Theory and Applications of Categories, Vol. 25, No. 8, 2011, pp. 189–246.

MODEL-CATEGORIES OF COALGEBRAS OVER OPERADS

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ABSTRACT. This paper constructs model structures on the categories of coalgebras and pointed irreducible coalgebras over an operad whose components are projective, finitely generated in each dimension, and satisfy a condition that allows one to take tensor products with a unit interval. The underlying chain-complex is assumed to be unbounded and the results for bounded coalgebras over an operad are derived from the unbounded case.

Contents

1. Introduction

Although the literature contains several papers on homotopy theories for *algebras* over operads—see [15], [18], and [19]—it is more sparse when one pursues similar results for coalgebras. In [21], Quillen developed a model structure on the category of 2-connected cocommutative coalgebras over the rational numbers. V. Hinich extended this in [14] to coalgebras whose underlying chain-complexes were unbounded (i.e., extended into negative dimensions). Expanding on Hinich's methods, K. Lefvre derived a model structure on the category of coassociative coalgebras—see [16]. In general, these authors use indirect methods, relating coalgebra categories to other categories with known model structures.

Our paper finds model structures for coalgebras over any operad fulfilling a basic requirement (condition 4.3). Since operads uniformly encode many diverse coalgebra structures (coassociative-, Lie-, Gerstenhaber-coalgebras, etc.), our results have wide applicability.

Received by the editors 2008-02-31 and, in revised form, 2011-02-17.
Transmitted by Ieke Moerdijk. Published on 2011-03-17.
2000 Mathematics Subject Classification: 18G55, 55U40.
Key words and phrases: operads, cofree coalgebras.

The author's intended application involves investigating the extent to which Quillen's results in rational homotopy theory ([21]) can be generalized to *integral* homotopy theory.

Several unique problems arise that require special techniques. For instance, constructing injective resolutions of coalgebras naturally leads into infinitely many negative dimensions. The resulting model structure—and even that on the underlying chain-complexes—fails to be cofibrantly generated (see [6]). Consequently, we cannot easily use it to induce a model structure on the category of coalgebras.

We develop the general theory for unbounded coalgebras, and derive the bounded results by applying a truncation functor.

In § 2, we define operads and coalgebras over operads. We also give a basic condition (see 4.3) on the operad under consideration that we assume to hold throughout the paper. Cofibrant operads always satisfy this condition and every operad is weakly equivalent to one that satisfies this condition.

In § 3, we briefly recall the notion of model structure on a category and give an example of a model structure on the category of unbounded chain-complexes.

In § 4, we define a model structure on categories of coalgebras over operads. When the operad is projective and finitely-generated in all dimensions, we verify that nearly free coalgebras satisfy Quillen's axioms of a model structure (see [20] or [12]).

Section 4.1 describes our model-structure —classes of cofibrations, fibrations and weak equivalences. Section 4.12 proves the first few axioms of a model-structure (CM 1 through CM 3, in Quillen's notation). Section 4.13 proves axiom CM 5, and section 4.22 proves CM 4.

A key step involves proving the existence of cofibrant and fibrant replacements for objects. In our model structure, all coalgebras are cofibrant (solving this half of the problem) and the hard part of is to find *fibrant* replacements.

We develop resolutions of coalgebras by cofree coalgebras—our so-called rug-resolutions—that solves the problem: see lemma 4.19 and corollary 4.20. This construction naturally leads into infinitely many negative dimensions and was the motivation for assuming underlying chain-complexes are unbounded.

All coalgebras are cofibrant and *fibrant* coalgebras are characterized as retracts of canonical resolutions called rug-resolutions (see corollary 4.21 and corollary 4.21)—an analogue to total spaces of Postnikov towers.

In the cocommutative case over the rational numbers, the model structure that we get is *not* equivalent to that of Hinich in [14]. He gives an example (9.1.2) of a coalgebra that is acyclic but not contractible. In our theory it *would* be contractible, since it is over the rational numbers and bounded.

In § 4.25, we discuss the (minor) changes to the methods in § 4 to handle coalgebras that are bounded from below. This involves replacing the cofree coalgebras by their truncated versions.

In § 5, we consider two examples over the rational numbers. In the rational, 2-connected, cocommutative, coassociative case, we recover the model structure Quillen defined in [21] —see example 5.2.

In appendix A, we study *nearly free* $\mathbb{Z}$-modules. These are modules whose countable submodules are all $\mathbb{Z}$-free. They take the place of free modules in our work, since the cofree coalgebra on a free modules is not free (but is nearly free).

In appendix B, we develop essential category-theoretic constructions, including equalizers (§ B.9), products and fibered products (§ B.14), and colimits and limits (§ B.21). The construction of limits in § B.21 was this project's most challenging aspect and consumed the bulk of the time spent on it. This section's key results are corollary B.28, which allows computation of inverse limits of coalgebras and theorem B.30, which shows that these inverse limits share a basic property with those of chain-complexes.

I am indebted to Professor Bernard Keller for several useful discussions.

2. Notation and conventions

Throughout this paper, R will denote a field or $\mathbb{Z}$.

2.1. Definition. *An R-module M will be called* nearly free *if every countable submodule is R-free.*

Remark. This condition is automatically satisfied unless $R = \mathbb{Z}$.

Clearly, any $\mathbb{Z}$-free module is also nearly free. The Baer-Specker group, $\mathbb{Z}^{\aleph_0}$, is a well-known example of a nearly free $\mathbb{Z}$-module that is *not* free—see [11], [1], and [24]. Compare this with the notion of $\aleph_1$-*free groups*—see [5].

By abuse of notation, we will often call chain-complexes nearly free if their underlying modules are (ignoring grading).

Nearly free $\mathbb{Z}$-modules enjoy useful properties that free modules do *not*. For instance, in many interesting cases, the cofree coalgebra of a nearly free chain-complex is nearly free.

2.2. Definition. *We will denote the closed symmetric monoidal category of unbounded, nearly free R-chain-complexes with R-tensor products by* $\mathbf{Ch}$. *We will denote the category of R*-free *chain chain-complexes that are* bounded from below *in dimension* 0 *by* $\mathbf{Ch}_0$.

The chain-complexes of $\mathbf{Ch}$ are allowed to extend into arbitrarily many negative dimensions and have underlying graded R-modules that are

- arbitrary if R is a field (but they will be free)
- *nearly free,* in the sense of definition 2.1, if $R = \mathbb{Z}$.

We make extensive use of the Koszul Convention (see [13]) regarding signs in homological calculations:

2.3. Definition. *If $f\colon C_1 \to D_1$, $g\colon C_2 \to D_2$ are maps, and $a \otimes b \in C_1 \otimes C_2$ (where a is a homogeneous element), then $(f \otimes g)(a \otimes b)$ is defined to be $(-1)^{\deg(g)\cdot\deg(a)} f(a) \otimes g(b)$.*

2.4. Remark. *If f_i, g_i are maps, it isn't hard to verify that the Koszul convention implies that* $(f_1 \otimes g_1) \circ (f_2 \otimes g_2) = (-1)^{\deg(f_2)\cdot\deg(g_1)}(f_1 \circ f_2 \otimes g_1 \circ g_2)$.

2.5. Definition. *The symbol I will denote the unit interval, a chain-complex given by*

$$\begin{aligned} I_0 &= R \cdot p_0 \oplus R \cdot p_1 \\ I_1 &= R \cdot q \\ I_k &= 0 \text{ if } k \neq 0, 1 \\ \partial q &= p_1 - p_0 \end{aligned}$$

Given $A \in \mathbf{Ch}$, we can define

$$A \otimes I$$

and

$$\mathrm{Cone}(A) = A \otimes I / A \otimes p_1$$

The set of morphisms of chain-complexes is itself a chain complex:

2.6. Definition. *Given chain-complexes $A, B \in \mathbf{Ch}$ define*

$$\mathrm{Hom}_R(A, B)$$

to be the chain-complex of graded R-morphisms where the degree of an element $x \in \mathrm{Hom}_R(A, B)$ is its degree as a map and with differential

$$\partial f = f \circ \partial_A - (-1)^{\deg f} \partial_B \circ f$$

As a R-module $\mathrm{Hom}_R(A, B)_k = \prod_j \mathrm{Hom}_R(A_j, B_{j+k})$.

Remark. Given $A, B \in \mathbf{Ch}^{S_n}$, we can define $\mathrm{Hom}_{RS_n}(A, B)$ in a corresponding way.

2.7. Definition. *If G is a discrete group, let $\mathbf{Ch}_0^G$ denote the category of chain-complexes equipped with a right G-action. This is again a closed symmetric monoidal category and the forgetful functor $\mathbf{Ch}_0^G \to \mathbf{Ch}_0$ has a left adjoint, $(-)[G]$. This applies to the symmetric groups, S_n, where we regard S_1 and S_0 as the trivial group. The* category of collections *is defined to be the product*

$$\mathrm{Coll}(\mathbf{Ch}_0) = \prod_{n \geq 0} \mathbf{Ch}_0^{S_n}$$

Its objects are written $\mathcal{V} = \{\mathcal{V}(n)\}_{n \geq 0}$. Each collection induces an endofunctor (also denoted $\mathcal{V}$) $\mathcal{V}\colon \mathbf{Ch}_0 \to \mathbf{Ch}_0$

$$\mathcal{V}(X) = \bigoplus_{n \geq 0} \mathcal{V}(n) \otimes_{RS_n} X^{\otimes n}$$

where $X^{\otimes n} = X \otimes \cdots \otimes X$ and S_n acts on $X^{\otimes n}$ by permuting factors. This endofunctor is a monad *if the defining collection has the structure of an* operad*, which means that $\mathcal{V}$ has a unit $\eta\colon R \to \mathcal{V}(1)$ and structure maps*

$$\gamma_{k_1,\dots,k_n}\colon \mathcal{V}(n) \otimes \mathcal{V}(k_1) \otimes \cdots \otimes \mathcal{V}(k_n) \to \mathcal{V}(k_1 + \cdots + k_n)$$

satisfying well-known equivariance, associativity, and unit conditions —see [23], [15].

We will call the operad $\mathcal{V} = \{\mathcal{V}(n)\}$ Σ-cofibrant *if* $\mathcal{V}(n)$ *is* RS_n*-projective for all* $n \geq 0$.

Remark. The operads we consider here correspond to *symmetric* operads in [23].

The term "unital operad" is used in different ways by different authors. We use it in the sense of Kriz and May in [15], meaning the operad has a 0-component that acts like an arity-lowering augmentation under compositions. Here $\mathcal{V}(0) = R$.

The term Σ-*cofibrant* first appeared in [3]. A simple example of an operad is:

2.8. Example. *For each* $n \geq 0$, $C(n) = \mathbb{Z}S_n$, *with structure-map induced by*

$$\gamma_{\alpha_1,\dots,\alpha_n}\colon S_n \times S_{\alpha_1} \times \cdots \times S_{\alpha_n} \to S_{\alpha_1+\cdots+\alpha_n}$$

defined by regarding each of the S_{α_i} *as permuting elements within the subsequence* $\{\alpha_1 + \cdots + \alpha_{i-1} + 1, \dots, \alpha_1 + \cdots + \alpha_i\}$ *of the sequence* $\{1, \dots, \alpha_1 + \cdots + \alpha_n\}$ *and making* S_n *permute these* n*-blocks. This operad is denoted* $\mathfrak{S}_0$. *In other notation, its* n^{th} *component is the* symmetric group-ring $\mathbb{Z}S_n$. *See [22] for explicit formulas.*

Another important operad is:

2.9. Example. *The Barratt-Eccles operad,* $\mathfrak{S}$, *is given by* $\mathfrak{S}(n) = \{C_*(\widetilde{K(S_n}, 1)\}$ *— where* $C_*(\widetilde{K(S_n}, 1)$ *is the normalized chain complex of the universal cover of the Eilenberg-Mac Lane space* $K(S_n, 1)$. *This is well-known (see [2] or [22]) to be a Hopf-operad, i.e. equipped with an operad morphism*

$$\delta\colon \mathfrak{S} \to \mathfrak{S} \otimes \mathfrak{S}$$

and is important in topological applications. See [22] for formulas for the structure maps.

For the purposes of this paper, the main example of an operad is

2.10. Definition. *Given any* $C \in \mathbf{Ch}$, *the associated* coendomorphism operad, $\mathrm{CoEnd}(C)$ *is defined by*

$$\mathrm{CoEnd}(C)(n) = \mathrm{Hom}_R(C, C^{\otimes n})$$

Its structure map

$$\gamma_{\alpha_1,\dots,\alpha_n}\colon \mathrm{Hom}_R(C, C^{\otimes n}) \otimes \mathrm{Hom}_R(C, C^{\otimes \alpha_1}) \otimes \cdots \otimes \mathrm{Hom}_R(C, C^{\otimes \alpha_n}) \to \mathrm{Hom}_R(C, C^{\otimes \alpha_1+\cdots+\alpha_n})$$

simply composes a map in $\mathrm{Hom}_R(C, C^{\otimes n})$ *with maps of each of the* n *factors of* C.

This is a non-unital operad, but if $C \in \mathbf{Ch}$ *has an augmentation map* $\varepsilon\colon C \to R$ *then we can regard* ε *as the generator of* $\mathrm{CoEnd}(C)(0) = R\cdot\varepsilon \subset \mathrm{Hom}_R(C, C^{\otimes 0}) = \mathrm{Hom}_R(C, R)$.

Given $C \in \mathbf{Ch}$ *with* subcomplexes $\{D_1, \dots, D_k\}$, *the* relative coendomorphism operad $\mathrm{CoEnd}(C; \{D_i\})$ *is defined to be the sub-operad of* $\mathrm{CoEnd}(C)$ *consisting of maps* $f \in \mathrm{Hom}_R(C, C^{\otimes n})$ *such that* $f(D_j) \subseteq D_j^{\otimes n} \subseteq C^{\otimes n}$ *for all* j.

We use the coendomorphism operad to define the main object of this paper:

2.11. Definition. *A* coalgebra over an operad $\mathcal{V}$ *is a chain-complex* $C \in \mathbf{Ch}$ *with an operad morphism* $\alpha: \mathcal{V} \to \mathrm{CoEnd}(C)$*, called its* structure map. *We will sometimes want to define coalgebras using the* adjoint structure map

$$\alpha: C \to \prod_{n \geq 0} \mathrm{Hom}_{RS_n}(\mathcal{V}(n), C^{\otimes n})$$

(in $\mathbf{Ch}$*) or even the set of chain-maps*

$$\alpha_n: C \to \mathrm{Hom}_{RS_n}(\mathcal{V}(n), C^{\otimes n})$$

for all $n \geq 0$*.*

We will sometimes want to focus on a particular class of $\mathcal{V}$-coalgebras: the *pointed, irreducible coalgebras*. We define this concept in a way that extends the conventional definition in [25]:

2.12. Definition. *Given a coalgebra over a unital operad* $\mathcal{V}$ *with adjoint structure-map*

$$\alpha_n: C \to \mathrm{Hom}_{RS_n}(\mathcal{V}(n), C^{\otimes n})$$

an element $c \in C$ *is called* group-like *if* $\alpha_n(c) = f_n(c^{\otimes n})$ *for all* $n > 0$*. Here* $c^{\otimes n} \in C^{\otimes n}$ *is the n-fold R-tensor product,*

$$f_n = \mathrm{Hom}_R(\epsilon_n, 1): \mathrm{Hom}_R(R, C^{\otimes n}) = C^{\otimes n} \to \mathrm{Hom}_{RS_n}(\mathcal{V}(n), C^{\otimes n})$$

and $\epsilon_n: \mathcal{V}(n) \to \mathcal{V}(0) = R$ *is the augmentation (which is n-fold composition with* $\mathcal{V}(0)$*).*

A coalgebra C over an operad $\mathcal{V}$ *is called* pointed *if it has a* unique *group-like element (denoted* 1*), and* pointed irreducible *if the intersection of any two sub-coalgebras contains this unique group-like element.*

Remark. Note that a group-like element generates a sub $\mathcal{V}$-coalgebra of C and must lie in dimension 0.

Although this definition seems contrived, it arises in "nature": The chain-complex of a pointed, simply-connected reduced simplicial set is naturally a pointed irreducible coalgebra over the Barratt-Eccles operad, $\mathfrak{S} = \{C(\widetilde{K(S_n, 1)})\}$ (see [22]). In this case, the operad action encodes the chain-level effect of Steenrod operations.

2.13. Definition. *We denote the* category *of nearly free coalgebras over* $\mathcal{V}$ *by* $\mathscr{S}_0$*. The terminal object in this category is* 0*, the null coalgebra.*

The category of nearly free pointed irreducible coalgebras *over* $\mathcal{V}$ *is denoted* $\mathscr{I}_0$*—this is only defined if* $\mathcal{V}$ *is unital.* Its *terminal object is the coalgebra whose underlying chain complex is R concentrated in dimension* 0 *with coproduct that sends* $1 \in R$ *to* $1^{\otimes n} \in R^{\otimes n}$*.*

It is not hard to see that these terminal objects are also the initial *objects of their respective categories.*

We also need:

2.14. Definition. *If $A \in \mathscr{C} = \mathscr{I}_0$ or $\mathscr{S}_0$, then $\lceil A \rceil$ denotes the underlying chain-complex in* **Ch** *of*

$$\ker A \to \bullet$$

where $\bullet$ denotes the terminal object in $\mathscr{C}$ — see definition 2.13. We will call $\lceil * \rceil$ the forgetful functor *from $\mathscr{C}$ to* **Ch**.

We can also define the analogue of an ideal:

2.15. Definition. *Let C be a coalgebra over the operad $\mathcal{U}$ with adjoint structure map*

$$\alpha\colon C \to \prod_{n \geq 0} \mathrm{Hom}_{RS_n}(\mathcal{U}(n), C^{\otimes n})$$

and let $D \subseteq \lceil C \rceil$ be a sub-chain complex that is a direct summand. Then D will be called a coideal *of C if the composite*

$$\alpha|D\colon D \to \prod_{n \geq 0} \mathrm{Hom}_{RS_n}(\mathcal{U}(n), C^{\otimes n}) \xrightarrow{\mathrm{Hom}_R(1_{\mathcal{U}}, p^{\otimes})} \prod_{n \geq 0} \mathrm{Hom}_{RS_n}(\mathcal{U}(n), (C/D)^{\otimes n})$$

vanishes*, where $p\colon C \to C/D$ is the projection to the quotient (in* **Ch***).*

Remark. Note that it is easier for a sub-chain-complex to be a coideal of a coalgebra than to be an ideal of an algebra. For instance, all sub-coalgebras of a coalgebra are also coideals. Consequently it is easy to form quotients of coalgebras and hard to form sub-coalgebras. This is dual to what occurs for algebras. We will use the concept of cofree coalgebra cogenerated by a chain complex:

2.16. Definition. *Let $C \in$* **Ch** *and let $\mathcal{V}$ be an operad. Then a $\mathcal{V}$-coalgebra G will be called* the cofree coalgebra cogenerated by C *if*

1. *there exists a morphism of chain-complexes $\varepsilon\colon G \to C$*
2. *given any $\mathcal{V}$-coalgebra D and any morphism of DG-modules $f\colon \lceil D \rceil \to C$, there exists a* unique *morphism of $\mathcal{V}$-coalgebras, $\hat{f}\colon D \to G$, that makes the diagram (of underlying chain-complexes)*

$$\begin{tikzcd} D \arrow[r, "\hat{f}"] \arrow[dr, "f"'] & G \arrow[d, "\varepsilon"] \\ & C \end{tikzcd}$$

commute.

