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CONGRUENCES OF MORITA EQUIVALENT SMALL CATEGORIES

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Abstract. Two categories are called Morita equivalent if the categories of functors from these categories to the category of sets are equivalent. We prove that congruence lattices of Morita equivalent small categories are isomorphic.

1. Preliminaries

Categories $\mathcal{A}$ and $\mathcal{B}$ are called **Morita equivalent** if the functor categories $\mathsf{Fun}(\mathcal{A}, \mathsf{Set})$ and $\mathsf{Fun}(\mathcal{B}, \mathsf{Set})$ are equivalent. The basic theory of Morita equivalent categories has been developed already in 1960s ([Artin, Grothendieck and Verdier, 1972]). One natural question to ask in Morita theory is that which properties are shared by Morita equivalent structures. For example, it is well known (see Proposition 21.11 in [Anderson and Fuller, 1974]) that ideal lattices of Morita equivalent rings are isomorphic. We shall prove an analogue of this result: Morita equivalent small categories have isomorphic congruence lattices. Although this result can be proved easily by topos-theoretic methods, we give an elementary proof using Cauchy completions.

In [Elkins and Zilber, 1976], a construction of a Cauchy completion $\overline{\mathcal{A}}$ of a small category $\mathcal{A}$ is given as follows:

- objects of $\overline{\mathcal{A}}$ are idempotents $e : A \to A$ of $\mathcal{A}$ (i.e. endomorphisms with $e^2 = e$);
- morphism sets are
$$\overline{\mathcal{A}}(e, e') = \{(e', a, e) \mid a \in \mathcal{A}(A, A'), e'ae = a\},$$
where $e : A \to A$ and $e' : A' \to A'$;
- composition is given by
$$(e'', b, e')(e', a, e) = (e'', ba, e).$$

The following result is well known.

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1.1. Theorem. *Two small categories are Morita equivalent if and only if their Cauchy completions are equivalent.*

We write $\mathcal{A}_0$ ($\mathcal{A}_1$) for the class of objects (morphisms) of a category $\mathcal{A}$.

1.2. Definition. [See [Mac Lane, 1998], p. 52]A **congruence** on a category $\mathcal{A}$ is a family $\rho = (\rho_{A,B})_{(A,B)\in\mathcal{A}_0^2}$ of equivalence relations $\rho_{A,B}$ on morphism sets $\mathcal{A}(A,B)$ that are compatible with the composition of morphisms.

If ρ is a congruence on a category $\mathcal{A}$ then one can form a **quotient category** $\mathcal{A}/\rho$, where $(\mathcal{A}/\rho)_0 = \mathcal{A}_0$, $(\mathcal{A}/\rho)(A,B) = \mathcal{A}(A,B)/\rho_{A,B}$ and $[g][f] = [gf]$ for every $f \in \mathcal{A}(A,B)$, $g \in \mathcal{A}(B,C)$.

We say that a congruence ρ **is contained** in a congruence σ (and denote $\rho \subseteq \sigma$) if $\rho_{A,B} \subseteq \sigma_{A,B}$ for all $A, B \in \mathcal{A}_0$. This is obviously an order relation, and it is not difficult to see that the set $\mathsf{Con}(\mathcal{A})$ of congruences on a small category $\mathcal{A}$ is a lattice.

2. Congruence lattices of equivalent small categories

2.1. Proposition. *If $\mathcal{A}$ and $\mathcal{B}$ are equivalent small categories then there is an isomorphism $\Gamma : \mathsf{Con}(\mathcal{A}) \to \mathsf{Con}(\mathcal{B})$ between their congruence lattices. Moreover, if $\rho \in \mathsf{Con}(\mathcal{A})$ then $\mathcal{A}/\rho$ is equivalent to $\mathcal{B}/\Gamma(\rho)$.*

