}_{\mathbb{R}}$ being the standard topology on $\mathbb{R}$). Let $c\colon \mathbb{R} \to \mathbb{R}$ be a curve such that $fc \in C^\infty(\mathbb{R}, \mathbb{R})$ for all $f \in \mathcal{F}_k$. It is clear that c is smooth since the identity map is in $\mathcal{F}_k$. Let us suppose, for a contradiction, that there is some $t \in \mathbb{R}$ such that $c(t) = 0$, that c passes through 0 at t, and that c is not flat at t. Let us assume for simplicity that $t = 0$, the general case follows by precomposition with a translation. These conditions imply that the first non-vanishing derivative of c at 0 is odd. Thus there is some odd l and $C \neq 0$ such that $c(s) = Cs^l + O(s^{l+1})$. Let $f \in \mathcal{F}_k$ be the function $f(x) = |x| x^k$ and consider fc. Expanding out, we see that

$$\begin{aligned}
(fc)(s) &= |Cs^l + O(s^{l+1})| (Cs^l + O(s^{l+1}))^k \\
&= |C| |s|^l |1 + O(s)| C^k s^{lk} (1 + O(s)) \\
&= |C| C^k |s| s^{l(k+1)-1} (1 + O(s)) && \text{as } l \text{ is odd.}
\end{aligned}$$

From this we deduce that the $l(k+1)$th derivative of fc does not exist at 0. Hence if $c\colon \mathbb{R} \to \mathbb{R}$ is such that $fc \in C^\infty(\mathbb{R}, \mathbb{R})$ for all $f \in \mathcal{F}_k$ then c can only pass through 0 when flat. This obviously generalises to more general plots.

The key point is that this condition is independent of k. Thus the underlying Smith space of $(\mathbb{R}, \mathcal{T}_{\mathbb{R}}, \mathcal{F}_k)$ is independent of k. It is clearly also not the standard Smith structure on $\mathbb{R}$, in fact it is the Smith structure on $\mathbb{R}$ with a kink at 0. Hence the functor $\mathfrak{Sm}\colon \mathcal{S}i \to \mathcal{S}m$ does not take colimits to colimits and so cannot be a left adjoint.

Now let us consider the action of $\mathfrak{Si}\colon \mathcal{S}m \to \mathcal{S}i$ on limits. For $t \in \mathbb{R}$ define a Smith space $(\mathbb{R}, \mathcal{T}_{\mathbb{R}}, \mathcal{F}_t)$ by putting a kink in $\mathbb{R}$ at t. There is an obvious morphism from each of these spaces to $\mathbb{R}$ with its standard Smith space structure. We consider the limit of this. The underlying set is again $\mathbb{R}$ and the underlying topology is the standard one. As we have said before, the plots into the Smith space structure are constant whence the functionals are all continuous functionals. Thus the limiting Smith space is $(\mathbb{R}, \mathcal{T}_{\mathbb{R}}, C(\mathbb{R}, \mathbb{R}))$.

Now let us transport this to $\mathcal{S}i$ via $\mathfrak{Si}$. The underlying set of the limit is again $\mathbb{R}$. The functionals in the Sikorski space structure on the limit must be the smallest family which contains all the $\mathcal{F}_t$. This is the family of piecewise-smooth maps, where we interpret "piecewise-smooth" to mean smooth except at a finite number of points and at those points all left and right derivatives exist. This is not $\mathfrak{Si}(\mathbb{R}, \mathcal{T}_{\mathbb{R}}, C(\mathbb{R}, \mathbb{R}))$ and hence $\mathfrak{Si}$ does not preserve limits.

Frölicher and Souriau. Let us consider $\mathfrak{Fr}\colon \mathcal{S}o \to \mathcal{F}r$. This is a left adjoint and so preserves colimits. Let us consider its action on limits.

Let $\gamma\colon \mathbb{R} \to \mathbb{R}^2$ be a smooth curve. Define a diffeology on $\mathbb{R}^2$, $\mathcal{D}_\gamma$, by starting with the family of smooth curves $\beta\colon U \to \mathbb{R}^2$ for which $\beta^{-1}(\overline{\gamma(\mathbb{R})})$ has void interior, then take the minimal diffeology containing such curves. The condition ensures that the resulting diffeology does not contain γ.

The family $\{\mathcal{D}_\gamma : \gamma \in C^\infty(\mathbb{R}, \mathbb{R}^2)\}$ defines a family of Souriau spaces. This has a limit which is formed by intersecting the diffeologies. As $\gamma \notin \mathcal{D}_\gamma$, we obtain the discrete diffeology containing only the constant maps. The underlying Frölicher structure of this is the corresponding discrete Frölicher structure.

Let us consider the Frölicher structure of $(\mathbb{R}^2, \mathcal{D}_\gamma)$. Clearly, a smooth map $f\colon \mathbb{R}^2 \to \mathbb{R}$ defines a morphism of Souriau spaces $(\mathbb{R}^2, \mathcal{D}_\gamma) \to (\mathbb{R}, \mathcal{D}_{\mathbb{R}})$. The converse also holds as can be proved by examining the crucial step in the proof of Boman's result in [Kriegl and Michor (1997), 3.4]. This step is the one labelled "(4) $\implies$ (3)". We need to show that a morphism $f\colon (\mathbb{R}^2, \mathcal{D}_\gamma) \to (\mathbb{R}, \mathcal{D}_{\mathbb{R}})$ is a C^∞-map. Clearly, it is smooth away from the image of γ. As the closure of $\operatorname{im}(\gamma)$ has void interior (since γ is smooth), we can choose the smooth curve, c, from [Kriegl and Michor (1997), 3.4] with the crucial property that $c^{-1}(\overline{\gamma(\mathbb{R})})$ has void interior. Thus f is a C^∞-map. Hence the underlying Frölicher structure of $(\mathbb{R}^2, \mathcal{D}_\gamma)$ is the standard Frölicher structure on $\mathbb{R}^2$. Thus the limit of the family $\{\mathfrak{Fr}(\mathbb{R}^2, \mathcal{D}_\gamma)\}$ is again the standard Frölicher structure on $\mathbb{R}^2$.

Hence $\mathfrak{Fr}$ does not preserve limits and so cannot be a right adjoint.

The functor $\mathfrak{So}\colon \mathcal{F}r \to \mathcal{S}o$ is a right adjoint and so preserves limits. Let us consider its action on colimits. With all the spaces having their natural Frölicher structures, consider the coequaliser of the maps $x, o\colon \mathbb{R} \to \mathbb{R}^2$ given by $x(t) = (t, 0)$ and $o(t) = (0, 0)$. The underlying set of this coequaliser is $\mathbb{R}^2$ "pinched" along the x-axis, let us call this X and write $q\colon \mathbb{R}^2 \to X$ for the projection. The Frölicher structure is given by taking those functionals $f\colon X \to \mathbb{R}$ which pull-back to C^∞-maps on $\mathbb{R}^2$. Let $(X, \mathcal{C}, \mathcal{F})$ be this Frölicher structure.