This universal property of cofree coalgebras implies that they are unique up to isomorphism if they exist.

The paper [23] gives a constructive proof of their existence in great generality (under the unnecessary assumption that chain-complexes are R-free). In particular, this paper defines cofree coalgebras $L_{\mathcal{V}}C$ and pointed irreducible cofree coalgebras $P_{\mathcal{V}}C$ cogenerated by a chain-complex C. There are several ways to define them:

1. $L_{\mathcal{V}}C$ is essentially the largest submodule of

$$C \oplus \prod_{k=1}^{\infty} \operatorname{Hom}_{RS_k}(\mathcal{V}(k), C^{\otimes k})$$

on which the coproduct defined by the dual of the composition-operations of $\mathcal{V}$ is well-defined.

2. If C is a coalgebra over $\mathcal{V}$, its image under the structure map

$$C \to C \oplus \prod_{k=1}^{\infty} \operatorname{Hom}_{RS_k}(\mathcal{V}(k), C^{\otimes k})$$

turns out to be a sub-coalgebra of the target—with a coalgebra structure that vanishes on the left summand (C) and is the dual of the structure-map of $\mathcal{V}$ on the right. We may define $L_{\mathcal{V}}C$ to be the sum of all coalgebras in $C \oplus \prod_{k=1}^{\infty} \operatorname{Hom}_{RS_k}(\mathcal{V}(k), C^{\otimes k})$ formed in this way. The classifying map of a coalgebra

$$C \to L_{\mathcal{V}}C$$

is just the structure map of the coalgebra structure.

The paper also defines the *bounded* cofree coalgebras:

- $M_{\mathcal{V}}C$ is essentially the largest sub-coalgebra of $L_{\mathcal{V}}C$ that is concentrated in nonnegative dimensions.

- $\mathscr{F}_{\mathcal{V}}C$ is the largest pointed irreducible sub-coalgebra of $P_{\mathcal{V}}C$ that is concentrated in nonnegative dimensions.

3. Model categories

We recall the concept of a model structure on a category $\mathcal{G}$. This involves defining specialized classes of morphisms called *cofibrations, fibrations,* and *weak equivalences* (see [20] and [12]). The category and these classes of morphisms must satisfy the conditions:

CM 1 $\mathcal{G}$ is closed under all finite limits and colimits

CM 2 Suppose the following diagram commutes in $\mathcal{G}$:

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "h"'] & & Y \\ & Z \arrow[ur, "f"'] & \end{tikzcd}$$

If any two of f, g, h are weak equivalences, so is the third.

CM 3 These classes of morphisms are closed under formation of *retracts:* Given a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & B \arrow[r] \arrow[d, "g"'] & A \arrow[d, "f"] \\ C \arrow[r] & D \arrow[r] & C \end{tikzcd}$$

whose horizontal composites are the identity map, if g is a weak equivalence, fibration, or cofibration, then so is f.

CM 4 Given a commutative solid arrow diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"'] & A \arrow[d, "p"] \\ W \arrow[r] \arrow[ur, dotted] & B \end{tikzcd}$$

where i is a cofibration and p is a fibration, the dotted arrow exists whenever i or p are trivial.

CM 5 Any morphism $f: X \to Y$ in $\mathcal{G}$ may be factored:

1. $f = p \circ i$, where p is a fibration and i is a trivial cofibration
2. $f = q \circ j$, where q is a trivial fibration and j is a cofibration

We also assume that these factorizations are *functorial*— see [10].

3.1. Definition. *An object, X, for which the map $\bullet \to X$ is a cofibration, is called* cofibrant. *An object, Y, for which the map $Y \to \bullet$ is a fibration, is called* fibrant.

The properties of a model category immediately imply that:

3.2. A model-category of chain-complexes Let **Ch** denote the category of unbounded chain-complexes over the ring R. The *absolute model structure* of Christensen and Hovey in [9], and Cole in [7] is defined via:

1. *Weak equivalences* are chain-homotopy equivalences: two chain-complexes C and D are weakly equivalent if there exist chain-maps: $f: C \to D$ and $g: D \to C$ and chain-homotopies $\varphi_1: C \to C$ and $\varphi_2: D \to D$ such that $d\varphi_1 = g \circ f - 1$ and $d\varphi_2 = f \circ g - 1$.

2. *Fibrations* are surjections of chain-complexes that are split (as maps of graded R-modules).

3. *Cofibrations* are injections of chain-complexes that are split (as maps of graded R-modules).

Remark. *All* chain complexes are fibrant and cofibrant in this model.

In this model structure, a quasi-isomorphism may *fail* to be a weak equivalence. It is well-known not to be cofibrantly generated (see [9]).

Since all chain-complexes are cofibrant, the properties of a model-category imply that all trivial fibrations are split as chain-maps. We will need this fact in somewhat more detail:

3.3. Lemma. *Let $f: C \to D$ be a trivial fibration of chain-complexes. Then there exists a chain-map $\ell: D \to C$ and a chain-homotopy $\varphi: C \to C$ such that*

1. $f \circ \ell = 1: D \to D$
2. $\ell \circ f - 1 = \partial\varphi$

Remark. See definition 2.6 for the notation $\partial\varphi$.

Proof. The hypotheses imply that there exist chain-maps

1. $f: C \to D$, $g: D \to C$ with $f \circ g - 1 = \partial\Phi_1$, $g \circ f - 1 = \partial\Phi_2$
2. f is split surjective, as a morphism of graded modules. This means that there exists a module morphism
$$v: D \to C$$
not necessarily a chain-map, such that $f \circ v = 1: D \to D$.

Now set
$$\ell = g - \partial(v \circ \Phi_1)$$
This is a chain-map, and if we compose it with f, we get
$$\begin{aligned} f \circ \ell &= f \circ g - f \circ \partial(v \circ \Phi_1) \\ &= f \circ g - \partial(f \circ v \circ \Phi_1) \\ &= f \circ g - \partial\Phi_1 \\ &= 1 \end{aligned}$$
Furthermore
$$\begin{aligned} \ell \circ f &= g \circ f - \partial(v \circ \Phi_1) \circ f \\ &= g \circ f - \partial(v \circ \Phi_1 \circ f) \\ &= 1 + \partial(\Phi_2) - \partial(v \circ \Phi_1 \circ f) \\ &= 1 + \partial(\Phi_2 - v \circ \Phi_1 \circ f) \end{aligned}$$
so we can set
$$\varphi = \Phi_2 - v \circ \Phi_1 \circ f$$
∎

We will need the following *relative* version of lemma 3.3 in the sequel:

3.4. Lemma. *Let*

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ C \arrow[r, "v"'] & D \end{tikzcd}$$

be a commutative diagram in **Ch** *such that*

- f *and* g *are trivial fibrations*
- v *is a cofibration*

Then there exists maps $\ell: C \to A$ *and* $m: D \to B$ *such that*

1. $f \circ \ell = 1: C \to C$, $g \circ m = 1: D \to D$
2. *the diagram*

$$\begin{tikzcd} A \arrow[r, "u"] & B \\ C \arrow[u, "\ell"] \arrow[r, "v"'] & D \arrow[u, "m"'] \end{tikzcd}$$

   commutes.

If u *is injective and split, there exist homotopies* $\varphi_1: A \otimes I \to A$ *from* 1 *to* $\ell \circ f$ *and* $\varphi_2: B \otimes I \to B$ *from* 1 *to* $m \circ g$*, respectively, such that the diagram*

$$\begin{tikzcd} A \otimes I \arrow[r, "u \otimes 1"] \arrow[d, "\varphi_1"'] & B \otimes I \arrow[d, "\varphi_2"] \\ A \arrow[r, "u"'] & B \end{tikzcd}$$

commutes.

Proof. We construct ℓ exactly as in lemma 3.3 and use it to create a commutative diagram

$$\begin{tikzcd} C \arrow[r, "u \circ \ell"] \arrow[d, "v"'] & B \arrow[d, "g"] \\ D \arrow[r, equal] & D \end{tikzcd}$$

Property CM 4 implies that we can complete this to a commutative diagram

$$\begin{tikzcd} C \arrow[r, "u \circ \ell"] \arrow[d, "v"'] & B \arrow[d, "g"] \\ D \arrow[r, equal] \arrow[ur, dotted, "m"] & D \end{tikzcd}$$

Let $\varphi_1': A \otimes I \to A$ and $\varphi_2': B \otimes I \to B$ be any homotopies from the identity map to $\ell \circ f$ and $m \circ g$, respectively. If there exists a chain-map $p: B \to A$ $p \circ u = 1: A \to A$ then $\varphi_1 = \varphi_1'$ and $\varphi_2 = u \circ \varphi_1' \circ (p \otimes 1) + \varphi_2' \circ (1 - (u \circ p) \otimes 1)$ have the required properties. ∎

We will also need one final property of cofibrations:

3.5. LEMMA. *Let* $\{p_i\colon C_{i+1} \to C_i\}$ *be an infinite sequence of chain-maps of chain-complexes and let*

$$f_i\colon A \to C_i$$

be cofibrations such that

$$\begin{tikzcd} & C_{i+1} \arrow[d, "p_i"] \\ A \arrow[ur, "f_{i+1}"] \arrow[r, "f_i"'] & C_i \end{tikzcd}$$

commutes for all $i \geq 0$. *Then*

$$\varprojlim f_i\colon A \to \varprojlim C_i$$

is a cofibration.

PROOF. Since $f_0\colon A \to C_0$ is a cofibration, there exists a map

$$\ell_0\colon C_0 \to A$$

of graded modules such that $\ell_0 \circ f_0 = 1\colon A \to A$. Now define $\ell_i = \ell_0 \circ p_0 \circ \cdots \circ p_{i-1}$. Since $p_i | \operatorname{im} A = 1\colon \operatorname{im} A \to \operatorname{im} A$,

$$\ell_i \circ f_i = 1$$

and we get a morphism of graded modules

$$\varprojlim \ell_i\colon \varprojlim C_i \to A$$

that splits $\varprojlim f_i$. ■

4. Model-categories of coalgebras

4.1. DESCRIPTION OF THE MODEL-STRUCTURE We will base our model-structure on that of the underlying chain-complexes in **Ch**. Definition 4.5 and definition 4.6 describe how we define cofibrations, fibrations, and weak equivalences.

We must allow non-R-free chain-complexes (when $R = \mathbb{Z}$) because the underlying chain complexes of the cofree coalgebras $P_{\mathcal{V}}(*)$ and $L_{\mathcal{V}}(*)$ are not known to be R-free. They certainly are if R is a field, but if $R = \mathbb{Z}$ their underlying abelian groups are subgroups of the Baer-Specker group, $\mathbb{Z}^{\aleph_0}$, which is $\mathbb{Z}$-torsion free but well-known *not* to be a free abelian group (see [24], [4] or the survey [8]).

4.2. PROPOSITION. *The forgetful functor (defined in definition 2.14) and cofree coalgebra functors define adjoint pairs*

$$\begin{aligned} P_{\mathcal{V}}(*)\colon \mathbf{Ch} &\leftrightarrows \mathscr{I}_0\colon \lceil * \rceil \\ L_{\mathcal{V}}(*)\colon \mathbf{Ch} &\leftrightarrows \mathscr{S}_0\colon \lceil * \rceil \end{aligned}$$

Remark. The adjointness of the functors follows from the universal property of cofree coalgebras—see [23].

4.3. Condition. Throughout the rest of this paper, *we assume that $\mathcal{V}$ is an operad equipped with a morphism of operads*

$$\delta\colon \mathcal{V} \to \mathcal{V} \otimes \mathfrak{S}$$

—where $\mathfrak{S}$ is the Barratt-Eccles operad (see example 2.9 —that makes the diagram

$$\begin{tikzcd} \mathcal{V} \arrow[r, "\delta"] \arrow[dr, equal] & \mathcal{V} \otimes \mathfrak{S} \arrow[d] \\ & \mathcal{V} \end{tikzcd}$$

commute. Here, the operad structure on $\mathcal{V} \otimes \mathfrak{S}$ is just the tensor product of the operad structures of $\mathcal{V}$ and $\mathfrak{S}$, and the vertical map is projection:e

$$\mathcal{V} \otimes \mathfrak{S} \xrightarrow{1 \otimes \hat{\epsilon}} \mathcal{V} \otimes T = \mathcal{V}$$

where T is the operad that is R in all arities and $\hat{\epsilon}\colon \mathfrak{S} \to T$ is defined by the augmentations:

$$\epsilon_n\colon \mathrm{R}S_n \to R$$

In addition, we assume that, for each $n \geq 0$, $\{\mathcal{V}(n)\}$ is an RS_n-projective chain-complex of finite type.

We also assume that the arity-1 component of $\mathcal{V}$ is equal to R, generated by the unit.

4.4. Remark. *Free and cofibrant operads (with each component of finite type) satisfy this condition. The condition that the chain-complexes are projective corresponds to the Berger and Moerdijk's condition of Σ-cofibrancy in [3].*

Now we define our model structure on the categories $\mathscr{I}_0$ and $\mathscr{S}_0$.

4.5. Definition. *A morphism $f\colon A \to B$ in $\mathscr{C} = \mathscr{S}_0$ or $\mathscr{I}_0$ will be called*

1. *a* weak equivalence *if $\lceil f \rceil\colon \lceil A \rceil \to \lceil B \rceil$ is a chain-homotopy equivalence in* **Ch**. *An object A will be called* contractible *if the augmentation map*

$$A \to \bullet$$

is a weak equivalence, where $\bullet$ denotes the terminal object in $\mathscr{C}$—see definition 2.13.

2. *a* cofibration *if $\lceil f \rceil$ is a cofibration in* **Ch**.
3. *a* trivial cofibration *if it is a weak equivalence and a cofibration.*

Remark. A morphism is a cofibration if it is a degreewise split monomorphism of chain-complexes. Note that all objects of $\mathscr{C}$ are cofibrant.

Our definition makes $f: A \to B$ a weak equivalence if and only if $\lceil f \rceil: \lceil A \rceil \to \lceil B \rceil$ is a weak equivalence in **Ch**.

4.6. Definition. *A morphism $f: A \to B$ in $\mathscr{S}_0$ or $\mathscr{I}_0$ will be called*

1. *a* fibration *if the dotted arrow exists in every diagram of the form*

$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"'] & A \arrow[d, "f"] \\ W \arrow[r] \arrow[ur, dotted] & B \end{tikzcd}$$

   in which $i: U \to W$ is a trivial cofibration.

2. *a* trivial fibration *if it is a fibration and a weak equivalence.*

Definition 4.5 explicitly described cofibrations and definition 4.6 defined fibrations in terms of them. We will verify the axioms for a model category (part of CM 4 and CM 5) and characterize fibrations.

We will occasionally need a stronger form of equivalence:

4.7. Definition. *Let $f, g: A \to B$ be a pair of morphisms in $\mathscr{S}_0$ or $\mathscr{I}_0$. A* strict homotopy *between them is a coalgebra-morphism (where $A \otimes I$ has the coalgebra structure defined in condition 4.3)*

$$F: A \otimes I \to B$$

such that $F|A \otimes p_0 = f: A \otimes p_0 \to B$ and $F|A \otimes p_1 = g: A \otimes p_{01} \to B$. A strict equivalence *between two coalgebras A and B is a pair of coalgebra-morphisms*

$$\begin{aligned} f: A &\to B \\ g: B &\to A \end{aligned}$$

and strict homotopies from $f \circ g$ to the identity of B and from $g \circ f$ to the identity map of A.

Remark. Strict equivalence is a direct translation of the definition of weak equivalence in **Ch** into the realm of coalgebras. Strict equivalences are weak equivalences but the converse is not true.

The reader may wonder why we didn't use strict equivalence in place of what is defined in definition 4.5. It turns out that in we are only able to prove CM 5 with the weaker notion of equivalence used here. In a few simple cases, describing fibrations is easy:

4.8. Proposition. *Let*

$$f: A \to B$$

be a fibration in **Ch**. *Then the induced morphisms*

$$\begin{aligned} P_{\mathcal{V}} f: P_{\mathcal{V}} A &\to P_{\mathcal{V}} B \\ L_{\mathcal{V}} f: L_{\mathcal{V}} A &\to L_{\mathcal{V}} B \end{aligned}$$

are fibrations in $\mathscr{I}_0$ *and* $\mathscr{S}_0$*, respectively.*

Proof. Consider the diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & P_{\mathcal{V}} A \arrow[d, "P_{\mathcal{V}} f"] \\ V \arrow[r] \arrow[ur, dotted] & P_{\mathcal{V}} B \end{tikzcd}$$

where $U \to V$ is a trivial cofibration—i.e., $\lceil U \rceil \to \lceil V \rceil$ is a trivial cofibration of *chain-complexes.* Then the dotted map exists by the the defining property of cofree coalgebras and by the existence of the lifting map in the diagram

$$\begin{tikzcd} \lceil U \rceil \arrow[r] \arrow[d] & A \arrow[d, "f"] \\ \lceil V \rceil \arrow[r] \arrow[ur, dotted] & B \end{tikzcd}$$

of chain-complexes. ■

4.9. Corollary. *All cofree coalgebras are fibrant.*

4.10. Proposition. *Let* C *and* D *be objects of* **Ch** *and let*

$$f_1, f_2: C \to D$$

be chain-homotopic morphisms via a chain-homotopy

$$F: C \otimes I \to D \tag{1}$$

Then the induced maps

$$\begin{aligned} P_{\mathcal{V}} f_i: P_{\mathcal{V}} C &\to P_{\mathcal{V}} D \\ L_{\mathcal{V}} f_i: L_{\mathcal{V}} C &\to L_{\mathcal{V}} D \end{aligned}$$

$i = 1, 2$*, are left-homotopic in* $\mathscr{I}_0$ *and* $\mathscr{S}_0$*, respectively via a strict chain homotopy*

$$F': P_{\mathcal{V}} f_i: (P_{\mathcal{V}} C) \otimes I \to P_{\mathcal{V}} D$$

If C and D are coalgebras over $\mathcal{V}$ and we equip $C \otimes I$ with a coalgebra structure using condition 4.3 and if F in 1 is strict then the diagram

$$\begin{tikzcd} C \otimes I \arrow[r, "F"] \arrow[d, "\alpha_C \otimes 1"'] & D \arrow[d, "\alpha_D"] \\ P_{\mathcal{V}}(\lceil C \rceil) \otimes I \arrow[r, "F'"'] & P_{\mathcal{V}}(\lceil D \rceil) \end{tikzcd}$$

commutes in the pointed irreducible case and the diagram

$$\begin{tikzcd} C \otimes I \arrow[r, "F"] \arrow[d, "\alpha_C \otimes 1"'] & D \arrow[d, "\alpha_D"] \\ L_{\mathcal{V}}(C) \otimes I \arrow[r, "F'"'] & L_{\mathcal{V}} D \end{tikzcd}$$

commutes in the general case. Here α_C and α_D are classifying maps of coalgebra structures.

