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A SMALL OBSERVATION ON CO-CATEGORIES

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Abstract. Various concerns suggest looking for internal co-categories in categories with strong logical structure. It turns out that in any coherent category $\mathcal{E}$, all co-categories are co-equivalence relations.

0.1. Definition. *Let $\mathcal{E}$ be any category. An (internal)* co-category $\mathbf{Q}$ *in $\mathcal{E}$ is an internal category in $\mathcal{E}^{op}$, i.e. objects and morphisms in $\mathcal{E}$*

$$\begin{tikzcd} Q^0 \arrow[r, shift left=2, "l"] \arrow[r, shift right=2, "r"'] & Q^1 \arrow[l, "i"'] \arrow[r, "q"] & Q^1 +_{Q^0} Q^1 \end{tikzcd}$$

such that the following diagrams commute:

$$\begin{tikzcd} Q^0 \arrow[r, "l"] \arrow[d, "l"'] & Q^1 \arrow[d, "q"] & A^0 \arrow[l, "r"'] \arrow[d, "r"] \\ Q^1 \arrow[r, "\nu_1"'] & Q^1 +_{Q^0} Q^1 & Q^1 \arrow[l, "\nu_2"] \end{tikzcd} \qquad \begin{tikzcd} Q^1 \arrow[r, "q"] \arrow[d, "q"'] & Q^1 +_{Q^0} Q^1 \arrow[d, "{[q,\nu_3]}"] \\ Q^1 +_{Q^0} Q^1 \arrow[r, "{[\nu_1,q]}"] & Q^1 +_{Q^0} Q^1 +_{Q^0} Q^1 \end{tikzcd}$$

$$\begin{tikzcd} Q^0 \arrow[r, "l"] \arrow[dr, "l"'] & Q^1 \arrow[d, "i"] & Q^0 \arrow[l, "r"'] \arrow[dl, "r"] \\ & Q^0 & \end{tikzcd} \qquad \begin{tikzcd} & Q^1 \arrow[dl, "1"'] \arrow[d, "q"] \arrow[dr, "1"] & \\ Q^1 & Q^1 +_{Q^0} Q^1 \arrow[l, "{[li,1]}"] \arrow[r, "{[1,ri]}"'] & Q^1 \end{tikzcd}$$

0.2. Definition. *A co-category $\mathbf{Q}$ is a* co-preorder *if the maps l, r are jointly epimorphic.*

A co-category $\mathbf{Q}$ is a co-groupoid *if there is a map $s : Q^1 \to Q^1$ satisfying the duals of the usual identities for the inverse map of a groupoid.*

A co-groupoid $\mathbf{Q}$ is a co-equivalence relation *if it is a co-preorder.*

0.3. Remark. *In a co-preorder, the co-composition q is uniquely determined by the maps l, r, i; likewise, in a co-groupoid, the co-inverse map s is determined by the rest of the structure.*

Together with the obvious maps, these give categories and full inclusions

$$\mathbf{CoEqRel}(\mathcal{E}) \hookrightarrow \mathbf{CoPreOrd}(\mathcal{E}) \hookrightarrow \mathbf{CoCat}(\mathcal{E}).$$

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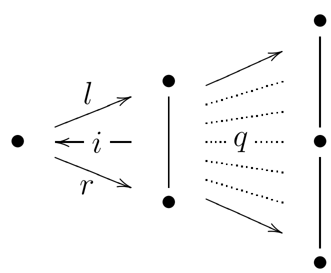
0.4. Example. If $\mathcal{E}$ has all (or enough) pushouts and $m : S \rightarrowtail A$ is any monomorphism, then the *co-kernel pair* of m is a co-equivalence relation

$$\begin{tikzcd} A \arrow[r, shift left=2, "\nu_1"] \arrow[r, shift right=2, "\nu_2"'] & A +_S A \arrow[l, "{[1_A, 1_A]}" description] \arrow[r, "{[\nu_1, \nu_3]}"] & A +_S A +_S A. \end{tikzcd}$$

This gives the object part of a functor $\mathbf{Mono}(\mathcal{E}) \to \mathbf{CoEqRel}(\mathcal{E})$, which (almost by definition) is one half of an equivalence whenever $\mathcal{E}$ is co-exact. (Here $\mathbf{Mono}(\mathcal{E})$ denotes the full subcategory of $\mathcal{E}^{\cdot \to \cdot}$ on monomorphisms.)

0.5. Example. A paradigmatic example is the interval $\mathbf{I}$ in **Top**, where I^0 is a singleton, I^1 is the unit interval, l and r are the endpoints, $I^1 +_{I^0} I^1$ is two copies of the interval joined end to end, and q is the obvious "stretching" map. Unfortunately, this is also of course not an actual co-category — the axioms hold only up to homotopy. However, it provides a very useful mental picture for the arguments below; and if we delete the interior of the interval, we obtain a genuine co-category. See also the examples below for more versions of the interval.



0.6. Definition. *A* coherent category *is a category with all finite limits, and images and unions that are stable under pullback.*

[Johnstone 2002, A1.3–4] gives various basic results on coherent categories, which we will use here without comment.

0.7. Definition. Coherent logic *is the fragment of first-order logic built up from atomic formulæ using finite con-/dis-junction and existential quantification.*

Coherent logic is discussed in [Johnstone 2002, D1.1–2]; the essential point is that coherent logic can be interpreted soundly in coherent categories, and so may be used as an internal language for working in them.