Proof. Let $\mathcal{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}$ be equivalence functors and let $\eta : 1_{\mathcal{A}} \Rightarrow GF$, $\varepsilon : FG \Rightarrow 1_{\mathcal{B}}$ be natural isomorphisms. We define mappings $\mathsf{Con}(\mathcal{A}) \underset{\Delta}{\overset{\Gamma}{\rightleftarrows}} \mathsf{Con}(\mathcal{B})$ by

$$
\begin{aligned}
\Gamma(\rho)_{B,B'} &:= \left\{\left(\varepsilon_{B'}F(f)\varepsilon_B^{-1}, \varepsilon_{B'}F(g)\varepsilon_B^{-1}\right) \mid (f,g) \in \rho_{G(B),G(B')}\right\},\\
\Delta(\sigma)_{A,A'} &:= \left\{\left(\eta_{A'}^{-1}G(k)\eta_A, \eta_{A'}^{-1}G(l)\eta_A\right) \mid (k,l) \in \sigma_{F(A),F(A')}\right\},
\end{aligned}
$$

$\rho \in \mathsf{Con}(\mathcal{A})$, $\sigma \in \mathsf{Con}(\mathcal{B})$, $A, A' \in \mathcal{A}_0$, $B, B' \in \mathcal{B}_0$. It is easy to see that $\Gamma(\rho)_{B,B'}$ is an equivalence relation. To prove its compatibility, take $(f,g) \in \rho_{G(B),G(B')}$ and $b \in \mathcal{B}(B', B'')$. Then $\varepsilon_{B''}^{-1} b \varepsilon_{B'} = F(a)$ for some $a : G(B') \to G(B'')$ and $(af, ag) \in \rho_{G(B),G(B'')}$. Consequently,

$$
\begin{aligned}
\left(b\varepsilon_{B'}F(f)\varepsilon_B^{-1}, b\varepsilon_{B'}F(g)\varepsilon_B^{-1}\right) &= \left(\varepsilon_{B''}\varepsilon_{B''}^{-1}b\varepsilon_{B'}F(f)\varepsilon_B^{-1}, \varepsilon_{B''}\varepsilon_{B''}^{-1}b\varepsilon_{B'}F(g)\varepsilon_B^{-1}\right)\\
&= \left(\varepsilon_{B''}F(af)\varepsilon_B^{-1}, \varepsilon_{B''}F(ag)\varepsilon_B^{-1}\right) \in \Gamma(\rho)_{B,B''}.
\end{aligned}
$$

$$
\begin{tikzcd}
B \arrow[d, "\varepsilon_B^{-1}"'] & B' \arrow[r, "b"] & B'' \arrow[d, "\varepsilon_{B''}^{-1}"] \\
(FG)(B) \arrow[r, shift left, "F(f)"] \arrow[r, shift right, "F(g)"'] & (FG)(B') \arrow[u, "\varepsilon_{B'}"] \arrow[r, "F(a)"'] & (FG)(B'')
\end{tikzcd}
$$

Hence $\Gamma(\rho)$ (and similarly $\Delta(\sigma)$) is a congruence. Clearly the mappings Γ and Δ preserve order. Note that

$$\rho_{(GF)(A),(GF)(A')} = \left\{\left(\eta_{A'} u \eta_A^{-1}, \eta_{A'} v \eta_A^{-1}\right) \mid (u,v) \in \rho_{A,A'}\right\}$$

for every $A, A' \in \mathcal{A}_0$. Using this and properties of equivalence functors we obtain

$$\begin{aligned}
&(\Delta\Gamma)(\rho)_{A,A'} = \Delta(\Gamma(\rho))_{A,A'}\\
&= \left\{\left(\eta_{A'}^{-1} G(k)\, \eta_A, \eta_{A'}^{-1} G(l)\, \eta_A\right) \mid (k,l) \in \Gamma(\rho)_{F(A),F(A')}\right\}\\
&= \left\{\left(\eta_{A'}^{-1} G\left(\varepsilon_{F(A')} F(f) \varepsilon_{F(A)}^{-1}\right) \eta_A, \eta_{A'}^{-1} G\left(\varepsilon_{F(A')} F(g) \varepsilon_{F(A)}^{-1}\right) \eta_A\right) \mid (f,g) \in \rho_{(GF)(A),(GF)(A')}\right\}\\
&= \left\{\left(\eta_{A'}^{-1} G\left(\varepsilon_{F(A')} F(\eta_{A'} u \eta_A^{-1}) \varepsilon_{F(A)}^{-1}\right) \eta_A, \eta_{A'}^{-1} G\left(\varepsilon_{F(A')} F(\eta_{A'} v \eta_A^{-1}) \varepsilon_{F(A)}^{-1}\right) \eta_A\right) \mid (u,v) \in \rho_{A,A'}\right\}\\
&= \left\{\left(\eta_{A'}^{-1} (GF)(u) \eta_A, \eta_{A'}^{-1} (GF)(v) \eta_A\right) \mid (u,v) \in \rho_{A,A'}\right\}\\
&= \left\{\left(\eta_{A'}^{-1} \eta_{A'} u, \eta_{A'}^{-1} \eta_{A'} v\right) \mid (u,v) \in \rho_{A,A'}\right\} = \rho_{A,A'}.
\end{aligned}$$

