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MONOIDAL FUNCTOR CATEGORIES AND GRAPHIC FOURIER TRANSFORMS

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ABSTRACT. This article represents a preliminary attempt to link Kan extensions, and some of their further developments, to Fourier theory and quantum algebra through $*$-autonomous monoidal categories and related structures. There is a close resemblance to convolution products and the Wiener algebra (of transforms) in functional analysis. The analysis term "kernel" (of a distribution) has also been adapted below in connection with certain special types of "distributors" (in the terminology of J. Bénabou) or "modules" (in the terminology of R. Street) in category theory. In using the term "graphic", in a very broad sense, we are clearly distinguishing the categorical methods employed in this article from standard Fourier and wavelet mathematics. The term "graphic" also applies to promultiplicative graphs, and related concepts, which can feature prominently in the theory.

Introduction

In the first section of this article symmetric $*$-autonomous monoidal categories $\mathcal{V}$ (in the sense of [1]) and enriched functor categories of the form $\mathcal{P}(\mathcal{A}) = [\mathcal{A}, \mathcal{V}]$ (cf. [13]), are used to describe aspects of the "graphic" upper and lower convolutions of functors from a promonoidal category $\mathcal{A}$ into $\mathcal{V}$ (in the sense of [6] for example), and their transforms. The particular $*$-autonomous monoidal structures that are of interest here have their tensor unit as the dualizing object: see §1.

A formal notion of categorical "Fourier" transform of a functor from a monoidal functor category $[\mathcal{A}, \mathcal{V}]$ is introduced in §3. This notion is a particular type of Kan extension based on the idea of a multiplicative kernel between promonoidal categories. Roughly speaking, multiplicative kernels correspond, via the Kan extension process, to tensor-preserving functors between the resulting enriched monoidal functor categories (into $\mathcal{V}$), the latter being in many ways analogous to function algebras into a ring. Then the transform of the convolution product of two functors becomes the (sometimes pointwise) tensor product of their transforms. Some basic examples of kernels are mentioned at the end of §2, including that of association schemes.

In §3 we also look at transforms of functors in the context of the abstract Weiner (or Joyal-Wiener) category, constructed by analogy with the Wiener algebra in functional analysis. This category of transforms is, under very simple conditions, a monoidal category

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equivalent to the original monoidal closed domain $[\mathcal{A}, \mathcal{V}]$. Of course, in A. Joyal's original example [12], the Wiener category becomes equivalent to the category of Joyal-analytic functors and weakly cartesian maps between them.

Thus, in general, the transform of a functor behaves a bit like the classical Fourier-type transform of a function, and there is usually a corresponding inversion process. For instance (see §4) the transforming functor under consideration may have a tentative left inverse (denoted Γ in the text), which then leads to the construction of the required (two-sided) inversion data.

These transforms also generalize the Fourier transforms of Hopf algebras (as discussed in [4] for example) which extend directly from Hopf algebras in $\mathcal{V}$ to the many-object Hopf algebroids of [10]; some mention of this is made in §4. Several other types of examples are described in §5

The original convolution construction on a functor category of the form $[\mathcal{A}, \mathcal{V}]$ for a small promonoidal structure $\mathcal{A}$, may be found in [6] and [7], where such categories are viewed in much the same light as function algebras. It is emphasised here that all the categorical concepts used below, such as "category", "functor", "natural transformation", etc., are $\mathcal{V}$-enriched (in the sense of [13]) over the given symmetric monoidal closed base category $\mathcal{V}$, unless otherwise mentioned in the text.

1. Upper and lower convolution

Let $\mathcal{V} = (\mathcal{V}_0, I, \otimes, [-,-], (-)^*)$ be a complete (hence cocomplete) $*$-autonomous symmetric monoidal closed category (in the sense of [1]) having I as the dualising object. Recall that

$$[X, Y] \cong (X \otimes Y^*)^*,$$

and if an object Z has a tensor dual $Z^\vee$ in $\mathcal{V}$, then the two notions of dual coincide

$$Z^\vee \cong [Z, I] = Z^*$$

in which case

$$[Z, X] \cong Z^* \otimes X \quad \text{and} \quad (Z \otimes X)^* \cong Z^* \otimes X^*$$

for all X in $\mathcal{V}$.

If $(\mathcal{A}, p, j)$ is a small promonoidal category over $\mathcal{V}$, then the *upper convolution* of f and g in the functor category $[\mathcal{A}, \mathcal{V}]$ is defined in [6] as

$$f \mathbin{\overline{*}} g = \int^{ab} f(a) \otimes g(b) \otimes p(a, b, -) \quad \text{and} \quad \overline{I} = j.$$

If $(\mathcal{A}^{\mathrm{op}}, p, j)$ is a small promonoidal category, then the *lower convolution* of f and g in $[\mathcal{A}, \mathcal{V}]$ is defined as

$$f \mathbin{\underline{*}} g = \int_{ab} (f(a)^* \otimes g(b)^* \otimes p(a, b, -))^* \quad \text{and} \quad \underline{I} = j^*.$$

(See [8] and [16] for example.)

Both products yield associative and unital monoidal structures on $[\mathcal{A},\mathcal{V}]$; the upper product preserves $\mathcal{V}$-colimits in each variable, while the lower product preserves $\mathcal{V}$-limits in each variable. The upper product $f \mathbin{\bar{*}} g$ in $[\mathcal{A},\mathcal{V}]$ using p on $\mathcal{A}$ gives $f \mathbin{\bar{*}} g$ on

$$[\mathcal{A},\mathcal{V}]^{\mathrm{op}} = [\mathcal{A}^{\mathrm{op}},\mathcal{V}^{\mathrm{op}}]$$

which transforms under the equivalence $\mathcal{V}^{\mathrm{op}} \simeq \mathcal{V}$ to the lower product $f^* \mathbin{\underline{*}} g^*$ in $[\mathcal{A}^{\mathrm{op}},\mathcal{V}]$ using the same p on $\mathcal{A}$, since

$$\begin{aligned}(f \mathbin{\bar{*}} g)^* &= \Big(\int^{ab} f(a)\otimes g(b)\otimes p(a,b,-)\Big)^* \\ &\cong \Big(\int^{ab} f(a)^{**}\otimes g(b)^{**}\otimes p(a,b,-)\Big)^* \\ &= f^* \mathbin{\underline{*}} g^*.\end{aligned}$$

An *antipode* S on a (promonoidal) category $(\mathcal{A},p,j)$ is a functor

$$S : \mathcal{A}^{\mathrm{op}} \to \mathcal{A}$$

such that $S^{\mathrm{op}} \dashv S$ with $S^2 \cong 1$. Then the resulting data $q(a,b,c) = p(Sa,Sb,Sc)$ and $k(c) = j(Sc)$ become part of an obvious promonoidal structure on $\mathcal{A}^{\mathrm{op}}$.