Remark. In other words, the cofree coalgebra functors map homotopies and weak equivalences in **Ch** to strict homotopies and strict equivalences, respectively, in $\mathscr{I}_0$ and $\mathscr{S}_0$.

If the homotopy in **Ch** was the result of applying the forgetful functor to a strict homotopy, then the generated strict homotopy is compatible with it.

Proof. We will prove this in the pointed irreducible case. The general case follows by a similar argument. The chain-homotopy between the f_i induces

$$P_{\mathcal{V}} F \colon P_{\mathcal{V}}(C \otimes I) \to P_{\mathcal{V}} D$$

Now we construct the map

$$H \colon (P_{\mathcal{V}} C) \otimes I \to P_{\mathcal{V}}(C \otimes I)$$

using the universal property of a cofree coalgebra and the fact that the coalgebra structure of $(P_{\mathcal{V}} C) \otimes I$ extends that of $P_{\mathcal{V}} C$ on both ends by condition 4.3. Clearly

$$P_{\mathcal{V}} F \circ H \colon (P_{\mathcal{V}} C) \otimes I \to P_{\mathcal{V}} D$$

is the required left-homotopy.

Now suppose C and D are coalgebras. If we define a coalgebra structure on $C \otimes I$ using condition 4.3, we get a diagram

$$\begin{tikzcd} C \otimes I \arrow[r, equal] \arrow[d, "\alpha_C \otimes 1"'] & C \otimes I \arrow[r, "F"] \arrow[d, "\alpha_{C \otimes I}"] & D \arrow[d, "\alpha_D"] \\ P_{\mathcal{V}}(\lceil C \rceil) \otimes I \arrow[r, "H"] \arrow[d, "\epsilon_C \otimes 1"'] & P_{\mathcal{V}}(\lceil C \otimes I \rceil) \arrow[r, "P_{\mathcal{V}} F"'] \arrow[d, "\epsilon_{C \otimes I}"] & P_{\mathcal{V}} \lceil D \rceil \\ C \otimes I \arrow[r, equal] & C \otimes I & \end{tikzcd}$$

where $\alpha_{C\otimes I}$ is the classifying map for the coalgebra structure on $C \otimes I$.

We claim that this diagram commutes. The fact that F is a coalgebra morphism implies that the upper right square commutes. The large square on the left (bordered by $C \otimes I$ on all four corners) commutes by the property of co-generating maps (of cofree coalgebras) and classifying maps. The two smaller squares on the left (i.e., the large square with the map H added to it) commute by the universal properties of cofree coalgebras (which imply that induced maps to cofree coalgebras are uniquely determined by their composites with co-generating maps). The diagram in the statement of the result is just the outer upper square of this diagram, so we have proved the claim. ■

This result implies a homotopy invariance property of the *categorical product,* $A_0 \boxtimes A_1$, defined explicitly in definition B.15 of appendix B.

4.11. Lemma. *Let $g: B \to C$ be a fibration in $\mathscr{I}_0$ and let $f: A \to C$ be a morphism in $\mathscr{I}_0$. Then the projection*

$$A \boxtimes^C B \to A$$

is a fibration.

Remark. The notation $A \boxtimes^C B$ denotes a fibered product—see definition B.18 in appendix B.14 for the precise definition. In other words, pullbacks of fibrations are fibrations.

Proof. Consider the diagram

$$\begin{tikzcd} U \arrow[r, "u"] \arrow[d, "i"'] & A \boxtimes^C B \arrow[d, "p_A"] \\ V \arrow[r, "v"'] \arrow[ur, dotted] & A \end{tikzcd} \tag{2}$$

where $U \to V$ is a trivial cofibration. The defining property of a categorical product implies that any map to $A \boxtimes^C B \subseteq A \boxtimes B$ is determined by its composites with the projections

$$\begin{aligned} p_A: A \boxtimes B &\to A \\ p_B: A \boxtimes B &\to B \end{aligned}$$

Consider the composite $p_B \circ u: U \to B$. The commutativity of the solid arrows in diagram 2 implies that the diagram

$$\begin{tikzcd} U \arrow[rr, "p_B \circ u"] \arrow[d, "i"'] \arrow[dr, "p_A \circ u"] & & B \arrow[d, "g"] \\ V \arrow[r, "v"'] & A \arrow[r, "f"'] & C \end{tikzcd}$$

commutes and this implies that the solid arrows in the diagram

$$\begin{tikzcd} U \arrow[r, "p_B \circ u"] \arrow[d, "i"'] & B \arrow[d, "g"] \\ V \arrow[r, "f \circ v"'] \arrow[ur, dotted] & C \end{tikzcd} \tag{3}$$

commute. The fact that $g: B \to C$ is a *fibration* implies that the dotted arrow exists in diagram 3, which implies the existence of a map $V \to A \boxtimes B$ whose composites with f and g agree. This defines a map $V \to A \boxtimes^C B$ that makes *all* of diagram 2 commute. The conclusion follows. ∎

4.12. Proof of CM 1 through CM 3 CM 1 asserts that our categories have all finite limits and colimits.

The results of appendix B prove that all *countable* limits and colimits exist—see theorem B.3 and theorem B.7.

CM 2 follows from the fact that we define weak equivalence the same was it is defined in **Ch**—so the model structure on **Ch** implies that this condition is satisfies on our categories of coalgebras. A similar argument verifies condition CM 3.

4.13. Proof of CM 5 We begin with:

4.14. Corollary. *Let $A \in \mathscr{I}_0$ be fibrant and let $B \in \mathscr{I}_0$. Then the projection*

$$A \boxtimes B \to B$$

is a fibration.

This allows us to verify CM 5, statement 2:

4.15. Corollary. *Let $f: A \to B$ be a morphism in $\mathscr{C} = \mathscr{I}_0$ or $\mathscr{S}_0$, and let*

$$Z = \begin{cases} P_{\mathscr{V}}\mathrm{Cone}(\lceil A \rceil) \boxtimes B & \text{when } \mathscr{C} = \mathscr{I}_0 \\ L_{\mathscr{V}}\mathrm{Cone}(\lceil A \rceil) \boxtimes B & \text{when } \mathscr{C} = \mathscr{S}_0 \end{cases}$$

Then f factors as

$$A \to Z \to B$$

where

1. $\mathrm{Cone}(\lceil A \rceil)$ *is the cone on* $\lceil A \rceil$ *(see definition 2.5) with the canonical inclusion* $i: \lceil A \rceil \to \mathrm{Cone}(\lceil A \rceil)$
2. *the morphism $i \boxtimes f: A \to Z$ is a cofibration*
3. *the morphism $Z \to B$ is projection to the second factor and is a fibration (by corollary 4.14).*

Proof. We focus on the pointed irreducible case. The general case follows by essentially the same argument. The existence of the (injective) morphism $A \to P_{\mathscr{V}}\mathrm{Cone}(\lceil A \rceil) \boxtimes B$ follows from the definition of $\boxtimes$. We claim that its image is a direct summand of $P_{\mathscr{V}}\mathrm{Cone}(\lceil A \rceil) \boxtimes B$ as a graded R-module (which implies that $i \boxtimes f$ is a cofibration). We clearly get a projection

$$P_{\mathscr{V}}\mathrm{Cone}(\lceil A \rceil) \boxtimes B \to P_{\mathscr{V}}\mathrm{Cone}(\lceil A \rceil)$$

and the composite of this with the co-generating map $\lceil P_{\mathcal{V}}\mathrm{Cone}(\lceil A\rceil)\rceil \to \mathrm{Cone}(\lceil A\rceil)$ gives rise a a morphism of chain-complexes

$$\lceil P_{\mathcal{V}}\mathrm{Cone}(\lceil A\rceil) \boxtimes B\rceil \to \mathrm{Cone}(\lceil A\rceil) \tag{4}$$

Now note the existence of a splitting map

$$\mathrm{Cone}(\lceil A\rceil) \to \lceil A\rceil$$

of *graded R-modules* (not coalgebras or *even* chain-complexes). Combined with the map in equation 4, we conclude that $A \to P_{\mathcal{V}}\mathrm{Cone}(\lceil A\rceil) \boxtimes B$ is a cofibration.

There is a weak equivalence $c\colon \mathrm{Cone}(\lceil A\rceil) \to \bullet$ in **Ch**, and 4.10 implies that it induces a strict equivalence $P_{\mathcal{V}}c\colon P_{\mathcal{V}}\mathrm{Cone}(\lceil A\rceil) \to \bullet$. Proposition B.17 implies that

$$c \boxtimes 1\colon P_{\mathcal{V}}\mathrm{Cone}(\lceil A\rceil) \boxtimes B \to \bullet \boxtimes B = B$$

is a strict equivalence. ■

The first part of CM 5 will be considerably more difficult to prove.

4.16. Definition. *Let pro-$\mathscr{I}_0$ and pro-$\mathscr{S}_0$ be the categories of inverse systems of objects of $\mathscr{I}_0$ and $\mathscr{S}_0$, respectively and let ind-$\mathscr{I}_0$ and ind-$\mathscr{S}_0$ be corresponding categories of direct systems. Morphisms are defined in the obvious way.*

Now we define the *rug-resolution* of a cofibration:

4.17. Definition. *Let $\mathcal{V} = \{\mathcal{V}(n)\}$ be a Σ-cofibrant (see definition 2.7) operad such that $\mathcal{V}(n)$ is of finite type for all $n \geq 0$. If $f\colon C \to D$ is a cofibration in $\mathscr{I}_0$ or $\mathscr{S}_0$, define*

$$\begin{aligned} G_0 &= D \\ f_0 = f\colon C &\to G_0 \\ G_{n+1} &= G_n \boxtimes^{L_{\mathcal{V}}\lceil H_n\rceil} L_{\mathcal{V}}\bar{H}_n \\ p_{n+1}\colon G_{n+1} &\to G_n \end{aligned}$$

for all n, where

1. *$\epsilon\colon C \to \bullet$ is the unique morphism.*
2. *H_n is the cofiber of f_n in the push-out*

$$\begin{tikzcd} C \arrow[r, "\epsilon"] \arrow[d, "f_n"'] & \bullet \arrow[d, dotted] \\ G_n \arrow[r, dotted] & H_n \end{tikzcd}$$

3. $G_n \to L_{\mathscr{V}}\lceil H_n \rceil$ *is the composite of the classifying map*

$$G_n \to L_{\mathscr{V}}\lceil G_n \rceil$$

with the map

$$L_{\mathscr{V}}\lceil G_n \rceil \to L_{\mathscr{V}}\lceil H_n \rceil$$

4. $\bar{H}_n = \Sigma^{-1}\mathrm{Cone}(\lceil H_n \rceil)$*—where* Σ^{-1} *denotes desuspension (in* $\mathbf{Ch}$*). It is contractible and comes with a canonical* $\mathbf{Ch}$*-fibration*

$$v_n \colon \bar{H}_n \to \lceil H_n \rceil \tag{5}$$

inducing the fibration

$$L_{\mathscr{V}}v_n \colon L_{\mathscr{V}}\bar{H}_n \to L_{\mathscr{V}}\lceil H_n \rceil \tag{6}$$

5. $p_{n+1} \colon G_{n+1} = G_n \boxtimes^{L_{\mathscr{V}}\lceil H_n \rceil} L_{\mathscr{V}}\bar{H}_n \to G_n$ *is projection to the first factor,*

6. *The map* $f_{n+1} \colon C \to G_n \boxtimes^{L_{\mathscr{V}}\lceil H_n \rceil} L_{\mathscr{V}}\bar{H}_n$ *is the* unique *morphism that makes the diagram*

$$\begin{tikzcd}
& G_n \boxtimes L_{\mathscr{V}}\bar{H}_n \arrow[dl] \arrow[dr] & \\
G_n & & L_{\mathscr{V}}\bar{H}_n \\
& C \arrow[ul, "f_n"] \arrow[ur, "\epsilon"'] \arrow[uu, dotted, "f_{n+1}"] &
\end{tikzcd}$$

commute (see lemma B.16), where the downwards maps are projections to factors. The map $\epsilon \colon C \to L_{\mathscr{V}}\bar{H}_n$ *is*

(a) *the map to the* basepoint *if the category is* $\mathscr{I}_0$ *(and* $L_{\mathscr{V}}\bar{H}_n$ *is replaced by* $P_{\mathscr{V}}\bar{H}_n$*),*

(b) *the* zero-*map if the category is* $\mathscr{S}_0$*.*

The commutativity of the diagram

$$\begin{tikzcd}
& L_{\mathscr{V}}H_n & \\
G_n \arrow[ur] & & L_{\mathscr{V}}\bar{H}_n \arrow[ul] \\
& C \arrow[ul, "f_n"] \arrow[ur, "\epsilon"'] &
\end{tikzcd}$$

implies that the image of f_{n+1} *actually lies in the fibered product,* $G_n \boxtimes^{L_{\mathscr{V}}\lceil H_n \rceil} L_{\mathscr{V}}\bar{H}_n$*.*

The rug-resolution *of* $f \colon C \to D$ *is the map of inverse systems* $\{f_i\} \colon \underline{\{C\}} \to \{G_i\} \to D$*, where* $\underline{\{C\}}$ *denotes the constant inverse system.*

Remark. Very roughly speaking, this produces something like a "Postnikov resolution" for $f: C \to D$. Whereas a Postnikov resolution's stages "push the trash upstairs," this one's "push the trash horizontally" or "under the rug"—something feasible because one has an infinite supply of rugs.

4.18. Proposition. *Following all of the definitions of 4.17 above, the diagrams*

$$\begin{tikzcd} & G_{n+1} \arrow[d, "p_{n+1}"] \\ C \arrow[ur, "f_{n+1}"] \arrow[r, "f_n"'] & G_n \end{tikzcd}$$

commute and induce maps $H_{n+1} \xrightarrow{p'_{n+1}} H_n$ *that fit into commutative diagrams of chain-complexes*

$$\begin{tikzcd} & \bar{H}_n \arrow[d, "e_n"] \\ H_{n+1} \arrow[ur, "u_n"] \arrow[r, "p'_{n+1}"'] & H_n \end{tikzcd}$$

It follows that the maps p'_{n+1} *are nullhomotopic for all* n.

Proof. Commutativity is clear from the definition of f_{n+1} in terms of f_n above.

To see that the induced maps are nullhomotopic, consider the diagram

$$\begin{tikzcd} G_n \boxtimes^{L_{\mathcal{V}}\lceil H_n \rceil} L_{\mathcal{V}}\bar{H}_n \arrow[rr] \arrow[d, "p_{n+1}"'] & & L_{\mathcal{V}}\bar{H}_n \arrow[r, "\varepsilon"] \arrow[d, "L_{\mathcal{V}}v_n"'] & \bar{H}_n \arrow[d, "v_n"] \\ G_n \arrow[r] & H_n \arrow[r, "\alpha"'] & L_{\mathcal{V}}H_n \arrow[r, "\varepsilon"'] & H_n \end{tikzcd}$$

where v_n is defined in equation 5, both ε-maps are cogenerating maps—see definition 2.16 —and $\alpha: H_n \to L_{\mathcal{V}}H_n$ is the classifying map.

The left square commutes by the definition of the *fibered* product, $G_n \boxtimes^{L_{\mathcal{V}}\lceil H_n \rceil} L_{\mathcal{V}}\bar{H}_n$ —see definition B.18. The right square commutes by the naturality of cogenerating maps.

Now, note that the composite $H_n \xrightarrow{\alpha} L_{\mathcal{V}}H_n \xrightarrow{\varepsilon} H_n$ is the *identity* map (a universal property of classifying maps of coalgebras). It follows that, *as a chain-map,* the composite

$$G_n \boxtimes^{L_{\mathcal{V}}\lceil H_n \rceil} L_{\mathcal{V}}\bar{H}_n \xrightarrow{p_{n+1}} G_n \to H_n$$

coincides with a chain-map that factors through the *contractible* chain-complex $\bar{H}_n$. ∎

Our main result is:

4.19. Lemma. *Let* $f: C \to D$ *be a cofibration as in definition 4.17 with rug-resolution* $\{f_i\}: \underline{\{C\}} \to \{G_i\} \to D$. *Then*

$$f_\infty = \varprojlim f_n: C \to \varprojlim G_n$$

is a trivial cofibration.

PROOF. We make extensive use of the material in appendix B.21 to show that the cofiber of
$$f_\infty\colon C \to \varprojlim G_n$$
is contractible. We focus on the category $\mathscr{S}_0$—the argument in $\mathscr{I}_0$ is very similar. In this case, the cofiber is simply the quotient. We will consistently use the notation $\bar{H}_n = \Sigma^{-1}\mathrm{Cone}(\lceil H_n\rceil)$

First, note that the maps
$$G_{n+1} \to G_n$$
induce compatible maps
$$\begin{array}{rcl} L_{\mathcal{V}}\lceil H_{n+1}\rceil & \to & L_{\mathcal{V}}\lceil H_n\rceil \\ L_{\mathcal{V}}\bar{H}_{n+1} & \to & L_{\mathcal{V}}\bar{H}_n \end{array}$$
so proposition B.22 implies that
$$\varprojlim G_n = (\varprojlim G_n) \boxtimes^{(\varprojlim L_{\mathcal{V}}\lceil H_n\rceil)} (\varprojlim L_{\mathcal{V}}\bar{H}_n)$$
and theorem B.3 implies that
$$\begin{array}{rcl} \varprojlim L_{\mathcal{V}}\lceil H_n\rceil & = & L_{\mathcal{V}}(\varprojlim \lceil H_i\rceil) \\ \varprojlim L_{\mathcal{V}}\bar{H}_n & = & L_{\mathcal{V}}(\varprojlim \bar{H}_i) = L_{\mathcal{V}}(\Sigma^{-1}\mathrm{Cone}(\varprojlim \lceil H_i\rceil)) \end{array}$$
from which we conclude
$$\varprojlim G_n = (\varprojlim G_n) \boxtimes^{L_{\mathcal{V}}(\varprojlim \lceil H_i\rceil)} L_{\mathcal{V}}(\Sigma^{-1}\mathrm{Cone}(\varprojlim \lceil H_i\rceil))$$

We claim that the projection
$$h\colon (\varprojlim G_n) \boxtimes^{L_{\mathcal{V}}(\varprojlim \lceil H_i\rceil)} L_{\mathcal{V}}(\Sigma^{-1}\mathrm{Cone}(\varprojlim \lceil H_i\rceil)) \to \varprojlim G_n \tag{7}$$
is *split* by a coalgebra morphism. To see this, first note that, by proposition 4.18, each of the maps
$$H_{n+1} \twoheadrightarrow H_n$$
is nullhomotopic via a nullhomotopy compatible with the maps in the inverse system $\{H_n\}$. This implies that
$$\varprojlim \lceil H_n\rceil$$
—the inverse limit of *chain complexes*—is contractible. It follows that the projection
$$\Sigma^{-1}\mathrm{Cone}(\varprojlim \lceil H_n\rceil) \twoheadrightarrow \varprojlim \lceil H_n\rceil$$
is a trivial fibration in **Ch**, hence split by a map
$$j\colon \varprojlim \lceil H_n\rceil \to \Sigma^{-1}\mathrm{Cone}(\varprojlim \lceil H_n\rceil) \tag{8}$$

This, in turn, induces a coalgebra morphism

$$L_{\mathcal{V}}j\colon L_{\mathcal{V}}(\varprojlim \lceil H_n \rceil) \to L_{\mathcal{V}}(\Sigma^{-1}\mathrm{Cone}(\varprojlim \lceil H_n \rceil))$$

splitting the canonical surjection

$$L_{\mathcal{V}}(\Sigma^{-1}\mathrm{Cone}(\varprojlim \lceil H_n \rceil)) \twoheadrightarrow L_{\mathcal{V}}(\varprojlim \lceil H_n \rceil)$$

and induces a map, g

$$\begin{aligned}\varprojlim G_n &= (\varprojlim G_n) \boxtimes^{L_{\mathcal{V}}(\varprojlim \lceil H_i \rceil)} L_{\mathcal{V}}(\varprojlim \lceil H_i \rceil)\\ &\xrightarrow{1\boxtimes L_{\mathcal{V}}j} (\varprojlim G_n) \boxtimes^{L_{\mathcal{V}}(\varprojlim \lceil H_i \rceil)} L_{\mathcal{V}}(\Sigma^{-1}\mathrm{Cone}(\varprojlim \lceil H_i \rceil))\end{aligned}$$

splitting the projection in formula 7. Since the image of $f_\infty(C)$ vanishes in $L_{\mathcal{V}}(\varprojlim \lceil H_i \rceil)$, it is not hard to see that $1 \boxtimes L_{\mathcal{V}}j$ is compatible with the inclusion of C in $\varprojlim G_i$.