Hence $\Delta\Gamma = 1_{\mathsf{Con}(\mathcal{A})}$ and, analogously, $\Gamma\Delta = 1_{\mathsf{Con}(\mathcal{B})}$.

To prove the second assertion, take $\rho \in \mathsf{Con}(\mathcal{A})$, denote $\sigma := \Gamma(\rho)$ and define a functor $F_\rho : \mathcal{A}/\rho \to \mathcal{B}/\Gamma(\rho)$ by

$$\begin{aligned}
F_\rho(A) &:= F(A),\\
F_\rho\left([f]_{\rho_{A,A'}}\right) &:= [F(f)]_{\sigma_{F(A),F(A')}},
\end{aligned}$$

$A, A' \in \mathcal{A}_0$, $f \in \mathcal{A}(A,A')$. First we show that, for every $f, g \in \mathcal{A}(A,A')$,

$$\begin{aligned}
&\left(\eta_{A'} f \eta_A^{-1}, \eta_{A'} g \eta_A^{-1}\right) \in \rho_{(GF)(A),(GF)(A')} \iff\\
&\left(\varepsilon_{F(A')} F\left(\eta_{A'} f \eta_A^{-1}\right) \varepsilon_{F(A)}^{-1}, \varepsilon_{F(A')} F\left(\eta_{A'} g \eta_A^{-1}\right) \varepsilon_{F(A)}^{-1}\right) \in \sigma_{F(A),F(A')}.
\end{aligned}$$

Suppose that the last holds. Then

$$\left(\varepsilon_{F(A')} F\left(\eta_{A'} f \eta_A^{-1}\right) \varepsilon_{F(A)}^{-1}, \varepsilon_{F(A')} F\left(\eta_{A'} g \eta_A^{-1}\right) \varepsilon_{F(A)}^{-1}\right) = \left(\varepsilon_{F(A')} F(f') \varepsilon_{F(A)}^{-1}, \varepsilon_{F(A')} F(g') \varepsilon_{F(A)}^{-1}\right)$$

for some $(f', g') \in \rho_{(GF)(A),(GF)(A')}$. This implies $F\left(\eta_{A'} f \eta_A^{-1}\right) = F(f')$, $F\left(\eta_{A'} g \eta_A^{-1}\right) = F(g')$, and hence $\left(\eta_{A'} f \eta_A^{-1}, \eta_{A'} g \eta_A^{-1}\right) = (f', g') \in \rho_{(GF)(A),(GF)(A')}$. The converse is obvious. Using the proved fact together with

$$(f,g) \in \rho_{A,A'} \iff \left(\eta_{A'} f \eta_A^{-1}, \eta_{A'} g \eta_A^{-1}\right) \in \rho_{(GF)(A),(GF)(A')}$$

and

$$\left(\varepsilon_{F(A')} F\left(\eta_{A'} f \eta_A^{-1}\right) \varepsilon_{F(A)}^{-1}, \varepsilon_{F(A')} F\left(\eta_{A'} g \eta_A^{-1}\right) \varepsilon_{F(A)}^{-1}\right) = (F(f), F(g))$$

we conclude that

$$(f,g) \in \rho_{A,A'} \iff (F(f), F(g)) \in \sigma_{F(A),F(A')}.$$

Thus F_ρ is well defined and faithful. It clearly is dense. To prove that F_ρ is full, take a morphism $k \in \mathcal{B}(F(A), F(A'))$. Then

$$\begin{aligned} F_\rho\left(\left[\eta_{A'}^{-1}G(k)\eta_A\right]_{\rho_{A,A'}}\right) &= \left[F(\eta_{A'}^{-1})(FG)(k)F(\eta_A)\right]_{\rho_{A,A'}} = \left[F(\eta_{A'})^{-1}(FG)(k)\varepsilon_{F(A)}^{-1}\right]_{\rho_{A,A'}} \\ &= \left[\varepsilon_{F(A')}\varepsilon_{F(A')}^{-1}k\right]_{\rho_{A,A'}} = [k]_{\rho_{A,A'}}. \end{aligned}$$

■

3. Congruence lattices of Morita equivalent small categories

Our main result is the following.