If the set of data $(\mathcal{A},p,j,S)$ is an *S-autonomous* category, in the sense that the cyclic condition

$$p(a,b,Sc) \cong p(b,c,Sa)$$

holds naturally in $a,b,c \in \mathcal{A}$, then the derived set $(\mathcal{A}^{\mathrm{op}},q,k,S^{\mathrm{op}})$ is S^{op}-autonomous since

$$p(Sa,Sb,S^2c) \cong p(Sb,Sc,S^2a);$$

that is

$$q(a,b,Sc) \cong q(b,c,Sa).$$

1.1. Theorem. *If $(\mathcal{A},p,j,S)$ is a S-autonomous promonoidal category and $(\mathcal{V},I,\otimes,(-)^*)$ is the $*$-autonomous base category, then the upper convolution structure on $[\mathcal{A},\mathcal{V}]$ is $*$-autonomous under the antipode defined by $f^*(a) = f(Sa)^*$ for f in $[\mathcal{A},\mathcal{V}]$.*

Proof. We have the natural isomorphisms

$$
\begin{aligned}
[f,g](c) &= \int_{ab} [f(a)\otimes p(c,a,b), g(b)] \text{ by definition of } [-,-] \text{ in } [\mathcal{A},\mathcal{V}],\\
&\cong \int_{ab} (f(a)\otimes p(c,a,b)\otimes g(b)^*)^* \text{ since } \mathcal{V} \text{ is } *\text{-autonomous monoidal},\\
&\cong (\int^{ab} f(a)\otimes g(b)^*\otimes p(c,a,b))^*,\\
&\cong (\int^{ab} f(a)\otimes g(Sb)^*\otimes p(c,a,Sb))^* \text{ since } S^2\cong 1,\\
&\cong (\int^{ab} f(a)\otimes g^*(b)\otimes p(a,b,Sc))^*\\
&\quad \text{since } p(c,a,Sb) = p(a,b,Sc) \text{ because } (\mathcal{A},p,j,S) \text{ is } S\text{-autonomous},\\
&= (f\mathbin{\overline{*}} g^*)^* \text{by definition of } \overline{*} \text{ on } [\mathcal{A},\mathcal{V}].
\end{aligned}
$$

■

The upper convolution $f \mathbin{\overline{*}_p} g$ is then related to the lower convolution $f \mathbin{\underline{*}_q} g$ on the functor category $[\mathcal{A},\mathcal{V}]$ using $q(a,b,c) = p(Sa,Sb,Sc)$ on $\mathcal{A}^{\mathrm{op}}$, the latter being naturally isomorphic to the product

$$(f^* \mathbin{\overline{*}_p} g^*)^*$$

on $[\mathcal{A},\mathcal{V}]$. This follows from the convolution calculation

$$
\begin{aligned}
(f^* \mathbin{\overline{*}_p} g^*)^*(c) &= (\int^{ab} f(Sa)^*\otimes g(Sb)^*\otimes p(a,b,Sc))^* \text{ by definition of } \overline{*}_p,\\
&\cong \int_{ab} (f(a)^*\otimes g(b)^*\otimes p(Sa,Sb,Sc))^* \text{ using } S^2\cong 1,\\
&\cong (\int^{ab} f(a)^*\otimes g(b)^*\otimes q(a,b,c))^* \text{ by definition of } q \text{ in } \mathcal{A}^{\mathrm{op}},\\
&= f(a) \mathbin{\underline{*}_q} g \text{ by definition of } \underline{*}_q.
\end{aligned}
$$

Note In the sequel we shall not go into the particular $*$-autonomous aspects of the theory in any great detail.

2. Multiplicative kernels

For the given base category $\mathcal{V}$, a *multiplicative kernel* from a promonoidal $\mathcal{A}$ to another promonoidal $\mathcal{X}$ is a $\mathcal{V}$-functor

$$K : \mathcal{A}^{\mathrm{op}}\otimes\mathcal{X}\to\mathcal{V}$$

together with two natural structure isomorphisms

$$\int^{yz} K(a,y) \otimes K(b,z) \otimes p(y,z,x) \cong \int^{c} K(c,x) \otimes p(a,b,c) \quad \text{and}$$
$$j(x) \cong \int^{c} K(c,x) \otimes j(c).$$

A $\mathcal{V}$-natural transformation

$$\sigma : H \Rightarrow K : \mathcal{A}^{\mathrm{op}} \otimes \mathcal{X} \to \mathcal{V}$$

is called *multiplicative* if it commutes with the structure isomorphisms of H and K.

As we shall see below, a $\mathcal{V}$-functor $K : A^{\mathrm{op}} \otimes \mathcal{X} \to \mathcal{V}$ has multiplicative kernel structure precisely when the induced cocontinuous $\mathcal{V}$-functor $\overline{K} : [\mathcal{A}, \mathcal{V}] \to [\mathcal{X}, \mathcal{V}]$ preserves up to natural isomorphism the tensor product and unit of the convolution monoidal structure.

A simple calculation with coends shows that the "module composite" of two multiplicative kernels is again a multiplicative kernel and, for the given $\mathcal{V}$, this leads to a monoidal bicategory, in the sense of [10], with promonoidal $\mathcal{V}$-categories as the objects (0-cells), multiplicative kernels $K : \mathcal{A}^{\mathrm{op}} \otimes \mathcal{X} \to \mathcal{V}$ as the 1-cells $\mathcal{A} \to \mathcal{X}$, and the multiplicative $\mathcal{V}$-natural transformations σ as the 2-cells.

Here are a few routine examples:

Examples

(a) In the case where $\mathcal{A}$ is monoidal and $\mathcal{X}$ is comonoidal, so that

$$p(a,b,c) = \mathcal{A}(a \otimes b, c)$$
$$j(c) = \mathcal{A}(I, c)$$

and

$$p(y,z,x) = \mathcal{X}(y,x) \otimes \mathcal{X}(z,x)$$
$$j(x) = I$$

the structure isomorphisms for a multiplicative $K : \mathcal{A}^{\mathrm{op}} \otimes \mathcal{X} \to \mathcal{V}$ reduce to isomorphisms

$$K(a,x) \otimes K(b,x) \xrightarrow{\cong} K(a \otimes b, x)$$
$$I \xrightarrow{\cong} K(I, x)$$

by the Yoneda lemma.

(b) If $\mathcal{A}$ and $\mathcal{X}$ are both monoidal then a $\mathcal{V}$-functor $\varphi : \mathcal{A} \to \mathcal{X}$ is multiplicative (i.e.,

$$\varphi(a) \otimes \varphi(b) \xrightarrow{\cong} \varphi(a \otimes b)$$
$$I \xrightarrow{\cong} \varphi(I))$$

if and only if the module

$$K(a,x) = \mathcal{X}(\varphi(a), x)$$

is a multiplicative kernel, again by Yoneda.

(c) For any $\mathcal{A}$ and $\mathcal{X}$ promonoidal and $\mathcal{V}$-functor $\psi : \mathcal{X} \to \mathcal{A}$ the conditions for the module

$$K(a,x) = \mathcal{A}(a, \psi(x))$$

to be a multiplicative kernel are precisely the conditions for restriction along ψ to be a multiplicative functor:

$$[\psi, 1] : [\mathcal{A}, \mathcal{V}] \to [\mathcal{X}, \mathcal{V}]$$

when $[\mathcal{A}, \mathcal{V}]$ and $[\mathcal{X}, \mathcal{V}]$ are given their upper convolution tensor products.

(d) If $\mathcal{A} = \mathcal{I}$ (the identity $\mathcal{V}$-category), then $K : \mathcal{X} \to \mathcal{V}$ is a multiplicative kernel if and only if $K \mathbin{\bar{*}} K \cong K$. In two trivial cases, take $\mathcal{V} = (0 \leq 1)$ (cartesian closed) and $\mathcal{A} = 1$. First let $\mathcal{X} = M$ be a module over a ring R, and define

$$p(x,y,z) = \begin{cases} 1 \text{ iff } z = rx + sy \text{ for some } r, s \in R \\ 0 \text{ else.} \end{cases}$$

Then $K : M \to (0 \leq 1)$ is a multiplicative kernel if and only if $K^{-1}(1) \subset M$ is a R-submodule. Secondly, let $\mathcal{X} = X$ be a convexity space with join $[x,y]$ of points $x, y \in \mathcal{X}$. Define

$$p(x,y,z) = \begin{cases} 1 \text{ iff } z \in [x,y] \\ 0 \text{ else;} \end{cases}$$

then the multiplicative kernels correspond to the convex subsets of X. The promonoidal structure on $\mathcal{X}$ has no identity in these two examples.