Now consider the diagram

$$\begin{tikzcd}
(\varprojlim G_n)/f_\infty(C) \arrow[r, "g"] \arrow[dddd, equal] & \left((\varprojlim G_n) \boxtimes^{L_{\mathcal{V}}(\varprojlim \lceil H_i \rceil)} L_{\mathcal{V}}(\Sigma^{-1}\mathrm{Cone}(\varprojlim \lceil H_i \rceil))\right)/f_\infty(C) \arrow[d, "q"] \\
 & (\varprojlim G_n)/f_\infty(C) \boxtimes^{L_{\mathcal{V}}(\varprojlim \lceil H_i \rceil)} L_{\mathcal{V}}(\Sigma^{-1}\mathrm{Cone}(\varprojlim \lceil H_i \rceil)) \arrow[d, equal] \\
 & (\varprojlim G_n/f_n(C)) \boxtimes^{L_{\mathcal{V}}(\varprojlim \lceil H_i \rceil)} L_{\mathcal{V}}(\Sigma^{-1}\mathrm{Cone}(\varprojlim \lceil H_i \rceil)) \arrow[d, equal] \\
 & (\varprojlim H_n) \boxtimes^{L_{\mathcal{V}}(\varprojlim \lceil H_i \rceil)} L_{\mathcal{V}}(\Sigma^{-1}\mathrm{Cone}(\varprojlim \lceil H_i \rceil)) \arrow[d, "p"] \\
(\varprojlim G_n)/f_\infty(C) \arrow[r, equal] & \varprojlim H_n
\end{tikzcd}$$

where:

1. The map

$$\begin{aligned} q\colon &\left((\varprojlim G_n) \boxtimes^{L_{\mathcal{V}}(\varprojlim \lceil H_i \rceil)} L_{\mathcal{V}}(\Sigma^{-1}\mathrm{Cone}(\varprojlim \lceil H_i \rceil))\right)/f_\infty(C)\\ &\to (\varprojlim G_n)/f_\infty(C) \boxtimes^{L_{\mathcal{V}}(\varprojlim \lceil H_i \rceil)} L_{\mathcal{V}}(\Sigma^{-1}\mathrm{Cone}(\varprojlim \lceil H_i \rceil))\end{aligned}$$

is induced by the projections

$$
\begin{tikzcd}
 & \varprojlim G_n \\
(\varprojlim G_n) \boxtimes^{L_{\mathcal{V}}(\varprojlim \lceil H_i \rceil)} L_{\mathcal{V}}(\Sigma^{-1}\mathrm{Cone}(\varprojlim \lceil H_i \rceil)) \arrow[ur] \arrow[dr] & \\
 & L_{\mathcal{V}}(\Sigma^{-1}\mathrm{Cone}(\varprojlim \lceil H_i \rceil))
\end{tikzcd}
$$

the fact that the image of f_∞ is effectively only in the factor $\varprojlim G_n$, and the defining property of fibered products.

2. The equivalence
$$\varprojlim G_n/f_\infty(C) = \varprojlim G_n/f_n(C)$$
follows from theorem B.30.
3. The vertical map on the left is the identity map because g splits the map h in formula 7.

We claim that the map (projection to the left factor)

$$\lceil p \rceil \colon \lceil (\varprojlim H_n) \boxtimes^{L_{\mathcal{V}}(\varprojlim \lceil H_i \rceil)} L_{\mathcal{V}}(\Sigma^{-1}\mathrm{Cone}(\varprojlim \lceil H_i \rceil))\rceil \to \lceil \varprojlim H_n \rceil$$

is *nullhomotopic* (as a **Ch**-morphism). This follows immediately from the fact that

$$\varprojlim H_n \hookrightarrow L_{\mathcal{V}}(\varprojlim \lceil H_n \rceil)$$

by corollary B.29, so that

$$
\begin{aligned}
&(\varprojlim H_n) \boxtimes^{L_{\mathcal{V}}(\varprojlim \lceil H_i \rceil)} L_{\mathcal{V}}(\Sigma^{-1}\mathrm{Cone}(\varprojlim \lceil H_i \rceil)) \\
&\subseteq L_{\mathcal{V}}(\varprojlim \lceil H_i \rceil) \boxtimes^{L_{\mathcal{V}}(\varprojlim \lceil H_i \rceil)} L_{\mathcal{V}}(\Sigma^{-1}\mathrm{Cone}(\varprojlim \lceil H_i \rceil)) \\
&= L_{\mathcal{V}}(\Sigma^{-1}\mathrm{Cone}(\varprojlim \lceil H_i \rceil))
\end{aligned}
$$

and $L_{\mathcal{V}}(\Sigma^{-1}\mathrm{Cone}(\varprojlim \lceil H_i \rceil))$ is contractible, by proposition 4.10 and the contractibility of $\Sigma^{-1}\mathrm{Cone}(\varprojlim \lceil H_i \rceil)$.

We conclude that
$$(\varprojlim G_n)/f_\infty(C) \xrightarrow{\mathrm{id}} (\varprojlim G_n)/f_\infty(C)$$
is nullhomotopic so $(\varprojlim G_n)/f_\infty(C)$ is contractible and

$$\lceil f_\infty \rceil \colon \lceil C \rceil \to \lceil \varprojlim G_n \rceil$$

is a weak equivalence in **Ch**, hence (by definition 4.5) f_∞ is a weak equivalence.

The map f_∞ is a *cofibration* due to the final statement of Theorem B.30. ∎

4.20. Corollary. *Let $\mathcal{V} = \{\mathcal{V}(n)\}$ be a Σ-cofibrant operad such that $\mathcal{V}(n)$ is of finite type for all $n \geq 0$. Let*

$$f: A \to B$$

be a morphism in $\mathscr{C} = \mathscr{S}_0$ or $\mathscr{I}_0$. Then there exists a functorial factorization of f

$$A \to Z(f) \to B$$

where

$$A \to Z(f)$$

is a trivial cofibration and

$$Z(f) \to B$$

is a fibration.

Remark. This is condition CM5, statement 1 in the definition of a model category at the beginning of this section. It, therefore, proves that the model structure described in 4.5 and 4.6 is well-defined.

By abuse of notation, we will call the $\{f_i\}: \underline{\{A\}} \to \{G_i\} \to L_{\mathcal{V}}(\lceil A \rceil) \boxtimes B \to B$ the *rug-resolution of the morphism* $A \to B$ (see the proof below), where $\{f_i\}: \underline{\{A\}} \to \{G_i\} \to L_{\mathcal{V}}(\lceil A \rceil) \boxtimes B$ is the rug-resolution of the cofibration $A \to L_{\mathcal{V}}(\lceil A \rceil) \boxtimes B$.

See proposition B.3 and corollary B.25 for the definition of inverse limit in the category $\mathscr{C} = \mathscr{S}_0$ or $\mathscr{I}_0$.

Proof. Simply apply definition 4.17 and lemma 4.19 to the cofibration

$$A \to L_{\mathcal{V}}(\lceil A \rceil) \boxtimes B$$

and project to the second factor. ∎

We can characterize fibrations now:

4.21. Corollary. *If $\mathcal{V} = \{\mathcal{V}(n)\}$ is a Σ-cofibrant operad such that $\mathcal{V}(n)$ is of finite type for all $n \geq 0$, then all fibrations are retracts of their rug-resolutions.*

Remark. This shows that rug-resolutions of maps contain canonical fibrations and all others are retracts of them.

Proof. Suppose $p: A \to B$ is some fibration. We apply corollary 4.20 to it to get a commutative diagram

$$\begin{tikzcd} A \arrow[d, "i"'] & \\ \bar{A} \arrow[r, "a_\infty"'] & B \end{tikzcd}$$

where $i: A \to \bar{A}$ is a trivial cofibration and $a_\infty: \bar{A} \to B$ is a fibration. We can complete this to get the diagram

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "i"'] & A \arrow[d, "p"] \\ \bar{A} \arrow[r, "a_\infty"'] \arrow[ur, dotted] & B \end{tikzcd}$$

The fact that $p: A \to B$ is a fibration and definition 4.6 imply the existence of the dotted arrow making the whole diagram commute. But this *splits* the inclusion $i: A \to \bar{A}$ and implies the result. ∎

The rest of this section will be spent on *trivial* fibrations —with a mind to proving the *second statement* in CM 4 in theorem 4.24. Recall that the first statement was a consequence of our *definition* of fibrations in $\mathscr{S}_0$ and $\mathscr{I}_0$.

4.22. PROOF OF CM 4 The first part of CM 4 is trivial: we have *defined* fibrations as morphisms that satisfy it—see definition 4.6. The proof of the second statement of CM 4 is more difficult and makes extensive use of the Rug Resolution defined in definition 4.17.

We begin by showing that a fibration of coalgebras becomes a fibration in **Ch** under the forgetful functor:

4.23. PROPOSITION. *Let $p: A \to B$ be a fibration in $\mathscr{C} = \mathscr{I}_0$ or $\mathscr{S}_0$. Then*

$$\lceil p \rceil: \lceil A \rceil \to \lceil B \rceil$$

is a fibration in $\mathbf{Ch}^+$ *or* $\mathbf{Ch}$*, respectively.*

PROOF. In the light of corollary 4.21, it suffices to prove this for rug-resolutions of fibrations.

Since they are iterated pullbacks of fibrations with contractible total spaces, it suffices to prove the result for something of the form

$$A \boxtimes^{L_\mathcal{V} B} L_\mathcal{V}(\Sigma^{-1}\mathrm{Cone}(B)) \to A$$

where $f: A \to L_\mathcal{V} B$ is some morphism. The fact that all morphisms are coalgebra morphisms implies the existence of a coalgebra structure on

$$Z = \lceil A \rceil \oplus^{\lceil L_\mathcal{V} B \rceil} \lceil L_\mathcal{V}(\Sigma^{-1}\mathrm{Cone}(B)) \rceil \subset \lceil A \boxtimes L_\mathcal{V}(\Sigma^{-1}\mathrm{Cone}(B)) \rceil \rightrightarrows \lceil L_\mathcal{V} B \rceil$$

where $\lceil A \rceil \oplus^{\lceil L_\mathcal{V} B \rceil} \lceil L_\mathcal{V}(\Sigma^{-1}\mathrm{Cone}(B)) \rceil$ is the fibered product in **Ch**. Since $L_\mathcal{V}(\Sigma^{-1}\mathrm{Cone}(B)) \to L_\mathcal{V} B$ is surjective, (because it is induced by the surjection, $\Sigma\mathrm{Cone}(B)) \to B$) it follows that the equalizer

$$\lceil A \boxtimes L_\mathcal{V}(\Sigma^{-1}\mathrm{Cone}(B)) \rceil \rightrightarrows \lceil L_\mathcal{V} B \rceil$$

surjects onto $\lceil A \rceil$. Since Z has a coalgebra structure, it is contained in the core,

$$\left\langle \lceil A \boxtimes L_\mathcal{V}(\Sigma^{-1}\mathrm{Cone}(B)) \rceil \rightrightarrows \lceil L_\mathcal{V} B \rceil \right\rangle = A \boxtimes^{L_\mathcal{V} B} L_\mathcal{V}(\Sigma^{-1}\mathrm{Cone}(B))$$

which also surjects onto A—so the projection

$$\lceil A \boxtimes^{L_{\mathcal{V}} B} L_{\mathcal{V}}(\Sigma^{-1}\mathrm{Cone}(B))\rceil \to \lceil A\rceil$$

is surjective and—as a map of graded R-modules—split. This is the definition of a fibration in **Ch**. ■

We are now in a position to prove the second part of CM 4:

4.24. Theorem. *Given a commutative solid arrow diagram*

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "i"'] & A \arrow[d, "p"] \\ W \arrow[r, "g"'] \arrow[ur, dotted] & B \end{tikzcd}$$

where i is a any cofibration and p is a trivial *fibration, the dotted arrow exists.*

Proof. Because of corollary 4.21, it suffices to prove the result for the *rug-resolution* of the trivial fibration $p: A \to B$. We begin by considering the diagram

$$\begin{tikzcd} \lceil U\rceil \arrow[r, "\lceil f\rceil"] \arrow[d, "\lceil i\rceil"'] & \lceil A\rceil \arrow[d, "\lceil p\rceil"] \\ \lceil W\rceil \arrow[r, "\lceil g\rceil"'] \arrow[ur, dotted, "\ell"] & \lceil B\rceil \end{tikzcd}$$

Because of proposition 4.23, $\lceil p\rceil$ is a trivial fibration and the dotted arrow exists in **Ch**.

If $\alpha: A \to L_{\mathcal{V}}A$ is the classifying map of A, $\hat{\ell}: W \to L_{\mathcal{V}}A$ is induced by $\ell: \lceil W\rceil \to \lceil A\rceil$, and $p_2: L_{\mathcal{V}}A \boxtimes B \to B$ is projection to the second factor, we get a commutative diagram

$$\begin{tikzcd} U \arrow[r, "(\alpha\circ f)\boxtimes(p\circ f)"] \arrow[d, "i"'] & L_{\mathcal{V}}A \boxtimes B \arrow[d, "p_2"] \\ W \arrow[r, "g"'] \arrow[ur, dotted, "\hat{\ell}\boxtimes g"'] & B \end{tikzcd} \tag{9}$$

It will be useful to build the rug-resolutions of $A \to L_{\mathcal{V}}A\boxtimes B = G_0$ and $B \to L_{\mathcal{V}}B\boxtimes B = \tilde{G}_0$ in parallel —denoted $\{G_n\}$ and $\{\tilde{G}_n\}$, respectively. Clearly the vertical morphisms in

$$\begin{tikzcd} A \arrow[r, "f_0"] \arrow[d, "p"'] & L_{\mathcal{V}}A \boxtimes B \arrow[d, "q_0"] \\ B \arrow[r, "\bar{f}_0"'] & L_{\mathcal{V}}B \boxtimes B \end{tikzcd} \tag{10}$$

are trivial fibrations and q_0 is a trivial fibration via a strict homotopy—see propositions 4.10 and B.17.

We prove the result by an induction that:

1. lifts the map $\hat{\ell}_1 = \hat{\ell} \boxtimes g: W \to L_{\mathcal{V}} A \boxtimes B$ to morphisms $\hat{\ell}_k: W \to G_k$ to successively higher stages of the rug-resolution of p. Diagram 10 implies the base case.
2. establishes that the vertical morphisms in

$$\begin{tikzcd} A \arrow[r, "f_n"] \arrow[d, "p"'] & G_n \arrow[d, "q_n"] \\ B \arrow[r, "\bar{f}_n"'] & \tilde{G}_n \end{tikzcd}$$

are trivial fibrations for all n, where the q_n are trivial via *strict* homotopies.

Lemma 3.4 implies that we can find splitting maps u and v such that

$$\begin{tikzcd} \lceil A \rceil \arrow[r, "f_n"] & \lceil G_n \rceil \\ \lceil B \rceil \arrow[r, "\bar{f}_n"'] \arrow[u, "u"] & \lceil \tilde{G}_n \rceil \arrow[u, "v"'] \end{tikzcd} \tag{11}$$

$p \circ u = 1: \lceil B \rceil \to \lceil B \rceil$, $q_n \circ v: \lceil \bar{G}_n \rceil \to \lceil \bar{G}_n \rceil$ and contracting homotopies Φ_1 and Φ_2 such that

$$\begin{tikzcd} \lceil A \rceil \otimes I \arrow[r, "f_n \otimes 1"] \arrow[d, "\Phi_1"'] & \lceil G_n \rceil \otimes I \arrow[d, "\Phi_2"] \\ \lceil A \rceil \arrow[r, "f_n"'] & \lceil G_n \rceil \end{tikzcd} \tag{12}$$

commutes, where $d\Phi_1 = u \circ p - 1$, and $d\Phi_2 = v \circ q_n - 1$ —where Φ_1 can be specified beforehand. Forming quotients gives rise to a commutative diagram

$$\begin{tikzcd} \lceil A \rceil \arrow[r, "f_n"] \arrow[d, "p"'] & \lceil G_n \rceil \arrow[r] \arrow[d, "q_n"] & H_n \arrow[d, "\hat{q}_n"] \\ \lceil B \rceil \arrow[r, "\bar{f}_n"'] & \lceil \tilde{G}_n \rceil \arrow[r] & \tilde{H}_n \end{tikzcd}$$

Furthermore the commutativity of diagrams 11 and 12 implies that v induces a splitting map $w: \tilde{H}_n \to H_n$ and Φ_2 induces a homotopy $\Xi: H_n \otimes I \to H_n$ with $d\Xi = w \circ \hat{q}_n - 1$— so $\hat{q}_n$ is a *weak equivalence* in **Ch**— even a trivial fibration.