Applying $\mathfrak{So}$ to this diagram of Frölicher spaces results in the standard diffeologies on $\mathbb{R}$ and $\mathbb{R}^2$. The underlying set of the coequaliser is again $\mathbb{R}^2$ "pinched" along the x-axis, i.e. X, and the diffeology consists of those plots which factor through the projection

$\mathbb{R}^2 \to X$. Let us write this as $(X, \mathcal{D})$. Let us compare $\mathcal{D}$ with the diffeology coming from applying $\mathfrak{So}$ to $(X, \mathcal{C}, \mathcal{F})$. By construction, $\mathcal{C}$ is in the diffeology of $\mathfrak{So}(X, \mathcal{C}, \mathcal{F})$.

Let $c_\flat \colon \mathbb{R} \to \mathbb{R}$ be a strictly increasing curve which is smooth, maps 0 to 0, and is flat at 0. Let $c_\natural \colon \mathbb{R} \to \mathbb{R}^2$ be the curve

$$c_\natural(t) = \begin{cases} (-1, c_\flat(t)) & t < 0, \\ (0,0) & t = 0, \\ (1, c_\flat(t)) & t > 0. \end{cases}$$

Let $c_\sharp = q c_\natural \colon \mathbb{R} \to X$. Clearly, any two lifts of $c_\sharp$ through q can only differ in their value at 0 since q is a bijection away from the x-axis. Thus there cannot be a continuous lift of $c_\sharp$ and so it cannot be in $\mathcal{D}$.

Let us show that $c_\sharp$ is in $\mathcal{C}$. Let $f \colon X \to \mathbb{R}$ be in $\mathcal{F}$. Then $f c_\sharp = f q c_\natural$ and $f q \colon \mathbb{R}^2 \to \mathbb{R}$ is a C^∞-map. Clearly, $f q c_\natural$ is smooth away from 0. Moreover, we see that all left and right limits of $f q c_\natural$ and its derivatives exist at 0. For $f q c_\natural$ itself, these limits are the same as they are $f q(-1, 0)$ and $f q(1, 0)$. For the derivatives, $f q$ is a C^∞-map whence, as $c_\flat$ is flat at 0, by the chain rule all the left and right limits of the derivatives at 0 are 0. Hence $f q c_\natural$ is a C^∞-map and so $c_\sharp$ is in $\mathcal{C}$, whence in the diffeology of $\mathfrak{So}(X, \mathcal{C}, \mathcal{F})$.

Hence $\mathfrak{So}$ does not take colimits to colimits and so cannot be a left adjoint.

Souriau and Chen. The functor $\mathfrak{So} \colon \mathcal{C}h \to \mathcal{S}o$ has both a left and a right adjoint. Let us show that neither of its adjoints has further adjoints.

Consider first $\mathfrak{Ch}^\sharp \colon \mathcal{S}o \to \mathcal{C}h$. This is a right adjoint and so preserves limits. Let us consider its action on colimits.

Let $(X, \mathcal{P})$ be a Chen space. From the axioms of a Chen structure, every plot $\phi \colon C \to X$ in $\mathcal{P}$ underlies a morphism of Chen spaces $\phi \colon (C, \mathcal{P}_C) \to (X, \mathcal{P})$. Since the identity lies in $\mathcal{P}_C$ we see that

$$(X, \mathcal{P}) = \varinjlim_{\mathcal{P}} (C, \mathcal{P}_C).$$

Moreover, $(C, \mathcal{P}_C) = \mathfrak{Ch}^\sharp(C, \mathcal{D}_C)$ where $\mathcal{D}_C$ is the standard diffeology on C. Hence every Chen space is the colimit of things in the image of $\mathfrak{Ch}^\sharp$. As $\mathfrak{Ch}^\sharp$ is not dense and $\mathcal{S}o$ is cocomplete, it cannot preserve colimits.

Now let us consider $\mathfrak{Ch}^\flat \colon \mathcal{S}o \to \mathcal{C}h$. This is a left adjoint and so preserves colimits. Let us consider its action on limits.

First we observe that if U is an open subset of some Euclidean space equipped with its standard diffeology then $\mathfrak{Ch}^\flat(U, \mathcal{D}_U) = \mathfrak{Ch}^\sharp(U, \mathcal{D}_U)$. Secondly we observe that the standard Chen structure and standard diffeology on $[0,1]$ are both limits in their respective categories of the family $\{(-\epsilon, 1+\epsilon)\}$, again with their standard structures. Since $\mathfrak{Ch}^\flat([0,1])$ is not the standard Chen structure on $[0,1]$ we see that $\mathfrak{Ch}^\flat$ does not preserve limits.

8. More on Equivalences

In the previous sections we have constructed a plethora of functors between the various categories of smooth spaces and shown that none of them are equivalences. However that does not exclude the possibility that there are other functors between these categories that are equivalences. In this section we consider this. Our conclusion is that these categories are not equivalent. Our strategy is to find things in each that may be termed *categorical invariants* which must, therefore, be preserved by any equivalence.

The first such invariant is the (rather, a) terminal object. This is obviously preserved (up to equivalence) by an equivalence of categories. The reason that this is an important invariant is that each of our categories is equipped with a faithful functor to the category of sets, establishing it as a "concrete category", and this functor is naturally isomorphic to that represented by a terminal object. As we shall show, that the terminal object is a categorical invariant means that any *arbitrary* equivalence defines an isomorphism of concrete categories.

The relevant properties of the categories are as follows. All but one of these are standard properties of concrete categories and can be found in standard category theory texts. References are given to [Adámek et al. (2006)].

Construct: That is, concrete over Set. ([Adámek et al. (2006), 5.1(2)].)

Amnestic: This means that if A_1 and A_2 are two objects in one of our categories with the same underlying set such that the identity on the underlying set lifts to morphisms $A_1 \to A_2$ and $A_2 \to A_1$ (necessarily isomorphisms), then $A_1 = A_2$. ([Adámek et al. (2006), 5.4(4)].)

Transportable: This means that if A is an object in one of our categories and S is a set isomorphic to the underlying set of A then there is an object, say A', in the category with underlying set S such that the isomorphism between S and the underlying set of A lifts to an isomorphism between A' and A. ([Adámek et al. (2006), 5.28].)

As the categories are amnestic, the object A' is unique.

Terminally concrete: By this we mean that the underlying-set functor is equivalent to the evaluation of the hom-functor on a terminal object.