(e) Association schemes [3, 18]. Given a promonoidal category $(\mathcal{A}, p, j)$ that has a unit object I to represent the identity j (i.e., $j(a) \cong \mathcal{A}(I, a)$) and given $\mathcal{X}$ of the form $\mathcal{B}^{\mathrm{op}} \otimes \mathcal{B}$ with the usual promultiplication corresponding to $\mathcal{B}$-bimodule composition "$\circ$", any functionally $\mathcal{A}$-indexed family of functors

$$M_a : \mathcal{B}^{\mathrm{op}} \otimes \mathcal{B} \to \mathcal{V}$$

with $M_I = \hom \mathcal{B}$, yields the multiplicative kernel

$$K : \mathcal{A}^{\mathrm{op}} \otimes (\mathcal{B}^{\mathrm{op}} \otimes \mathcal{B}) \to \mathcal{V}$$

(where $K(a,x,y) = M_a(x,y)$) if and only if there exists a natural "structure" isomorphism

$$M_a \circ M_b \cong \int^c p(a,b,c) \otimes M_c.$$

Again this follows directly from the Yoneda lemma applied to the multiplicative kernel criteria for this particular example.

By also defining an antipode

$$T : \mathcal{B}^{\mathrm{op}} \to \mathcal{B}$$

on $\mathcal{B}$, one can incorporate the notion of transpose matrix into this setting by defining

$$M_a^T(x, y) = M_a(Ty, Tx).$$

Finally, in the case of an association scheme, one has that (for $\mathcal{V} = \mathbf{Set}$) the category $\mathcal{B}^{\mathrm{op}} \otimes \mathcal{B}$ corresponds to the cartesian product $X \times X$ of a set X with itself, while $\mathcal{A}$ is the discrete category with objects the members of the given partition of $X \times X$, the cardinals of the respective promultiplication values $p(a, b, c)$ being the structure constants of the association scheme, and j being represented by the identity relation on X. Here $(\mathcal{A}, p, j)$ has $p(a, b, c) \cong p(b^*, a^*, c^*)$ under the antipode $Sa = a^*$ (the reverse relation on a), while T is just the identity function on X.

3. Transforms and analytic functors

Given $K : \mathcal{A}^{\mathrm{op}} \otimes \mathcal{X} \to \mathcal{V}$ as before, define the *(K-)transform* $\overline{K}(f) : \mathcal{X} \to \mathcal{V}$ of $f : \mathcal{A} \to \mathcal{V}$ as the (left) Kan extension [13]

$$\overline{K}(f)(x) = \int^a K(a, x) \otimes f(a),$$

and similarly the *dual* transform of h in $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ is defined as the (right) Kan extension

$$\overline{K}^{\vee}(h)(x) = \int_a [K(a, x), h(a)]$$

— especially used here when $\mathcal{V}$ is $*$-autonomous, in which case we have

$$\begin{aligned}\overline{K}^{\vee}(h) &= \int_a (K(a, x) \otimes h(a)^*)^* \\ &\cong (\int^a K(a, x) \otimes h(a)^*)^* \\ &= \overline{K}(h^*)^*.\end{aligned}$$

Then multiplicative functors (that is, functors between monoidal categories that preserve tensor product up to natural isomorphism without coherence conditions) of the forms $\overline{K}$ and $\overline{K}^{\vee}$, are generated by their corresponding multiplicative kernels K as in:

3.1. Theorem. *If K is multiplicative, then*

(i) *$\overline{K}$ preserves upper convolution, and*

(ii) *$\overline{K}^{\vee}$ preserves lower convolution.*

PROOF.

(i)

$$\begin{aligned}
\overline{K}(f \,\overline{*}\, g) &= \overline{K}\Big(\int^{ab} f(a) \otimes g(b) \otimes p(a,b,-)\Big) \\
&= \int^{c} K(c,-) \otimes \int^{ab} f(a) \otimes g(b) \otimes p(a,b,c) \\
&\cong \int^{ab}\int^{yz} K(a,y) \otimes K(b,z) \otimes p(y,z,-) \otimes f(a) \otimes g(b) \\
&\cong \int^{ab}\int^{yz} K(a,y) \otimes f(a) \otimes K(b,z) \otimes g(b) \otimes p(y,z,-) \\
&= \int^{yz} \Big(\int^{a} K(a,y) \otimes f(a)\Big) \otimes \Big(\int^{b} K(b,z) \otimes g(b)\Big) \otimes p(y,z,-) \\
&= \overline{K}(f) \,\overline{*}\, \overline{K}(g).
\end{aligned}$$

(ii) For functors $h = f^*$ and $k = g^*$ in $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$, we have

$$\begin{aligned}
\overline{K}^{\vee}(h \,\underline{*}\, k) &\cong \overline{K}((h \,\underline{*}\, k)^*)^* \\
&\cong \overline{K}(f \,\overline{*}\, g)^* && \text{since } h \,\underline{*}\, k = f^* \,\underline{*}\, g^* \cong (f \,\overline{*}\, g)^*, \\
&\cong (\overline{K}(f) \,\overline{*}\, \overline{K}(g))^* && \text{since } \overline{K} \text{ preserves } \overline{*} \text{ by (i)}, \\
&\cong \overline{K}(f)^* \,\underline{*}\, \overline{K}(g)^* \\
&\cong \overline{K}(h^*)^* \,\underline{*}\, \overline{K}(k^*)^* \\
&= \overline{K}^{\vee}(h) \,\underline{*}\, \overline{K}^{\vee}(k).
\end{aligned}$$

■

Each $\mathcal{V}$-functor $K : \mathcal{A}^{\mathrm{op}} \otimes \mathcal{X} \to \mathcal{V}$ yields the standard Kan adjunction

$$(\epsilon, \eta) : \overline{K} \dashv \underline{K} : [\mathcal{X}, \mathcal{V}] \to [\mathcal{A}, \mathcal{V}],$$

where

$$\underline{K}(f)(a) = \int_x [K(a,x), f(x)].$$

In the case where K is a multiplicative kernel we call $\overline{K}$ an "analytic" or "Fourier" transformation if and only if η is an equaliser (i.e., a regular monomorphism) in $[\mathcal{A}, \mathcal{V}]$.

NOTE Some of the examples in Section 4 have $\overline{K}$ fully faithful which means η is an isomorphism, i.e.,

$$[f, g] \cong [\overline{K}(f), \overline{K}(g)].$$

Then, if $\mathcal{V}$ is $*$-autonomous, there results a dual isomorphism involving

$$\langle f, g\rangle = \int^a f(a)^* \otimes g(a),$$

namely

$$\langle f, g\rangle \cong \langle \overline{K}(f), \overline{K}(g)\rangle.$$

This follows from

$$\begin{aligned}
\langle f, g\rangle^* &= (\int^a f(a)^* \otimes g(a))^* \\
&= \int_a (g(a) \otimes f(a)^*)^* \\
&= \int_a [g(a), f(a)] \\
&= \int_x [\overline{K}(g)(x), \overline{K}(f)(x)] && \text{since } \overline{K} \text{ is fully faithful,} \\
&= \int_x (\overline{K}(g)(x) \otimes \overline{K}(f)(x)^*)^* \\
&= (\int^x \overline{K}(f)(x)^* \otimes \overline{K}(g)(x))^* \\
&= \langle \overline{K}(f), \overline{K}(g)\rangle^*
\end{aligned}$$

which is a kind of Parseval relation.