If we assume that the lifting has been carried out to the n^{th} stage, we have a map

$$\ell_n: W \to G_n$$

making

$$\begin{tikzcd} \lceil W \rceil \arrow[r, "\ell_n"] \arrow[d, "g"'] & \lceil G_n \rceil \arrow[r] \arrow[d, "q_n"] & H_n \arrow[d, "\hat{q}_n"] \\ \lceil B \rceil \arrow[r, "\bar{f}_n"'] & \lceil \tilde{G}_n \rceil \arrow[r] & \tilde{H}_n \end{tikzcd}$$

commute. Since the image of B in $\tilde{H}_n$ vanishes (by the way H_n and $\tilde{H}_n$ are constructed—see statement 2 in definition 4.17), it follows that the image of W in H_n lies in the *kernel* of $\hat{q}_n$— a trivial fibration in **Ch**. We conclude that the inclusion of W in H_n is null-homotopic, hence *lifts* to $\bar{H}_n = \Sigma^{-1}(\mathrm{Cone}(H_n))$ in such a way that

$$\begin{tikzcd}
 & & \bar{H}_n \arrow[d] \\
\lceil W \rceil \arrow[urr, bend left, "r"] \arrow[r, "\ell_n"] \arrow[d, "g"'] & \lceil G_n \rceil \arrow[r, "t"] \arrow[d, "q_n"] & H_n \arrow[d, "\hat{q}_n"] \\
\lceil B \rceil \arrow[r, "\bar{f}_n"'] & \lceil \tilde{G}_n \rceil \arrow[r] & \tilde{H}_n
\end{tikzcd}$$

commutes—as a diagram of *chain-complexes*. Now note that G_{n+1} is the fibered product $G_n \boxtimes^{L_{\mathcal{V}}(G_n/A)} L_{\mathcal{V}}(\Sigma^{-1}(\mathrm{Cone}(G_n/A)))$ —and that the chain-maps r and $t \circ \ell_n$ induce c*oalgebra morphisms* making the diagram

$$\begin{tikzcd}
 & \bar{H}_n \arrow[d] \\
W \arrow[ur, "L_{\mathcal{V}} r"] \arrow[dr, "\ell_n"'] & H_n \\
 & G_n \arrow[u, "L_{\mathcal{V}} t"']
\end{tikzcd}$$

commute—thereby inducing a coalgebra-morphism

$$\ell_{n+1} \colon W \to G_n \boxtimes^{L_{\mathcal{V}}(H_n)} L_{\mathcal{V}} \Sigma^{-1} \mathrm{Cone}(H_n) = G_{n+1}$$

that makes the diagram

$$\begin{tikzcd}
 & G_{n+1} \arrow[d, "p_{n+1}"] \\
W \arrow[ur, "\ell_{n+1}"] \arrow[r, "\ell_n"'] & G_n
\end{tikzcd}$$

commute (see statement 5 of definition 4.17). This proves assertion 1 in the induction step.

To prove assertion 2 in our induction hypothesis, note that the natural homotopy in diagram 12 induces (by passage to the quotient) a natural homotopy, Φ', that makes the diagram of chain-complexes

$$\begin{tikzcd}
\lceil G_n \rceil \otimes I \arrow[r, "t \otimes 1"] \arrow[d, "\Phi_2"'] & H_n \otimes I \arrow[d, "\Phi'"] \\
\lceil G_n \rceil \arrow[r, "t"'] & H_n
\end{tikzcd}$$

commute. This can be expanded to a commutative diagram

$$\begin{tikzcd}
\lceil G_n \rceil \otimes I \arrow[r, "t\otimes 1"] \arrow[d, "\Phi_2"'] & H_n \otimes I \arrow[d, "\Phi'"] & \bar{H}_n \otimes I \arrow[l] \arrow[d, "\bar{\Phi}"] \\
\lceil G_n \rceil \arrow[r, "t"'] & H_n & \bar{H}_n \arrow[l]
\end{tikzcd}$$

The conclusion follows from the fact that Φ' and $\bar{\Phi}$ induce *strict* homotopies (see definition 4.7) *after the cofree coalgebra-functor* is applied (see proposition 4.10) and proposition B.19.

We conclude that $G_{n+1} \to \tilde{G}_{n+1}$ is a trivial fibration.

Induction shows that we can define a lifting

$$\ell_\infty : W \to \varprojlim G_n$$

that makes the diagram

$$\begin{tikzcd}
U \arrow[r, "\iota\circ f"] \arrow[d, "i"'] & \varprojlim G_n \arrow[d, "p_\infty"] \\
W \arrow[r, "g"'] \arrow[ur, dotted, "\ell_\infty"] & B
\end{tikzcd}$$

commute. ∎

4.25. The bounded case In this section, we develop a model structure on a category of coalgebras whose underlying chain-complexes are b*ounded from below.*

4.26. Definition. *Let:*

1. $\mathbf{Ch}_0$ *denote the subcategory of* $\mathbf{Ch}$ *bounded at dimension* 1*. If* $A \in \mathbf{Ch}_0$*, then* $A_i = 0$ *for* $i < 1$*.*
2. $\mathscr{I}_0^+$ *denote the category of pointed irreducible coalgebras,* C*, over* $\mathcal{V}$ *such that* $\lceil C \rceil \in \mathbf{Ch}_0$*. This means that* $C_i = 0$*,* $i < 1$*. Note, from the definition of* $\lceil C \rceil$ *as the kernel of the augmentation map, that the underlying chain-complex of* C *is equal to* R *in dimension* 0*.*

There is clearly an inclusion of categories

$$\iota : \mathbf{Ch}_0 \to \mathbf{Ch}$$

compatible with model structures.

Now we define our model structure on $\mathscr{I}_0^+$:

4.27. DEFINITION. *A morphism $f: A \to B$ in $\mathscr{I}_0^+$ will be called*

1. *a* weak equivalence *if $\lceil f \rceil: \lceil A \rceil \to \lceil B \rceil$ is a weak equivalence in $\mathbf{Ch}_0$ (i.e., a chain homotopy equivalence). An object A will be called* contractible *if the augmentation map*

$$A \to R$$

is a weak equivalence.

2. *a* cofibration *if $\lceil f \rceil$ is a cofibration in $\mathbf{Ch}_0$.*

3. *a* trivial cofibration *if it is a weak equivalence and a cofibration.*

REMARK. A morphism is a cofibration if it is a degreewise split monomorphism of chain-complexes. Note that all objects of $\mathscr{I}_0^+$ are cofibrant.

If R is a *field*, all modules are vector spaces therefore free. Homology equivalences of bounded free chain-complexes induce chain-homotopy equivalence, so our notion of weak equivalence becomes the *same* as homology equivalence (or quasi-isomorphism).

4.28. DEFINITION. *A morphism $f: A \to B$ in $\mathscr{I}_0^+$ will be called*

1. *a* fibration *if the dotted arrow exists in every diagram of the form*

$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"'] & A \arrow[d, "f"] \\ W \arrow[r] \arrow[ur, dotted] & B \end{tikzcd}$$

in which $i: U \to W$ is a trivial cofibration.

2. *a* trivial fibration *if it is a fibration and a weak equivalence.*

4.29. COROLLARY. *If $\mathcal{V} = \{\mathcal{V}(n)\}$ is an operad satisfying condition 4.3, the description of cofibrations, fibrations, and weak equivalences given in definitions 4.27 and 4.28 satisfy the axioms for a model structure on $\mathscr{I}_0^+$.*

PROOF. We carry out all of the constructions of § 4 and appendix B while consistently replacing cofree coalgebras by their truncated versions (see [23]). This involves substituting $M_{\mathcal{V}}(*)$ for $L_{\mathcal{V}}(*)$ and $\mathscr{F}_{\mathcal{V}}(*)$ for $P_{\mathcal{V}}(*)$. ∎

5. Examples

We will give a few examples of the model structure developed here. In all cases, we will make the simplifying assumption that R is a field (this is *not* to say that interesting applications *only* occur when R is a field). We begin with coassociative coalgebras over the rationals:

5.1. Example. *Let $\mathcal{V}$ be the operad with component n equal to $\mathbb{Q}S_n$ with the obvious S_n-action —and we consider the category of pointed, irreducible coalgebras, $\mathscr{I}_0$. Coalgebras over this $\mathcal{V}$ are coassociative coalgebras. In this case $P_{\mathcal{V}}C = T(C)$, the graded tensor algebra with coproduct*

$$c_1 \otimes \cdots \otimes c_n \mapsto \sum_{k=0}^{n} (c_1 \otimes \cdots \otimes c_k) \otimes (c_{k+1} \otimes \cdots \otimes c_n) \tag{13}$$

where $c_1 \otimes \cdots \otimes c_0 = c_{n+1} \otimes \cdots \otimes c_n = 1 \in C^0 = \mathbb{Q}$. The n-fold coproducts are just composites of this 2-fold coproduct and the "higher" coproducts vanish identically. We claim that this makes

$$A \boxtimes B = A \otimes B \tag{14}$$

This is due to the well-known identity $T(\lceil A \rceil \oplus \lceil B \rceil) = T(\lceil A \rceil) \otimes T(\lceil B \rceil)$. The category $\mathscr{I}_0^+$ is a category of 1-connected coassociative coalgebras where weak equivalence is equivalent to homology equivalence.

If we assume coalgebras to be cocommutative we get:

5.2. Example. *Suppose $R = \mathbb{Q}$ and $\mathcal{V}$ is the operad with all components equal to $\mathbb{Q}$, concentrated in dimension 0, and equipped with trivial symmetric group actions. Coalgebras over $\mathcal{V}$ are just cocommutative, coassociative coalgebras and $\mathscr{I}_0^+$ is a category of 1-connected coalgebras similar to the one Quillen considered in [21]. Consequently, our model structure for $\mathscr{I}_0^+$ induces the model structure defined by Quillen in [21] on the subcategory of 2-connected coalgebras.*

In this case, $P_{\mathcal{V}}C$ is defined by

$$P_{\mathcal{V}}C = \bigoplus_{n \geq 0} (C^{\otimes n})^{S_n}$$

where $(C^{\otimes n})^{S_n}$ is the submodule of

$$\underbrace{C \otimes \cdots \otimes C}_{n \text{ factors}}$$

invariant under the S_n-action. The assumption that the base-ring is $\mathbb{Q}$ implies a canonical isomorphism

$$P_{\mathcal{V}}C = \bigoplus_{n \geq 0} (C^{\otimes n})^{S_n} \cong S(C)$$

Since $S(\lceil A \rceil \oplus \lceil B \rceil) \cong S(\lceil A \rceil) \otimes S(\lceil B \rceil)$, we again get $A \boxtimes B = A \otimes B$.

A. Nearly free modules

In this section, we will explore the class of nearly free $\mathbb{Z}$-modules —see definition 2.1. We show that this is closed under the operations of taking direct sums, tensor products, countable products and cofree coalgebras. It appears to be fairly large, then, and it would be interesting to have a direct algebraic characterization.

A.1. Remark. *A module must be torsion-free (hence flat) to be nearly free. The converse is not true, however: $\mathbb{Q}$ is flat but* not *nearly free.*

The definition immediately implies that:

A.2. Proposition. *Any submodule of a nearly free module is nearly free.*

Nearly free modules are closed under operations that preserve free modules:

A.3. Proposition. *Let M and N be $\mathbb{Z}$-modules. If they are nearly free, then so are $M \oplus N$ and $M \otimes N$.*

Infinite direct sums of nearly free modules are nearly free.

Proof. If $F \subseteq M \oplus N$ is countable, so are its projections to M and N, which are free by hypothesis. It follows that F is a countable submodule of a free module.

The case where $F \subseteq M \otimes N$ follows by a similar argument: The elements of F are finite linear combinations of monomials $\{m_\alpha \otimes n_\alpha\}$ —the set of which is countable. Let

$$\begin{aligned} A &\subseteq M \\ B &\subseteq N \end{aligned}$$

be the submodules generated, respectively, by the $\{m_\alpha\}$ and $\{n_\alpha\}$. These will be countable modules, hence $\mathbb{Z}$-free. It follows that

$$F \subseteq A \otimes B$$

is a free module.

Similar reasoning proves the last statement, using the fact that any direct sum of free modules is free. ∎

A.4. Proposition. *Let $\{F_n\}$ be a countable collection of $\mathbb{Z}$-free modules. Then*

$$\prod_{n=1}^{\infty} F_n$$

is nearly free.

Proof. In the case where $F_n = \mathbb{Z}$ for all n

$$B = \prod_{n=1}^{\infty} \mathbb{Z}$$

is the Baer-Specker group, which is well-known to be nearly free— see [1], [11, vol. 1, p. 94 Theorem 19.2], and [5]. It is also well-known *not* to be $\mathbb{Z}$-free—see [24] or the survey [8].

First suppose each of the F_n are countably generated. Then

$$F_n \subseteq B$$

and

$$\prod F_n \subseteq \prod B = B$$

which is nearly-free.

In the general case, any countable submodule, C, of $\prod F_n$ projects to a countably-generated submodule, A_n, of F_n under all of the projections

$$\prod F_n \to F_n$$

and, so is contained in

$$\prod A_n$$

which is nearly free, so C must be $\mathbb{Z}$-free. ∎

A.5. Corollary. *Let $\{N_k\}$ be a countable set of nearly free modules. Then*

$$\prod_{k=1}^{\infty} N_k$$

is also nearly free.

Proof. Let

$$F \subset \prod_{k=1}^{\infty} N_k$$

be countable. If F_k is its projection to factor N_k, then $F_k$will be countable, hence free. It follows that

$$F \subset \prod_{k=1}^{\infty} F_k$$

and the conclusion follows from proposition A.4. ∎

A.6. Corollary. *Let A be nearly free and let F be $\mathbb{Z}$-free of countable rank. Then*

$$\mathrm{Hom}_{\mathbb{Z}}(F, A)$$

is nearly free.

Proof. This follows from corollary A.5 and the fact that

$$\mathrm{Hom}_{\mathbb{Z}}(F, A) \cong \prod_{k=1}^{\mathrm{rank}(F)} A$$

∎

A.7. Corollary. *Let $\{F_n\}$ be a sequence of countably-generated $\mathbb{Z}S_n$-projective modules and and let A be nearly free. Then*

$$\prod_{n=1}^{\infty} \mathrm{Hom}_{\mathbb{Z}S_n}(F_n, A^{\otimes n})$$

is nearly free.

Proof. This is a direct application of the results of this section and the fact that

$$\mathrm{Hom}_{\mathbb{Z}S_n}(F_n, A^{\otimes n}) \subseteq \mathrm{Hom}_{\mathbb{Z}}(F_n, A^{\otimes n}) \subseteq \mathrm{Hom}_{\mathbb{Z}}(\hat{F}_n, A^{\otimes n})$$

where $\hat{F}_n$ is a $\mathbb{Z}S_n$-free module of which F_n is a direct summand. ∎

A.8. Theorem. *Let C be a nearly free $\mathbb{Z}$-module and let $\mathcal{V} = \{\mathcal{V}(n)\}$ be a Σ-finite operad with $\mathcal{V}(n)$ of finite type for all $n \geq 0$. Then*

$$\begin{gathered} \lceil L_{\mathcal{V}} C \rceil \\ \lceil M_{\mathcal{V}} C \rceil \\ \lceil P_{\mathcal{V}} C \rceil \\ \lceil \mathscr{F}_{\mathcal{V}} C \rceil \end{gathered}$$

are all nearly free.

Proof. This follows from theorem B.8 which states that all of these are submodules of

$$\prod_{n \geq 0} (\mathcal{V}(n), A^{\otimes n})$$

and the fact that near-freeness is inherited by submodules. ∎

B. Category-theoretic constructions

In this section, we will study general properties of coalgebras over an operad. Some of the results will require coalgebras to be pointed irreducible. We begin by recalling the structure of cofree coalgebras over operads in the pointed irreducible case.

B.1. Cofree-coalgebras We will make extensive use of *cofree coalgebras* over an operad in this section—see definition 2.16.

If they exist, it is not hard to see that cofree coalgebras must be *unique* up to an isomorphism.

The paper [23] gave an explicit construction of $L_{\mathcal{V}} C$ when C was an R-free chain complex. When R is a field, all chain-complexes are R-free, so the results of the present paper are already true in that case.

Consequently, we will restrict ourselves to the case where $R = \mathbb{Z}$.

B.2. Proposition. *The forgetful functor (defined in definition 2.14) and cofree coalgebra functors define adjoint pairs*

$$\begin{aligned} P_{\mathcal{V}}(*)\colon \mathbf{Ch} &\leftrightarrows \mathscr{I}_0\colon \lceil * \rceil \\ L_{\mathcal{V}}(*)\colon \mathbf{Ch} &\leftrightarrows \mathscr{S}_0\colon \lceil * \rceil \end{aligned}$$

Remark. The adjointness of the functors follows from the universal property of cofree coalgebras—see [23]. The Adjoints and Limits Theorem in [17] implies that:

B.3. Theorem. *If $\{A_i\} \in$ ind-$\mathbf{Ch}$ and $\{C_i\} \in$ ind-$\mathscr{I}_0$ or ind-$\mathscr{S}_0$ then*

$$\begin{aligned} \varprojlim P_{\mathcal{V}}(A_i) &= P_{\mathcal{V}}(\varprojlim A_i) \\ \varprojlim L_{\mathcal{V}}(A_i) &= L_{\mathcal{V}}(\varprojlim A_i) \\ \lceil \varinjlim C_i \rceil &= \varinjlim \lceil C_i \rceil \end{aligned}$$

Remark. This implies that colimits in $\mathscr{I}_0$ or $\mathscr{S}_0$ are the same as colimits of underlying chain-complexes.

B.4. Proposition. *If $C \in \mathbf{Ch}$, let $\mathscr{G}(C)$ denote the lattice of countable subcomplexes of C. Then*

$$C = \varinjlim \mathscr{G}(C)$$

Proof. Clearly $\varinjlim \mathscr{G}(C) \subseteq C$ since all of the canonical maps to C are inclusions. Equality follows from every element $x \in C_k$ being contained in a finitely generated subcomplex, C_x, defined by

$$(C_x)_i = \begin{cases} R \cdot x & \text{if } i = k \\ R \cdot \partial x & \text{if } i = k-1 \\ 0 & \text{otherwise} \end{cases}$$

∎

B.5. Lemma. *Let $n > 1$ be an integer, let F be a finitely-generated projective (non-graded) RS_n-module, and let $\{C_\alpha\}$ a direct system of modules. Then the natural map*

$$\varinjlim \operatorname{Hom}_{RS_n}(F, C_\alpha) \to \operatorname{Hom}_{RS_n}(F, \varinjlim C_\alpha)$$

is an isomorphism.

If F and the $\{C_\alpha\}$ are graded, the corresponding statement is true if F is finitely-generated and RS_n-projective in each dimension.

Proof. We will only prove the non-graded case. The graded case follows from the fact that the maps of the $\{C_\alpha\}$ preserve grade.

In the non-graded case, finite generation of F implies that the natural map

$$\bigoplus_\alpha \operatorname{Hom}_{RS_n}(F, C_\alpha) \to \operatorname{Hom}_{RS_n}(F, \bigoplus_\alpha C_\alpha)$$

is an isomorphism, where α runs over any indexing set. The projectivity of F implies that $\operatorname{Hom}_{RS_n}(F, *)$ is exact, so the short exact sequence defining the filtered colimit is preserved. ∎

B.6. Proposition. *Let $\mathcal{V} = \{\mathcal{V}(n)\}$ be an operad satisfying condition 4.3, and let C be a chain-complex with $\mathscr{G}(C) = \{C_\alpha\}$ a family of flat subcomplexes ordered by inclusion that is closed under countable sums. In addition, suppose*

$$C = \varinjlim C_\alpha$$

Then

$$\prod_{n\geq 0} \mathrm{Hom}_{RS_n}(\mathcal{V}(n), C^{\otimes n}) = \varinjlim \prod_{n\geq 0} \mathrm{Hom}_{RS_n}(\mathcal{V}(n), C_\alpha^{\otimes n})$$

Proof. Note that C, as the limit of flat modules, it itself flat.