8.1. Proposition. *Let $\mathcal{A}$ and $\mathcal{B}$ be amnestic, transportable constructs that are terminally concrete. If $\mathcal{A}$ and $\mathcal{B}$ are equivalent then they are isomorphic as constructs.*

Proof. Let us write $|\cdot|$ for the underlying set functors of both $\mathcal{A}$ and $\mathcal{B}$. Let $\mathfrak{F} \colon \mathcal{A} \to \mathcal{B}$ and $\mathfrak{G} \colon \mathcal{B} \to \mathcal{A}$ be two functors giving an equivalence of categories. As these are equivalences, they preserve categorical constructions up to natural isomorphism. Since anything isomorphic to a terminal object is again a terminal object, if $*$ is a terminal object for $\mathcal{A}$ then $\mathfrak{F}(*)$ is a terminal object for $\mathcal{B}$.

As $\mathcal{A}$ and $\mathcal{B}$ are terminally concrete, we have natural isomorphisms

$$|A| \cong \mathcal{A}(*, A) \cong \mathcal{B}(\mathfrak{F}(*), \mathfrak{F}(A)) \cong |\mathfrak{F}(A)|$$

Using this natural isomorphism, we can define a new functor $\hat{\mathfrak{F}} \colon \mathcal{A} \to \mathcal{B}$ with the property that $|\hat{\mathfrak{F}}(A)| = |A|$.

We start by defining $\hat{\mathfrak{F}}$ on objects. As $\mathcal{B}$ is amnestic and transportable, for each object, A, in $\mathcal{A}$ there is a unique object in $\mathcal{B}$, which we shall write as $\hat{\mathfrak{F}}(A)$, with underlying set $|A|$ such that the isomorphism $|A| \cong |\mathfrak{F}(A)|$ lifts to an isomorphism $\hat{\mathfrak{F}}(A) \cong \mathfrak{F}(A)$. This defines $\hat{\mathfrak{F}}$ on objects. To define $\hat{\mathfrak{F}}$ on morphisms, we use the isomorphism $\hat{\mathfrak{F}}(A) \cong \mathfrak{F}(A)$ to transfer $\mathfrak{F}(f) \colon \mathfrak{F}(A_1) \to \mathfrak{F}(A_2)$ to $\hat{\mathfrak{F}}(A_1) \to \hat{\mathfrak{F}}(A_2)$.

The isomorphisms $\hat{\mathfrak{F}}(A) \cong \mathfrak{F}(A)$ then fit together to define a natural isomorphism $\hat{\mathfrak{F}} \cong \mathfrak{F}$. The resulting functor $\hat{\mathfrak{F}} \colon \mathcal{A} \to \mathcal{B}$ is then a concrete functor as it preserves the underlying sets.

A similar construction results in $\hat{\mathfrak{G}} \colon \mathcal{B} \to \mathcal{A}$ with $|\hat{\mathfrak{G}}(B)| = |B|$. Since $\hat{\mathfrak{F}}$ is equivalent to $\mathfrak{F}$ and $\hat{\mathfrak{G}}$ to $\mathfrak{G}$, $\hat{\mathfrak{F}}$ and $\hat{\mathfrak{G}}$ still define equivalences of categories.

The composition $\hat{\mathfrak{G}}\hat{\mathfrak{F}}$ therefore has the property that for each object, A, in $\mathcal{A}$, $|\hat{\mathfrak{G}}\hat{\mathfrak{F}}(A)| = |A|$. Similarly, for each object, B, in $\mathcal{B}$, $|\hat{\mathfrak{F}}\hat{\mathfrak{G}}(B)| = |B|$. Thus $\mathfrak{F}$ and $\mathfrak{G}$ define bijections on the classes of objects of the two categories.

As $\hat{\mathfrak{G}}$ and $\hat{\mathfrak{F}}$ define an equivalence of categories, for each pair of objects, A_1 and A_2, in $\mathcal{A}$, $\mathcal{A}(A_1, A_2) \to \mathcal{B}(\hat{\mathfrak{F}}(A_1), \hat{\mathfrak{F}}(A_2))$ is a bijection. Hence $\hat{\mathfrak{F}}$ is such that it is a bijection on objects and for each pair of objects is a bijection on the corresponding morphism set. Thus it is an isomorphism of categories. As it is a concrete functor, it is an isomorphism of concrete categories, as required. ∎

The above says that if we wish to look for equivalences between our categories then it is sufficient to look for isomorphisms of constructs. Another way of saying that is to say that if two of our categories have underlying category Set then they are equivalent via an arbitrary equivalence if and only if they are equivalent via one which preserves the underlying category.

To show that this is also true when the underlying category is the category of topological spaces we need to analyse the "topology classifier": $\{0, 1\}$ with its order topology.

8.2. Proposition. *Let $\mathcal{A}$ be a category of smooth spaces constructed as in Section 3 using the category of topological spaces as the underlying category. Any set–preserving isomorphism of $\mathcal{A}$ preserves the underlying topological spaces.*

Proof. To see this we observe that the topology on a set is determined by the set of continuous functions to $\{0, 1\}$ equipped with the order topology $\{\emptyset, \{0\}, \{0, 1\}\}$, which we shall denote by $\mathcal{T}_0$.

We define the structure of an object in $\mathcal{A}$ on $(\{0, 1\}, \mathcal{T}_0)$ by taking the indiscrete structure. This has all continuous maps from test objects to $(\{0, 1\}, \mathcal{T}_0)$ as input test functions and then the appropriate output test functions to satisfy the forcing condition for $\mathcal{A}$. This has the property that any continuous map $|A| \to (\{0, 1\}, \mathcal{T}_0)$ lifts to an $\mathcal{A}$-morphism. Thus the topology on the underlying object in Top of an object, A, in $\mathcal{A}$ is determined by the set of $\mathcal{A}$-morphisms from A to the indiscrete $\mathcal{A}$–structure on $(\{0, 1\}, \mathcal{T}_0)$.

A set–preserving automorphism of $\mathcal{A}$ takes the indiscrete $\mathcal{A}$–structure on $(\{0,1\},\mathcal{T}_0)$ to another object in $\mathcal{A}$ with underlying set $\{0,1\}$. Now $\{0,1\}$ has four topologies: discrete, indiscrete, and the two order topologies. Each of these has a corresponding indiscrete $\mathcal{A}$–structure. As these indiscrete structures are expressible as colimits, a set–preserving automorphism of $\mathcal{A}$ must take indiscrete structures to indiscrete structures. Thus it defines an order–automorphism of the lattice consisting of the indiscrete $\mathcal{A}$–structures on the four topologies on $\{0,1\}$. In particular, $(\{0,1\},\mathcal{T}_0)$ with the indiscrete $\mathcal{A}$–structure either maps to itself or to the indiscrete $\mathcal{A}$–structure on the opposite order topology.

Suppose that there is a set–preserving automorphism of $\mathcal{A}$ which sends the indiscrete structure on $(\{0,1\},\mathcal{T}_0)$ to that on the other order topology.