Returning to the case where η is merely a regular monomorphism, we have that

$$\begin{tikzcd} f \arrow[r, "\eta"'] & \underline{K}\overline{K}(f) \arrow[r, shift left, "\eta\underline{K}\overline{K}"] \arrow[r, shift right, "\underline{K}\overline{K}\eta"'] & \underline{K}\overline{K}\underline{K}\overline{K}(f) \end{tikzcd}$$

is an equaliser diagram in $[\mathcal{A}, \mathcal{V}]$ (see [2] and [14] for example).

For each such kernel K one can construct a “Joyal-Wiener” category, here denoted $\mathbf{Joy}(K)$, as follows. A map

$$\alpha : \overline{K}(f) \Rightarrow \overline{K}(g)$$

in $[\mathcal{X}, \mathcal{V}]$ is called *regular* when

$$\overline{K}\underline{K}(\alpha)\overline{K}(\eta) = \overline{K}(\eta)\alpha;$$

then using the equalizer hypothesis on η, each such regular α equals $\overline{K}(\beta)$ for a unique $\beta : f \Rightarrow g$ in $[\mathcal{A}, \mathcal{V}]$. With this in mind, the $\mathcal{V}$-category $\mathbf{Joy}(K)$ is defined to be that subcategory of the Kleisli category $[\mathcal{A}, \mathcal{V}]_T$ for the monad

$$T = (\underline{K}\overline{K}, \underline{K}\epsilon\overline{K}, \eta),$$

with the same objects as $[\mathcal{A},\mathcal{V}]_T \subset [\mathcal{X},\mathcal{V}]$ but with the equaliser equations

$$\begin{tikzcd}
\mathbf{Joy}(K)(f,g) \arrow[r] & {[\mathcal{X},\mathcal{V}](\overline{K}(f),\overline{K}(g))} \arrow[r] \arrow[dr, "{(1,\overline{K}(\eta))}"'] & {[\mathcal{X},\mathcal{V}](\overline{K}\underline{K}\overline{K}(f),\overline{K}\underline{K}\overline{K}(g))} \arrow[d, "{(\overline{K}\eta,1)}"] \\
& & {[\mathcal{X},\mathcal{V}](\overline{K}(f),\overline{K}\underline{K}\overline{K}(g))}
\end{tikzcd}$$

in $\mathcal{V}$ defining its respective ($\mathcal{V}$-enriched) homs.

As a result, we obtain the usual Kleisli factorisation

$$\begin{tikzcd}
{[\mathcal{A},\mathcal{V}]_T} \arrow[rr, hook] & & {[\mathcal{X},\mathcal{V}]}\ , \\
& {[\mathcal{A},\mathcal{V}]} \arrow[ul, "\overline{K}_T"] \arrow[ur, "\overline{K}"'] &
\end{tikzcd}$$

where $\overline{K}_T$ is conservative, and $\mathbf{Joy}(K)$ is a $\mathcal{V}$-category with an equivalence

$$[\mathcal{A},\mathcal{V}] \xrightarrow{\simeq} \mathbf{Joy}(K)$$

and a conservative embedding into $[\mathcal{X},\mathcal{V}]$.

To relate this to the work of Joyal [12], we now consider the special case where $\mathcal{A}$ is promonoidal and $\mathcal{X} = k_*\mathcal{V}_0$ (comonoidal). Denote a fixed kernel E by

$$X(a) = E(a,X)$$

and think of the coend

$$F(X) = \int^a f(a) \otimes X(a)$$

as being an "E-analytic" functor of $X \in k_*\mathcal{V}_0$ with "coefficients" f in $[\mathcal{A},\mathcal{V}]$ and values in $\mathcal{V}$. Thus we obtain obvious types of extensions of A. Joyal's original notion of analytic functor [12]. That is, if we take $\mathcal{A}$ to be the free $\mathcal{V}$-category on the groupoid {finite sets and bijections} and let $E(a,X) = \bigotimes^a X$, then an E-analytic functor $F : k_*\mathcal{V}_0 \to \mathcal{V}$ takes the form

$$\begin{aligned} F(X) &= \int^a f(a) \otimes X(a) && (X(a) = E(a,X)) \\ &\cong \sum_a f(a) \underset{\mathrm{Aut}(a)}{\otimes} \left(\bigotimes^a X\right). \end{aligned}$$

Note that, for $\mathcal{V} = \mathbf{Set}$ and $\mathcal{V} = \mathbf{Vect}_k$, the units η_f of the adjunction $\overline{E} \dashv \underline{E}$ are equalisers in $[\mathcal{A},\mathcal{V}]$ because the diagram

$$\begin{tikzcd}
f(b) \arrow[r, "\eta_{f,b}"] & \int_X [E(b,X), \int^a E(a,X) \otimes f(a)] \arrow[d, "t"] \\
\int^a \mathcal{A}(a,b) \otimes f(a) \arrow[u, "Y"] \arrow[r, "\int^a m\otimes 1"'] & \int^a E(a,b) \otimes f(a)
\end{tikzcd}$$

commutes, where t is the composite map

$$\int_x [E(b,x), \int^a E(a,x) \otimes f(a)] \xrightarrow{\text{proj}} [E(b,b), \int^a E(a,b) \otimes f(a)]$$
$$\xrightarrow{[\mathrm{id},1]} [I, \int^a E(a,b) \otimes f(a)] \cong \int^a E(a,b) \otimes f(a),$$

and $\int^a m \otimes 1$ is a monomorphism in $\mathcal{V}$ since

$$m : \mathcal{A}(a,b) \to E(a,b)$$

is a cartesian natural monomorphism ($\mathcal{A}$ being a groupoid). Hence

$$[\mathcal{A}, \mathcal{V}] \xrightarrow{\simeq} \mathbf{Joy}(E)$$

so that, for $\mathcal{V} = \mathbf{Set}$, $\mathbf{Joy}(E)$ is equivalent to the category of all Joyal-analytic functors on **Set** and weakly cartesian maps between them, as one might expect.

Finally, even in the case of general $\mathcal{V}$, one can define the convolution product of two E-analytic functors

$$F(X) = \int^a f(a) \otimes X(a) \quad \text{and} \quad G(X) = \int^a g(a) \otimes X(a)$$

by

$$F * G(X) = \int^{ab} f(a) \otimes g(b) \otimes \int^c p(a,b,c) \otimes X(c)$$
$$\cong \int^c (\int^{ab} f(a) \otimes g(b) \otimes p(a,b,c)) \otimes X(c).$$

Also, the Hadamard product [5] can be defined as

$$F \times G(X) = \int^a (f(a) \otimes g(a)) \otimes X(a)$$

if $\mathcal{A}$ has a comonoidal structure as well as its initial promonoidal structure. In fact, if $K : \mathcal{A}^{\mathrm{op}} \otimes \mathcal{A} \to \mathcal{V}$ is multiplicative with respect to these two structures, then the K-transform of F, defined by

$$\overline{K}(F)(X) = \int^b (\int^a f(a) \otimes K(a,b)) \otimes X(b),$$

has the property

$$\overline{K}(F * G)(X) \cong (\overline{K}(F) \times \overline{K}(G))(X).$$

4. Examples where $\overline{K}$ is fully faithful

In this section we provide explicit inverses to various examples of $\mathcal{V}$-functors of the form

$$\overline{K} : [\mathcal{A}, \mathcal{V}] \to [\mathcal{X}, \mathcal{V}].$$

In these examples, $\mathcal{X}$ may be a large $\mathcal{V}$-category so that the functor category $[\mathcal{X}, \mathcal{V}]$ does not exist in the $\mathcal{V}$-universe; however the underlying "category" $[\mathcal{X}, \mathcal{V}]_0$ makes some sense.