The $\mathbb{Z}$-flatness of C implies that any $y \in C^{\otimes n}$ is in the image of

$$C_\alpha^{\otimes n} \hookrightarrow C^{\otimes n}$$

for some $C_\alpha \in \mathscr{G}(C)$ and any $n \geq 0$. The finite generation and projectivity of the $\{\mathcal{V}(n)\}$ in every dimension implies that any map

$$x_i \in \mathrm{Hom}_{RS_n}(\mathcal{V}(n), C^{\otimes n})_j$$

lies in the image of

$$\mathrm{Hom}_{RS_n}(\mathcal{V}(n), C_{\alpha_i}^{\otimes n}) \hookrightarrow \mathrm{Hom}_{RS_n}(\mathcal{V}(n), C^{\otimes n})$$

for some $C_{\alpha_i} \in \mathscr{G}(C)$. This implies that

$$x \in \mathrm{Hom}_{RS_n}(\mathcal{V}(n), C^{\otimes n})$$

lies in the image of

$$\mathrm{Hom}_{RS_n}(\mathcal{V}(n), C_\alpha^{\otimes n}) \hookrightarrow \mathrm{Hom}_{RS_n}(\mathcal{V}(n), C^{\otimes n})$$

where $C_\alpha = \sum_{i=0}^{\infty} C_{\alpha_i}$, which is still a subcomplex of the lattice $\mathscr{G}(C)$.

If

$$x = \prod x_n \in \prod_{n\geq 0} \mathrm{Hom}_{RS_n}(\mathcal{V}(n), C^{\otimes n})$$

then each x_n lies in the image of

$$\mathrm{Hom}_{RS_n}(\mathcal{V}(n), C_{\alpha_n}^{\otimes n}) \hookrightarrow \mathrm{Hom}_{RS_n}(\mathcal{V}(n), C^{\otimes n})$$

where $C_{\alpha_n} \in \mathscr{G}(C)$ and x lies in the image of

$$\prod_{n\geq 0} \mathrm{Hom}_{RS_n}(\mathcal{V}(n), C_\alpha^{\otimes n}) \hookrightarrow \prod_{n\geq 0} \mathrm{Hom}_{RS_n}(\mathcal{V}(n), C^{\otimes n})$$

where $C_\alpha = \sum_{n\geq 0} C_{\alpha_n}$ is countable.

The upshot is that

$$\prod_{n\geq 0} \mathrm{Hom}_{RS_n}(\mathcal{V}(n), C^{\otimes n}) = \varinjlim \prod_{n\geq 0} \mathrm{Hom}_{RS_n}(\mathcal{V}(n), C_\alpha^{\otimes n})$$

as C_α runs over all subcomplexes of the lattice $\mathscr{G}(C)$. ∎

B.7. Theorem. *Let $\mathcal{V} = \{\mathcal{V}(n)\}$ be an operad satisfying condition 4.3.*
If C is a $\mathcal{V}$-coalgebra whose underlying chain-complex is nearly free, then

$$C = \varinjlim C_\alpha$$

where $\{C_\alpha\}$ ranges over all the countable sub-coalgebras of C.

Proof. To prove the statement, we show that every

$$x \in C$$

is contained in a countable sub-coalgebra of C.

Let

$$a\colon C \to \prod_{n\geq 0} \mathrm{Hom}_{RS_n}(\mathcal{V}(n), C^{\otimes n})$$

be the adjoint structure-map of C, and let $x \in C_1$, where C_1 is a countable sub-chain-complex of $\lceil C \rceil$.

Then $a(C_1)$ is a countable subset of $\prod_{n\geq 0} \mathrm{Hom}_{RS_n}(\mathcal{V}(n), C^{\otimes n})$, each element of which is defined by its value on the countable set of RS_n-projective generators of $\{\mathcal{V}_n\}$ for all $n > 0$. It follows that the targets of these projective generators are a countable set of elements

$$\{x_j \in C^{\otimes n}\}$$

for $n > 0$. If we enumerate all of the $c_{i,j}$ in $x_j = c_{1,j} \otimes \cdots \otimes c_{n,j}$ and their differentials, we still get a countable set. Let

$$C_2 = C_1 + \sum_{i,j} R \cdot c_{i,j}$$

This will be a countable sub-chain-complex of $\lceil C \rceil$ that contains x. By an easy induction, we can continue this process, getting a sequence $\{C_n\}$ of countable sub-chain-complexes of $\lceil C \rceil$ with the property

$$a(C_i) \subseteq \prod_{n\geq 0} \mathrm{Hom}_{RS_n}(\mathcal{V}(n), C_{i+1}^{\otimes n})$$

arriving at a countable sub-chain-complex of $\lceil C \rceil$

$$C_\infty = \bigcup_{i=1}^{\infty} C_i$$

that is closed under the coproduct of C. It is not hard to see that the induced coproduct on C_∞ will *inherit* the identities that make it a $\mathcal{V}$-coalgebra. ∎

B.8. Corollary. *Let $\mathcal{V} = \{\mathcal{V}(n)\}$ be a Σ-cofibrant operad such that $\mathcal{V}(n)$ is of finite type for all $n \geq 0$. If C is nearly-free, then the cofree coalgebras*

$$L_{\mathcal{V}}C,\ P_{\mathcal{V}}C,\ M_{\mathcal{V}}C,\ \mathscr{F}_{\mathcal{V}}C$$

are well-defined and

$$\left.\begin{array}{r} L_{\mathcal{V}}C = \varinjlim L_{\mathcal{V}}C_\alpha \\ P_{\mathcal{V}}C = \varinjlim P_{\mathcal{V}}C_\alpha \\ M_{\mathcal{V}}C = \varinjlim M_{\mathcal{V}}C_\alpha \\ \mathscr{F}_{\mathcal{V}}C = \varinjlim \mathscr{F}_{\mathcal{V}}C_\alpha \end{array}\right\} \subseteq \prod_{n\geq 0} \mathrm{Hom}_{RS_n}(\mathcal{V}(n), C^{\otimes n})$$

where C_α ranges over the countable sub-chain-complexes of C.

Proof. The near-freeness of C implies that the C_α are all $\mathbb{Z}$-free when $R = \mathbb{Z}$, so the construction in [23] gives cofree coalgebras $L_{\mathcal{V}}C_\alpha$.

Since (by theorem B.7)

$$C = \varinjlim C_\alpha$$

where C_α ranges over countable sub-coalgebras of C, we get coalgebra morphisms

$$b_\alpha : C_\alpha \to L_{\mathcal{V}}\lceil C_\alpha \rceil$$

inducing a coalgebra morphism

$$b: C \to \varinjlim L_{\mathcal{V}}\lceil C_\alpha \rceil$$

We claim that $L_{\mathcal{V}}\lceil C\rceil = \varinjlim L_{\mathcal{V}}\lceil C_\alpha\rceil$. We first note that $\varinjlim L_{\mathcal{V}}\lceil C_\alpha\rceil$ depends only on $\lceil C\rceil$ and not on C . If D is a $\mathcal{V}$-coalgebra with $\lceil C\rceil = \lceil D\rceil$ then, by theorem B.7, $D = \varinjlim D_\beta$ where the D_β are the countable sub-coalgebras of D.

We also know that, in the poset of sub-chain-complexes of $\lceil C\rceil = \lceil D\rceil$, $\{\lceil C_\alpha\rceil\}$ and $\{\lceil D_\beta\rceil\}$ are both cofinal. This implies the cofinality of $\{L_{\mathcal{V}}\lceil C_\alpha\rceil\}$ and $\{L_{\mathcal{V}}\lceil D_\beta\rceil\}$, hence

$$\varinjlim L_{\mathcal{V}}\lceil C_\alpha\rceil = \varinjlim L_{\mathcal{V}}\lceil D_\beta\rceil$$

This unique $\mathcal{V}$-coalgebra has all the categorical properties of the cofree-coalgebra

$$L_{\mathcal{V}}\lceil C\rceil$$

which proves the first part of the result.

The statement that

$$L_{\mathcal{V}}\lceil C\rceil \subseteq \prod_{n\geq 0} \mathrm{Hom}_{RS_n}(\mathcal{V}(n), C^{\otimes n})$$

follows from

1. The canonical inclusion
$$L_{\mathcal{V}}C_{\alpha} \subseteq \prod_{n\geq 0} \mathrm{Hom}_{RS_n}(\mathcal{V}(n), C_{\alpha}^{\otimes n})$$
in [23], and
2. the fact that the hypotheses imply that
$$\prod_{n\geq 0} \mathrm{Hom}_{RS_n}(\mathcal{V}(n), C^{\otimes n}) = \varinjlim \prod_{n\geq 0} \mathrm{Hom}_{RS_n}(\mathcal{V}(n), C_{\alpha}^{\otimes n})$$
—see proposition B.6.

Similar reasoning applies to $P_{\mathcal{V}}C$, $M_{\mathcal{V}}C$, $\mathscr{F}_{\mathcal{V}}C$. ∎

B.9. Core of a module

B.10. Lemma. *Let $A, B \subseteq C$ be sub-coalgebras of $C \in \mathscr{C} = \mathscr{S}_0$ or $\mathscr{I}_0$. Then $A + B \subseteq C$ is also a sub-coalgebra of C.*

In particular, given any sub-DG-module
$$M \subseteq \lceil C \rceil$$
there exists a maximal sub-coalgebra $\langle M\rangle$—called the core of M—with the universal property that any sub-coalgebra $A \subseteq C$ with $\lceil A \rceil \subseteq M$ is a sub-coalgebra of $\langle M\rangle$.

This is given by
$$\alpha(\langle M\rangle) = \alpha(C) \cap P_{\mathcal{V}}M \subseteq P_{\mathcal{V}}C$$
where
$$\alpha: C \to P_{\mathcal{V}}C$$
is the classifying morphism of C.

Proof. The first claim is clear—$A + B$ is clearly closed under the coproduct structure. This implies the second claim because we can always form the sum of any set of sub-coalgebras contained in M.

The second claim follows from:

The fact that
$$\langle M\rangle = \alpha^{-1}(\alpha(C) \cap P_{\mathcal{V}}M)$$
implies that it is the inverse image of a coalgebra (the intersection of two coalgebras) under an injective map (α), so it is a subcoalgebra of C with $\lceil\langle M\rangle\rceil \subseteq M$.

Given any subcoalgebra $A \subseteq C$ with $\lceil A \rceil \subseteq M$, the diagram

$$\begin{tikzcd} A \arrow[r, hook] & C \arrow[r, "\alpha"] \arrow[dr, equal] & P_{\mathcal{V}}C \arrow[d, "\epsilon"] \\ & & C \end{tikzcd}$$

where $\epsilon\colon P_{\mathcal{V}}C \to C$ is the cogeneration map, implies that

$$\begin{aligned} \alpha(A) &\subseteq \alpha(C) \\ \epsilon(\alpha(A)) &\subseteq \epsilon(P_{\mathcal{V}}M) \end{aligned}$$

which implies that $A \subseteq \langle M\rangle$, so $\langle M\rangle$ has the required universal property. ∎

B.11. Corollary. *Let $C \in \mathscr{C} = \mathscr{S}_0$ or $\mathscr{I}_0$ and $M \subseteq \lceil C\rceil$ a sub-DG-module and suppose*

$$\Phi\colon C \otimes I \to C$$

is a coalgebra morphism with the property that $\Phi(M \otimes I) \subseteq M$. Then

$$\Phi(\langle M\rangle \otimes I) \subseteq \langle M\rangle$$

Proof. The hypotheses imply that the diagrams

$$\begin{tikzcd} C \otimes I \arrow[r, "\alpha\otimes 1"] \arrow[d, equal] & (L_{\mathcal{V}}C) \otimes I \arrow[d] \\ C \otimes I \arrow[r] \arrow[d, "\Phi"'] & L_{\mathcal{V}}(C \otimes I) \arrow[d, "L_{\mathcal{V}}\Phi"] \\ C \arrow[r, "\alpha"'] & L_{\mathcal{V}}C \end{tikzcd}$$

and

$$\begin{tikzcd} (L_{\mathcal{V}}M) \otimes I \arrow[r] \arrow[d] & (L_{\mathcal{V}}C) \otimes I \arrow[d] \\ L_{\mathcal{V}}(M \otimes I) \arrow[r] \arrow[d, "L_{\mathcal{V}}(\Phi|M\otimes I)"'] & L_{\mathcal{V}}(C \otimes I \arrow[d, "L_{\mathcal{V}}\Phi"] \\ L_{\mathcal{V}}M \arrow[r] & L_{\mathcal{V}}C \end{tikzcd}$$

commute. Lemma B.10 implies the result. ∎

This allows us to construct *equalizers* in categories of coalgebras over operads:

B.12. Corollary. *If*

$$f_i\colon A \to B$$

with i running over some index set, is a set of morphisms in $\mathscr{C} = \mathscr{S}_0$ or $\mathscr{S}_0$, then the equalizer of the $\{f_i\}$ is

$$\langle M\rangle \subseteq A$$

where M is the equalizer of $\lceil f_i\rceil\colon \lceil A\rceil \to \lceil B\rceil$ in **Ch**.

Remark. Roughly speaking, it is easy to construct coequalizers of coalgebra morphisms and hard to construct equalizers—since the kernel of a morphism is not necessarily a sub-coalgebra. This is dual to what holds for *algebras* over operads.

Proof. Clearly $f_i|\langle M\rangle = f_j|\langle M\rangle$ for all i, j. On the other hand, any sub-DG-algebra with this property is contained in $\langle M\rangle$ so the conclusion follows. ∎

B.13. Proposition. *Let $C \in \mathscr{I}_0$ and let $\{A_i\}$, $i \geq 0$ be a descending sequence of sub-chain-complexes of $\lceil C \rceil$—i.e., $A_{i+1} \subseteq A_i$ for all $i \geq 0$. Then*

$$\left\langle \bigcap_{i=0}^{\infty} A_i \right\rangle = \bigcap_{i=0}^{\infty} \langle A_i \rangle$$

Proof. Clearly, any intersection of coalgebras is a coalgebra, so

$$\bigcap_{i=0}^{\infty} \langle A_i \rangle \subseteq \left\langle \bigcap_{i=0}^{\infty} A_i \right\rangle$$

On the other hand

$$\lceil \left\langle \bigcap_{i=0}^{\infty} A_i \right\rangle \rceil \subseteq \bigcap_{i=0}^{\infty} A_i \subseteq A_n$$

for any $n > 0$. Since $\langle \bigcap_{i=0}^{\infty} A_i \rangle$ is a *coalgebra* whose underlying chain complex is contained in A_n, we must actually have

$$\lceil \left\langle \bigcap_{i=0}^{\infty} A_i \right\rangle \rceil \subseteq \lceil \langle A_n \rangle \rceil$$

which implies that

$$\left\langle \bigcap_{i=0}^{\infty} A_i \right\rangle \subseteq \bigcap_{i=0}^{\infty} \langle A_i \rangle$$

and the conclusion follows. ∎

B.14. Categorical products We can use cofree coalgebras to explicitly construct the categorical product in $\mathscr{I}_0$ or $\mathscr{S}_0$:

B.15. Definition. *Let A_i, $i = 0, 1$ be objects of $\mathscr{C} = \mathscr{S}_0$ or $\mathscr{I}_0$. Then*

$$A_0 \boxtimes A_1 = \langle M_0 \cap M_1 \rangle \subseteq Z = \begin{cases} L_{\mathcal{V}}(\lceil A_0 \rceil \oplus \lceil A_1 \rceil) & \text{if } \mathscr{C} = \mathscr{S}_0 \\ P_{\mathcal{V}}(\lceil A_0 \rceil \oplus \lceil A_1 \rceil) & \text{if } \mathscr{C} = \mathscr{I}_0 \end{cases}$$

where

$$M_i = p_i^{-1}(\lceil \operatorname{im} A_i \rceil)$$

under the projections

$$p_i \colon Z \to \begin{cases} L_{\mathcal{V}}(\lceil A_i \rceil) & \text{if } \mathscr{C} = \mathscr{S}_0 \\ P_{\mathcal{V}}(\lceil A_i \rceil) & \text{if } \mathscr{C} = \mathscr{I}_0 \end{cases}$$

induced by the canonical maps $\lceil A_0 \rceil \oplus \lceil A_1 \rceil \to \lceil A_i \rceil$. The $\operatorname{im} A_i$ are images under the canonical morphisms

$$A_i \to \left\{ \begin{array}{cc} L_{\mathcal{V}} \lceil A_i \rceil & \text{if } \mathscr{C} = \mathscr{S}_0 \\ P_{\mathcal{V}}(\lceil A_i \rceil) & \text{if } \mathscr{C} = \mathscr{I}_0 \end{array} \right\} \to Z$$

classifying coalgebra structures—see definition 2.16.

Remark. By identifying the A_i with their canonical images in Z, we get canonical projections to the factors

$$A_0 \boxtimes A_1 \to A_i$$

B.16. Lemma. *The operation, $\boxtimes$, defined above is a categorical product in $\mathscr{C} = \mathscr{S}_0$ or $\mathscr{I}_0$. In other words, morphisms $f_i: C \to A_i$, $i = 0, 1$ in $\mathscr{C}$ induce a unique morphism $\bar{f}: C \to A_0 \boxtimes A_1$ that makes the diagram*

$$\begin{tikzcd} & A_0 & \\ C \arrow[ur, "f_0"] \arrow[rr, "\bar{f}"'] \arrow[dr, "f_1"'] & & A_0 \boxtimes A_1 \arrow[ul] \arrow[dl] \\ & A_1 & \end{tikzcd}$$

commute.

Proof. The morphisms f_i induce morphisms $\lceil f_i \rceil: \lceil C \rceil \to \lceil C_i \rceil$ which in turn induce a morphism

$$\lceil f_0 \rceil \oplus \lceil f_1 \rceil: \lceil C \rceil \to \lceil A_0 \rceil \oplus \lceil A_1 \rceil$$

This induces a *unique* coalgebra morphism

$$C \to L_{\mathcal{V}}(\lceil A_0 \rceil \oplus \lceil A_1 \rceil)$$

(when $\mathscr{C} = \mathscr{S}_0$) and the diagram

$$\begin{tikzcd} & A_0 \arrow[r, hook] & L_{\mathcal{V}}\lceil A_0 \rceil & \\ C \arrow[ur, "f_0"] \arrow[rrr, "\bar{f}"'] \arrow[dr, "f_1"'] & & & L_{\mathcal{V}}(\lceil A_0 \rceil \oplus \lceil A_1 \rceil) \arrow[ul] \arrow[dl] \\ & A_1 \arrow[r, hook] & L_{\mathcal{V}}\lceil A_1 \rceil & \end{tikzcd}$$

shows that its image actually lies in $A_0 \boxtimes A_1 \subseteq L_{\mathcal{V}}(\lceil A_0 \rceil \oplus \lceil A_1 \rceil)$. The case where $\mathscr{C} = \mathscr{I}_0$ is entirely analogous. ∎

B.17. Proposition. *Let $F: A \otimes I \to A$ and $G: B \otimes I \to B$ be strict homotopies in $\mathscr{C} = \mathscr{I}_0$ or $\mathscr{S}_0$ (see definition 4.7). Then there is a strict homotopy*

$$(A \boxtimes B) \otimes I \xrightarrow{F \hat{\boxtimes} G} A \boxtimes B$$

that makes the diagrams

$$\begin{tikzcd}
(A \boxtimes B) \otimes I \arrow[r, "F\hat{\boxtimes}G"] \arrow[d] & A \boxtimes B \arrow[d] \\
A \otimes I \arrow[r, "F"'] & A
\end{tikzcd} \tag{15}$$

and

$$\begin{tikzcd}
(A \boxtimes B) \otimes I \arrow[r, "F\hat{\boxtimes}G"] \arrow[d] & A \boxtimes B \arrow[d] \\
B \otimes I \arrow[r, "G"'] & B
\end{tikzcd} \tag{16}$$

commute. If

$$\begin{aligned} f_1, f_2 \colon A &\to A' \\ g_1, g_2 \colon B &\to B' \end{aligned}$$

are strictly homotopic morphisms with respective strict homotopies

$$\begin{aligned} F \colon A \otimes I &\to A' \\ G \colon B \otimes I &\to B' \end{aligned}$$

then $F\hat{\boxtimes}G$ *is a strict homotopy between* $f_1 \boxtimes g_1$ *and* $f_2 \boxtimes g_2$.