3.1. Theorem. *If $\mathcal{A}$ and $\mathcal{B}$ are Morita equivalent small categories then there is an isomorphism $\Pi : \mathsf{Con}(\mathcal{A}) \to \mathsf{Con}(\mathcal{B})$ between their congruence lattices. Moreover, if $\rho \in \mathsf{Con}(\mathcal{A})$ then $\mathcal{A}/\rho$ is Morita equivalent to $\mathcal{B}/\Pi(\rho)$.*

It follows from Theorem 1.1, Proposition 2.1 and the next proposition.

3.2. Proposition. *If $\mathcal{A}$ is a small category then there is an isomorphism $\Pi : \mathsf{Con}(\mathcal{A}) \to \mathsf{Con}(\overline{\mathcal{A}})$ between congruence lattices. Moreover, if $\rho \in \mathsf{Con}(\mathcal{A})$ then $\mathcal{A}/\rho$ is Morita equivalent to $\overline{\mathcal{A}}/\Pi(\rho)$.*

Proof. Let E be the set of idempotents of $\mathcal{A}$. Define a mapping $\Pi : \mathsf{Con}(\mathcal{A}) \to \mathsf{Con}(\overline{\mathcal{A}})$ by

$$\Pi(\rho)_{e,e'} := \{((e', e'se, e), (e', e'te, e)) \mid (s,t) \in \rho_{A,A'}\},$$

where $\rho \in \mathsf{Con}(\mathcal{A})$, $e, e' \in E$, $e : A \to A$ and $e' : A' \to A'$. It is not difficult to see that $\Pi(\rho)_{e,e'}$ is a reflexive and symmetric relation on $\overline{\mathcal{A}}(e, e')$. To prove transitivity of $\Pi(\rho)_{e,e'}$, take $(p,q), (q,r) \in \Pi(\rho)_{e,e'}$. Then there exist $(s,t), (u,v) \in \rho_{A,A'}$ such that $(p,q) = ((e', e'se, e), (e', e'te, e))$ and $(q,r) = ((e', e'ue, e), (e', e've, e))$, in particular $e'te = e'ue$. Since ρ is a congruence, we have $e'se \;\rho_{A,A'}\; e'te = e'ue \;\rho_{A,A'}\; e've$, and hence $(e'se, e've) \in \rho_{A,A'}$. Consequently,

$$(p,r) = ((e', e'se, e), (e', e've, e)) = ((e', e'(e'se)e, e), (e', e'(e've)e, e)) \in \Pi(\rho)_{e,e'}.$$

If $(s,t) \in \rho_{A,A'}$ and $(e'', a, e') : e' \to e''$ in $\overline{\mathcal{A}}$, where $e'' : A'' \to A''$, then $a : A' \to A''$ in $\mathcal{A}$, hence $(as, at) \in \rho_{A,A''}$ and

$$\begin{aligned} ((e'', a, e')(e', e'se, e), (e'', a, e')(e', e'te, e)) &= ((e'', ae'se, e), (e'', ae'te, e)) \\ &= ((e'', e''ase, e), (e'', e''ate, e)) \in \Pi(\rho)_{e,e''}. \end{aligned}$$

Similarly $\Pi(\rho)$ is compatible with precomposition and thus $\Pi(\rho) \in \mathsf{Con}(\overline{\mathcal{A}})$.