The method is to firstly find a left inverse Γ to $\overline{K}$, then show that Γ_0 is faithful when restricted to the full image of $\overline{K}_0$ in $[\mathcal{X}, \mathcal{V}]_0$. For convenience, we shall here suppose that the unit $I \in \mathcal{V}_0$ generates $\mathcal{V}_0$, so that Γ is then $\mathcal{V}$-faithful on the full image of $\overline{K}$; by virtue of this monomorphism, the full image of $\overline{K}$ is usually a genuine $\mathcal{V}$-category. Thus, since

$$\begin{tikzcd} {[f,g]} \arrow[r, "\overline{K}"] \arrow[dr, "\cong"'] & {[\overline{K}(f), \overline{K}(g)]} \arrow[d, "\Gamma"] \\ & {[\Gamma\overline{K}(f), \Gamma\overline{K}(g)]} \end{tikzcd}$$

commutes, one has that Γ is an isomorphism, hence that $\overline{K}$ is fully faithful. In other words

$$[\mathcal{A}, \mathcal{V}] \xrightarrow{\simeq} \mathbf{Joy}(K)$$

where $\mathbf{Joy}(K) \subset [\mathcal{X}, \mathcal{V}]$ is a full embedding, not merely a conservative one.

In this part we shall suppose that $\mathcal{V}$ is a complete and cocomplete symmetric monoidal closed category. Of course the upper transformation $\overline{K}$ has a corresponding lower $\overline{K}^{\vee}$ if $\mathcal{V}$ happens to be $*$-autonomous also, in which case

$$\overline{K}^{\vee}(h) \cong \overline{K}(h^*)^*$$

so that $\Gamma\overline{K} \cong 1$ if and only if

$$\Gamma^{\vee}\overline{K}^{\vee}(h) = \Gamma(\overline{K}^{\vee}(h)^*)^* \cong h$$

for all h in $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$.

EXAMPLE 1 One of the simplest examples of a Fourier transform is that obtained from a Hopf algebroid in the sense of [10]. First $\mathcal{A}$ is given the promonoidal structure

$$p(a, b, c) = \mathcal{A}(a, Sb) \otimes \mathcal{A}(b, c) \quad \text{and} \quad j(a) = I,$$

where S denotes the antipode of the algebroid, so that convolution on $[\mathcal{A}, \mathcal{V}]$ becomes

$$\begin{aligned} f * g(c) &= \int^{a,b} f(a) \otimes g(b) \otimes p(a, b, c) \\ &\cong \int^{b} \int^{a} f(a) \otimes \mathcal{A}(a, Sb) \otimes g(b) \otimes \mathcal{A}(b, c) \\ &\cong \int^{b} f(Sb) \otimes g(b) \otimes \mathcal{A}(b, c) \end{aligned}$$

on applying the Yoneda lemma.

Secondly, let $\mathcal{X}$ be $\mathcal{A}$ also; then the pointwise tensor on $[\mathcal{X}, \mathcal{V}]$ is the usual

$$f \otimes g(c) = f(c) \otimes g(c).$$

If $K : \mathcal{A}^{\mathrm{op}} \otimes \mathcal{X} \to \mathcal{V}$ is taken to be the hom functor of $\mathcal{A}$, we then get an isomorphism

$$\overline{K} : [\mathcal{A}, \mathcal{V}] \cong [\mathcal{X}, \mathcal{V}] \quad \text{with} \quad \overline{K}(f * g) \cong f \otimes g$$

because, for such a Hopf algebroid, the so-called "fusion" natural isomorphism

$$\mathcal{A}(a, Sb) \otimes \mathcal{A}(b, c) \xrightarrow{\cong} \mathcal{A}(a, c) \otimes \mathcal{A}(b, c)$$

is always available. In the particular case where $\mathcal{A}$ is the single Hopf algebra H, the Fourier isomorphism is the composite

$$\begin{tikzcd} H \otimes H \arrow[rr, "\cong"] \arrow[d, "1\otimes\delta"'] & & H \otimes H \\ H \otimes H \otimes H \arrow[rr, "1\otimes S\otimes 1"'] & & H \otimes H \otimes H \arrow[u, "\mu\otimes 1"'] \end{tikzcd}$$

with inverse the fusion map $(\mu \otimes 1)(1 \otimes \delta)$; see [4, §2.3].

Example 2 Given any (small) promonoidal $\mathcal{V}$-category $(\mathcal{A}, p, j)$, let $\mathcal{A}$ be $\mathcal{A}$ itself,

$$K = p : \mathcal{A}^{\mathrm{op}} \otimes \mathcal{A}^{\mathrm{op}} \otimes \mathcal{A} \to \mathcal{V}$$

and $\mathcal{X} = \mathcal{A}^{\mathrm{op}} \otimes \mathcal{A}$, promonoidal via bimodule composition in $[\mathcal{A}^{\mathrm{op}} \otimes \mathcal{A}, \mathcal{V}]$. Then

$$\overline{K} : [\mathcal{A}, \mathcal{V}] \to [\mathcal{A}^{\mathrm{op}} \otimes \mathcal{A}, \mathcal{V}]$$

is given by

$$\overline{K}(f) = \int^{a} f(a) \otimes p(a, -, -).$$

and is always conservative.

Since $f * g = \int^{xy} f(x) \otimes g(y) \otimes p(x, y, -)$, we obtain

$$\begin{aligned} \overline{K}(f * g)(u, v) &= \int^{xyz} f(x) \otimes g(y) \otimes p(x, y, z) \otimes p(z, u, v) \\ &\cong \int^{z} \int^{x} f(x) \otimes p(x, z, v) \otimes \int^{y} g(y) \otimes p(y, u, z) \\ &= \int^{z} \overline{K}(f)(z, v) \otimes \overline{K}(g)(u, z) \\ &= (\overline{K}(f) \circ \overline{K}(g))(u, v), \end{aligned}$$

so K is multiplicative. A tentative Γ for this $\overline{K}$ is given by

$$\Gamma(F)(b) = \int^a F(a,b) \otimes j(a)$$

since

$$\begin{aligned}\Gamma\overline{K}(f)(b) &= \int^a \overline{K}(f)(a,b) \otimes j(a) \\ &= \int^a \int^x f(x) \otimes p(x,a,b) \otimes j(a) \\ &\cong \int^x f(x) \otimes \mathcal{A}(x,b) \qquad \text{since } p * j = \mathcal{A}(-,-) \\ &\cong f(b) \qquad \text{by Yoneda.}\end{aligned}$$

4.1. Proposition. *If $\mathcal{V} = \mathbf{Vect}_k$ and $j(d)$ is finite dimensional for all d, then $F \simeq [j, \Gamma(F)]$ if $j : \mathcal{A} \to \mathcal{V}$ is fully faithful.*

Proof. We require $F \to [j, F * j]$ to be an isomorphism for all F in $[\mathcal{A}^{\mathrm{op}} \otimes \mathcal{A}, \mathcal{V}]$. But

$$\begin{aligned}{[j, F * j](d,c)} &= [j(d), F * j(c)] \\ &= [j(d), \int^x F(x,c) \otimes j(x)] \\ &\cong \int^x F(x,c) \otimes [j(d), j(x)]\end{aligned}$$

and

$$\begin{aligned}F(d,c) &\cong \int^x F(x,c) \otimes \mathcal{A}(d,x) \quad \text{(Yoneda)} \\ &\simeq \int^x F(x,c) \otimes [j(d), j(x)]\end{aligned}$$

if j is $\mathcal{V}$-fully faithful, in which case $\overline{K}$ is fully faithful. ■

Example 3 Let $(\mathcal{A}, p, j)$ be a small promonoidal $\mathcal{V}$-category. A functor

$$\varphi : \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$$

is called multiplicative if there are natural isomorphisms

$$\int^c \varphi(c) \otimes p(a,b,c) \cong \varphi(a) \otimes \varphi(b)$$

$$\int^c \varphi(c) \otimes j(c) \cong I$$

(in the sense of [9]). A natural transformation between two such functors is called multiplicative if it commutes with these isomorphisms.