Let X be an infinite set and let $\mathcal{T}$ be the topology whose closed sets are the finite subsets. Equip $(X,\mathcal{T})$ with its indiscrete $\mathcal{A}$–structure. The $\mathcal{A}$-morphisms between indiscrete $\mathcal{A}$–structures are simply the continuous maps on the underlying topological spaces and thus the $\mathcal{A}$-morphisms from the indiscrete $\mathcal{A}$–structure on $(X,\mathcal{T})$ to that on $(\{0,1\},\mathcal{T}_0)$ are the characteristic functions of the closed sets.

Under the supposed set–preserving automorphism of $\mathcal{A}$, the indiscrete structure on $(X,\mathcal{T})$ is sent to some object, A, in $\mathcal{A}$ with underlying set X and the property that the $\mathcal{A}$-morphisms from A to the indiscrete $\mathcal{A}$–structure on $(\{0,1\},\mathcal{T}_1)$ are precisely the characteristic functions of the closed sets in $\mathcal{T}$.

Now A has underlying topological space $(X,\mathcal{T}')$ and $\mathcal{A}$-morphisms are continuous. This means that a closed set in $\mathcal{T}$ is open in $\mathcal{T}'$. Hence singleton sets are open and so $\mathcal{T}'$ is the discrete topology. Then *all* maps $X\to\{0,1\}$ lift to $\mathcal{A}$-morphisms from A to the indiscrete $\mathcal{A}$–structure on $(\{0,1\},\mathcal{T}_1)$. As X is infinite, this contradicts the statement that the $\mathcal{A}$-morphisms are the characteristic functions of the closed sets in $\mathcal{T}$.

Hence any set–preserving automorphism of $\mathcal{A}$ also preserves the underlying topologies. ∎

To further distinguish between the categories we note that under a set–preserving isomorphism, not only the underlying set is preserved but also the endomorphism monoid of the object (as a submonoid of the endomorphism monoid of the underlying set).

The first application of this enables us to distinguish the category of Frölicher spaces from those of Chen and Souriau spaces.

8.3. Proposition. *The only Frölicher structures on $\mathbb{R}$ whose endomorphism monoid contains $C^\infty(\mathbb{R},\mathbb{R})$ are the standard, the discrete, and the indiscrete structures. In particular, the only Frölicher structure on $\mathbb{R}$ whose endomorphism monoid is precisely $C^\infty(\mathbb{R},\mathbb{R})$ is the standard structure.*

The proof of this depends on a modification of a result of Takens [Takens (1979)]. (Interestingly, this paper was motivated by reading about what we are calling Souriau spaces.)

8.4. Theorem. *[Takens (1979), Theorem 1] Let $\Phi\colon M_1\to M_2$ be a bijection between two smooth n–manifolds such that $\lambda\colon M_2\to M_2$ is a diffeomorphism iff $\Phi^{-1}\circ\lambda\circ\Phi$ is a diffeomorphism. Then Φ is a diffeomorphism.* ∎

Proof of Proposition 8.3 Obviously those three structures have the property that their endomorphism monoids contain $C^\infty(\mathbb{R},\mathbb{R})$. Thus, to show the converse, let $(\mathbb{R},\mathcal{C},\mathcal{F})$ be a Frölicher space with endomorphism monoid containing $C^\infty(\mathbb{R},\mathbb{R})$ which is neither discrete nor indiscrete.

Then for a smooth map $\psi\colon \mathbb{R}\to\mathbb{R}$, $\phi\psi\alpha\in C^\infty(\mathbb{R},\mathbb{R})$ for all $\phi\in\mathcal{F}$ and $\alpha\in\mathcal{C}$. In particular, this implies that if $\psi\in C^\infty(\mathbb{R},\mathbb{R})$ and $\alpha\in\mathcal{C}$ then $\psi\alpha\in\mathcal{C}$, similarly for $\mathcal{F}$.

We start by showing that $\mathcal{C}$ consists of continuous maps. Suppose, for a contradiction, that $\alpha\in\mathcal{C}$ is not continuous at, say, $x\in\mathbb{R}$. Then there is a sequence $(x_n)\to x$ such that $\alpha(x_n)\not\to\alpha(x)$. By passing to a subsequence if necessary, we can assume that there is some $\epsilon>0$ such that $|\alpha(x_n)-\alpha(x)|\geq\epsilon$ for all n. Let ψ be a bump function at $\alpha(x)$ which vanishes outside $(\alpha(x)-\epsilon,\alpha(x)+\epsilon)$. Then $\psi\alpha\in\mathcal{C}$ but $\psi\alpha(x)=1$ and $\psi\alpha(x_n)=0$ for all n. Now for $\phi\in\mathcal{F}$, $\phi\psi\alpha\in C^\infty(\mathbb{R},\mathbb{R})$. Thus $\phi\psi\alpha$ is continuous and so $(\phi\psi\alpha(x_n))\to\phi\psi\alpha(x)$. Hence $\phi(0)=\phi(1)$. There was nothing special about 0 and 1 and thus ϕ is constant. As this holds for arbitrary ϕ, $\mathcal{F}$ consists only of constant functions. However, the only Frölicher structure on $\mathbb{R}$ for which this holds is the indiscrete one which we assumed was not the structure under consideration. Thus the curves in $\mathcal{C}$ are continuous functions.

Now we want to show that at least one curve in $\mathcal{C}$ is injective on some interval. Suppose, for another contradiction, that this is not true. Then for every curve $\alpha\in\mathcal{C}$ and interval $(a,b)\subseteq\mathbb{R}$, α is not injective on (a,b). Let $\phi\in\mathcal{F}$. Then $\phi\alpha$ has the same property. As $\phi\alpha$ is smooth, this implies that $\phi\alpha$ has zero differential everywhere and hence is constant. As this holds for all ϕ and all α, either $\mathcal{C}$ or $\mathcal{F}$ must consist only of constant functions. This results in either the discrete or indiscrete Frölicher structures, which again are ruled out by assumption.

Thus there is some $\alpha\in\mathcal{C}$ which is injective on some interval. By restricting first to a compact subinterval and then to an open subinterval of this, we see that α is a homeomorphism from some open interval to another open interval, say (a,b). By precomposing with an appropriate smooth function, we obtain a homeomorphism $\beta\colon\mathbb{R}\to(a,b)$ in $\mathcal{C}$. Choose a diffeomorphism $\psi\colon(a,b)\to\mathbb{R}$. We claim that $\psi\beta\in\mathcal{C}$. To see this, note that for any bounded interval I in $\mathbb{R}$, there is a smooth function $\psi_I\colon\mathbb{R}\to\mathbb{R}$ such that $\psi_I\beta$ and $\psi\beta$ agree on I. By our assumption on $(\mathbb{R},\mathcal{C},\mathcal{F})$, $\psi_I\beta\in\mathcal{C}$ and thus $\psi\beta\in\mathcal{C}$.

Hence $\mathcal{C}$ contains a homeomorphism, say γ.