Consequently, if $f \colon A \to A'$ *and* $g \colon B \to B'$ *are strict equivalences, then*

$$f \boxtimes g \colon A \boxtimes B \to A' \boxtimes B'$$

is a strict equivalence.

Proof. The projections

$$\begin{tikzcd}
A \boxtimes B \arrow[r] \arrow[d] & B \\
A &
\end{tikzcd}$$

induce projections

$$\begin{tikzcd}
(A \boxtimes B) \otimes I \arrow[r] \arrow[d] & B \otimes I \\
A \otimes I &
\end{tikzcd}$$

and the composite

$$(A \boxtimes B) \otimes I \to (A \otimes I) \boxtimes (B \otimes I) \xrightarrow{F \boxtimes G} A \boxtimes B$$

satisfies the first part of the statement.

Note that diagrams 15 and 16 —and the fact that maps to $A' \boxtimes B'$ are *uniquely determined* by their composites with the projections $A' \boxtimes B' \rightrightarrows A', B'$ (the defining universal property of $\boxtimes$)—implies that $F\hat{\boxtimes}G$ is a strict homotopy between $f_1 \boxtimes g_1$ and $f_2 \boxtimes g_2$. The final statement is also clear. ■

In like fashion, we can define categorical *fibered* products of coalgebras:

B.18. Definition. *Let*

$$\begin{tikzcd} F \arrow[r] \arrow[d, "i"'] & A \arrow[d, "f"] \\ B \arrow[r] & C \end{tikzcd}$$

be a diagram in $\mathscr{S}_0$ or $\mathscr{I}_0$. Then the fibered product with respect to this diagram, $A \boxtimes^C B$, *is defined to be the equalizer*

$$F \to A \boxtimes B \rightrightarrows C$$

by the maps induced by the projections $A \boxtimes B \to A$ and $A \boxtimes B \to B$ composed with the maps in the diagram.

We have an analogue to proposition B.17:

B.19. Proposition. *Let $A \xrightarrow{f} B \xleftarrow{g} C$, $A' \xrightarrow{f'} B' \xleftarrow{g'} C'$ be diagrams in $\mathscr{C} = \mathscr{I}_0$ or $\mathscr{S}_0$ and let*

$$\begin{tikzcd} A \otimes I \arrow[r, "f\otimes 1"] \arrow[d, "H_A"'] & B \otimes I \arrow[d, "H_B"] & C \otimes I \arrow[l, "g\otimes 1"'] \arrow[d, "H_C"] \\ A' \arrow[r, "f'"'] & B' & C \arrow[l, "g'"] \end{tikzcd}$$

commute, where the H_α are strict homotopies. Then there exists a strict homotopy

$$(A \boxtimes^B C) \otimes I \xrightarrow{H_A \hat{\boxtimes}^{H_B} H_C} A' \boxtimes^{B'} C'$$

between the morphisms

$$(H_A | A \otimes p_i) \boxtimes (H_C \otimes p_i) \colon A \boxtimes^B C \to A' \boxtimes^{B'} C'$$

for $i = 0, 1$.

Proof. The morphism $H_A \hat{\boxtimes}^{H_B} H_C$ is constructed exactly as in proposition B.17. The conclusion follows by the same reasoning used to prove the final statement of that result. ∎

B.20. Proposition. *Let U, V and W be objects of* **Ch** *and let Z be the fibered product of*

$$\begin{tikzcd} & V \arrow[d, "g"] \\ U \arrow[r, "f"'] & W \end{tikzcd}$$

in **Ch***—i.e., Z is the equalizer*

$$Z \to U \oplus V \rightrightarrows W$$

in **Ch**. *Then* $P_{\mathcal{V}}Z$ *is the fibered product of*

$$\begin{tikzcd} & P_{\mathcal{V}}V \arrow[d, "P_{\mathcal{V}}g"] \\ P_{\mathcal{V}}U \arrow[r, "P_{\mathcal{V}}f"'] & P_{\mathcal{V}}W \end{tikzcd} \tag{17}$$

in $\mathscr{I}_0$ *and* $L_{\mathcal{V}}Z$ *is the fibered product of*

$$\begin{tikzcd} & L_{\mathcal{V}}V \arrow[d, "L_{\mathcal{V}}g"] \\ L_{\mathcal{V}}U \arrow[r, "L_{\mathcal{V}}f"'] & L_{\mathcal{V}}W \end{tikzcd} \tag{18}$$

in $\mathscr{S}_0$.

PROOF. We prove this in the pointed irreducible case. The other case follows by an analogous argument.

The universal properties of cofree coalgebras imply that $P_{\mathcal{V}}(U \oplus V) = P_{\mathcal{V}}U \boxtimes P_{\mathcal{V}}V$. Suppose F is the fibered product of diagram 17. Then

$$P_{\mathcal{V}}Z \subseteq F$$

On the other hand, the composite

$$F \to P_{\mathcal{V}}U \boxtimes P_{\mathcal{V}}V = P_{\mathcal{V}}(U \oplus V) \to U \oplus V$$

where the rightmost map is the co-generating map, has composites with f and g that are equal to each other—so it lies in $Z \subseteq U \oplus V$. This induces a *unique* coalgebra morphism

$$j \colon F \to P_{\mathcal{V}}Z$$

left-inverse to the inclusion

$$i \colon P_{\mathcal{V}}Z \subseteq F$$

The uniqueness of induced maps to cofree coalgebras implies that $j \circ i = i \circ j = 1$. ■

B.21. LIMITS AND COLIMITS We can use cofree coalgebras and adjointness to the forgetful functors to define categorical limits and colimits in $\mathscr{I}_0$ and $\mathscr{S}_0$.

Categorical reasoning implies that

B.22. PROPOSITION. *Let*

$$\begin{tikzcd} & \{B_i\} \arrow[d, "b_i"] \\ \{A_i\} \arrow[r, "a_i"'] & \{C_i\} \end{tikzcd}$$

be a diagram in pro-$\mathscr{I}_0$ or pro-$\mathscr{S}_0$. Then

$$\varprojlim (A_i \boxtimes^{C_i} B_i) = (\varprojlim A_i) \boxtimes^{(\varprojlim C_i)} (\varprojlim B_i)$$

See definition B.18 for the fibered product notation.

Theorem B.3 implies that colimits in $\mathscr{I}_0$ or $\mathscr{S}_0$ are the same as colimits of underlying chain-complexes. The corresponding statement for limits is not true except in a special case:

B.23. Proposition. *Let $\{C_i\} \in$ pro-$\mathscr{I}_0$ or pro-$\mathscr{S}_0$ and suppose that all of its morphisms are injective. Then*

$$\lceil \varprojlim C_i \rceil = \varprojlim \lceil C_i \rceil$$

Remark. In this case, the limit is an intersection of coalgebras. This result says that to get the limit of $\{C_i\}$, one

1. forms the limit of the underlying chain-complexes (i.e., the intersection) and
2. equips that with the coalgebra structure in induced by its inclusion into any of the C_i

That this constructs the limit follows from the *uniqueness* of limits.

B.24. Definition. *Let $A = \{A_i\} \in$ pro-$\mathscr{I}_0$. Then define the* normalization of A, *denoted $\hat{A} = \{\hat{A}_i\}$, as follows:*

1. *Let $V = P_{\mathcal{V}}(\varprojlim \lceil A_i \rceil)$ with canonical maps*
$$q_n \colon P_{\mathcal{V}}(\varprojlim \lceil A_i \rceil) \to P_{\mathcal{V}}(\lceil A_n \rceil)$$
for all $n > 0$.
2. *Let $f_n \colon A_n \to P_{\mathcal{V}}(\lceil A_n \rceil)$ be the coalgebra classifying map—see definition 2.16.*

Then $\hat{A}_n = \langle q_n^{-1}(f_n(A_n)) \rangle$, and $\hat{A}_{n+1} \subseteq \hat{A}_n$ for all $n > 0$. Define $\hat{A} = \{\hat{A}_n\}$, with the injective structure maps defined by inclusion.

If $A = \{A_i\} \in$ pro-$\mathscr{S}_0$ then the corresponding construction holds, where we consistently replace $P_{\mathcal{V}}(*)$ by $L_{\mathcal{V}}(*)$.

Normalization reduces the *general* case to the case dealt with in proposition B.23.

B.25. Corollary. *Let $C = \{g_i \colon C_i \to C_{i-1}\}$ in pro-$\mathscr{I}_0$ or pro-$\mathscr{S}_0$. Then*

$$\varprojlim C_i = \varprojlim \hat{C}_i$$

where $\{\hat{C}_i\}$ is the normalization of $\{C_i\}$. In particular, if C is in $\mathscr{I}_0$

$$\varprojlim C_i = \left\langle \bigcap_{i=0}^{\infty} \lceil p_i \rceil^{-1} \lceil \alpha_i \rceil (\lceil C_i \rceil) \right\rangle \subseteq P_{\mathcal{V}}(\varprojlim \lceil C_i \rceil)$$

where $p_n \colon P_{\mathcal{V}}(\varprojlim \lceil C_i \rceil) \to P_{\mathcal{V}}(\lceil C_n \rceil)$ and $\alpha_n \colon C_n \to P_{\mathcal{V}}(\lceil C_n \rceil)$ are as in definition B.24, and the corresponding statement holds if C is in pro-$\mathscr{S}_0$ with $P_{\mathcal{V}}(*)$ replaced by $L_{\mathcal{V}}(*)$.

Proof. Assume the notation of definition B.24. Let

$$f_i\colon C_i \to \left\{ \begin{array}{c} P_{\mathcal{V}}(\lceil C_i \rceil) \\ L_{\mathcal{V}}\lceil C_i \rceil \end{array} \right\}$$

be the classifying maps in $\mathscr{I}_0$ or $\mathscr{S}_0$, respectively —see definition 2.16. We deal with the case of the category $\mathscr{I}_0$—the other case is entirely analogous. Let

$$q_n\colon P_{\mathcal{V}}(\varprojlim \lceil C_i \rceil) \to P_{\mathcal{V}}(\lceil C_n \rceil)$$

be induced by the canonical maps $\varprojlim \lceil C_i \rceil \to \lceil C_n \rceil$.

We verify that

$$X = \left\langle \bigcap_{i=0}^{\infty} \lceil q_i \rceil^{-1} \lceil f_i \rceil (\lceil C_i \rceil) \right\rangle = \varprojlim \hat{C}_i$$

has the category-theoretic properties of an inverse limit. We must have morphisms

$$p_i\colon X \to C_i$$

making the diagrams

$$\begin{tikzcd} X \arrow[r, "p_i"] \arrow[dr, "p_{i-1}"'] & C_i \arrow[d, "g_i"] \\ & C_{i-1} \end{tikzcd} \tag{19}$$

commute for all $i > 0$. Define $p_i = f_i^{-1} \circ q_i\colon X \to C_i$ —using the fact that the classifying maps $f_i\colon C_i \to P_{\mathcal{V}}\lceil C_i \rceil$ are always injective (see [23] and the definition 2.16). The commutative diagrams

$$\begin{tikzcd} C_i \arrow[r, "\alpha_i"] \arrow[d, "g_i"'] & P_{\mathcal{V}}\lceil C_i \rceil \arrow[d, "P_{\mathcal{V}}\lceil g_i \rceil"] \\ C_{i-1} \arrow[r, "\alpha_{i-1}"'] & P_{\mathcal{V}}\lceil C_{i-1} \rceil \end{tikzcd}$$

and

$$\begin{tikzcd} \varprojlim P_{\mathcal{V}}\lceil C_i \rceil \arrow[r, "p_i"] \arrow[dr, "p_{i-1}"'] & P_{\mathcal{V}}\lceil C_i \rceil \arrow[d, "P_{\mathcal{V}}\lceil g_i \rceil"] \\ & P_{\mathcal{V}}\lceil C_{i-1} \rceil \end{tikzcd}$$

together imply the commutativity of the diagram with the diagrams 19. Consequently, X is a *candidate* for being the inverse limit, $\varprojlim C_i$.

We must show that any other candidate Y possesses a *unique* morphism $Y \to X$, making appropriate diagrams commute. Let Y be such a candidate. The morphism of inverse systems defined by classifying maps (see definition 2.16)

$$C_i \to P_{\mathcal{V}}(\lceil C_i \rceil)$$

implies the existence of a *unique* morphism

$$Y \to \varprojlim P_{\mathcal{V}} \lceil C_i \rceil = P_{\mathcal{V}} \varprojlim \lceil C_i \rceil$$

The commutativity of the diagrams

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & P_{\mathcal{V}} \varprojlim \lceil C_i \rceil \arrow[d, "p_i"] \\ C_i \arrow[r, "\alpha_i"'] & P_{\mathcal{V}} \lceil C_i \rceil \end{tikzcd}$$

for all $i \geq 0$ implies that $\lceil \operatorname{im} Y \rceil \subseteq \lceil p_i \rceil^{-1} \lceil \alpha_i \rceil (\lceil C_i \rceil)$. Consequently

$$\lceil \operatorname{im} Y \rceil \subseteq \bigcap_{i=0}^{\infty} \lceil p_i \rceil^{-1} \lceil \alpha_i \rceil (\lceil C_i \rceil)$$

Since Y is a coalgebra, its image must lie within the maximal sub-coalgebra contained within $\bigcap_{i=0}^{\infty} \lceil p_i \rceil^{-1} \lceil \alpha_i \rceil (\lceil C_i \rceil)$, namely $X = \langle \bigcap_{i=0}^{\infty} \lceil p_i \rceil^{-1} \lceil \alpha_i \rceil (\lceil C_i \rceil) \rangle$. This proves the first claim. Proposition B.13 implies that $X = \bigcap_{i=0}^{\infty} \hat{C}_i = \varprojlim \hat{C}_i$. ∎

B.26. Lemma. *Let $\{g_i : C_i \to C_{i-1}\}$ be an inverse system in* **Ch**. *If $n > 0$ is an integer, then the natural map*

$$\left(\varprojlim C_i\right)^{\otimes n} \to \varprojlim C_i^{\otimes n}$$

is injective.

Proof. Let $A = \varprojlim C_i$ and $p_i : A \to C_i$ be the natural projections. If

$$W_k = \ker p_k^{\otimes n} : \left(\varprojlim C_i\right)^{\otimes n} \to C_k^{\otimes n}$$

we will show that

$$\bigcap_{k=1}^{\infty} W_k = 0$$

If $K_i = \ker p_i$, then

$$\bigcap_{i=1}^{\infty} K_i = 0$$

and

$$W_i = \sum_{j=1}^{n} \underbrace{A \otimes \cdots \otimes K_i \otimes \cdots \otimes A}_{j^{\text{th}} \text{ position}}$$

Since all modules are nearly-free, hence, flat (see remark A.1), we have

$$W_{k+1} \subseteq W_k$$

for all k, and

$$\bigcap_{i=1}^{m} W_i = \sum_{j=1}^{n} \underbrace{A \otimes \cdots \otimes \left(\bigcap_{i=1}^{m} K_i\right) \otimes \cdots \otimes A}_{j^{\text{th}}\text{ position}}$$

from which the conclusion follows. ■

B.27. Proposition. *Let $\{C_i\} \in$ pro-$\mathscr{I}_0$, and suppose $\mathcal{V} = \{\mathcal{V}(n)\}$ is a Σ-cofibrant operad with $\mathcal{V}(n)$ of finite type for all $n \geq 0$. Then the projections*

$$\lceil P_{\mathcal{V}}(\varprojlim \lceil C_i \rceil)\rceil \to \lceil P_{\mathcal{V}}(\lceil C_n \rceil)\rceil$$

for all $n > 0$, induce a canonical injection

$$\mu\colon \lceil P_{\mathcal{V}}(\varprojlim \lceil C_i \rceil)\rceil \hookrightarrow \varprojlim \lceil P_{\mathcal{V}}(\lceil C_i \rceil)\rceil$$

In addition, the fact that the structure maps

$$\alpha_i\colon C_i \to P_{\mathcal{V}}(\lceil C_i \rceil)$$

of the $\{C_i\}$ are coalgebra morphisms implies the existence of an injective **Ch***-morphism*

$$\hat{\alpha}\colon \varprojlim \lceil C_i \rceil \hookrightarrow \varprojlim \lceil P_{\mathcal{V}}(\lceil C_i \rceil)\rceil$$

Corresponding statements hold for pro-$\mathscr{S}_0$ and the functors $L_{\mathcal{V}}(*)$.

Proof. We must prove that

$$\mu\colon \lceil P_{\mathcal{V}}(\varprojlim \lceil C_i \rceil)\rceil \to \varprojlim \lceil P_{\mathcal{V}}(\lceil C_i \rceil)\rceil$$

is injective. Let $K = \ker \mu$. Then

$$K \subset \lceil P_{\mathcal{V}}(\varprojlim \lceil C_i \rceil)\rceil \subseteq \prod_{n\geq 0} \mathrm{Hom}_{RS_n}(\mathcal{V}(n), D^{\otimes n})$$

where $D = \varprojlim \lceil C_i \rceil$ (see [23]). If $n \geq 0$, let

$$p_n\colon \prod_{n\geq 0} \mathrm{Hom}_{RS_n}(\mathcal{V}(n), D^{\otimes n}) \to \mathrm{Hom}_{RS_n}(\mathcal{V}(n), D^{\otimes n})$$

denote the canonical projections. The diagrams

$$\begin{tikzcd}
\prod_{n\geq 0} \mathrm{Hom}_{RS_n}(\mathcal{V}(n), D^{\otimes n}) \arrow[r, "\prod \mathrm{Hom}_R(1, b_k^{\otimes n})"] \arrow[d, "p_n"'] & \prod_{n\geq 0} \mathrm{Hom}_{RS_n}(\mathcal{V}(n), C_k^{\otimes n}) \arrow[d, "q_n"] \\
\mathrm{Hom}_{RS_n}(\mathcal{V}(n), D^{\otimes n}) \arrow[r, "\mathrm{Hom}_R(1, b_k^{\otimes n})"'] & \mathrm{Hom}_{RS_n}(\mathcal{V}(n), C_k^{\otimes n})
\end{tikzcd}$$

commute for all k and $n \geq 0$, where q_n is the counterpart of p_n and $b_k: \varprojlim \lceil C_i \rceil \to \lceil C_k \rceil$ is the canonical map. It follows that

$$p_k(K) \subseteq \ker \mathrm{Hom}_R(1, b_k^{\otimes n})$$

for all $n \geq 0$, or

$$p_k(K) \subseteq \bigcap_{k>0} \ker \mathrm{Hom}_R(1, b_k^{\otimes n})$$

We claim that

$$\bigcap_{n>0} \ker \mathrm{Hom}_R(1, b_k^{\otimes n}) = \mathrm{Hom}_R(1, \bigcap_{k>0} \ker b_k^{\otimes n}) = 0$$

The equality on the left follows from the left-exactness of Hom_R and filtered limits (of chain-complexes). The equality on the right follows from the fact that

1. $\bigcap_{k>0} \ker b_k = 0$
2. the left exactness of $\otimes$ for *R-flat* modules (see remark A.1).
3. Lemma B.26.