The (naturally) comonoidal $\mathcal{V}$-category

$$\mathcal{X} = \mathcal{M}(\mathcal{A}^{\mathrm{op}}, \mathcal{V})$$

is defined to be the free $\mathcal{V}$-category in the ordinary category of all multiplicative functors from $\mathcal{A}^{\mathrm{op}}$ to $\mathcal{V}$, and multiplicative natural transformations between them. The kernel

$$K : \mathcal{A}^{\mathrm{op}} \otimes \mathcal{M}(\mathcal{A}^{\mathrm{op}}, \mathcal{V}) \to \mathcal{V}$$

is given by evaluation $K(a, \varphi) = \varphi(a)$ so that

$$\overline{K}(f)(\varphi) = \int^{a} f(a) \otimes \varphi(a).$$

4.2. Remark. In fact, it is quite rare for each representable functor on $\mathcal{A}^{\mathrm{op}}$ to be multiplicative in any sense. Rather, if we don't assume this and suppose that the canonical fork

$$\int^{c} \varphi(a) \otimes \varphi^{*}(c) \otimes \varphi(c) \otimes \varphi^{*}(b) \underset{1\otimes e}{\overset{e\otimes 1}{\rightrightarrows}} \varphi(a) \otimes \varphi^{*}(b) \xrightarrow{e} \mathcal{A}(a, b)$$

(where $\varphi^{*} = \int_{x} [\varphi(x), \mathcal{A}(x, -)]$ is the conjugate of φ and e is the evident evaluation map) is always a coequalizer in $\mathcal{V}$ for at least one φ in $\mathcal{M}(\mathcal{A}^{\mathrm{op}}, \mathcal{V})$, then $\overline{K}$ is conservative and we can proceed from there (see §5).

If each representable functor $\mathcal{A}(-, a)$ is multiplicative, we can define a functor

$$\Gamma : [\mathcal{M}(\mathcal{A}^{\mathrm{op}}, \mathcal{V}), \mathcal{V}] \to [\mathcal{A}, \mathcal{V}]$$

by

$$\Gamma(F)(d) = F(\mathcal{A}(-, d)),$$

so that $\Gamma\overline{K} \cong 1$ by the Yoneda lemma. But this Γ is faithful on the full subcategory of $[\mathcal{M}(\mathcal{A}^{\mathrm{op}}, \mathcal{V}), \mathcal{V}]$ consisting of those functors F which preserve the Yoneda colimit

$$\varphi \cong \int^{a} \varphi(a) \otimes \mathcal{A}(-, a).$$

Moreover, each $\overline{K}(f)$ is clearly an F with this property, so that $\overline{K}$ is a full embedding.

By using the assumption that each φ is multiplicative the kernel K can be seen to be multiplicative.

Example 4 Let $\mathcal{C}$ be a monoidal $\mathcal{V}$-category and let $\mathcal{A} \subset \mathcal{C}$ be a small Cauchy dense promonoidal $\mathcal{V}$-subcategory. Let $\mathcal{X} = \mathcal{C} \otimes \mathcal{V}$ with the product monoidal structure from $\mathcal{C}$ and $\mathcal{V}$, and let

$$K : \mathcal{A}^{\mathrm{op}} \otimes (\mathcal{C} \otimes \mathcal{V}) \to \mathcal{V}$$

be given by

$$K(a, c, x) = \mathcal{C}(a, c) \otimes x.$$

Then

$$\overline{K}(f)(c, x) = \int^a f(a) \otimes \mathcal{C}(a, c) \otimes x \quad \text{and} \quad \Gamma(F)(a) = F(a, I)$$

since

$$\begin{aligned} \Gamma\overline{K}(f)(b) &= \overline{K}(f)(b, I) \\ &= \int^a f(a) \otimes \mathcal{C}(a, b) \otimes I \\ &\cong \int^a f(a) \otimes \mathcal{A}(a, b) \\ &\cong f(b). \end{aligned}$$

Further application of the Yoneda lemma, and the calculus of coends, shows that $\overline{K}$ is multiplicative because

$$\begin{aligned} \int^a K(a, z, x) \otimes p(b, c, a) &\cong \int^a (\mathcal{C}(a, z) \otimes x) \otimes \mathcal{C}(b \otimes c, a) \quad \text{since } p(b, c, a) = \mathcal{C}(b \otimes c, a) \text{ on } \mathcal{A}, \\ &\cong \mathcal{C}(b \otimes c, z) \otimes x \quad \text{since } \mathcal{A} \subset \mathcal{C} \text{ is Cauchy dense}, \\ &\cong \int^{z', z'', x', x''} (\mathcal{C}(b, z') \otimes x') \otimes (\mathcal{C}(c, z'') \otimes x'') \otimes (\mathcal{C}(z' \otimes z'', z) \otimes [x' \otimes x'', x], \\ &= \int^{z', z'', x', x''} K(b, z', x') \otimes K(c, z'', x'') \otimes p((z', x'), (z'', x''), (z, x)) \end{aligned}$$

as required. Moreover, $\overline{K}$ lands in the full subcategory of $[\mathcal{C} \otimes \mathcal{V}, \mathcal{V}]$ consisting of those F for which the canonical map

$$x \otimes F(a, I) \xrightarrow{\cong} F(a, x)$$

is an isomorphism, on which Γ is clearly faithful, so that $\overline{K}$ is fully faithful.

Example 5 Suppose $\mathcal{A}$ is a locally finite small monoidal $\mathcal{V}$-category with a given natural isomorphism

$$\mathcal{A}(a, b) \cong \mathcal{A}(b, a)^*.$$

Let $\mathcal{X} = [\mathcal{A}, \mathcal{V}_f]^{\mathrm{op}}$ (monoidal under convolution) and define

$$K : \mathcal{A}^{\mathrm{op}} \otimes [\mathcal{A}, \mathcal{V}_f]^{\mathrm{op}} \to \mathcal{V}$$

by $K(a,g) = g(a)^*$. Then

$$\overline{K}(f)(g) = \int^a f(a) \otimes g(a)^* = \langle g, f \rangle$$

where

$$\overline{K} : [\mathcal{A}, \mathcal{V}] \to [[\mathcal{A}, \mathcal{V}_f]^{\mathrm{op}}, \mathcal{V}]$$

so that $\Gamma(F)(a) = F(\mathcal{A}(a, -))$ gives

$$\begin{aligned} \Gamma\overline{K}(f)(b) &= \int^a f(a) \otimes \mathcal{A}(b,a)^* \\ &\cong \int^a f(a) \otimes \mathcal{A}(a,b) \\ &\cong f(b). \end{aligned}$$

To show that $\overline{K}$ is multiplicative, we first prove:

4.3. LEMMA. *$g(a)^* \cong [\mathcal{A}, \mathcal{V}](\mathcal{A}(a,-), g)^* \cong [\mathcal{A}, \mathcal{V}](g, \mathcal{A}(a,-))$ for all g in $[\mathcal{A}, \mathcal{V}]$.*

PROOF.

$$\begin{aligned} \int_b [g(b), \mathcal{A}(a,b)] &= \int_b (g(b) \otimes \mathcal{A}(a,b)^*)^* \\ &\cong (\int^b g(b) \otimes \mathcal{A}(b,a))^* \\ &\cong g(a)^* \quad \text{by Yoneda.} \end{aligned}$$

∎

4.4. PROPOSITION. *$\overline{K}$ is multiplicative.*

PROOF. We require a natural isomorphism

$$\int^{g,h} K(a,g) \otimes K(b,h) \otimes [\mathcal{A}, \mathcal{V}](k, g * h) \cong \int^c K(c,k) \otimes p(a,b,c).$$