We also define a mapping $\Omega : \mathsf{Con}(\overline{\mathcal{A}}) \to \mathsf{Con}(\mathcal{A})$ by

$$\Omega(\tau)_{A,B} := \{(a,b) \mid ((1_B, a, 1_A), (1_B, b, 1_A)) \in \tau_{1_A, 1_B}\},$$

where $\tau \in \mathsf{Con}(\overline{\mathcal{A}})$, $A, B \in \mathcal{A}_0$. Clearly $\Omega(\tau)_{A,B}$ is an equivalence relation. Let us show that $\Omega(\tau)$ is compatible with precomposition. Take $(a,b) \in \Omega(\tau)_{A,B}$ and $c \in \mathcal{A}(C,A)$. Note that

$$((1_B, ac, 1_C), (1_B, bc, 1_C)) = ((1_B, a, 1_A)(1_A, c, 1_C), (1_B, b, 1_A)(1_A, c, 1_C)) \in \tau_{1_C, 1_B},$$

because $\tau \in \mathsf{Con}(\overline{\mathcal{A}})$. Consequently, $(ac, bc) \in \Omega(\tau)_{C,B}$. The proof that $\Omega(\tau)$ is compatible with postcomposition is symmetric. Hence $\Omega(\tau) = (\Omega(\tau)_{A,B})_{(A,B)\in\mathcal{A}_0^2} \in \mathsf{Con}(\mathcal{A})$.

Clearly Π and Ω preserve order.

Now, for $\rho \in \mathsf{Con}(\mathcal{A})$ and $A, B \in \mathcal{A}$,

$$\begin{aligned}(\Omega\Pi)(\rho)_{A,B} &= \Omega(\Pi(\rho))_{A,B} = \{(a,b) \mid ((1_B, a, 1_A), (1_B, b, 1_A)) \in \Pi(\rho)_{1_A, 1_B}\} \\ &= \{(a,b) \mid (a,b) \in \rho_{A,B}\} = \rho_{A,B}.\end{aligned}$$

On the other hand, if $\tau \in \mathsf{Con}(\overline{\mathcal{A}})$ and $e, e' \in E$, $e : A \to A$, $e' : A' \to A'$, then

$$(\Pi\Omega)(\tau)_{e,e'} = \Pi(\Omega(\tau))_{e,e'} = \{((e', e'se, e), (e', e'te, e)) \mid (s,t) \in \Omega(\tau)_{A,A'}\}.$$

Let us prove the inclusion $(\Pi\Omega)(\tau)_{e,e'} \subseteq \tau_{e,e'}$. Note that $(s,t) \in \Omega(\tau)_{A,A'}$ if and only if $((1_{A'}, s, 1_A), (1_{A'}, t, 1_A)) \in \tau_{1_A, 1_{A'}}$. Composing with $(e', e', 1_{A'})$ and $(1_A, e, e)$ we obtain $((e', e'se, e), (e', e'te, e)) \in \tau_{e,e'}$.

To prove the converse, let $((e', u, e), (e', v, e)) \in \tau_{e,e'}$. Then $((1_{A'}, u, 1_A), (1_{A'}, v, 1_A)) \in \tau_{1_A, 1_{A'}}$ implies $(u,v) \in \Omega(\tau)_{A,A'}$. Thus $((e', u, e), (e', v, e)) = ((e', e'ue, e), (e', e've, e)) \in (\Pi\Omega)(\tau)_{e,e'}$, and we have proven the equality $(\Pi\Omega)(\tau) = \tau$.

Let us now prove the second assertion. We denote $\widetilde{\mathcal{A}} := \mathcal{A}/\rho$, $\widetilde{\overline{\mathcal{A}}} := \overline{\mathcal{A}}/\Pi(\rho)$ and $\tau := \Pi(\rho)$. Thus $\widetilde{\mathcal{A}}(A,B) = \mathcal{A}(A,B)/\rho_{A,B}$ and $\widetilde{\overline{\mathcal{A}}}(e,e') = \overline{\mathcal{A}}(e,e')/\tau_{e,e'}$, $A, B \in \mathcal{A}_0$, $e, e' \in E$.

We shall show that the Cauchy completions $\overline{\widetilde{\mathcal{A}}}$ and $\overline{\widetilde{\overline{\mathcal{A}}}}$ (constructed as in Section 1) are equivalent.