It follows that $p_n(K) = 0$ for all $n \geq 0$ and $K = 0$.

The map

$$\hat{\alpha}: \varprojlim \lceil C_i \rceil \hookrightarrow \varprojlim \lceil P_{\mathcal{V}}(\lceil C_i \rceil) \rceil$$

is induced by *classifying maps* of the coalgebras $\{C_i\}$, which induce a morphism of limits because the structure maps $C_k \to C_{k-1}$ are coalgebra morphisms, making the diagrams

$$\begin{tikzcd} C_k \arrow[r] \arrow[d] & C_{k-1} \arrow[d] \\ P_{\mathcal{V}}(\lceil C_k \rceil) \arrow[r] & P_{\mathcal{V}}(\lceil C_{k-1} \rceil) \end{tikzcd}$$

commute for all $k > 0$. ■

B.28. Corollary. *Let $C = \{g_i: C_i \to C_{i-1}\} \in$ pro-$\mathscr{I}_0$, and suppose $\mathcal{V} = \{\mathcal{V}(n)\}$ is a Σ-cofibrant operad with $\mathcal{V}(n)$ of finite type for all $n \geq 0$. Then*

$$\varprojlim C_i = \bigcap_{i=1}^{\infty} q_i^{-1}(\alpha_i(C_i)) \subseteq L_{\mathcal{V}}(\varprojlim \lceil C_i \rceil) \tag{20}$$

with the coproduct induced from $L_{\mathcal{V}}(\varprojlim \lceil C_i \rceil)$, and where

$$q_i: L_{\mathcal{V}}(\varprojlim \lceil C_i \rceil) \to L_{\mathcal{V}}(\lceil C_i \rceil)$$

is the projection and

$$\alpha_i: C_i \to L_{\mathcal{V}}(\lceil C_i \rceil) \tag{21}$$

is the classifying map, for all i. In addition, the sequence

$$0 \to \lceil \varprojlim C_i \rceil \to \varprojlim \lceil C_i \rceil \xrightarrow{\hat{\alpha}} \frac{\varprojlim \lceil L_{\mathcal{V}}(\lceil C_i \rceil) \rceil}{\mu(\lceil L_{\mathcal{V}}(\varprojlim \lceil C_i \rceil) \rceil)}$$

$$\to \frac{\varprojlim (\lceil L_{\mathcal{V}}(\lceil C_i \rceil) / \lceil \alpha_i(C_i) \rceil \rceil)}{\operatorname{im} \mu(\lceil L_{\mathcal{V}}(\varprojlim \lceil C_i \rceil) \rceil)} \to \varprojlim{}^1 \lceil C_i \rceil \to 0$$

is exact in **Ch**, *where the injection*

$$\lceil \varprojlim C_i \rceil \to \varprojlim \lceil C_i \rceil$$

is induced by the projections

$$p_i \colon \varprojlim C_i \to C_i$$

and

$$\hat{\alpha} \colon \varprojlim \lceil C_i \rceil \to \varprojlim \lceil L_{\mathcal{V}}(\lceil C_i \rceil) \rceil$$

is induced by the $\{\alpha_i\}$ in equation 21. The map

$$\mu \colon \lceil L_{\mathcal{V}}(\varprojlim \lceil C_i \rceil \rceil \hookrightarrow \varprojlim \lceil L_{\mathcal{V}}(\lceil C_i \rceil \rceil$$

is constructed in Proposition B.27.

If $C \in$ pro-$\mathscr{S}_0$, then the corresponding statements apply, where $L_{\mathcal{V}}(*)$ is replaced by $P_{\mathcal{V}}(*)$.

Remark. The first statement implies that the use of the $\langle * \rangle$-functor in corollary B.25 is unnecessary—at least if $\mathcal{V}$ is projective in the sense defined above.

The remaining statements imply that $\varprojlim C_i$ is the *largest* sub-chain-complex of $\varprojlim \lceil C_i \rceil$ upon which one can define a coproduct that is compatible with the maps

$$\varprojlim C_i \to C_i$$

Proof. First, consider the projections

$$q_i \colon L_{\mathcal{V}}(\varprojlim \lceil C_i \rceil) \to L_{\mathcal{V}}(\lceil C_i \rceil)$$

The commutativity of the diagram

$$\begin{tikzcd} L_{\mathcal{V}}(\varprojlim \lceil C_i \rceil) \arrow[r, hook] \arrow[d, "q_i"'] & \varprojlim L_{\mathcal{V}}(C_i) \arrow[d] \\ L_{\mathcal{V}}(C_i) \arrow[r, equal] & L_{\mathcal{V}}(C_i) \end{tikzcd}$$

implies that

$$\varprojlim \ker q_i = \bigcap_{i=1}^{\infty} \ker q_i = 0$$

Now, consider the exact sequence

$$0 \to \ker q_i \to q_i^{-1}(\alpha_i(C_i)) \to \lceil C_i \rceil \to 0$$

and pass to inverse limits. We get the standard 6-term exact sequence for inverse limits (of $\mathbb{Z}$-modules):

$$\begin{aligned} 0 \to \varprojlim \ker q_i &\to \varprojlim q_i^{-1}(\alpha_i(\lceil C_i \rceil)) \to \varprojlim \lceil C_i \rceil \\ &\to \varprojlim{}^1 \ker q_i \to \varprojlim{}^1 q_i^{-1}(\alpha_i(C_i)) \to \varprojlim{}^1 \lceil C_i \rceil \to 0 \end{aligned}$$

which, with the fact that $\varprojlim \ker q_i = 0$, implies that

$$\bigcap_{i=1}^{\infty} q_i^{-1}(\alpha_i(C_i)) = \varprojlim q_i^{-1}(\alpha_i(C_i)) \hookrightarrow \varprojlim \lceil C_i \rceil$$

The conclusion follows from the fact that

$$\varprojlim C_i = \left\langle \bigcap_{i=1}^{\infty} q_i^{-1}(\alpha_i(C_i)) \right\rangle \subseteq \bigcap_{i=1}^{\infty} q_i^{-1}(\alpha_i(C_i))$$

It remains to prove the claim in equation 20, which amounts to showing that

$$J = \bigcap_{i=1}^{\infty} q_i^{-1}(\alpha_i(C_i)) \subseteq L_{\mathcal{V}}(\varprojlim \lceil C_i \rceil)$$

is closed under the coproduct of $L_{\mathcal{V}}(\varprojlim \lceil C_i \rceil)$ —i.e., it is a coalgebra even *without* applying the $\langle * \rangle$-functor. If $n \geq 0$, consider the diagram

$$\begin{tikzcd}
q_j^{-1}(\alpha_j(C_j)) \arrow[r, "\alpha_j^{-1} \circ q_j"] \arrow[d, hook] & C_j \arrow[d, hook, "\alpha_j"'] \arrow[dddd, bend left, "c_{n,j}"] \\
L_{\mathcal{V}}(\varprojlim \lceil C_i \rceil) \arrow[r, "q_j"'] \arrow[d, "\hat{\delta}_n"'] & L_{\mathcal{V}}(\lceil C_j \rceil) \arrow[d, "\delta_{j,n}"'] \\
\mathrm{Hom}_{RS_n}(\mathcal{V}(n), L_{\mathcal{V}}(\varprojlim \lceil C_i \rceil)^{\otimes n})) \arrow[r, "r_{n,j}"] \arrow[d, hook, "\hat{\mu}_n"'] & \mathrm{Hom}_{RS_n}(\mathcal{V}(n), (L_{\mathcal{V}}(\lceil C_j \rceil)^{\otimes n})) \\
\mathrm{Hom}_{RS_n}(\mathcal{V}(n), (\varprojlim L_{\mathcal{V}}(\lceil C_i \rceil))^{\otimes n}) \arrow[ur, "\mathrm{Hom}_R(1, p_j^{\otimes n})"'] \arrow[d, hook] & \\
\prod_{n \geq 0} \mathrm{Hom}_{RS_n}(\mathcal{V}(n), L_{\mathcal{V}}(\lceil C_i \rceil)^{\otimes n}) \arrow[uur, bend right, "\pi_j"'] & \mathrm{Hom}_{RS_n}(\mathcal{V}(n), C_j^{\otimes n}) \arrow[uu, "s_{n,j}"']
\end{tikzcd}$$

where:

1. the δ_i and $\hat{\delta}$-maps are coproducts and the α_i are coalgebra morphisms.
2. $r_{n,j} = \mathrm{Hom}_R(1, q_j^{\otimes n})$,
3. The map $\hat{\mu}_n$ is defined by

$$\begin{aligned}\hat{\mu}_n = \mathrm{Hom}_R(1, \mu^{\otimes n})\colon &\mathrm{Hom}_{RS_n}(\mathcal{V}(n), (L_{\mathcal{V}}(\varprojlim \lceil C_i \rceil)^{\otimes n})) \\ &\hookrightarrow \mathrm{Hom}_{RS_n}(\mathcal{V}(n), (\varprojlim L_{\mathcal{V}}(\lceil C_i \rceil)^{\otimes n})\end{aligned}$$

   where $\mu\colon L_{\mathcal{V}}(\varprojlim \lceil C_i \rceil) \hookrightarrow \varprojlim L_{\mathcal{V}}(\lceil C_i \rceil)$ is the map defined in Proposition B.27.
4. $s_{n,j} = \mathrm{Hom}_R(1, \alpha_j^{\otimes n})$, and $\alpha_j\colon C_j \to L_{\mathcal{V}}(\lceil C_j \rceil)$ is the classifying map.
5. $p_j\colon \varprojlim L_{\mathcal{V}}(\lceil C_i \rceil) \to L_{\mathcal{V}}(\lceil C_j \rceil)$ is the canonical projection.
6. $c_{n,j}\colon C_j \to \mathrm{Hom}_{RS_n}(\mathcal{V}(n), C_j^{\otimes n})$ is the coproduct.

This diagram and the projectivity of $\{\mathcal{V}(n)_*\}$ and the near-freeness of $L_{\mathcal{V}}(\varprojlim \lceil C_i \rceil)$ (and flatness: see remark A.1) implies that

$$\hat{\delta}_n \left(q_j^{-1}(\alpha_j(C_j)) \right) \subseteq \mathrm{Hom}_{RS_n}(\mathcal{V}(n), L_{n,j})$$

where $L_{n,j} = q_j^{-1}(\alpha_j(C_j))^{\otimes n} + \ker r_{n,j}$ and

$$\bigcap_{j=1}^{\infty} L_{n,j} = \left(\bigcap_{j=1}^{\infty} q_j^{-1}(\alpha_j(C_j)) \right)^{\otimes n} + \ker \hat{\mu}_n = J^{\otimes n}$$

so J is closed under the coproduct for $L_{\mathcal{V}}(\varprojlim \lceil C_i \rceil)$.

Now, we claim that the exact sequence B.28 is just B.21 in another form—we have expressed the $\varprojlim{}^1$ terms as quotients of limits of *other* terms.

The exact sequences

$$0 \to \ker q_k \to \lceil L_{\mathcal{V}}(\varprojlim \lceil C_i \rceil) \rceil \xrightarrow{q_k} \lceil L_{\mathcal{V}}(\lceil C_k \rceil) \rceil \to 0$$

for all k, induces the sequence of limits

$$\begin{array}{c} 0 \to \varprojlim \ker q_k \to \lceil L_{\mathcal{V}}(\varprojlim \lceil C_i \rceil) \rceil \to \varprojlim \lceil L_{\mathcal{V}}(\lceil C_i \rceil) \rceil \to \varprojlim{}^1 \ker q_k \to 0 \\ \Big\| \\ 0 \end{array}$$

which implies that

$$\varprojlim{}^1 \ker q_k = \frac{\varprojlim \lceil L_{\mathcal{V}}(\lceil C_i \rceil) \rceil}{\lceil L_{\mathcal{V}}(\varprojlim \lceil C_i \rceil) \rceil}$$

In like fashion, the exact sequences

$$0 \to q_k^{-1}(\alpha_k(C_k)) \to \lceil L_{\mathcal{V}}(\varprojlim \lceil C_i \rceil) \rceil \to \lceil L_{\mathcal{V}}(\lceil C_k \rceil)/\alpha_k(C_k) \rceil \to 0$$

imply that

$$\varprojlim{}^1 q_i^{-1}(\alpha_i(C_i)) = \frac{\varprojlim (\lceil L_{\mathcal{V}}(\lceil C_i \rceil)/\lceil \alpha_i(C_i) \rceil \rceil)}{\operatorname{im} \lceil L_{\mathcal{V}}(\varprojlim \lceil C_i \rceil) \rceil}$$

■

B.29. Corollary. *Let $\{C_i\} \in$ pro-$\mathscr{I}_0$, and suppose $\mathcal{V} = \{\mathcal{V}(n)\}$ is a Σ-cofibrant operad with $\mathcal{V}(n)$ of finite type for all $n \geq 0$. If*

$$\alpha_i \colon C_i \to P_{\mathcal{V}}(\lceil C_i \rceil)$$

are the classifying maps with

$$\hat{\alpha} \colon \varprojlim \lceil C_i \rceil \to \varprojlim \lceil P_{\mathcal{V}}(\lceil C_i \rceil) \rceil$$

the induced map, and if

$$\mu \colon \lceil P_{\mathcal{V}}(\varprojlim \lceil C_i \rceil) \rceil \to \varprojlim \lceil P_{\mathcal{V}}(\lceil C_i \rceil) \rceil$$

is the inclusion defined in proposition B.27, then

$$\mu\left(\lceil \varprojlim C_i \rceil\right) = \mu\left(\lceil P_{\mathcal{V}}(\varprojlim \lceil C_i \rceil) \rceil\right) \cap \hat{\alpha}\left(\varprojlim \lceil C_i \rceil\right) \subseteq \varprojlim \lceil P_{\mathcal{V}}(\lceil C_i \rceil) \rceil$$

A corresponding results holds in the category pro-$\mathscr{S}_0$ after consistently replacing the functor $P_{\mathcal{V}}(*)$ by $L_{\mathcal{V}}(*)$.

Remark. The naive way to construct $\varprojlim C_i$ is to try to equip $\varprojlim \lceil C_i \rceil$ with a coproduct—a process that fails because we only get a map

$$\varprojlim \lceil C_i \rceil \to \prod_{n \geq 0} \operatorname{Hom}_{RS_n}(\mathcal{V}(n), \varprojlim (C_i^{\otimes n})) \neq \prod_{n \geq 0} \operatorname{Hom}_{RS_n}(\mathcal{V}(n), (\varprojlim C_i)^{\otimes n})$$

which is not a true coalgebra structure.

Corollary B.29 implies that this naive procedure *almost* works. Its failure is precisely captured by the degree to which

$$\lceil P_{\mathcal{V}}(\varprojlim \lceil C_i \rceil) \rceil \neq \varprojlim \lceil P_{\mathcal{V}}(\lceil C_i \rceil) \rceil$$

Proof. This follows immediately from the exact sequence B.28. ■

Our main result

B.30. Theorem. *Let* $\{f_i\}\colon \{A\} \to \{C_i\}$ *be a morphism in pro-*$\mathscr{I}_0$ *over a* Σ*-cofibrant operad* $\mathcal{V} = \{\mathcal{V}(n)\}$ *with* $\mathcal{V}(n)$ *of finite type for all* $n \geq 0$*. Let*

1. $\{A\}$ *be the constant object*
2. *the* $\{f_i\}$ *be cofibrations for all* i

Then $\{f_i\}$ *induces a cofibration* $f = \varprojlim \{f_i\}\colon A \to \varprojlim \{C_i\}$ *and the sequence*

$$0 \to \lceil A \rceil \xrightarrow{\varprojlim f_i} \lceil \varprojlim C_i \rceil \to \lceil \varprojlim (C_i/A) \rceil \to 0$$

is exact. In particular, if $\lceil \varprojlim (C_i/A) \rceil$ *is contractible, then* $\varprojlim f_i$ *is a weak equivalence.*

Proof. We will consider the case of $\mathscr{S}_0$—the other case follows by a similar argument.

The inclusion

$$\lceil \varprojlim C_i \rceil \subseteq \varprojlim \lceil C_i \rceil$$

from corollary B.28, and the left-exactness of filtered limits in **Ch** implies the left-exactness of the filtered limits in pro-$\mathscr{I}_0$, and that the inclusion

$$\lceil A \rceil \hookrightarrow \lceil \varprojlim C_i \rceil$$

is a cofibration in **Ch**.

The fact that

$$\lceil \varprojlim C_i \rceil = \lceil \bigcap_{i=1}^{\infty} q_i^{-1}(\alpha_i(C_i)) \rceil \subseteq \lceil L_{\mathcal{V}}(\varprojlim \lceil C_i \rceil) \rceil$$

from the same corollary and the diagram

$$\begin{tikzcd}
q_j^{-1}(\alpha_j(C_j)) \arrow[r, "h"] \arrow[d, hook] & u_j^{-1}(\alpha'_j(C_j/A)) \arrow[d, hook] \\
L_{\mathcal{V}}(\varprojlim \lceil C_i \rceil) \arrow[r, two heads] \arrow[d, "q_j"] & L_{\mathcal{V}}(\varprojlim \lceil C_i/A \rceil) \arrow[d, "u_j"] \\
L_{\mathcal{V}}(\lceil C_j \rceil) \arrow[r, two heads] & L_{\mathcal{V}}(\lceil C_j/A \rceil) \\
C_j \arrow[r, two heads] \arrow[u, "\alpha_j"] & C_j/A \arrow[u, "\alpha'_j"']
\end{tikzcd}$$

shows that the map h is surjective.

The final statement follows from Lemma 3.5 and the fact that

$$\lceil \varprojlim C_i \rceil \subseteq \varprojlim \lceil C_i \rceil$$

(by Corollary B.28) so the splitting map $\varprojlim \lceil C_i \rceil \to \lceil A \rceil$ induces a splitting map $\lceil \varprojlim C_i \rceil \to \lceil A \rceil$.

The conclusion follows. ∎

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