By the lemma, the left side is isomorphic to

$$\begin{aligned} &\int^{g,h} [\mathcal{A}, \mathcal{V}](g, \mathcal{A}(a,-)) \otimes [\mathcal{A}, \mathcal{V}](h, \mathcal{A}(b,-)) \otimes [\mathcal{A}, \mathcal{V}](k, g * h) \\ &\quad \cong [\mathcal{A}, \mathcal{V}](k, \mathcal{A}(a,-) * \mathcal{A}(b,-)) \quad \text{by Yoneda,} \\ &\quad \cong [\mathcal{A}, \mathcal{V}](\mathcal{A}(a \otimes b, -), k)^* \quad \text{by the Lemma,} \\ &\quad \cong k(a \otimes b)^* \\ &\quad \cong \int^c K(c,k) \otimes p(a,b,c), \end{aligned}$$

as required. ∎

Also Γ is faithful on the full image of $\overline{K}$ since

$$\begin{aligned}\overline{K}(f)(g)^* &\cong \int_a [f(a), g(a)] \\ &\cong \int_a [\overline{K}(f)(\mathcal{A}(a,-)), g(a)].\end{aligned}$$

so that $\overline{K}$ is fully faithful.

Example 6 Let $\mathcal{V} = \mathbf{Vect}_k$ for a fixed field k, and let $\mathcal{X}$ be a finite promonoidal $\mathcal{V}$-category. Take

$$\mathcal{A} = [\mathcal{X}, \mathbf{Vect}_f]^{\mathrm{op}} \quad \text{(which is monoidal under convolution)}$$

and let

$$K : \mathcal{A}^{\mathrm{op}} \otimes \mathcal{X} \to \mathcal{V} \quad \text{be evaluation.}$$

Then

$$\overline{K}(f)(x) = \int^a f(a) \otimes a(x) \cong f(\mathcal{X}(x,-)),$$

and we choose

$$\Gamma(F)(a) = \int_x [a(x), F(x)].$$

Then

$$\begin{aligned}\Gamma\overline{K}(f)(b) &= \int_x [b(x), \int^a a(x) \otimes f(a)] \\ &\cong \int^a \int_x b(x)^* \otimes a(x) \otimes f(a) \quad \text{since } \int^a \text{ is left exact,} \\ &\cong \int^a \mathcal{A}(b,a) \otimes f(a) \\ &\cong f(b) \quad \text{by Yoneda.}\end{aligned}$$

Thus Γ is faithful on the full image of $\overline{K}$ since

$$\begin{aligned}\Gamma\overline{K}(f)(\mathcal{X}(x,-)) &\cong f(\mathcal{X}(x,-)) \\ &\cong \overline{K}f(x);\end{aligned}$$

moreover, $\overline{K}$ is multiplicative by Yoneda, as required.

Example 7 Let $\mathcal{C}$ be a small braided compact closed category, let $\mathcal{A} = \mathcal{C}^{\mathrm{op}}$ with the monoidal structure induced from $\mathcal{C}$, and let $\mathcal{X}$ be a (finite) Cauchy dense full subcategory of $\mathcal{A}$ (cf. [11] and [15]).

Then $\mathcal{X}$ is promonoidal with respect to the structure induced by $\mathcal{A}$, namely

$$\begin{aligned} p(x,y,z) &= \mathcal{C}(z, x\otimes y) \\ j(z) &= \mathcal{C}(z, I); \end{aligned}$$

the associative and unital axioms follow from the Cauchy density of $\mathcal{X} \subset \mathcal{A}$. Moreover, here the unit η of the $\overline{K} \dashv \underline{K}$ adjunction, where $K(a,x) = \mathcal{C}(x,a)$, is an isomorphism since each component

$$\eta_f : f(a) \to \int_x [K(a,x), \int^b K(b,x) \otimes f(b)]$$

gives

$$\begin{tikzcd} f(a) \arrow[r] \arrow[dr] & \int_x [\mathcal{C}(x,a), \int^b \mathcal{C}(x,b) \otimes f(b)] \arrow[d, "\cong"] \\ & \int_x [\mathcal{C}(x,a), f(x)] \end{tikzcd}$$

by the Yoneda lemma, which becomes the isomorphism

$$f(a) \xrightarrow{\cong} \int_x [\mathcal{C}(x,a), f(x)]$$

by the Cauchy density of $\mathcal{X} \subset \mathcal{A}$. Hence $\overline{K}$ is fully faithful.

Also, $\overline{K}$ is multiplicative because the kernel K satisfies the conditions

$$\begin{aligned} \int^{xy} K(a,x) \otimes K(b,y) \otimes p(x,y,z) &= \int^{xy} \mathcal{C}(x,a) \otimes \mathcal{C}(y,b) \otimes \mathcal{C}(z, x\otimes y) \\ &\cong \mathcal{C}(z, a\otimes b) \quad \text{by Cauchy density of } \mathcal{X} \subset \mathcal{A}, \end{aligned}$$

while

$$\begin{aligned} \int^c K(c,z) \otimes \mathcal{C}(c, a\otimes b) &= \int^c \mathcal{C}(z,c) \otimes \mathcal{C}(c, a\otimes b) \\ &\cong \mathcal{C}(z, a\otimes b) \end{aligned}$$

by the Yoneda lemma applied to $c \in \mathcal{C}$, and

$$j(x) \cong \int^c \mathcal{C}(x,c) \otimes j(c) = \int^c K(c,x) \otimes j(c)$$

for the same reason.

5. When $\overline{K}$ is conservative

Sometimes $\overline{K}$ is only conservative; i.e., the unit of the $\overline{K} \dashv \underline{K}$ adjunction is merely an equaliser, not an isomorphism. Then, as pointed out earlier, we obtain an equivalence

$$[\mathcal{A}, \mathcal{V}] \simeq \mathbf{Joy}(K)$$

with a conservative multiplicative embedding of $\mathbf{Joy}(K)$ into $[\mathcal{X}, \mathcal{V}]$. Examples are available if we suppose that the composite of two equalisers is an equaliser in $\mathcal{V}$ and that, for the given functor K, the maps

1. coevaluation
$$f(a) \to \int_x [K(a,x), K(a,x) \otimes f(a)]$$
and
2. coprojection
$$K(a,x) \otimes f(a) \to \int^b K(b,x) \otimes f(b)$$

are both equalisers in $\mathcal{V}$ for all f in $[\mathcal{A}, \mathcal{V}]$.

The conservatism of $\overline{K}$ then follows from consideration of the diagram

$$\begin{tikzcd}
f(a) \arrow[r, "\mathrm{coev}"] \arrow[dr, "\eta_f"'] & \int_x [K(a,x), K(a,x) \otimes f(a)] \arrow[d, "{[1,\mathrm{coprojn}]}"] \\
& \int_x [K(a,x), \int^b K(b,x) \otimes f(b)]
\end{tikzcd}$$

which can be seen to commute by applying the Yoneda lemma to the variable $f \in [\mathcal{A}, \mathcal{V}]$.