Observe that the the objects of $\overline{\widetilde{\mathcal{A}}}$ are the idempotents of $\widetilde{\mathcal{A}}$, that is, the equivalence classes of endomorphisms $e : A \to A$ in $\mathcal{A}$ such that $e^2 \, \rho_{A,A} \, e$. The morphism sets are

$$\overline{\widetilde{\mathcal{A}}}([e]_{\rho_{A,A}}, [e']_{\rho_{A',A'}}) = \{([e']_{\rho_{A',A'}}, [a]_{\rho_{A,A'}}, [e]_{\rho_{A,A}}) \mid e, e', a \in \mathcal{A}_1, e^2 \, \rho \, e, (e')^2 \, \rho \, e', e'ae \, \rho \, a\}.$$

The objects of $\widetilde{\overline{\mathcal{A}}}$ are the idempotents of $\mathcal{A}$ and the morphisms are the equivalence classes $[e', a, e]_{\tau_{e,e'}}$ of morphisms (e', a, e) in $\overline{\mathcal{A}}$, where $e^2 = e$, $(e')^2 = e'$ and $e'ae = a$ in $\mathcal{A}$. Hence the idempotents of $\widetilde{\overline{\mathcal{A}}}$ (i.e. the objects of $\overline{\widetilde{\overline{\mathcal{A}}}}$) are the classes $[e, a, e]_{\tau_{e,e}}$ such that $e^2 = e$, $eae = a$ in $\mathcal{A}$ and $(e, a^2, e) \, \tau_{e,e} \, (e, a, e)$. If $[e, a, e]_{\tau_{e,e}}$ and $[e', a', e']_{\tau_{e',e'}}$ are two objects of $\overline{\widetilde{\overline{\mathcal{A}}}}$ then the morphism set $\overline{\widetilde{\overline{\mathcal{A}}}}\left([e, a, e]_{\tau_{e,e}}, [e', a', e']_{\tau_{e',e'}}\right)$ is

$$\left\{\left([e', a', e']_{\tau_{e',e'}}, [e', x, e]_{\tau_{e,e'}}, [e, a, e]_{\tau_{e,e}}\right) \mid x \in \mathcal{A}_1, e'xe = x, (e', a'xa, e) \, \tau_{e,e'} \, (e', x, e)\right\}.$$

Note that if $e : A \to A$ and $e' : A' \to A'$ in $\mathcal{A}$ then

$$(e', u, e)\, \tau_{e,e'}\, (e', v, e) \Longrightarrow u\, \rho_{A,A'}\, v. \tag{1}$$

Indeed, $(e', u, e)\, \tau_{e,e'}\, (e', v, e)$ if and only if there exists $(s, t) \in \rho_{A,A'}$ such that $e'se = u$ and $e'te = v$. Due to compatibility, also $(u, v) = (e'se, e'te) \in \rho_{A,A'}$.

Define a functor $F : \overline{\widetilde{\overline{\mathcal{A}}}} \to \overline{\widetilde{\mathcal{A}}}$ by the assignment

$$\begin{tikzcd}
{[e,a,e]_{\tau_{e,e}}} \arrow[r, mapsto] \arrow[d, "{([e',a',e']_\tau, [e',x,e]_\tau, [e,a,e]_\tau)}"'] & {[a]_{\rho_{A,A}}} \arrow[d, "{([a']_\rho, [x]_\rho, [a]_\rho)}"] \\
{[e',a',e']_{\tau_{e',e'}}} \arrow[r, mapsto] & {[a']_{\rho_{A',A'}}}
\end{tikzcd}$$

where $e : A \to A$ and $e' : A' \to A'$ in $\mathcal{A}$. If $[e, a, e]_{\tau_{e,e}}$ is an object of the category $\overline{\widetilde{\overline{\mathcal{A}}}}$ then $(e, a^2, e)\, \tau_{e,e}\, (e, a, e)$, hence $a^2\, \rho_{A,A}\, a$ by (1), and so $[a]_{\rho_{A,A}}$ is an object of $\overline{\widetilde{\mathcal{A}}}$. If $(e, a, e)\, \tau_{e,e}\, (e, a_1, e)$ then again $a\, \rho_{A,A}\, a_1$, and therefore F is well defined on objects. A similar argument shows that the definition of F on morphisms does not depend on the choice of the representatives of τ-classes. Moreover, if $\left([e', a', e']_{\tau_{e',e'}}, [e', x, e]_{\tau_{e,e'}}, [e, a, e]_{\tau_{e,e}}\right)$ is a morphism in $\overline{\widetilde{\overline{\mathcal{A}}}}$ then $(e', a'xa, e)\, \tau_{e,e'}\, (e', x, e)$, so $a'xa\, \rho_{A,A'}\, x$ and $\left([a']_{\rho_{A',A'}}, [x]_{\rho_{A,A'}}, [a]_{\rho_{A,A}}\right)$ is indeed a morphism in $\overline{\widetilde{\mathcal{A}}}$. Straightforward calculations show that F is a functor.