Now we turn our attention to $\mathcal{F}$. For $\phi\in\mathcal{F}$, $\phi\gamma\in C^\infty(\mathbb{R},\mathbb{R})$. As γ is a homeomorphism, we can write ϕ as $\phi\gamma\gamma^{-1}$. Thus $\mathcal{F}\subseteq C^\infty(\mathbb{R},\mathbb{R})\gamma^{-1}$. Since $\mathcal{F}$ must contain a non-constant map, there is some non-constant $\psi\in C^\infty(\mathbb{R},\mathbb{R})$ with $\psi\gamma^{-1}\in\mathcal{F}$. As ψ is non-constant, there are bounded intervals, say $I,J\subseteq\mathbb{R}$, such that ψ restricts to a diffeomorphism $I\to J$. By restricting if necessary, we can assume that the inverse is the restriction of a smooth map $\theta\colon\mathbb{R}\to\mathbb{R}$. Then $\theta\psi$ is the identity on I and so $\theta\psi\gamma^{-1}$ equals γ^{-1} on $\gamma(I)(=(\gamma^{-1})^{-1}(I))$. Since $\mathcal{F}$ is invariant under translation (as $C^\infty(\mathbb{R},\mathbb{R})$ acts on it on the *right*), we see that γ^{-1} is locally in $\mathcal{F}$ and hence in $\mathcal{F}$. Thus $\mathcal{F}=C^\infty(\mathbb{R},\mathbb{R})\gamma^{-1}$.

We therefore see that $\mathcal{C}=\gamma C^\infty(\mathbb{R},\mathbb{R})$. Thus the Frölicher structure is

$$(\mathbb{R},\gamma C^\infty(\mathbb{R},\mathbb{R}),C^\infty(\mathbb{R},\mathbb{R})\gamma^{-1}).$$

The endomorphism monoid of this structure is $\gamma C^\infty(\mathbb{R},\mathbb{R})\gamma^{-1}$. Thus $\gamma\colon\mathbb{R}\to\mathbb{R}$ is a homeomorphism with the property that $\gamma C^\infty(\mathbb{R},\mathbb{R})\gamma^{-1} \supseteq C^\infty(\mathbb{R},\mathbb{R})$. Equivalently, that $\gamma^{-1} C^\infty(\mathbb{R},\mathbb{R})\gamma \subseteq C^\infty(\mathbb{R},\mathbb{R})$. This does not quite fit the hypotheses for Takens' theorem as that requires equality here. However, careful examination of [Takens (1979)] shows that the proof still applies. It seems that the "if and only if" part of the hypotheses of Takens' theorem is required to prove that the bijections are homeomorphisms. As we already have that property, we can assume weaker conditions.

In detail, the proof of [Takens (1979), Lemma 3.1] shows that if (using notation of [Takens (1979)]) $\Phi \colon \mathbb{R} \to \mathbb{R}$ is a homeomorphism such that $\Phi^{-1}(\Phi(y) + c)$ is differentiable for all $y, c \in \mathbb{R}$ then Φ and Φ^{-1} are differentiable everywhere. Then the rest of the proof of [Takens (1979), Theorem 2] for the case $n = 1$ applies as stated. Hence our Frölicher structure on $\mathbb{R}$ is the standard one. ∎

8.5. Corollary. *The only set–preserving automorphism of the category of Frölicher spaces is the identity.* ∎

The proof of the proposition involves much of the structure specific to Frölicher spaces and thus would seem difficult to generalise to the other categories of interest, save perhaps for Smith spaces. However, there is still something that can be said without too much extra work by looking at underlying Frölicher spaces.

Thus in each category we look at those objects with underlying set $\mathbb{R}$. The endomorphism monoid of one of these objects will be a submonoid of $\mathrm{Set}(\mathbb{R},\mathbb{R})$. We focus our attention on those objects with endomorphism monoid $C^\infty(\mathbb{R},\mathbb{R})$.

For Frölicher spaces there is only one such object: $\mathbb{R}$ with its standard structure. We strongly suspect that this is true for the other categories as well (though note that it is not true when the forcing conditions are very weak) but even without direct proof of this we can still show that in each case, the standard structure on $\mathbb{R}$ is special.

Let $\mathcal{A}$ be one of our categories and A an object in $\mathcal{A}$ with underlying set $\mathbb{R}$ and endomorphism monoid $C^\infty(\mathbb{R},\mathbb{R})$. Because the functor $\mathcal{A} \to \mathcal{F}r$ is faithful and set-preserving, the Frölicher space corresponding to A has underlying set $\mathbb{R}$ and endomorphism monoid containing $C^\infty(\mathbb{R},\mathbb{R})$. Thus by Proposition 8.3 there are only three choices for this Frölicher space. As the next proposition shows, this is enough to test whether or not our original object, A, was $\mathbb{R}$ with its standard structure. In the statement of this proposition, we write $\mathrm{Const}(\mathbb{R},\mathbb{R})$ for the set of constant functions from $\mathbb{R}$ to itself.

8.6. Proposition. *In each of our categories, we consider an object $\mathbb{R}_\alpha$ which has underlying set $\mathbb{R}$ and endomorphism monoid $C^\infty(\mathbb{R},\mathbb{R})$.*

In the categories of Chen and Souriau spaces, the underlying Frölicher space of $\mathbb{R}_\alpha$ is either the standard structure on $\mathbb{R}$ or the discrete structure. It is the standard structure only if $\mathbb{R}_\alpha = \mathbb{R}$. In particular, if $\mathbb{R}$ denotes the "standard" $\mathbb{R}$ then

$$\mathcal{A}(\mathbb{R}_\alpha, \mathbb{R}) = \begin{cases} C^\infty(\mathbb{R},\mathbb{R}) & \textit{if } \mathbb{R}_\alpha = \mathbb{R}, \\ \mathrm{Const}(\mathbb{R},\mathbb{R}) & \textit{otherwise} \end{cases}$$

where $\mathcal{A}$ is either $\mathcal{C}h$ or $\mathcal{S}o$.

In the categories of Smith and Sikorski spaces, the underlying Frölicher space of $\mathbb{R}_\alpha$ is either the standard structure or the indiscrete structure. It is the standard structure only if $\mathbb{R}_\alpha = \mathbb{R}$. In particular, if $\mathbb{R}$ denotes the "standard" $\mathbb{R}$ then

$$\mathcal{A}(\mathbb{R}, \mathbb{R}_\alpha) = \begin{cases} C^\infty(\mathbb{R}, \mathbb{R}) & \text{if } \mathbb{R}_\alpha = \mathbb{R}, \\ \operatorname{Const}(\mathbb{R}, \mathbb{R}) & \text{otherwise} \end{cases}$$

where $\mathcal{A}$ is either $\mathcal{S}i$ or $\mathcal{S}m$.

Proof. The arguments are essentially the same for all four cases. We shall consider Souriau spaces in detail and then indicate the necessary changes for the others.