0.8. Proposition. *In a coherent category $\mathcal{E}$, every co-category $\mathbf{Q}$ is a co-equivalence relation.*

Proof. First, we show that any co-category $\mathbf{Q}$ is a co-preorder.

Arguing in the internal logic: given x in Q^1, consider $q(x)$, in $Q^1 +_{Q_0} Q^1$. Either there is some y in Q^1 with $q(x) = \nu_1(y)$, or else some y with $q(x) = \nu_2(y)$. In the first case, we then have $x = [li, 1]q(x) = li(y)$; in the second, $x = ri(y)$. Thus any x in Q^1 is in the image of either l or r, i.e. l and r are jointly covering, hence epi. (Indeed, in the first case $x = li(y) = l(il)i(y) = li(li(y)) = li(x)$, and in the second, $x = ri(x)$.)

Restating this diagrammatically: $Q^1 +_{Q^0} Q^1$ is the union of the subobjects $\nu_j : Q^1 \to Q^1 +_{Q^0} Q^1$, so Q^1 is the union of the subobjects $m_j = q^*(\nu_j)$:

$$\begin{tikzcd} P_j \arrow[r, "q_j"] \arrow[d, "m_j"] & Q^1 \arrow[d, "\nu_j"] \\ Q^1 \arrow[r, "q"] & Q^1 +_{Q^0} Q^1 \end{tikzcd}$$

So m_1, m_2 are jointly covering. But by the co-unit identities, $liq_1 = [li, 1]\nu_1 q_1 = [li, 1]qm_1 = m_1$, and $riq_2 = m_2$. Thus liq_1, riq_2 are jointly covering, and hence so are l, r.

Now, we check that any co-preorder is a co-equivalence relation. (We give only the diagrammatic version. Exercise: restate this in the internal logic!) We want to define $s : Q^1 \to Q^1$ with $sl = r$, $sr = l$. Since l, r are monos with union Q^1, the pullback square

$$\begin{tikzcd} \bullet \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Q^0 \arrow[d, "r"] \\ Q^0 \arrow[r, "l"] & Q^1 \end{tikzcd}$$

is also a pushout, so to construct s as above, it is enough to show that $r\pi_1 = l\pi_2$. But $\pi_1 = il\pi_1 = ir\pi_2 = \pi_2$, so $r\pi_1 = r\pi_2 = l\pi_1 = l\pi_2$, and we are done. ■

0.9. Corollary. *If $\mathcal{E}$ is coherent and has co-kernel pairs of monos, then* $\boldsymbol{CoCat}(\mathcal{E}) \simeq \boldsymbol{Mono}(\mathcal{E})$. *(In particular, this holds if $\mathcal{E}$ is a pretopos [Johnstone 2002, A1.4.8].)*

Proof. A coherent category is co-effective, so if it has co-kernel pairs, it is co-exact. ■

0.10. Corollary. *For any topos $\mathcal{E}$,* $\boldsymbol{CoCat}(\mathcal{E}) \simeq (\mathcal{E}/\Omega)_{colax}$. ■

(A *colax* map $(A, \varphi) \to (B, \psi)$ is a map $f : A \to B$ such that $\varphi \leq_\Omega \psi f$.)

In particular, inspecting this equivalence, we see that in this case there is a universal internal co-category in $\mathcal{E}$, from which every co-category in $\mathcal{E}$ may be obtained uniquely by pullback: it is the co-kernel pair of $\top : 1 \to \Omega$.

0.11. Example. The condition that unions are preserved by pullback is crucial: **AbGp**, for instance, is regular, and has unions, but there is a non-co-preorder co-category corresponding to the interval pictured above, given by the objects

$$Q^0 = \langle v_0 \rangle \qquad Q^1 = \langle v_0, e_1, v_1 \rangle \qquad Q^1 +_{Q^0} Q^1 = \langle v_0, e_1, v_1, e_2, v_2 \rangle$$

(with the natural maps making this a pushout), and maps given by the matrices

$$l = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad r = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad i = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \quad q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This example may be given more structure; it is, for instance, the total space of an natural co-category in **Ch**(**AbGp**). Since all the underlying groups are free and of finite rank, dualising by transposing matrices also gives corresponding categories in **AbGp** and **Ch**(**AbGp**).

However, any category in a Mal'cev category is a groupoid (this has been observed by various authors, e.g. in [CKP 1993]), so any co-category in a co-Mal'cev category (e.g. in an Abelian category, or a topos [Bourn 1996]) is a co-groupoid.

0.12. Example. An example of a non-co-groupoid co-category is the interval **I** in **Cat**, with $I^0 = (\cdot)$, $I^1 = (\cdot \to \cdot)$; seen as a co-simplicial object, this is just the usual inclusion functor $\Delta \hookrightarrow$ **PreOrd** $\hookrightarrow$ **Cat**.

Indeed, the functor **Cat** $\to$ **SSet** $\to$ **SAbGp** $\to$ **Ch**(**AbGp**) $\to$ **Ch**(**AbGp**) "take nerve; take free abelian groups; normalise to a complex; quotient out by subcomplex generated in degrees ≥ 2" sends **I** to the co-category in **Ch**(**AbGp**) of the previous example.

Co-categories arise as candidate "interval objects" when using 2-categories to model intensional type theory [Awodey, Warren, 2009]. There, one seeks them in categories with some sort of "weakened" logical structure; the present result confirms the suspicion that examples in classical "strict" logical categories are necessarily fairly trivial.

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