If $[a]_{\rho_{A,A}}$ is an object in $\overline{\widetilde{\mathcal{A}}}$ then $a^2\, \rho_{A,A}\, a$, hence $(1_A, a^2, 1_A)\, \tau_{1_A,1_A}\, (1_A, a, 1_A)$. Consequently, $[1_A, a, 1_A]_{\tau_{1_A,1_A}}$ is an object of $\overline{\widetilde{\overline{\mathcal{A}}}}$ which maps to $[a]_{\rho_{A,A}}$. So F is dense.

To prove that F is full, let $[e, a, e]_{\tau_{e,e}}, [e', a', e']_{\tau_{e',e'}}$ be two objects in $\overline{\widetilde{\overline{\mathcal{A}}}}$ and consider a morphism $\left([a']_{\rho_{A',A'}}, [x]_{\rho_{A,A'}}, [a]_{\rho_{A,A}}\right)$ in $\overline{\widetilde{\mathcal{A}}}$. Then $a'xa\, \rho_{A,A'}\, x$, $e^2 = e$, $(e')^2 = e'$, $eae = a$, $e'a'e' = a'$, $(e, a^2, e)\, \tau_{e,e}\, (e, a, e)$ and $(e', (a')^2, e')\, \tau_{e',e'}\, (e', a', e')$. We have $e'(e'xe)e = e'xe$ and $(e', e'a'xae, e) = (e', a'(e'xe)a, e)\, \tau_{e,e'}\, (e', e'xe, e)$, because $a'xa\, \rho_{A,A'}\, x$ implies $(e', e'(a'xa)e, e)\, \tau_{e,e'}\, (e', e'xe, e)$. Also, $e'xe\, \rho_{A,A'}\, e'a'xae = a'xa\, \rho_{A,A'}\, x$. Thus we have shown that $\left([e', a', e']_{\tau_{e',e'}}, [e', x, e]_{\tau_{e,e'}}, [e, a, e]_{\tau_{e,e}}\right)$ is a morphism in $\overline{\widetilde{\overline{\mathcal{A}}}}$ which maps to $\left([a']_{\rho_{A',A'}}, [e'xe]_{\rho_{A,A'}}, [a]_{\rho_{A,A}}\right) = \left([a']_{\rho_{A',A'}}, [x]_{\rho_{A,A'}}, [a]_{\rho_{A,A}}\right)$.

To show that F is faithful, suppose that $\left([e', a', e']_{\tau_{e',e'}}, [e', x, e]_{\tau_{e,e'}}, [e, a, e]_{\tau_{e,e}}\right)$ and $\left([e', a', e']_{\tau_{e',e'}}, [e', x', e]_{\tau_{e,e'}}, [e, a, e]_{\tau_{e,e}}\right)$ are two morphisms in $\overline{\widetilde{\overline{\mathcal{A}}}}$ that map to the same morphism $\left([a']_{\rho_{A',A'}}, [x]_{\rho_{A,A'}}, [a]_{\rho_{A,A}}\right) = \left([a']_{\rho_{A',A'}}, [x']_{\rho_{A,A'}}, [a]_{\rho_{A,A}}\right)$ in $\overline{\widetilde{\mathcal{A}}}$. Then $x\, \rho_{A,A'}\, x'$ and hence $(e', e'xe, e)\, \tau_{e,e'}\, (e', e'x'e, e)$. But $e'xe = x$ and $e'x'e = x'$, so $[e', x, e]_{\tau_{e,e'}} = [e', x', e]_{\tau_{e,e'}}$. ∎

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