When forming a Frölicher space from a Souriau space one only adds functions from $\mathbb{R}$ to those already in the plots. The output functions stay the same. Thus if a Souriau space maps to a discrete Frölicher space, the plots from $\mathbb{R}$ in the Souriau structure must have been constant plots. Therefore the plots from any test object must have been locally constant and so the Souriau structure must have been the discrete one. In particular, only the discrete Souriau structure on $\mathbb{R}$ maps to the discrete Frölicher structure on $\mathbb{R}$ and the discrete Souriau structure does not have endomorphism monoid $C^\infty(\mathbb{R}, \mathbb{R})$.

Now suppose that $\mathbb{R}_\alpha$ is as in the statement of the proposition and assume that the underlying Frölicher structure on $\mathbb{R}_\alpha$ is the standard Frölicher structure on $\mathbb{R}$. By the above, the Souriau structure cannot have been the discrete one and thus there is a non–constant map $\mathbb{R} \to \mathbb{R}$ in the plots of $\mathbb{R}_\alpha$. As this becomes a morphism in the Frölicher category, it must be a C^∞-map. It is therefore a diffeomorphism when restricted to some interval. By precomposition with an appropriate C^∞-map, we can obtain the identity map on some interval in the Souriau structure on $\mathbb{R}_\alpha$. As the endomorphism monoid of $\mathbb{R}_\alpha$ is $C^\infty(\mathbb{R}, \mathbb{R})$, the Souriau structure must be translation invariant. The identity on $\mathbb{R}$ is therefore locally in the Souriau structure of $\mathbb{R}_\alpha$ and hence, by the sheaf condition, the identity itself is in the structure. Thus the Souriau structure on $\mathbb{R}_\alpha$ contains $C^\infty(\mathbb{R}, \mathbb{R})$. On the other hand, it can contain no more than this as it maps to the standard Frölicher structure on $\mathbb{R}$. Hence $\mathbb{R}_\alpha$ is the standard Souriau structure on $\mathbb{R}$.

If $\mathbb{R}_\alpha$ is as in the statement but is not the standard structure then it must therefore map to the indiscrete Frölicher structure. As the functor from Souriau spaces to Frölicher spaces is faithful, the set of morphisms from $\mathbb{R}_\alpha$ to $\mathbb{R}$ must be contained in the set of morphisms on the underlying Frölicher spaces. However, the only morphisms from the indiscrete Frölicher structure on $\mathbb{R}$ to the standard one are the constant maps.

The argument is exactly the same for Chen spaces. For Smith and Sikorski, one swaps the rôles of the input and output functions in the various arguments. ∎

The key in this proof is obviously that the forcing conditions in each case are sufficient to guarantee that if the identity on $\mathbb{R}$ is locally a test function then it is a test function.

This result singles out the standard structure on $\mathbb{R}$ in each category as either a sort of "initial" (Chen and Souriau) or "final" (Smith and Sikorski) object in the set of objects with underlying set $\mathbb{R}$ and endomorphism monoid $C^\infty(\mathbb{R}, \mathbb{R})$. To see this, say for Souriau spaces, suppose that $\mathbb{R}_\alpha$ has underlying set $\mathbb{R}$ and endomorphism monoid $C^\infty(\mathbb{R}, \mathbb{R})$ but that $\mathbb{R}_\alpha \neq \mathbb{R}$. Then, from the above proposition, $\mathcal{S}o(\mathbb{R}_\alpha, \mathbb{R}) = \operatorname{Const}(\mathbb{R}, \mathbb{R})$. But as $\mathbb{R}_\alpha$ is not the discrete Souriau structure on $\mathbb{R}$ the set $\mathcal{S}o(\mathbb{R}, \mathbb{R}_\alpha)$ must contain something other than constant morphisms. Thus we can identify the standard structure on $\mathbb{R}$ as being that which has endomorphism monoid $C^\infty(\mathbb{R}, \mathbb{R})$ and which has non-constant morphisms to every other such object, but with no non-constant morphisms in the other direction.

Now that we can identify $\mathbb{R}$ with its standard structure in each of the categories, we can show that much of the structure of these categories must be preserved under equivalence. Enough structure to show that no pairs of the categories that we are considering are equivalent.

8.7. Corollary. *Any set–preserving isomorphism between two of the categories of Chen, Souriau, or Frölicher spaces preserves* $\mathbb{R}$ *with its standard structure. Similarly between two of the categories of Sikorski, Smith, or Frölicher spaces.*

Proof. The two cases are formally similar so we shall just do one. Let $\mathcal{A}$ and $\mathcal{B}$ be two of the categories of Chen, Souriau, or Frölicher spaces and suppose that $\mathfrak{F} \colon \mathcal{A} \to \mathcal{B}$ is a set–preserving isomorphism. Let us write $\mathbb{R}$ for the standard structure on $\mathbb{R}$ in each category.

Let $\mathbb{R}_\alpha$ be such that $\mathfrak{F}(\mathbb{R}_\alpha) = \mathbb{R}$. Then

$$\mathcal{A}(\mathbb{R}_\alpha, \mathbb{R}) = \mathcal{B}(\mathfrak{F}(\mathbb{R}_\alpha), \mathfrak{F}(\mathbb{R})) = \mathcal{B}(\mathbb{R}, \mathfrak{F}(\mathbb{R})).$$

Now $\mathfrak{F}(\mathbb{R})$ has underlying set $\mathbb{R}$ and endomorphism monoid $C^\infty(\mathbb{R}, \mathbb{R})$ so it is not discrete and thus the hom-set $\mathcal{B}(\mathbb{R}, \mathfrak{F}(\mathbb{R}))$ must contain something non-trivial. Thus $\mathcal{A}(\mathbb{R}_\alpha, \mathbb{R}) \neq \operatorname{Const}(\mathbb{R}, \mathbb{R})$ and so $\mathbb{R}_\alpha = \mathbb{R}$. Hence $\mathfrak{F}$ preserves the "standard" $\mathbb{R}$. ∎

8.8. Corollary. *Any set–preserving isomorphism between two of the categories of Chen, Souriau, or Frölicher spaces on the one hand, or between two of the categories of Sikorski, Smith, or Frölicher spaces on the other, preserves the test objects.*

Proof. Once $\mathbb{R}$ with its standard structure is fixed then so are the product structures on the spaces $\mathbb{R}^n$ and the subspace structures on any subsets, in particular on the test objects. ∎

8.9. Corollary. *There are no equivalences between any of the categories of Chen, Souriau, and Frölicher spaces. Nor are there any between any of the categories of Sikorski, Smith, and Frölicher spaces.*

Proof. The argument is the same in all cases so let us take Chen and Souriau as an example. Any equivalence between these two defines a set–preserving isomorphism which must preserve the test objects. We therefore find that if $\mathfrak{F} \colon \mathcal{S}o \to \mathcal{C}h$ is set–preserving isomorphism then for a object, D, in $\mathcal{S}o$ and test object, T,

$$\mathfrak{I}_D(T) = \mathcal{S}o(\mathfrak{S}(T), D) = \mathcal{C}h(\mathfrak{S}(T), \mathfrak{F}(D))$$

and thus applying the restriction functor to $\mathfrak{F}(D)$ yields D again. However, the restriction functor is not an isomorphism as there are at least two distinct Chen structures on $[0,1]$ which restrict to the same Souriau structure. Therefore $\mathfrak{F}$ cannot exist.