Example 8 Note that all the coprojections

$$K(a,x) \otimes f(a) \to \int^b K(b,x) \otimes f(b)$$

are coretractions (hence equalisers) when $\mathcal{A}$ is separable in the sense that there exists a natural transformation

$$\mathcal{A}(a,b) \xrightarrow{\delta} \mathcal{A}(a,c) \otimes \mathcal{A}(c,b)$$

for which the composites

$$\mathcal{A}(a,a) \xrightarrow{\delta} \mathcal{A}(a,a) \otimes \mathcal{A}(a,a) \xrightarrow{\mu} \mathcal{A}(a,a)$$

are isomorphisms for all $a \in \mathcal{A}$, where μ denotes the composition in $\mathcal{A}$. The verification of this is straightforward on observing that the separability of $\mathcal{A}$ implies the existence of a canonical comparison map

$$\int^b K(b,x) \otimes f(b) \xrightarrow{\hat{\delta}} \int_b K(b,x) \otimes f(b).$$

Example 9 Let $\mathcal{C}$ be a cocomplete monoidal category for which each functor of the form

$$- \otimes C : \mathcal{C} \to \mathcal{C}$$

preserves colimits, and suppose that there exists a full embedding

$$N : \mathcal{A}^{\mathrm{op}} \subset \mathcal{C}$$

where $\{Nx; x \in \mathcal{A}\}$ is a small dense set of projectives in $\mathcal{C}$. Suppose also that $I = NI$ for some $I \in \mathcal{A}$.

Define the functor $U : \mathcal{C} \to [\mathcal{A}^{\mathrm{op}} \otimes \mathcal{A}, \mathcal{V}]$ by

$$U(C)(x, y) = \mathcal{C}(Ny, Nx \otimes C).$$

If $[\mathcal{A}^{\mathrm{op}} \otimes \mathcal{A}, \mathcal{V}]$ is given the monoidal structure of bimodule composition, then U becomes a conservative multiplicative functor and there is also an induced monoidal equivalence

$$\overline{N} : \mathcal{C} \simeq [\mathcal{A}, \mathcal{V}]$$

given by $\overline{N}(C)(x) = \mathcal{C}(Nx, C)$, where $\mathcal{A}$ has the promonoidal structure for which

$$p : \mathcal{A}^{\mathrm{op}} \otimes \mathcal{A}^{\mathrm{op}} \otimes \mathcal{A} \to \mathcal{V} \quad \text{corresponds to} \quad UN : \mathcal{A}^{\mathrm{op}} \to [\mathcal{A}^{\mathrm{op}} \otimes \mathcal{A}, \mathcal{V}]$$

and

$$j = \mathcal{A}(I, -) : \mathcal{A} \to \mathcal{V},$$

and where $[\mathcal{A}, \mathcal{V}]$ has the resulting convolution monoidal structure.

Example 10 ($\mathcal{V}$ = commutative monoids in **Set**; see [17]) Let G be a finite group, let $\mathcal{A} = \mathbf{Span}(\text{G-}\mathbf{Set}_f)$, and let $\mathcal{X} = \mathbf{Rep}_f(G)$ (for a fixed field) viewed as $\mathcal{V}$-categories. Define the kernel

$$K : \mathcal{A}^{\mathrm{op}} \otimes \mathcal{X} \to \mathcal{V}$$

by $K(a, x) = \mathcal{X}(Fa, x)$ where

$$F : \mathcal{A} \to \mathcal{X} = \mathbf{Rep}_f(G)$$

is the extension of the canonical functor

$$F : \text{G-}\mathbf{Set}_f \to \mathcal{X},$$

defined by "summing over the fibres" (see [17, §10] for example), and define the functor

$$U : \mathcal{X} \to \mathbf{Span}(\text{G-}\mathbf{Set})$$

by $U(x)$ is the large induced G-Set of the representation $x : G \to \mathbf{Vect}_f$ in $\mathcal{X}$.

Then we have a canonical natural transformation

$$\tau : \mathcal{X}(Fb, Fa) \to \mathbf{Span}(\text{G-}\mathbf{Set})(b, UFa)$$

in $\mathcal{V}$, and a natural coretraction

$$\beta : a \to UF(a)$$

in **Span**(G-**Set**). The fact that $\overline{K}$ is conservative now follows from commutativity of the diagram

$$\begin{tikzcd}
f(a) \arrow[r, "\eta_f"] \arrow[dd, "\cong"'] & \int_x [\mathcal{X}(Fa,x), \int^b \mathcal{X}(Fb,x) \otimes f(b)] \\
& \int^b \mathcal{X}(Fb,Fa) \otimes f(b) \arrow[u, "\cong"'] \arrow[d, "\int^b \tau\otimes 1"] \\
\int^b \mathcal{A}(b,a) \otimes f(b) \arrow[r, "\rho"'] & \int^b \mathbf{Span}(\text{G-}\mathbf{Set})(b, UFa) \otimes f(b)
\end{tikzcd}$$

where ρ is the coretraction induced by β, and the isomorphisms shown in the diagram are again by the Yoneda lemma.

The main result of [9] applies to show that $\overline{K}$ is multiplicative.

EXAMPLE 11 In Example 2 of §4 we have that

$$\overline{K} : [\mathcal{A}, \mathcal{V}] \to [\mathcal{A}^{\mathrm{op}} \otimes \mathcal{A}, \mathcal{V}]$$

is conservative if $(\mathcal{A}, p, j)$ is a closed category; i.e., if

$$p(a,x,y) = \mathcal{A}(a,[x,y]) \quad \text{and} \quad j(y) = \mathcal{A}(I,y).$$

In fact, here η_f is a coretraction since

$$\begin{aligned}
\eta_f : f(a) \to & \int_{xy} [p(a,x,y), \int^b p(b,x,y) \otimes f(b)] \\
\cong & \int_{xy} [\mathcal{A}(a,[x,y]), f([x,y])] \quad \text{by Yoneda,}
\end{aligned}$$

where

$$\begin{tikzcd}
f(a) \arrow[r, "\eta_f"] \arrow[d, "1"'] & \int_{xy} [\mathcal{A}(a,[x,y]), f([x,y])] \arrow[d, "x=I, y=z"] \\
f(a) \arrow[r, "\cong"] & \int_z [\mathcal{A}(a,z), f(z)]
\end{tikzcd}$$

commutes. Moreover, if there is a natural isomorphism $[a,[b,c]] \cong [b,[a,c]]$ then this $\overline{K}$ is multiplicative.

Appendix

The term "graphic" applied to coends, convolution products, transforms, etc., in the above context is intended to relate especially to base categories like $\mathcal{V} = \mathbf{Set}$ and $\mathcal{V} = \mathbf{Vect}_k$, where there is a definite notion of finite graph. In such cases we call a given $\mathcal{V}$-subgraph $\mathcal{X}$ of the $\mathcal{V}$-category $\mathcal{A}$ a finite $\mathcal{V}$-graph if $\mathrm{ob}(\mathcal{X})$ is finite and each $\mathcal{V}$-object $\mathcal{X}(x, y)$ of edges is finite (or finite dimensional).

Then, if the category $\mathcal{A}$ is the directed union of all its finite $\mathcal{V}$-subgraphs $\mathcal{X}_\varphi$ (or some convenient subset of these), there results a canonical isomorphism

$$\operatorname*{colim}_{\varphi} \int^{x} T_\varphi(x, x) \xrightarrow{\cong} \int^{a} T(a, a),$$

where T_φ denotes the restriction of each $\mathcal{V}$-functor

$$T : \mathcal{A}^{\mathrm{op}} \otimes \mathcal{A} \to \mathcal{V}$$

to the $\mathcal{V}$-graph $\mathcal{X}_\varphi^{\mathrm{op}} \otimes \mathcal{X}_\varphi$, and where each finite "coend" $\int^x T_\varphi(x, x)$ over $x \in \mathcal{X}_\varphi$ is computed in the same way as the usual coend of a functor over any (small) category. In particular, any finite $\mathcal{V}$-limit which commutes with each finite $\int^x$, also commutes with their filtered colimit $\int^a$ over $a \in \mathcal{A}$.

However, we note that for many practical purposes the finite "coend" $\int^x$ can be replaced by the (usual) coend over the corresponding full subcategory of $\mathcal{A}$ determined by $\mathrm{ob}(\mathcal{X})$.

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