This strategy works for each pair: we find two objects in the "higher" category that restrict to the same object in the "lower" one.

For Souriau and Frölicher we take $\mathbb{R}^2$ with its standard Souriau structure and the "wire" structure where all input test functions factor through $\mathbb{R}$.

For Smith and Frölicher we take $\mathbb{R}$ with its usual topology and all continuous maps, and $\mathbb{R}$ with its discrete topology and all maps.

For Sikorski and Smith we take $\mathbb{R}$ with its usual topology and all continuous maps, and $\mathbb{R}$ with its usual topology and all piecewise–smooth maps. ∎

There is one final situation to rule out. Our strategy of focussing on $\mathbb{R}$ works well so long as we restrict ourselves to one "side" at a time. It does not rule out an equivalence between, say, Souriau spaces and Sikorski spaces. Indeed, swapping "sides" like this is a more complicated issue and so we shall take the easy way out by using the fact that in our particular examples we can use the underlying categories to rule out any equivalences.

8.10. Proposition. *There are no equivalences between either the category of Chen spaces or the category of Souriau spaces and the category of Sikorski spaces or the category of Smith spaces.*

Proof. It is sufficient to look for a set–preserving isomorphism. Any such isomorphism must preserve the number of structures on a given set. In the categories of Smith and Sikorski spaces there are four objects with underlying set $\{0, 1\}$. These are:

1. $\{0, 1\}$ with the indiscrete topology and constant functions,
2. $\{0, 1\}$ with the order topology and constant functions,
3. $\{0, 1\}$ with the reverse order topology and constant functions,
4. $\{0, 1\}$ with the discrete topology and all functions.

The functions in the first three are forced by the fact that all three have only constant continuous functions to $\mathbb{R}$. The forcing conditions for both Smith and Sikorski spaces force the functions for the discrete topology to be all functions.

In the categories of Chen, Souriau, and Frölicher spaces there are only two objects with underlying set $\{0, 1\}$. These are:

1. $\{0, 1\}$ with only constant (input) functions, and
2. $\{0, 1\}$ with all (input) functions.

As $4 \neq 2$, there cannot be an equivalence between one of the categories of Smith or Sikorski spaces on the one hand, and one of the categories of Chen, Souriau, or Frölicher spaces on the other. ∎

In this section we have proceeded on a mixture of general results and case–by–case analysis. It is entirely possible that a fuller analysis of the general structure would lead to an elimination of the more ad hoc aspects, but as the goal was to analyse the specific examples of these categories where a simple ad hoc argument sufficed we deemed it more appropriate to give it than to search for what could be a more complicated but more general result.

9. Topology

Two of the main categories under consideration and three of Chen's definitions involve topology directly. This proves to be somewhat of a nuisance. If one takes seriously the maxim that anything related to smooth structure must be detectable by test functions, one must make certain assumptions as to the relationship between the topology and the test functions. In simple language, if we cannot detect the difference between two topologies with test functions then we should not distinguish between them.

This does not mean that topology does not have a rôle to play in the theory of smooth structures. Rather, it means that the topology must depend on the smooth structure and not the other way around. It is easy to consistently define a topology on a Frölicher space. When comparing Smith spaces and Frölicher spaces in Section 5 we described two such topologies: the curvaceous topology, being the finest topology such that the curves are continuous, and the functional topology, being the coarsest topology such that the functions are continuous. However it is defined, the resulting topology may not be the one first thought of.

By looking at the evolution of Chen's definitions, it is clear that he came to this opinion on the rôle of topology and he eventually disposed of it. Indeed, one can avoid the issue of topology in Chen's definitions by ensuring that it is sufficiently weak that any "reasonable" map is continuous. The only situation where this might cause a problem is in Chen's first definition where he used Hausdorff spaces. Fortunately he quickly dropped this condition and, eventually, dropped the topology condition altogether.

However, topology is used in a more intricate fashion in Smith spaces and Sikorski spaces. In both there is the requirement that all the functionals be continuous. This puts a limit on how weak the topology could be. If this were the only place that the topology was used then we could remove it again by ensuring that it is sufficiently strong that any "reasonable" map is continuous. However the other use of the topology in each definition puts a limit on how strong the topology can be. Nonetheless, it is still possible to modify the definitions to remove the topology.

For Smith spaces it is obvious that if one removes the topology from the picture, and thus the requirement that all the maps involved be continuous, then one simply gets Frölicher spaces. The completeness axiom is, by Boman's result [Boman (1967)], equivalent to that of Frölicher.

For Sikorski spaces it is less obvious what to do. The axioms that the functionals should form an algebra and be closed under post-composition by smooth functions do not involve the topology and so can be left as they are. The interaction between the family of functions and the topology occurs in two places: the continuity of the functionals (referred to above) and the fact that the functionals are locally detectable. Let us consider these.

Let X be a set and suppose that $\mathcal{F}$ is a family of functionals on X for which there exists some topology $\mathcal{T}$ making $(X, \mathcal{T}, \mathcal{F})$ into a Sikorski space. Let $\mathcal{T}_f$ denote the topology on X induced by the family of functionals, $\mathcal{F}$. As the functionals in $\mathcal{F}$ must be continuous, the topology $\mathcal{T}$ is at least as fine as $\mathcal{T}_f$. If $f \colon X \to \mathbb{R}$ is $\mathcal{T}_f$–locally in $\mathcal{F}$ then, since $\mathcal{T}_f \subseteq \mathcal{T}$, it is $\mathcal{T}$–locally in $\mathcal{F}$ and hence, as $(X, \mathcal{T}, \mathcal{F})$ is a Sikorski space, in $\mathcal{F}$. Thus $(X, \mathcal{T}_f, \mathcal{F})$ is a Sikorski space. In conclusion, if $\mathcal{F}$ is a family of functions on X for which *some* topology exists making it into a Sikorski space then we can take the topology to be that defined by $\mathcal{F}$.

We can use this to drop the topology from the definition. First, the fact that the functionals in $\mathcal{F}$ are continuous is now a tautology. This leaves us with locally detectable.

Let $f \colon X \to \mathbb{R}$ and $x \in X$ be such that there is a neighborhood V of x and $g \in \mathcal{F}$ with $f|_V = g|_V$. As $\mathcal{F}$ is an algebra and is closed under post-composition by smooth functions, a basis for the functional topology is given by the sets $h^{-1}(\mathbb{R} \smallsetminus \{0\})$ for $h \in \mathcal{F}$. Thus there is some $h \in \mathcal{F}$ such that $h(x) \neq 0$ and $h^{-1}(\mathbb{R} \smallsetminus \{0\}) \subseteq V$. As h is zero outside V, $hf = hg$ as functions on X. As $\mathcal{F}$ is an algebra, this means that $hf \in \mathcal{F}$.

Conversely, suppose that $f \colon X \to \mathbb{R}$ and $x \in X$ are such that there is some $h \in \mathcal{F}$ with $h(x) \neq 0$ and $hf \in \mathcal{F}$. As $h(x) \neq 0$ we can find a function $g \in \mathcal{F}$ such that $g(y) = h(y)^{-1}$ in a neighborhood of x. Then $ghf \in \mathcal{F}$ and ghf agrees with f in a neighborhood of x.

Thus a suitable non-topological replacement for the "locally detectable" axiom of a Sikorski space would be: if $f \colon X \to \mathbb{R}$ is a functional such that for each $x \in X$ there is some $f_x \in \mathcal{F}$ with $f_x(x) \neq 0$ and $f f_x \in \mathcal{F}$ then $f \in \mathcal{F}$.

However, note that this condition does not fit into our general picture. It cannot be tested by diagrams of the form

$$\xrightarrow{x} X \dashrightarrow T \xrightarrow{t}$$

Rather we have to extend the notion of a trial to include diagrams of the form

$$\begin{array}{ccccc} & X & \dashrightarrow & T & \\ \xrightarrow{x} & \times & & \times & \xrightarrow{t} \\ & X' & \xrightarrow[\phi]{} & T' & \end{array}$$

This is clearly doable, but given that the only example that requires it is this of de-topologised Sikorski spaces we have elected not to further burden the rest of the paper.

10. Non-Set–Based Theories

Our general recipe does not cover everything that one would wish to treat as a smooth manifold. The common property of all our categories is that they are fundamentally set–based theories. However, this approach does not cover other extensions such as synthetic and non-commutative differential geometry where objects do not have underlying sets. It is, nonetheless, easy to extend part of the general recipe to non-set–based theories. Instead of starting with triples $(U, \mathfrak{I}, \mathfrak{O})$ one starts with pairs $(\mathfrak{I}, \mathfrak{O}, \circ)$ where $\mathfrak{I}$ and $\mathfrak{O}$ are, respectively, a contravariant and a covariant functor on the test category and $\circ$ is a natural transformation of bifunctors from $\mathfrak{I} \times \mathfrak{O}$ to the hom-functor on the test category.

Providing the test category is essentially small, these are the objects of a category which is known, in folklore at least, as the *Isbell Envelope* of the test category. The idea behind this dates back to Lawvere's thesis, reprinted in [Lawvere (2004)], and was further developed by Isbell under the name *double envelope* (the name *Isbell envelope* was later coined by Lawvere). It can also be understood in terms of profunctors. The two functors $\mathfrak{I} \colon \mathcal{T} \to \mathrm{Set}$ and $\mathfrak{O} \colon \mathcal{T} \to \mathrm{Set}$ are profunctors $\mathcal{T} \to 1$ and $1 \to \mathcal{T}$ respectively. Their composition, as profunctors, is the product $\mathfrak{I} \times \mathfrak{O}$ and the natural transformation $\mathfrak{I} \times \mathfrak{O} \to \mathcal{T}(-,-)$ is a morphism of profunctors. Thus we obtain what may be termed a *lax factorisation of* $\mathcal{T}(-,-)$ *through* 1.

However it is described, the main difficulty is in selecting the appropriate generalisation of the forcing conditions. With an underlying category, the forcing conditions are answers to the question "Which of the *available* morphisms between the underlying objects should be viewed as smooth?". Without an underlying category, the forcing conditions need to answer a more subtle question. What this question should be is something that will become evident with further study of the extant examples of non-set–based theories of generalised smooth objects; a study that we defer to another paper.

References

[Adámek et al. (2006)] Jiří Adámek, Horst Herrlich, and George E. Strecker. Abstract and concrete categories: the joy of cats. *Repr. Theory Appl. Categ.*, page 1–507, 2006, http://www.tac.mta.ca/tac/reprints/articles/17/tr17.pdf.

[Baez and Hoffnung] John Baez and Alex Hoffnung. Convenient categories of smooth spaces. *Trans. Amer. Math. Soc.*, (to appear), http://arxiv.org/abs/0807.1704v2.

[Boman (1967)] Jan Boman. Differentiability of a function and of its compositions with functions of one variable. *Math. Scand.*, 20:249–268, 1967.

[Chen (1973)] Kuo Tsai Chen. Iterated integrals of differential forms and loop space homology. *Ann. of Math. (2)*, 97:217–246, 1973.

[Chen (1975)] Kuo Tsai Chen. Iterated integrals, fundamental groups and covering spaces. *Trans. Amer. Math. Soc.*, 206:83–98, 1975.

[Chen (1977)] Kuo Tsai Chen. Iterated path integrals. *Bull. Amer. Math. Soc.*, 83:831–879, 1977.

[Frölicher (1982)] Alfred Frölicher. Smooth structures. volume 962 of *Lecture Notes in Math.*, page 69–81, Berlin, 1982. Springer.

[Iglesias-Zemmour] Patrick Iglesias-Zemmour. Diffeology, http://math.huji.ac.il/~piz/Site/The%20Book/The%20Book.html.

[Isbell (1960)] J. R. Isbell. Adequate subcategories. *Illinois J. Math.*, 4:541–552, 1960.

[Kriegl and Michor (1997)] Andreas Kriegl and Peter W. Michor. The convenient setting of global analysis. Providence, RI, 1997.

[Lawvere (2004)] F. William Lawvere. Functorial semantics of algebraic theories and some algebraic problems in the context of functorial semantics of algebraic theories. *Repr. Theory Appl. Categ.*, page 1–121, 2004.

[Mostow (1979)] Mark A. Mostow. The differentiable space structures of Milnor classifying spaces, simplicial complexes, and geometric realizations. *J. Differential Geom.*, 14:255–293, 1979.

[Sikorski (1971)] Roman Sikorski. Differential modules. *Colloq. Math.*, 24:45–79, 1971.

[Smith (1966)] J. Wolfgang Smith. The de Rham theorem for general spaces. *Tôhoku Math. J. (2)*, 18:115–137, 1966.

[Souriau (1980)] J.-M. Souriau. Groupes différentiels. volume 836 of *Lecture Notes in Math.*, page 91–128, Berlin, 1980. Springer.

[Takens (1979)] Floris Takens. Characterization of a differentiable structure by its group of diffeomorphisms. *Bol. Soc. Brasil. Mat.*, 10:17–25, 1979.

Department of Mathematics,
Norwegian University of Science and Technology
7491 Trondheim, Norway.
Email: andrew.stacey@math.ntnu.no

This article may be accessed at http://www.tac.mta.ca/tac/ or by anonymous ftp at ftp://ftp.tac.mta.ca/pub/tac/html/volumes/25/4/25-04.{dvi,ps,pdf}